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Silvia M. Bianchi, Graciela L. Nasini & Paola B. Tolomei

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The minor inequalities in the description of the set covering polyhedron of circulant matrices

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Abstract In this work we give a complete description of the set covering polyhedron of circulant matrices C_{sk}^{k} with $s = 2, 3$ and $k \geq 3$ by linear inequalities. In particular, we prove that every non boolean facet defining inequality is associated with a circulant minor of the matrix. We also give a polynomial time separation algorithm for inequalities involved in the description.

Keywords Polyhedral combinatorics · Set covering · Circulant matrices

1 Introduction

The *weighted set covering problem* can be stated as

$$\begin{aligned} \min c^T x \\ Ax \geq \mathbf{1} \\ x \in 0, 1^n \end{aligned} \tag{SC}$$

S. M. Bianchi (✉) · G. L. Nasini · P. B. Tolomei
Departamento de Matemática, Facultad de Ciencias Exactas, Ingeniería y Agrimensura,
Universidad Nacional de Rosario, 2000 Rosario, Santa Fe, Argentina
e-mail: sbianchi@fceia.unr.edu.ar

G. L. Nasini · P. B. Tolomei
CONICET, Buenos Aires, Argentina
e-mail: nasini@fceia.unr.edu.ar

P. B. Tolomei
e-mail: ptolomei@fceia.unr.edu.ar

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where A is an $m \times n$ matrix with 0, 1 entries, c is an n-vector and $\mathbf{1}$ is the m-vector of all ones.

These types of problems are relevant in practice, but hard to solve in general. One often successful way to tackle such problems is the polyhedral approach involving the solution space of the problem (Sassano 1989).

The *set covering polyhedron* (SCP), denoted by $Q(A)$, is the convex hull of integer points in $Q_R(A) = \{x \in [0, 1]^n : Ax \geq \mathbf{1}\}$.

If for some matrix A it holds that $Q_R(A) = Q(A)$, the matrix is called *ideal* and this would enable us to solve SC as a linear program using the constraints $x \geq \mathbf{0}$ instead of the integrality requirements. However, when A is nonideal, finding a description for $Q(A)$ in terms of linear restrictions is, in general, as hard as solving SC.

Cornuéjols and Novick (1994) studied the SCP on a particular class of matrices, called *circulant matrices* and denoted as C_n^k with $1 \leq k \leq n-1$. They identified all the ideal circulant matrices which are C_6^3, C_9^3, C_8^4 and C_n^2, for even $n \geq 4$. They also provide sufficient conditions a given submatrix must satisfy in order to be a *circulant minor*. Circulant matrices and circulant minors will be formally defined in the next section.

Using these results, Argiroffo and Bianchi (2009) obtained a family of facets of $Q(C_n^k)$ associated with some particular circulant minors. Previously, Bouchakour et al. (2008), when working on the dominating set polyhedron of cycles, they obtained the complete description of $Q(C_n^3)$ for every $n \geq 5$. Interestingly, all the non boolean non rank constraints involved in this description belong to this family of inequalities associated with circulant minors.

Later, Aguilera (2010) completely identified all circulant minors that a circulant matrix may have. This result allowed us to obtain in Bianchi et al. (2009) a wider class of valid inequalities associated with circulant minors, which we call *minor inequalities*.

In this paper, we present two new families of circulant matrices for which the SCP can be described by boolean facets and minor inequalities. We also give a polynomial time separation algorithm for these inequalities.

A preliminary version of this work appeared without proofs in Bianchi et al. (2010).

2 Notations, definitions and preliminary results

Given a 0, 1 matrix A, we say that a row v of A is a *dominating row* if $v \geq u$ for some u row of A, $u \neq v$. In this work, every matrix has 0, 1 entries, no zero columns and no dominating rows.

Also, every time we state $S \subseteq \mathbb{Z}_n$ for some $n \in \mathbb{N}$, we consider $S \subseteq \{0, \ldots, n-1\}$ and the addition between the elements of S is taken modulo n. Rows and columns of an $m \times n$ matrix A are indexed by $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Two matrices A and A' are *isomorphic* if A' can be obtained from A by permutation of rows and columns.

Given $N \subset \mathbb{Z}_n$, the *minor of A obtained by contraction of N* and denoted by A/N, is the submatrix of A that results after removing all columns indexed in N and all the dominating rows that may occur.

In this work, when we refer to *a minor* of A we always consider a minor obtained by contraction.

Considering the one-to-one correspondence between a vector $x \in \{0,1\}^n$ and the subset $S_x \subseteq \mathbb{Z}_n$ whose characteristic vector is x itself, we agree to abuse of notation by writing x instead of S_x.

A *cover* of a matrix A is a vector $x \in \{0,1\}^n$ such that $Ax \geq \mathbf{1}$. A cover x of A is *minimal* if there is no other cover $\tilde{x}$ such that $\tilde{x} \subset x$. A cover x is *minimum* if $|x| = \sum_{i \in \mathbb{Z}_n} x_i$ is minimum and in this case $|x|$ is called the *covering number* of the matrix A, denoted by $\tau(A)$.

Since every cover of a minor of A is a cover of A, $\tau(A/N) \geq \tau(A)$ for all $N \subset \mathbb{Z}_n$.

We denote by $(a,b)_n$ the $\mathbb{Z}_n$-cyclic open interval of points strictly between a and b and analogous meanings for $[a,b)_n$, $(a,b]_n$ and $[a,b]_n$.

Given n and k with $1 \leq k \leq n-1$, the *circulant* matrix C_n^k is the square matrix whose ith row is the incidence vector of $C^i = [i, i+k)_n$.

It is not hard to see that, for every $i \in \mathbb{Z}_n$,

$$x^i = \left\{ i + hk : 0 \leq h \leq \left\lceil \frac{n}{k} \right\rceil - 1 \right\} \subset \mathbb{Z}_n$$

is a cover of C_n^k of size $\left\lceil \frac{n}{k} \right\rceil$. It is also clear that $\tau(C_n^k) \geq \left\lceil \frac{n}{k} \right\rceil$ and then $\tau(C_n^k) = \left\lceil \frac{n}{k} \right\rceil$. Let us also observe that for every minimal cover x of C_n^k and any $i \in \mathbb{Z}_n$, $|x \cap C^i| \leq 2$.

We say that a minor of C_n^k is a *circulant minor* if it is isomorphic to a circulant matrix. In Cornuéjols and Novick (1994), the authors give sufficient conditions for a subset $N \subset \mathbb{Z}_n$ to ensure that C_n^k/N is a circulant minor of C_n^k. These conditions are obtained in terms of simple dicycles in a particular digraph.

Indeed, given C_n^k, the digraph $G(C_n^k)$ has vertex set $\mathbb{Z}_n$ and (i,j) is an arc of $G(C_n^k)$ if $j \in \{i+k, i+k+1\}$. We say that an arc $(i, i+k)$ has length k and an arc $(i, i+k+1)$ has length $k+1$.

If D is a simple dicycle in $G(C_n^k)$, and n_2 and n_3 denote the number of arcs of length k and $k+1$ respectively, $kn_2 + (k+1)n_3 = n_1 n$ for some unique positive integer n_1. We say that n_1, n_2 and n_3 are the *parameters associated with* the dicycle D.

