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Some advances on the set covering polyhedron of circulant matrices

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ABSTRACT

Studying the set covering polyhedron of consecutive ones circulant matrices, Argiroffo and Bianchi found a class of facet defining inequalities, induced by a particular family of circulant minors. In this work we extend these results to inequalities associated with every circulant minor. We also obtain polynomial separation algorithms for particular classes of such inequalities.

1. Introduction

The well-known concept of domination in graphs was introduced by Berge [6] in 1962, modeling many utility location problems in operations research.

Given a graph $G = (V, E)$ a *dominating set* is a subset $D \subset V$ such that every node outside D is adjacent to at least one node in D. Given a cost vector $w \in \mathbb{R}^{|V|}$, the *Minimum Weight Dominating Set Problem* (MWDSP for short), consists of finding a dominating set D such that $\sum_{v \in D} w_v$ is minimum. MWDSP arises in many applications, involving the strategic placement of men or pieces on the nodes of a network. As an example, consider a computer network in which one wishes to choose a smallest set of computers that are able to transmit messages to all the remaining computers [18]. Many other interesting examples include sets of representatives, school bus routing, (r, d)-configurations, radio stations, social network theory, kernels of games, etc. [15].

The MWDSP is NP-hard for general graphs and has been extensively investigated from an algorithmic point of view ([7,11,12,14] among others). The cardinality version (that is when the weights are 0 and 1) has been shown to be polynomially solvable in several classes of graphs such as cactus graphs [16] and the class of series–parallel graphs [17].

However, a few results on the MWDSP derived from the polyhedral point of view are known. An interesting result in this context can be found in [10], working on the problem when the underlying graph is a cycle.

Actually, the MWDSP corresponds to particular instances of the Minimum Weighted Set Covering Problem (MWSCP).

Indeed, given an $m \times n$ 0, 1 matrix A, a *cover of* A is a vector $x \in \{0, 1\}^n$ such that $Ax \geq \mathbf{1}$, where $\mathbf{1}$ is the vector with all components at value one. Given a cost function $w \in \mathbb{R}^n$, the MWSCP consists of solving the integer program

$$\min\{wx : Ax \geq \mathbf{1},\ x \in \{0, 1\}^n\}.$$

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This is equivalent to solving the problem

$$\min\{wx : x \in Q^*(A)\}$$

where $Q^*(A)$ is the convex hull of points in $\{x \in \{0,1\}^n : Ax \geq \mathbf{1}\}$. The set $Q^*(A)$ is usually called the *set covering polyhedron* associated with A.

In particular, given a graph $G = (V, E)$, if A is a matrix such that each row corresponds to the characteristic vector of the closed neighborhood of a node $v \in V$ (i.e., A is the *closed neighborhood matrix of* G) then every cover of A is the characteristic vector of a dominating set of G and conversely. Therefore, solving the MWSCP on A is equivalent to solve the MWDSP on G.

It is easy to see that the closed neighborhood matrix of a cycle is a *circulant* matrix. Hence, the findings in [10] correspond to obtaining the complete description of the set covering polyhedron for the $0, 1$ $n \times n$ matrices having three consecutive ones per row, known as the family of circulant matrices C_n^3.

In general, the closed neighborhood of a *web* graph is a circulant matrix. Web graphs have been thoroughly studied in the literature (see [21,23,24]).

The main goal of this work is the study of the MWSCP on circulant matrices and its direct consequences on the MWDSP when the underlying graph is a web graph.

Previous results on the set covering polyhedron of circulant matrices can be found in [2,3,13,19,20]. In [13] it was observed that if A is a circulant matrix then every set $\{x \in [0,1]^n : Ax \geq \mathbf{1},\ x_i = 1\}$ for $i = 1, \dots, n$ is an integer polyhedron. Then it holds that solving the MWSCP on a circulant matrix can be thought as solving at most n linear programs. Hence, the MWSCP on circulant matrices results in a polynomial problem.