In Aguilera (2008) it is proved that the existence of nonnegative integers n_1, n_2 and n_3 satisfying the conditions $n_1 n = kn_2 + (k+1)n_3$ and $\gcd(n_1, n_2, n_3) = 1$ are also sufficient for the existence of a simple dicycle in $G(C_n^k)$ with n_2 arcs of length k and n_3 arcs of length $k+1$. Moreover, the same author completely characterized in Aguilera (2010) subsets N of $\mathbb{Z}_n$ for which C_n^k/N is a circulant minor in terms of dicycles in the digraph $G(C_n^k)$. We rewrite Theorem 3.10 of Aguilera (2010) in order to suit the current notation in the following way:

Theorem 1 *Let n, k be integers verifying $2 \leq k \leq n-1$ and let $N \subset \mathbb{Z}_n$ such that $1 \leq |N| \leq n-2$. Then, the following are equivalent:*

1. *C_n^k/N is isomorphic to $C_{n'}^{k'}$.*
2. *N induces in $G(C_n^k)$ $d \geq 1$ disjoint simple dicycles $D_0, \ldots, D_{d-1}$, each of them having the same parameters n_1, n_2 and n_3 and such that $|N| = d(n_2+n_3)$, $n' = n - d(n_2+n_3) \geq 1$ and $k' = k - dn_1 \geq 1$.*

Thus, whenever we refer to a circulant minor of C_n^k with parameters d, n_1, n_2 and n_3, we mean the non negative integers whose existence is guaranteed by the previous theorem. In addition, for each $j \in \mathbb{Z}_d$, N^j refers to the subset of $\mathbb{Z}_n$ inducing the simple dicycle D^j in $G(C_n^k)$, $W^j = \{i \in N^j : i-(k+1) \in N^j\}$ and $W = \cup_{j\in\mathbb{Z}_d} W^j$. Then, for all $j \in \mathbb{Z}_d$, $|W^j| = n_3$ and $|N^j| = n_2 + n_3$.

In Bianchi et al. (2012) it was proved that given $W \subset \mathbb{Z}_n$ corresponding to a circulant minor of C_n^k, we can rebuilt N such that C_n^k/N is such a minor. So, in what follows, we usually refer to a circulant minor *defined by* W. We also say that W *defines* a circulant minor.

Circulant minors, or equivalently subsets $W \subset \mathbb{Z}_n$ inducing them, play an important role in the description of the set covering polytope of circulant matrices.

It is known that $Q(C_n^k)$ is a full dimensional polyhedron. Also, for every $i \in \mathbb{Z}_n$, the constraints $x_i \geq 0$, $x_i \leq 1$ and $\sum_{j\in C^i} x_j \geq 1$ are facet defining inequalities of $Q(C_n^k)$ (Balas and Ng 1989) and we call them *boolean facets*. In addition, it is known that every non boolean facet of $Q(C_n^k)$, in the form $ax \geq \alpha$ with $\alpha > 0$, has positive coefficients (Argiroffo and Bianchi 2009).

The inequality $\sum_{i=1}^n x_i \geq \left\lceil \frac{n}{k} \right\rceil$, called the *rank constraint*, is always valid for $Q(C_n^k)$ and defines a facet if and only if n is not a multiple of k (see Sassano 1989).

However, for most circulant matrices these constraints are not enough to obtain their corresponding SCP (Argiroffo and Bianchi 2009).

Actually, in Theorem 8 of Bianchi et al. (2012) we obtained a new family of non boolean and non rank facet defining inequalities of the SCP of circulant matrices associated with circulant minors.

Theorem 2 (Bianchi et al. 2012) *Let $W \subset \mathbb{Z}_n$ define a circulant minor of C_n^k isomorphic to $C_{n'}^{k'}$. Then, the inequality*

$$\sum_{i\in W} 2x_i + \sum_{i\notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{1}$$

is a valid inequality for $Q(C_n^k)$. Moreover, if $2 \leq k' \leq n'-2$, $\left\lceil \frac{n'}{k'} \right\rceil > \left\lceil \frac{n}{k} \right\rceil$ and $n' = 1 (\mathrm{mod}\, k')$, then the inequality (1) defines a facet of $Q(C_n^k)$.

From now on, we say that inequality (1) is the *minor inequality corresponding to W* or the *minor inequality corresponding to the minor defined by W*.

Observe that if $\left\lceil \frac{n'}{k'} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$, the minor inequality is dominated by the rank constraint. In addition, in Bianchi et al. (2012) it is proved that if n' is a multiple of k' then the corresponding inequality is valid for $Q_R(C_n^k)$. As our main interest are the relevant constraints in the description of $Q(C_n^k)$, we call *relevant minors* those minors isomorphic to $C_{n'}^{k'}$ with $n' \neq 0\,(\mathrm{mod}\, k')$ and $\left\lceil \frac{n'}{k'} \right\rceil > \left\lceil \frac{n}{k} \right\rceil$. Inequalities associated with relevant minors will be *relevant minor inequalities*.

In Bianchi et al. (2009) we stated the following conjecture.

Conjecture 1 (Bianchi et al. 2009) *A relevant minor inequality corresponding to a minor of C_n^k isomorphic to $C_{n'}^{k'}$ defines a facet of $Q(C_n^k)$ if and only if $n' = 1\,(\mathrm{mod}\, k')$.*

It can be seen that every non boolean and non rank facet defining inequality of $Q(C_n^3)$ obtained in Bouchakour et al. (2008) is a relevant minor inequality satisfying Conjecture 1 (see section 6 of Argiroffo and Bianchi 2009).

Our goal is to enlarge the family of circulant matrices for which the same holds. For this purpose in Sect. 3 we give necessary conditions for an inequality to be a non boolean non rank facet defining inequality of $Q(C_n^k)$. In Sect. 4, we focus on matrices of the form C_{sk}^k and show that every facet defining inequality with right hand side $s+1$ is a minor inequality satisfying Conjecture 1. Moreover, we prove that this inequalities can be separated in polytime. Finally, in Sect. 5 we prove that $Q(C_{2k}^k)$ and $Q(C_{3k}^k)$ are described in terms of boolean facets and minor inequalities with right hand side $s+1$.

3 Properties of facets of $Q(C_n^k)$

Let $ax \geq \alpha$ be a non boolean, non rank facet defining inequality of $Q(C_n^k)$ with integer coefficients. A root $\tilde{x}$ of $ax \geq \alpha$ is a cover of C_n^k satisfying $a\tilde{x} = \alpha$. Since the inequality has positive coefficients $\tilde{x}$ is a minimal cover of C_n^k.

We define $a^0 = \min\{a_i : i \in \mathbb{Z}_n\}$ and $W = \{i \in \mathbb{Z}_n : a_i > a^0\}$. Clearly, $a^0 \geq 1$ and $W \neq \mathbb{Z}_n$. Moreover, $W \neq \emptyset$ since otherwise $ax \geq \alpha$ would be dominated by the rank inequality. By denoting $\overline{W} = \{i \in \mathbb{Z}_n : i \notin W\}$, $ax \geq \alpha$ can be written as

$$\sum_{i \in W} a_i x_i + a^0 \sum_{i \in \overline{W}} x_i \geq \alpha. \tag{2}$$

Observe that, for every cover $\tilde{x}$ of C_n^k,

$$a\tilde{x} = \sum_{i \in \tilde{x} \cap W} a_i + a^0 |\tilde{x} \cap \overline{W}| \geq a^0 |\tilde{x}|.$$

Since (2) is not the rank inequality, it has a root $\tilde{x}$ that is not a minimum cover and then

$$\alpha = a\tilde{x} \geq a^0|\tilde{x}| \geq a^0(\tau(C_n^k) + 1) = a^0\left(\left\lceil \frac{n}{k} \right\rceil + 1\right),$$

and every minimum cover x must satisfy $x \cap W \neq \emptyset$ since otherwise, it would violate (2).

For the sequel it is convenient to make the next observation:

Remark 1 Every non boolean non rank facet defining inequality of $Q(C_n^k)$ is of the form (2) with $a^0 \geq 1$, $\emptyset \subsetneq W \subsetneq \mathbb{Z}_n$, $a_i \geq a^0 + 1$ for all $i \in W$ and $\alpha \geq a^0(\lceil \frac{n}{k} \rceil + 1)$. Moreover, $|x \cap W| \geq 1$ for every minimum cover x of C_n^k.

We have the following results:

Lemma 1 *Let* (2) *be a non boolean non rank facet defining inequality of* $Q(C_n^k)$. *Then:*

1. For every $i \in \mathbb{Z}_n$ there exists

(a) a root $\tilde{x}$ such that $i \in \tilde{x}$,
(b) a root $\tilde{x}$ such that $i \notin \tilde{x}$,
(c) a root $\tilde{x}$ such that $|\tilde{x} \cap C^i| = 2$.

2. Let $i \in W$ and $\tilde{x}$ a root such that $i \in \tilde{x}$.

(a) If there exists $j \neq i$ such that $j \in \tilde{x} \cap C^{i-k+1}$, then $[i, j+k]_n \subset W$.
(b) If there exists $j \neq i$ such that $j \in \tilde{x} \cap C^i$, then $[j-k, i]_n \subset W$.

Proof Let $i \in \mathbb{Z}_n$.

If for every root $\tilde{x}$ of (2) $i \notin \tilde{x}$ ($i \in \tilde{x}$), then every root of (2) is also a root of the boolean facet defined by $x_i \geq 0$ ($x_i \leq 1$), a contradiction. Then, items 1.(a) and (b) hold.

Let us observe that every root $\tilde{x}$ of (2) satisfying $|\tilde{x} \cap C^i| = 1$ is also a root of the boolean facet defined by the inequality $\sum_{j \in C^i} x_j \geq 1$.

Then, we conclude that there exists a root $\tilde{x}$ such that $|\tilde{x} \cap C^i| \geq 2$. Recalling that $\tilde{x}$ is a minimal cover, $|\tilde{x} \cap C^i| \leq 2$ and then item 1.(c) follows.

In order to prove item 2., let $i \in W$ (i.e., $a_i > a^0$) and $\tilde{x}$ be a root of (2) such that $i \in \tilde{x}$.

Assume $j \neq i$ such that $j \in \tilde{x} \cap C^{i-k+1}$. Observe that for any $h \in [i, j+k]_n$, $\hat{x} = \tilde{x} \setminus \{i\} \cup \{h\}$ is a cover of C_n^k satisfying (2). Then, we have

$$a\hat{x} = a\tilde{x} - a_i + a_h = \alpha - a_i + a_h \geq \alpha$$

implying $a_h \geq a_i > a^0$, i.e., $h \in W$.

Now, using similar arguments when $j \neq i$ such that $j \in \tilde{x} \cap C^i$ we obtain $a_h \geq a_i > a^0$ for $h \in [j-k, i]_n$ and the lemma follows. □

From the previous results, we obtain the following relevant properties of facet defining inequalities of $Q(C_n^k)$.

Theorem 3 *Let (2) be a non boolean non rank facet defining inequality of $Q(C_n^k)$. Then,*

1. for every $i \in \mathbb{Z}_n$, $|C^i \cap \overline{W}| \geq 2$.
2. for every $i \in W$, $a_i \leq 2a^0$.

Proof Suppose that $|C^i \cap \overline{W}| = 0$ for some $i \in \mathbb{Z}_n$. Let consider a root x of (2) such that $|x \cap C^i| = 2$ that exists according to Lemma 1 item 1.(c). From Lemma 1 item 2.(a) we get $i+k \in W$ and then $|C^{i+1} \cap \overline{W}| = 0$. Iteratively using the same argument, we arrive to $W = \mathbb{Z}_n$, a contradiction. Thus, we have proved that $|C^i \cap \overline{W}| \geq 1$ for all $i \in \mathbb{Z}_n$.

Now suppose that $C^i \cap \overline{W} = \{h\}$ for some $i \in \mathbb{Z}_n$. Let x be again a root of (2) such that $x \cap C^i = \{s, s'\}$ with $s \in [i, s')_n$. If $h \in [i, s]_n$ then $s' \in W$ and by Lemma 1 item 2.(a), $C^{s+1} \subset W$. Similarly, if $h \in [s', i+k-1]_n$, $s \in W$ and $C^{s'-k} \subset W$. In both cases, we obtain a contradiction with $|C^i \cap \overline{W}| \geq 1$ for all $i \in \mathbb{Z}_n$ as we have already noted. We conclude that $h \in (s, s')_n$.

In particular, we have proved that if $C^i \cap \overline{W} = \{h\}$ for some $i \in \mathbb{Z}_n$, then $i \neq h$.

Moreover, since $h \in (s, s')_n$, $s' \in W$. Applying Lemma 1 item 2.(a) we have that $[s', s+k]_n \subset W$ and then $C^{i+1} \cap \overline{W} = \{h\}$. Iteratively using the same argument, we arrive to $C^h \cap \overline{W} = \{h\}$ contradicting the previous observation.

Then, $|C^i \cap \overline{W}| \geq 2$ for all $i \in \mathbb{Z}_n$.

In order to prove item 2., let $i \in W$ and $\tilde{x}$ be a root of (2) such that $i \in \tilde{x}$. By item 1., $C^{i-k} \cap \overline{W} \neq \emptyset$ and there is $h \in C^{i-k} \cap \overline{W}$ such that $(h, i]_n \subset W$. Let $\ell \in C^h \cap \overline{W}$ then $i \in (h, \ell)_n$ and hence $\hat{x} = \tilde{x} \setminus \{i\} \cup \{h, \ell\}$ is a cover of C_n^k. Therefore, $a\hat{x} = a\tilde{x} - a_i + 2a^0 = \alpha - a_i + 2a^0 \geq \alpha$ or equivalently, $a_i \leq 2a^0$. $\square$

4 Minor inequalities of $Q(C_{sk}^k)$

In this section we consider the circulant matrices C_n^k for which n is a multiple of k. Firstly, we will see that almost every circulant matrix can be thought as a minor of such a matrix.