In Section 2 of this work, we present basic definitions and preliminaries needed for the remaining sections. In Section 3 we introduce a family of valid inequalities for the set covering polyhedron of circulant matrices. We obtain sufficient conditions for a valid inequality to define a facet of the polyhedron. We also conjecture that this condition is also necessary. In Section 4 we prove that a subfamily of the inequalities presented in Section 3 can be separated in polynomial time.

A preliminary version of this work appeared without proofs in [8].

2. Definitions, notations and preliminary results

In what follows, every time we state $S \subset \mathbb{Z}_n$ for some $n \in \mathbb{N}$, we consider $S \subset \{0, \dots, n-1\}$ and the addition between the elements of S is taken modulo n.

Given a set F of vectors in $\{0,1\}^n$, we say $y \in F$ is a dominating vector (of F) if there exist $x \in F$ such that $x \leq y$. It can be also said that x is dominated by y.

From now on, every matrix has 0,1 entries, no zero columns and no dominating rows. If A is such an $m \times n$ matrix, its rows and columns are indexed by $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. Two matrices A and A' are *isomorphic* and we write $A \approx A'$, if A' can be obtained from A by permutation of rows and columns.

If $S \subset \mathbb{Z}_m$ and $T \subset \mathbb{Z}_n$, let $A_{S,T}$ be the submatrix of A with entries a_{ij} where $i \in S$ and $j \in T$.

Given $N \subset \mathbb{Z}_n$, let us denote by $R(N) = \{j \in \mathbb{Z}_m : j$ is a dominating row of $A_{\mathbb{Z}_m, \mathbb{Z}_n - N}\}$. A *minor of* A *obtained by contraction of* N, denoted by A/N, is the matrix $A_{\mathbb{Z}_m - R(N), \mathbb{Z}_n - N}$. In this work, when we refer to *a minor* of A we are always considering a minor obtained by contraction.

Observe that there exists a one-to-one correspondence between a vector $x \in \{0,1\}^n$ and the subset $S_x \subset \mathbb{Z}_n$ whose characteristic vector is x itself. Hence, we agree to abuse of notation by writing x instead of S_x. In this way, if $x \in \{0,1\}^n$, we write $i \in x$ meaning that $x_i = 1$. Also, if x is dominated by $y \in \{0,1\}^n$ then we write $x \subset y$.

Remind that a *cover* of a matrix A is a vector $x \in \{0,1\}^n$ such that $Ax \geq \mathbf{1}$. In addition, the *cardinality* of a cover x is denoted by $|x|$ and equals $\mathbf{1}x$. A cover x is *minimum* if it has the minimum cardinality and in this case $|x|$ is called the *covering number* of the matrix A, denoted by $\tau(A)$. Observe that every cover of a minor of A is a cover of A and then, for all $N \subset \mathbb{Z}_n$, it holds that $\tau(A/N) \geq \tau(A)$.

Recall that the set covering polyhedron of A, denoted by $Q^*(A)$, is defined as the convex hull of its covers. The polytope $Q(A) = \{x \in [0,1]^n : Ax \geq \mathbf{1}\}$ is known as the *linear relaxation of* $Q^*(A)$. When $Q^*(A) = Q(A)$ the matrix A is ideal and the MWSCP can be solved in polynomial time (in the size of A).

Given n and k with $2 \leq k \leq n-2$, for every $i \in \mathbb{Z}_n$ let $C^i = \{i, i+1, \dots, i+(k-1)\} \subset \mathbb{Z}_n$. The *circulant* matrix C_n^k is the square matrix whose i-th row is the incidence vector of C^i. Observe that, for $j \in \mathbb{Z}_n$, the j-th column of C_n^k is the incidence vector of C^{j-k+1}.

We say that a minor of C_n^k is a *circulant minor* if it is isomorphic to a circulant matrix.

Remark 1. Let C_n^k be a circulant matrix and let $x = \{i_j : j \in \mathbb{Z}_r\} \subset \mathbb{Z}_n$ with $0 \leq i_0 < i_1 < \cdots < i_{r-1} \leq n-1$. The following propositions are equivalent:

(i) x is a cover of C_n^k,
(ii) $i_{j+1} - 1 \in C^{i_j}$ for all $j \in \mathbb{Z}_r$,
(iii) $i_{j-1} \in C^{i_j - k}$ for all $j \in \mathbb{Z}_r$.

It is not hard to see that $\tau(C_n^k) \geq \lceil \frac{n}{k} \rceil$. Moreover, for every $i \in \mathbb{Z}_n$

$$x^i = \left\{ i + hk : 0 \leq h \leq \left\lfloor \frac{n}{k} \right\rfloor \right\} \subset \mathbb{Z}_n$$

is a cover of C_n^k of size $\lceil \frac{n}{k} \rceil$, and then $\tau(C_n^k) = \lceil \frac{n}{k} \rceil$.

Also, the set $\{x^i : i \in \mathbb{Z}_n\}$ is linearly independent if and only if n is not a multiple of k. Thus the inequality $\sum_{i=1}^n x_i \geq \lceil \frac{n}{k} \rceil$ that is always valid for $Q^*(C_n^k)$, defines a facet if and only if n is not a multiple of k (see [20]). This inequality will be called the *rank constraint*.

In addition, for every $i \in \mathbb{Z}_n$, the constraints $x_i \geq 0$ and $\sum_{j \in C^i} x_j \geq 1$ are facet defining inequalities of $Q^*(C_n^k)$ (see [4,20] for further details). We call them *Boolean facets*.

It is also known that if $ax \geq \beta$ is a non-Boolean facet defining inequality of $Q^*(C_n^k)$ then $a > \mathbf{0}$ [3].

We say that $\alpha x \geq \beta$ is a *Chvátal–Gomory* inequality if there exists an inequality $\gamma x \geq \delta$, valid for $Q(C_n^k)$ such that $\lceil \gamma_i \rceil = \alpha_i$, for all $i = 1, \ldots, n$ and $\lceil \delta \rceil = \beta$. Clearly, every Chvátal–Gomory inequality is valid for $Q^*(C_n^k)$.

Ideal circulant matrices have been completely identified by Cornuéjols et al. in [13]. Many of the ideas and results obtained in this seminal paper inspired further results presented in this work.

In fact, the authors in [13] characterize ideal circulant matrices in terms of a nonideal circulant minor and give sufficient conditions for a subset $N \subset \mathbb{Z}_n$ to ensure that C_n^k/N is a circulant minor. These conditions are obtained in terms of simple dicycles in a particular digraph.

Indeed, given C_n^k, the digraph $G(C_n^k)$ has node set $\mathbb{Z}_n$ and (i, j) is an arc of $G(C_n^k)$ if $j \in \{i + k, i + k + 1\}$. In this way, we will say that an arc $(i, i + k)$ has length k and an arc $(i, i + k + 1)$ has length $k + 1$.

If D is a simple dicycle in $G(C_n^k)$, and n_2 and n_3 denote the number of arcs of length k and $k + 1$ respectively, then there must be a positive integer $n_1 \geq 1$ such that $n_1 n = kn_2 + (k + 1)n_3$ and $\gcd(n_1, n_2, n_3) = 1$ (gcd means greatest common divisor). Moreover, the conditions $n_1 n = kn_2 + (k + 1)n_3$ and $\gcd(n_1, n_2, n_3) = 1$ are not only necessary but also sufficient for the existence of a simple dicycle in $G(C_n^k)$ (see [1] for further details).

We say that n_1, n_2 and n_3 are the *parameters associated with* the dicycle.