In Theorem 2.10 of Aguilera (2010) it was proved that a matrix C_n^k has a minor isomorphic to $C_{n'}^{k'}$ if and only if

$$\frac{k'}{k} \leq \frac{n'}{n} \leq \frac{k'+1}{k+1}. \tag{3}$$

As a consequence we have:

Lemma 2 *Let $n' = hk' + r$ with $1 \leq r \leq k' - 1$. Then, there exist s and k such that C_{sk}^k has a minor isomorphic to $C_{n'}^{k'}$ if and only if $r \leq h - 1$.*

Proof Let s and k be such that C_{sk}^k has a minor isomorphic to $C_{hk'+r}^{k'}$. Then, according to (3), we have $\frac{k'}{k} \leq \frac{hk'+r}{sk}$ and $s \leq h$.

In addition, $\frac{hk'+r}{sk} \leq \frac{k'+1}{k+1}$, is equivalent to

$$n' = hk' + r \leq [s(k'+1) - (hk'+r)]k$$

Since $n' \geq 1$, $s(k'+1) - (hk'+r) > 0$ and then, $s > \frac{hk'+r}{k'+1}$.

In summary, we have that

$$\frac{hk'+r}{k'+1} < s \leq h$$

and then $r \leq h - 1$.

Conversely, if $r \leq h - 1$, it is easy to see that by taking $s = h$ and $k \geq \frac{hk'+r}{h-r}$ the condition (3) holds and by Theorem 2.10 of Aguilera (2010), C_{sk}^k has a minor isomorphic to $C_{sk'+r}^{k'}$. $\square$

Hence, for a fixed k', except for a finite number of values of n', matrix $C_{n'}^{k'}$ is isomorphic to a minor of some matrix C_{sk}^{k}.

Let us start the study of polyhedra $Q(C_{sk}^{k})$, for $s \geq 2$. Observe that matrices C_{3s}^{3} have already been studied in Bouchakour et al. (2008). Moreover, for $k = 4$ we take $s \geq 3$ since $Q(C_{8}^{4})$ is described by boolean inequalities (see Cornuéjols and Novick 1994).

Remind that if $\tilde{x}$ is a minimum cover of C_{sk}^{k}, $\tilde{x} = x^{j} = \{j + rk, r \in \mathbb{Z}_s\}$ for some $j \in \mathbb{Z}_{sk}$. Therefore, $x^{i} = x^{i+k}$ for any $i \in \mathbb{Z}_{sk}$ and there are exactly k minimum covers (x^{i} with $i \in \mathbb{Z}_k$) defining a partition of $\mathbb{Z}_{sk}$.

From now on, we consider facet defining inequalities of $Q(C_{sk}^{k})$ in the form

$$\sum_{i \in W} a_i x_i + a^0 \sum_{i \in \overline{W}} x_i \geq (s+1)a^0 \tag{4}$$

for some $a^0 \geq 1$, $\emptyset \subsetneq W \subsetneq \mathbb{Z}_{sk}$ and $2a^0 \geq a_i \geq a^0 + 1$ for every $i \in W$. Recall that, from the results in Theorem 3 we also know that $|\overline{W} \cap C^{i}| \geq 2$.

Inequalities in the form (4) will be referred as $(s+1)$-*inequalities* of $Q(C_{sk}^{k})$.

We will prove that every facet defining $(s+1)$-inequality of $Q(C_{sk}^{k})$ is a minor inequality.

Observe that every root $\tilde{x}$ of (4) has cardinality s or $s+1$. Moreover, $\tilde{x} \cap W \neq \emptyset$ if and only if $\tilde{x}$ is a minimum cover of C_{sk}^{k}. Thus, if $i \in \tilde{x} \cap W$, $\tilde{x} = x^{i}$.

Hence,

Lemma 3 *Let* (4) *be a facet defining* $(s+1)$-*inequality of* $Q(C_{sk}^{k})$. *Then, for every* $i \in \mathbb{Z}_{sk}$, *the following hold:*

1. $|x^{i} \cap W| = 1$. *Moreover,* $|W| = k$.
2. *If* $i \in W$, *then* $a_i = 2a^0$.

Proof Let $i \in \mathbb{Z}_{sk}$. From the previous observations $|x^{i} \cap W| \geq 1$.

Let assume that there exist $j \neq \ell$ such that $\{j, \ell\} \subset x^{i} \cap W$. Then $x^{i} = x^{j} = x^{\ell}$. So, given a root x of (4), x contains j if and only if it also contains ℓ. Therefore every root of the inequality (4) lies in the hyperplane $x_\ell - x_j = 0$, a contradiction.

Then $|x^{i} \cap W| = 1$ and, recalling that $\{x^{i} : i \in \mathbb{Z}_k\}$ defines a partition of $\mathbb{Z}_{sk}$, $|W| = k$.

Hence, if $i \in W$, $x^{i} \cap W = \{i\}$ and x^{i} is a root of (4). As a consequence, we have:

$$ax^{i} = a_i + (s-1)a^0 \geq (s+1)a^0$$

and then, $a_i \geq 2a^0$. By Theorem 3 item 2., we have that $a_i = 2a^0$. □

We have proved that every facet defining $(s+1)$-inequality of $Q(C_{sk}^{k})$ can be written as

$$2\sum_{i \in W} x_i + \sum_{i \in \overline{W}} x_i \geq s+1 \tag{5}$$

where W verifies $|W \cap x^{i}| = 1$ and $|\overline{W} \cap C^{i}| \geq 2$, for every $i \in \mathbb{Z}_{sk}$.

The next theorem proves that the class of minor inequalities of $Q(C^k_{sk})$ includes facet defining $(s+1)$-inequalities.

Theorem 4 *Let $W \subset \mathbb{Z}_{sk}$ such that $|W \cap x^j| = 1$ for every $j \in \mathbb{Z}_{sk}$. Then, the inequality (5) is valid for $Q(C^k_{sk})$. Moreover, if $|\overline{W} \cap C^j| \geq 1$ for every $j \in \mathbb{Z}_{sk}$, it is a minor inequality.*

Proof Observe that, if $|\overline{W} \cap C^i| = 0$ for some $i \in \mathbb{Z}_{sk}$, then inequality (5) is the sum of the rank constraint and the boolean facet $\sum_{j \in C^i} x_j \geq 1$.

For the other cases, we have to prove that W defines a circulant minor of C^k_{sk}. In particular, we will prove that W corresponds to a simple dicycle D in $G(C^k_{sk})$ which verifies item 2. of Theorem 1.

In order to obtain D we proceed in the following way. As $|W \cap x^j| = 1$ for every $j \in \mathbb{Z}_k$, let $i_j \in \mathbb{Z}_{sk}$ such that $x^j \cap W = \{i_j\}$ and let t_j such that $0 \leq t_j \leq s-1$ and $i_j = j + t_j k$.

For every $j \in \mathbb{Z}_k$, let define the $i_j i_{j+1}$-dipath P_j in the digraph $G(C^k_{sk})$ induced by the set $V_j \cup \{i_{j+1}\}$, where

$$V_j = \{i_j + rk : 0 \leq r \leq n_2^j\} \subset x^j,$$

and n_2^j is defined according to the following cases.