Later, Aguilera in [2] completely characterized subsets N of $\mathbb{Z}_n$ for which C_n^k/N is a circulant minor in terms of dicycles in the digraph $G(C_n^k)$. We rewrite Theorem 3.10 of [2] in the following way:

Theorem 1. *Let n, k be positive integers verifying $2 \leq k \leq n - 2$ and let $N \subset \mathbb{Z}_n$. Then, the following assertions are equivalent:*

(i) $C_n^k/N \approx C_{n'}^{k'}$.

(ii) *N induces d disjoint simple dicycles $D_0, \ldots, D_{d-1}$ in $G(C_n^k)$, each of them having the same parameters n_1, n_2 and n_3 such that $n = n' - d(n_2 + n_3)$ and $k' = k - dn_1$.*

Thus, whenever we refer to a circulant minor of C_n^k with parameters d, n_1, n_2 and n_3, we are referring to the non-negative integers whose existence is guaranteed by the previous theorem. In addition, N^j, with $j \in \mathbb{Z}_d$ refers to each of the subsets inducing a simple dicycle D^j in $G(C_n^k)$. Moreover, we call $W^j = \{i \in N^j : i - (k + 1) \in N^j\}$, for $j \in \mathbb{Z}_d$ and $W = \cup_{j \in \mathbb{Z}_d} W^j$. Then, $|W^j| = n_3$ and $|N^j| = n_2 + n_3$ for all $j \in \mathbb{Z}_d$.

Observe that, the parameters d, n_1, n_2 and n_3 are not enough to identify the minor itself. For example, C_9^4 has nine different minors with parameters $d = n_1 = n_2 = n_3 = 1$. Indeed, for every $i \in \mathbb{Z}_9$, $C_9^4/\{i, i + 4\} \approx C_7^3$.

Let us remark that starting from $W \subset \mathbb{Z}_n$ corresponding to a circulant minor M of C_n^k we can obtain the set $N \subset \mathbb{Z}_n$ such that $M \approx C_n^k/N$. In order to see this, it is enough to observe that, given $j \in W$, we can construct the set N^j inducing the simple dicycle in $G(C_n^k)$ with $j \in N^j$. Indeed, let $N^j := \{j, j - (k + 1)\}$ and $i = j - (k + 1)$. While $i \neq j$ we repeat the next step: if $i \in W$ then we add $i - (k + 1)$ to N^j and set $i := i - (k + 1)$ else we add $i - k$ to N^j and set $i := i - k$. Once we obtain N^j, it is clear that we also obtain the parameters n_1, n_2 and n_3 associated with the dicycle induced by N^j. Also, considering $|W| = dn_3$, we can obtain the parameter d. Hence, we compute $n' = n - d(n_2 + n_3)$ and $k' = k - dn_1$.

So, in what follows, we usually refer to a circulant minor *defined by* $W \subset \mathbb{Z}_n$. We will also refer to *the dicycle of* $G(C_n^k)$ *induced by* W^j, considering the dicycle induced by the corresponding subset N^j.

Remark 2. Let $W \subset \mathbb{Z}_n$.

(i) If $W = \{w_i : i \in \mathbb{Z}_{|W|}\}$ with $0 \leq w_0 < \cdots < w_{|W|-1} \leq n - 1$, then W defines a circulant minor with parameters $d = n_1 = 1$ if and only if $w_{i+1} - w_i = 1 \pmod{k}$ and $w_{i+1} - w_i \geq k + 1$, for all $i \in \mathbb{Z}_{|W|}$.

(ii) W defines a circulant minor with parameters $d \geq 2$ and $n_1 = 1$ if and only if $W = \cup_{j \in \mathbb{Z}_d} W^j$, for all $j \in \mathbb{Z}_d$, W^j defines a circulant minor with parameters $d^j = n_1^j = 1$ and for all $r, j \in \mathbb{Z}_d$ with $r \neq j$, $N^r \cap N^j = \emptyset$.

In the next section, we will see that subsets $W \subset \mathbb{Z}_n$ that define circulant minors, play an important role in the description of the set covering polytope of circulant matrices.