If $j \neq k-1$ then

$$n_2^j = \begin{cases} (s-1) - (t_j - t_{j+1}) & \text{if } \; t_j - t_{j+1} \geq 0 \\ (t_{j+1} - t_j) - 1 & \text{otherwise} \end{cases}$$

else

$$n_2^{k-1} = \begin{cases} t_0 - t_{k-1} - 2 & \text{if } \; t_{k-1} - t_0 \leq -2 \\ (s-2) - (t_{k-1} - t_0) & \text{if } \; -1 \leq t_{k-1} - t_0 \leq s-2 \\ s-1 & \text{if } \; t_{k-1} - t_0 = s-1. \end{cases}$$

Observe that subsets V_j with $j \in \mathbb{Z}_k$ are mutually disjoint and for any $j \in \mathbb{Z}_k$, the $i_j i_{j+1}$-dipath P_j induced by $V_j \cup \{i_{j+1}\}$ in the digraph $G(C^k_{sk})$ has exactly n_2^j arcs of length k and one arc of length $k+1$. Moreover, $\bigcup_{j \in \mathbb{Z}_k} V_j$ induces a simple dicycle D in $G(C^k_{sk})$ (see Fig. 1 as example).

Let us analyze the parameters n_1, n_2 and n_3 associated with D such that $n_1(sk) = n_2 k + n_3(k+1)$.

Clearly, $n_2 = \sum_{j \in \mathbb{Z}_k} n_2^j$, $n_3 = k$ and then $n_1 s = n_2 + k + 1$.

In order to verify item 2. of Theorem 1, it only remains to prove that $n_1 \leq k-1$. We claim that the latter is equivalent to prove that $n_2 \leq k(s-1) - 2$.

Actually, since $n_1 s = n_2 + k + 1$ we have that $n_2 \leq k(s-1) - 2$ implies $n_1 s \leq k(s-1) - 2 + k + 1 = ks - 1$ and therefore $n_1 \leq k-1$. Conversely, if $n_1 \leq k-1$ then $n_1 s \leq (k-1)s = ks - s \leq ks - 1$. Again, using the fact that $n_1 s = n_2 + k + 1$ we have $n_2 + k + 1 \leq ks - 1$ and hence $n_2 \leq k(s-1) - 2$.

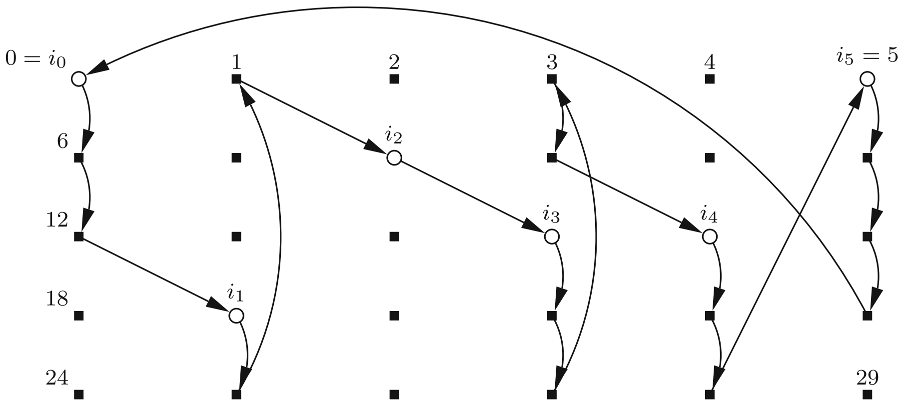

Fig. 1 The dicycle D associated with $W = \{0, 5, 8, 15, 16, 19\} \subset \mathbb{Z}_{sk}$ with $s = 5$ and $k = 6$

Let us now prove that $n_2 \leq k(s-1) - 2$.

From the definition, $n_2^j \leq s - 1$ for every $j \in \mathbb{Z}_k$.

Suppose that $n_2^j = s - 1$ for every $j \in \mathbb{Z}_k$, $j \neq \ell$.

If $\ell = k - 1$, then $t_j = t_{j+1}$ for all $0 \leq j \leq k - 2$ and then $t_0 = t_j$ for all $1 \leq j \leq k - 1$. In this case, $C^{i_0} = W$ contradicting the fact that $|C^{i_0} \cap \overline{W}| \geq 1$.

Now, if $\ell \neq k - 1$ then $t_j = t_\ell$ for all $0 \leq j \leq \ell$ and $t_j = t_{\ell+1}$ for all $\ell + 1 \leq j \leq k - 1$. Since $n_2^{k-1} = s - 1$ then either $t_{k-1} = t_0 - 1$ and $t_0 \neq 0$ or $t_{k-1} = s - 1$ and $t_0 = 0$. In any case $C^{i_\ell} = W$ again contradicting $|C^{i_\ell} \cap \overline{W}| \geq 1$.

As a consequence, there must be at least two different values of j with $n_2^j \leq s - 2$ and the theorem follows. □

Corollary 1 *Every facet defining* $(s + 1)$*-inequality of* $Q(C_{sk}^k)$ *is a relevant minor inequality verifying Conjecture* 1.

Proof From the proof of the previous theorem every facet defining $(s + 1)$-inequality of $Q(C_{sk}^k)$ is associated with a relevant minor $C_{n'}^{k'}$ with $n' = sk - (n_2 + k)$, $k' = k - n_1$ and $n_1 s = n_2 + k + 1$.

Then, $n' = sk - (n_1 s - 1) = s(k - n_1) + 1 = sk' + 1$ and Conjecture 1 holds. □

In the remainder of this section we will prove that the $(s+1)$-inequalities in the form (5) with W such that $|x^i \cap W| = 1$ for every $i \in \mathbb{Z}_k$ can be separated in polynomial time. The strategy is similar to the one in Bouchakour et al. (2008), i.e., we reduce the separation problem to a shortest path problem in an acyclic digraph. Also, we use ideas in Bouchakour et al. (2008).

Any of these inequalities can be written as

$$\sum_{i \in W} x_i + \sum_{i \in \mathbb{Z}_{sk}} x_i \geq s + 1$$

or equivalently

$$\sum_{i \in W} x_i \geq s + 1 - \sum_{i \in \mathbb{Z}_{sk}} x_i. \tag{6}$$

Defining $L(x) := s + 1 - \sum_{i \in \mathbb{Z}_{sk}} x_i$, the separation problem for these inequalities can be stated as follows: given $\hat{x} \in \mathbb{R}^n$, decide if there exists $W \subset \mathbb{Z}_{sk}$ with $|x^i \cap W| = 1$ for all $i \in \mathbb{Z}_k$ such that

$$\sum_{i \in W} \hat{x}_i < L(\hat{x}).$$

We will reduce this problem to a shortest path problem in an acyclic digraph.
For this purpose let us define the digraph $D(C^k_{sk}) = (V, A)$, where

$$V = \left(\bigcup_{i \in \mathbb{Z}_k} V_i \right) \cup \{r, t\}$$

with $V_i = x^i = \{i, i + k, \ldots, i + (s - 1)k\}$ for $i \in \mathbb{Z}_k$ and

$$A = \left(\bigcup_{i \in \mathbb{Z}_{k-1}} A_i \right) \cup A_r \cup A_t$$

with $A_i = \{(l, m) : l \in V_i,\ m \in V_{i+1}\}$ for $i \in \mathbb{Z}_{k-1}$, $A_r = \{(r, m) : m \in V_0\}$ and $A_t = \{(l, t) : l \in V_{k-1}\}$. For illustration see Fig. 2.

Observe that there is a one-to-one correspondence between rt-paths in $D(C^k_{sk})$ and subsets $W \subset \mathbb{Z}_{sk}$ with $|x^i \cap W| = 1$ for every $i \in \mathbb{Z}_k$.

Indeed, let $W \subset \mathbb{Z}_{sk}$ such that $x^j \cap W = \{i_j\}$ for every $j \in \mathbb{Z}_k$. Clearly, $\{r, i_0, i_1, \ldots, i_{k-1}, t\}$ induces an rt-path in $D(C^k_{sk})$. Conversely, if P is an rt-path in $D(C^k_{sk})$, by construction, $|V(P) \cap V_i| = 1$ for all $i \in \mathbb{Z}_k$, and

$$W = \bigcup_{i \in \mathbb{Z}_k} (V(P) \cap V_i)$$

verifies $|W \cap x^i| = 1$ for every $i \in \mathbb{Z}_k$.

Then, we have the following result.

Theorem 5 *Given $s \geq 2$ and $k \geq 4$, the inequalities (5) with $|x^i \cap W| = 1$ for every $i \in \mathbb{Z}_k$, can be separated in polynomial time.*

Proof Let $\hat{x} \in \mathbb{R}^n$ and $D(C^k_{sk})$ be the digraph previously defined. For each arc $(i, j) \in A \setminus A_t$, assign length $\hat{x}_j$ otherwise if $(i, j) \in A_t$ assign length zero (see Fig. 2).

In this way, given $W \subset \mathbb{Z}_{sk}$ with $|W \cap x^i| = 1$, the length of the rt-path corresponding to W is $\sum_{i \in W} \hat{x}_i$.

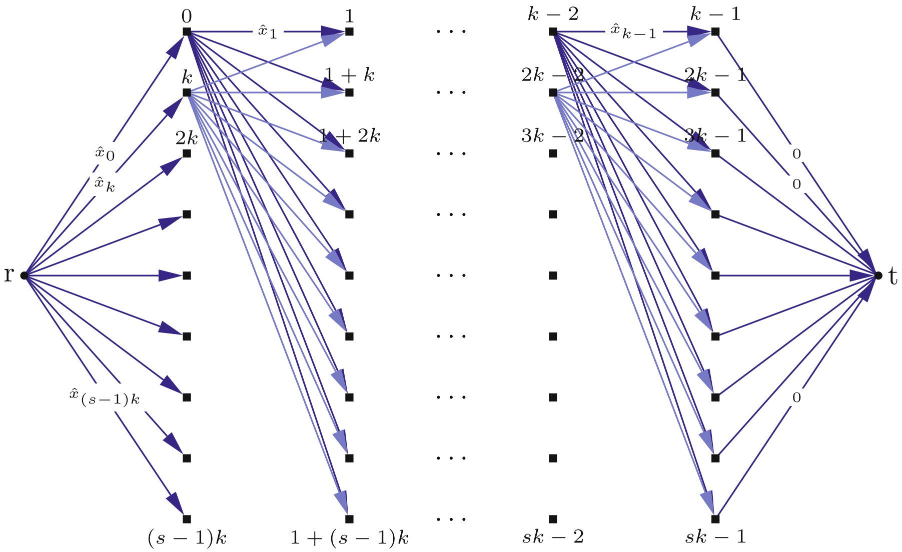

Fig. 2 The digraph $D(C^k_{sk})$

Then, the separation problem can be reduced to decide if there exists an rt-path in $D(C^k_{sk})$ with length less than $L(\hat{x}) = s+1-\sum_{i\in\mathbb{Z}_{sk}} \hat{x}_i$ or, equivalently, if the shortest path in $D(C^k_{sk})$ has length less than $L(\hat{x})$.

Since $D(C^k_{sk})$ is acyclic, computing the shortest path can be done in polynomial time using for instance Bellman algorithm (Bellman 1958). $\square$

Let us observe that, if $\hat{x} \in Q_R(C^k_{sk})$, the separation problem for inequalities of the form (5) such that $|x^i \cap W| = 1$ is equivalent to the separation problem for relevant minor $(s+1)$-inequalities of $Q(C^k_{sk})$.

5 The set covering polyhedron of C^k_{2k} and C^k_{3k}

In this section we prove that minor inequalities together with the boolean facets completely describe the set covering polyhedron of C^k_{2k} and C^k_{3k} for $k \geq 2$. Actually, if $k = 2$ the matrices are ideal and if $k = 3$ the result follows from Theorem 3.11 in Bouchakour et al. (2008) and Theorem 4.

The key is to prove that for every non boolean non rank facet defining inequality there exists a cover $\tilde{x}$ of C^k_{sk}, with $|\tilde{x}| = s+1$ and $\tilde{x} \cap W = \emptyset$. Provided such a cover exists, then,

$$\alpha \leq \sum_{i\in W} a_i \tilde{x}_i + \sum_{i\in \overline{W}} a^0 \tilde{x}_i = a^0|\tilde{x}| = a^0(s+1).$$

Hence, $\alpha = (s+1)a^0$ and (2) is a facet defining $(s+1)$-inequality. By Theorem 4, it is a minor inequality. Since it is a facet defining inequality, it is a relevant minor inequality.

Recall that if (2) is a facet defining inequality of $Q(C_{sk}^k)$ for some $W \subset \mathbb{Z}_{sk}$ and $a^0 \geq 1$, $|x^i \cap W| \geq 1$ for all $i \in \mathbb{Z}_k$ and, from Theorem 3 item 1, $|C^i \cap \overline{W}| \geq 2$, for all $i \in \mathbb{Z}_k$.

Let us start with the case $s = 2$.

Theorem 6 *For every $k \geq 3$, every non boolean facet defining inequality of $Q(C_{2k}^k)$ is a relevant minor inequality.*

Proof Let $W \subset \mathbb{Z}_{2k}$ such that (2) is a facet defining inequality of $Q(C_{2k}^k)$. As we have already observed, it is enough to prove that there is a cover $\tilde{x}$ such that $|\tilde{x}| = 3$ and $\tilde{x} \subset \overline{W}$. W.l.o.g., we can assume that $0 \in \overline{W}$ and since $|x^0 \cap W| \geq 1$ we have that $k \in W$.

Considering that $|C^0 \cap \overline{W}| \geq 2$, we can define $t = \max\{s : 1 \leq s \leq k-1\,,\ s \in \overline{W}\}$ that makes $(t, k]_{2k} \subset W$. Since $|W \cap x^t| \geq 1$ and $t \in \overline{W}$, $t + k \in W$.

Since $|C^t \cap \overline{W}| \geq 2$ and $(t, k]_{2k} \subset W$, there exists $t' \in \overline{W} \cap (k, t+k)_{2k}$. We have that $\tilde{x} = \{0, t, t'\} \subset \overline{W}$ is a cover of C_{2k}^k and the proof is complete. □

In what follows we prove that for every inequality (2) defining a facet of $Q(C_{3k}^k)$ there exists a cover $\tilde{x}$ of C_{3k}^k with $|\tilde{x}| = 4$ and $\tilde{x} \cap W = \emptyset$. For this purpose, let us call $\mathcal{W}$ the family of $W \subset \mathbb{Z}_{3k}$ such that, for every $i \in \mathbb{Z}_{3k}$, $|x^i \cap W| \geq 1$ and $|C^i \cap \overline{W}| \geq 2$. Clearly, every W associated with a non boolean facet defining inequality of $Q(C_{3k}^k)$ is in $\mathcal{W}$.

For every $W \in \mathcal{W}$ and $i \in \overline{W}$ we define

$$\omega(i) := \min\{t \geq 0 : i + k + t \in \overline{W}\}.$$

Since $|C^{i+k} \cap \overline{W}| \geq 2$, $\omega(i) \leq k - 2$ for all $i \in \overline{W}$.

First we prove the following:

Lemma 4 *Let $W \in \mathcal{W}$ and $i \in \overline{W}$ such that $\overline{W} \cap [i + 2k, i + 2k + \omega(i)]_{3k} \neq \emptyset$. Then there exists a cover $\tilde{x}$ of C_{3k}^k with $|\tilde{x}| = 4$ and $\tilde{x} \cap W = \emptyset$.*

Proof Let us first observe that $\omega(i) \geq 1$. Indeed, if $\omega(i) = 0$, $i + k \in \overline{W}$ and $[i + 2k, i + 2k + \omega(i)]_n = \{i + 2k\}$. Then, by hypothesis, $i + 2k \in \overline{W}$ contradicting the fact that $|x^i \cap W| \geq 1$.

Let $\ell \in \overline{W} \cap [i + 2k, i + 2k + \omega(i)]_{3k}$. Since $|C^{i+w(i)} \cap \overline{W}| \geq 2$ and $[i + k, i + k + \omega(i))_{3k} \subset W$, there exists $t \in \overline{W} \cap [i + w(i), i + k)_{3k}$. Then, $\tilde{x} = \{i, t, i + k + \omega(i), \ell\} \subset \overline{W}$ is a cover of C_{3k}^k. □

As an immediate consequence we get that if $W \in \mathcal{W}$ and $|W \cap x^j| = 1$ for some $j \in \mathbb{Z}_{3k}$ then C_{3k}^k admits a cover $\tilde{x} \subset \overline{W}$ with $|\tilde{x}| = 4$. Indeed, w.l.o.g., we can assume that $j \in \overline{W}$ and $j + k \in W$. Then, $\overline{W} \cap [j + 2k, j + 2k + \omega(j)]_{3k} \neq \emptyset$ and we can apply the above lemma.

Therefore, it only remains to consider subsets $W \in \mathcal{W}$ such that, for every $i \in \mathbb{Z}_{3k}$, $|x^i \cap W| \geq 2$ and for every $i \in \overline{W}$, $[i + 2k, i + 2k + \omega(i)]_{3k} \subset W$. W.l.o.g., we can assume that $0 \in \overline{W}$. Let us call $\mathcal{W}^*$ the family of all subsets $W \in \mathcal{W}$ satisfying these conditions.

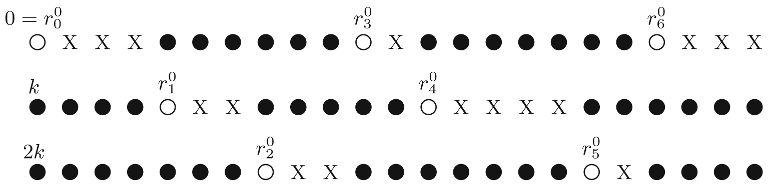

Fig. 3 A subset $\widetilde{W} \in \mathcal{W}$

Given $W \in \mathcal{W}$ and $i \in \overline{W}$ let us recursively define the sequence $r^i = \{r_t^i\}_{t=0}^{\infty} \subset \mathbb{Z}_{3k}$ as follows:

- $r_0^i = i$
- $r_t^i = r_{t-1}^i + k + \omega(r_{t-1}^i)$, for $t \geq 1$.

According to the sequence r^i, we define $p^i = \max\left\{t : \sum_{j=0}^{t-1} \omega(r_j^i) \leq k-1\right\}$.

Observe that, given $i \in \overline{W}$, by definition of $\omega(i)$, $r^i \subset \overline{W}$ and $[r_t^i + k, r_{t+1}^i)_{3k} \subset W$ for all t. Moreover, if $W \in \mathcal{W}^*$, $[r_t^i + 2k, r_t^i + 2k + \omega(r_t^i)]_{3k} = [r_t^i + 2k, r_{t+1}^i + k]_{3k} \subset W$ and, since $[r_{t+1}^i + k, r_{t+2}^i)_{3k} \subset W$, we have that $[r_t^i + 2k, r_{t+2}^i)_{3k} \subset W$.

From this observation we can conclude that for every $W \in \mathcal{W}^*$ and every $i \in \overline{W}$, $p^i \neq 0 \pmod 3$.

In Fig. 3 we sketch a subset $\widetilde{W} \in \mathcal{W}$, for $k = 23$.

Each node corresponds to an element in $\mathbb{Z}_{69}$. Black nodes and white nodes correspond to elements in and out $\widetilde{W}$, respectively. Crosses correspond to elements that may or may not belong to $\widetilde{W}$. We also show the first seven elements of the sequence r^0 that allow us to see that $p^0 = 6$, i.e., $p^0 = 0 \pmod 3$. We will show that $\widetilde{W} \notin \mathcal{W}^*$.

In fact, observe that $r_7^0 = r_2^0$ and the sequence r^0 cycles. Then, $r_0^0 \in [r_6^0 + 2k, r_7^0 + k)_{69}$ and $r_0^0 \notin \widetilde{W}$. Since for all $W \in \mathcal{W}^*$ and every t, $[r_t^i + 2k, r_{t+1}^i + k]_{3k} \subset W$, we have that $\widetilde{W} \notin \mathcal{W}^*$.

In general, given $W \in \mathcal{W}$ such that $p^i = 0 \pmod 3$ for some $i \in \overline{W}$, following the same reasoning we can arrive to $r_{p^i+1}^i = r_2^i$ and $r_0^i \in \overline{W} \cap [r_{p^i}^i + 2k, r_{p^i+1}^i + k)_{3k}$ and then $W \notin \mathcal{W}^*$. Then, we have:

Lemma 5 *For every $W \in \mathcal{W}^*$, $p^i \neq 0 \pmod 3$ for every $i \in \overline{W}$.*

Moreover, we have the following result:

Lemma 6 *Let $W \in \mathcal{W}^*$. Then, there exists $j \in \overline{W}$ such that $p^j = 1 \pmod 3$.*

Proof Let $i \in \overline{W}$ and let us call $j = r_{p^i}^i$.

According to the previous lemma, $p^i \neq 0 \pmod 3$ and then, consider $p^i = 2 \pmod 3$. W.l.o.g., we can assume that $i = 0$ (for illustration consider the example in Fig. 4).

We have that $[j + k, k)_{3k} \subset W$. In addition, we know that $[r^0 + k, r_1^0)_{3k} = [k, r_1^0)_{3k} \subset W$. Then, $[j + k, r_1^0)_{3k} \subset W$ implying that $r_1^j = r_1^0$ and $r_t^j = r_t^0$ for all $t \geq 1$. Moreover,

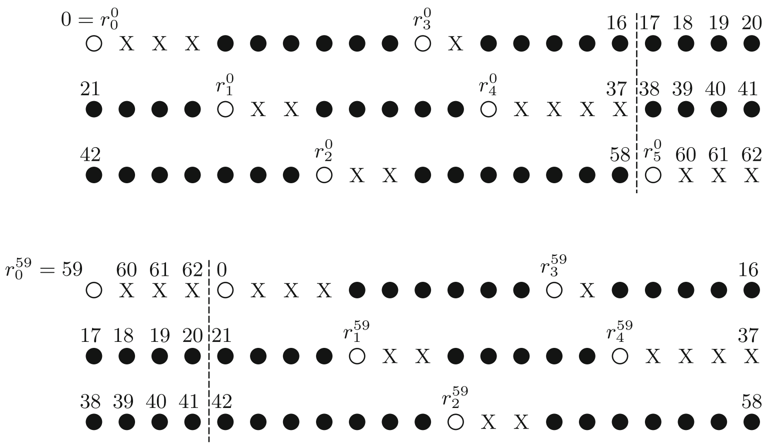

Fig. 4 $W \in \mathcal{W}^*$ with $p^0 = 5 = 2 \pmod 3$, $j = r^0_{p^0} = 59$ and $p^{59} = 4 = 1 \pmod 3$

$$\omega(r_0^j) = \omega(j) = k - \sum_{t=0}^{p^0-1} \omega(r_t^0) + \omega(r_0^0).$$

Then,

$$\sum_{t=0}^{p^0-2} \omega(r_t^j) \le k - 1 \quad \text{and} \quad \sum_{t=0}^{p^0-1} \omega(r_t^j) > k - 1.$$

Therefore, $p^j = p^0 - 1$ and $p^j = 1 \pmod 3$. □

Finally, we can prove:

Theorem 7 *Let $W \in \mathcal{W}^*$ associated with a facet defining inequality (2). Then, there exists a cover $\tilde{x}$ of C^k_{3k} with $|\tilde{x}| = 4$ and $\tilde{x} \cap W = \emptyset$.*

Proof By the previous observation, w.l.o.g., we can assume that $p^0 = 1 \pmod 3$. It is not hard to see that if $p^0 = 1$, $C^{2k} \subset W$, a contradiction. Then, $p^0 \ge 4$. We will prove that $p^0 \ne 4$.

Suppose that $p^0 = 4$ (see Fig. 5 as example).

In this case, the set $\{r_0^0, r_3^0, r_1^0, r_4^0, r_2^0\} \subset \overline{W}$ results in a cover of C^k_{3k} with cardinality five, $\alpha \le 5a^0$ and every root of (2) has cardinality four or five.

From Lemma 1 item 1.(c), there exists a root $\tilde{x}$ of (2) such that $\tilde{x} \cap C^{2k} = \{s, t\}$ with $s \in [2k, t)_{3k}$. We will prove that $\{s, t\} \subset \overline{W}$.

Observe that $C^{2k} \cap \overline{W} \subset [r_2^0, r_3^0 + 2k)_{3k}$. If $t \in W$, applying Lemma 1 item 2.(a), we have that $0 \in W$, a contradiction. Hence, $t \in \overline{W}$.

Again applying Lemma 1 item 2.(a), if $s \in W$ then $[t-k, 2k)_{3k} \cap C^k \subset W$ implying $r_4^0 \in W$, a contradiction. Then, $s \in \overline{W}$.

Then, $\{s, t\} \subset [r_2^0, r_2^0 + \omega(r_2^0))_{3k}$.

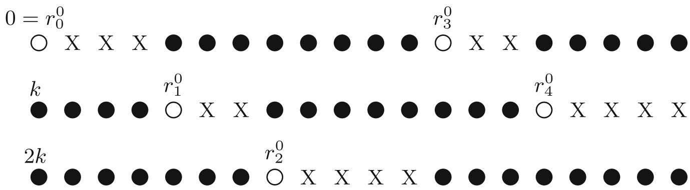

Fig. 5 A subset $W \in \mathcal{W}^*$ with $p^0 = 4$

Since $\tilde{x}$ is a minimal cover, there exist $\ell, \ell' \in \tilde{x}$ such that $\ell \in [s-k, t-k) \subset W$ and $\ell' \in (s+k, t+k] \subset W$.

Recall that $|\tilde{x}| = 4$ or $|\tilde{x}| = 5$. Moreover, $|\tilde{x}| = 5$ if and only if $\tilde{x} \subset \overline{W}$. Then, $|\tilde{x}| = 4$ and $\tilde{x} = \{s, t, \ell, \ell'\}$.

Since $\tilde{x}$ is a root of (2), $\alpha = 2a^0 + a_\ell + a_{\ell'}$.

Observe that $\hat{x} = \{r_1^0, r_4^0, \ell', t\}$ is a cover of C_{3k}^k which violates (2) since $3a^0 + a_{\ell'} < \alpha = 2a^0 + a_\ell + a_{\ell'}$.

Therefore, $p^0 \neq 4$ and then $p^0 \geq 7$.

Finally, $\tilde{x} = \{r_0^0, r_6^0, r_4^0, r_2^0\} \subset \overline{W}$ is a cover of C_{3k}^k. □

As a consequence of Lemma 4 and Theorem 7, we have the following:

Theorem 8 *For every $k \geq 3$, every non boolean facet defining inequality of $Q(C_{3k}^k)$ is a relevant minor inequality.*

Let us observe that, from Theorems 5, 6 and 8 we have a polynomial time algorithm based on linear programming that solves the Set Covering Problem for matrices C_{2k}^k and C_{3k}^k, for every $k \geq 3$.

On the other hand, Proposition 5.5 in Aguilera (2010) proves that, for every k, the set covering polyhedron of C_{2k+1}^k, C_{3k+1}^k, C_{3k+2}^k is described by means of boolean facets and the rank constraint. Lemma 2 and Theorems 6 and 8 give an alternative proof of the same fact.

Indeed, remind that the rank constraint is always a facet defining inequality of $Q(C_{sk+r}^k)$ when $1 \leq r \leq k-1$ and it has right hand side $s+1$. Hence, if $Q(C_{sk+r}^k)$ with $s = 2, 3$ and $1 \leq r \leq s-1$ has a non boolean non rank facet defining inequality of the form (2) it must have right hand side at least $s+2$. But, applying Lemma 2, C_{sk+r}^k is a minor of $C_{sk'}^{k'}$ for some k' and then there would exist some facet defining inequality for $Q(C_{sk'}^{k'})$ with right hand side different from $s+1$ contradicting Theorems 6 and 8.

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