3. Relevant minor inequalities

Cornuéjols and Novick in [13] obtained sufficient conditions under which a circulant matrix has a circulant minor. More precisely and according to our current notation, Lemma 4.5 in [13] gives conditions on parameters n_1, n_2 and n_3 in order to ensure that C_n^k has a circulant minor with parameter $d = 1$. Using this result, the authors in [3] found a Chvátal–Gomory inequality associated with each one of these particular minors. More specifically, if W induces a circulant minor isomorphic to $C_{n'}^{k'}$ with parameter $d = 1$ then

$$\sum_{i\in W} 2x_i + \sum_{i\notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil$$

is a Chvátal–Gomory inequality.

Moreover, the authors proved that if $n' = 1 \pmod{k'}$ and $\left\lceil \frac{n'}{k'} \right\rceil > \left\lceil \frac{n}{k} \right\rceil$, the inequality defines a facet.

In addition, the results in [10] imply that these inequalities, together with the Boolean facets and the rank constraint completely describe $Q^*(C_n^3)$.

Later and as we have already mentioned in Theorem 1, Aguilera in [2] characterized all circulant minors of circulant matrices. Moreover, according to our current notation, results Lemma 2.4 and Theorem 2.5 in [2] can be stated as follows:

Lemma 2. *Let $N \subset \mathbb{Z}_n$ be such that $C_n^k/N \approx C_{n'}^{k'}$. Then,*

(i) $R(N) = \{i + 1 : i \in N\}$.
(ii) $\left|C^i - N\right| = k' + 1$ *if* $i + k \in W$ *and* $\left|C^i - N\right| = k'$ *otherwise.*

From these results, we can prove that Theorem 6.9 in [3] holds for every W associated with any circulant minor of C_n^k. Formally:

Theorem 3. *Let $W \subset \mathbb{Z}_n$ be a subset defining a minor isomorphic to $C_{n'}^{k'}$. Then, the inequality*

$$\sum_{i\in W} 2x_i + \sum_{i\notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{1}$$

is a valid inequality for $Q^*(C_n^k)$. Moreover, it is a Chvátal–Gomory inequality.

Proof. Let $N \subset \mathbb{Z}_n$ be the subset defining the minor i.e. $C_n^k/N \approx C_{n'}^{k'}$ and let us call A the row submatrix of C_n^k defined by rows not in $R(N)$, i.e. $A = (C_n^k)_{\mathbb{Z}_n - R(N), \mathbb{Z}_n}$. Recall that the i-th column of C_n^k is the incidence vector of C^{i-k+1}. By Lemma 2(i), the number of entries at value one in the i-th column of A is the number of times an index of the form $j + 1$ with $j \notin N$ belongs to C^{i-k+1}, i.e. $\left|C^{i-k} - N\right|$. On the other hand, Lemma 2(ii) states that $\left|C^{i-k} - N\right| \in \{k', k' + 1\}$ and $\left|C^{i-k} - N\right| = k' + 1$ if and only if $i \in W$. In summary, each column of A has k' or $k' + 1$ entries at value one. Moreover, the i-th column has $k' + 1$ entries at value one if and only if $i \in W$. Thus if we add up all the rows of submatrix A we get:

$$\sum_{i\in W} (k' + 1)x_i + \sum_{i\notin W} k'x_i \geq n'. \tag{2}$$

Then, if we divide all the coefficients by k' and round up, we obtain the inequality (1). □

From now on, we say that inequality (1) is the *minor inequality* corresponding to the minor defined by W.

Remind that if M is a minor of C_n^k isomorphic to $C_{n'}^{k'}$ then $\left\lceil \frac{n'}{k'} \right\rceil \geq \left\lceil \frac{n}{k} \right\rceil$. Observe that when $\left\lceil \frac{n'}{k'} \right\rceil = \left\lceil \frac{n}{k} \right\rceil$ the minor inequality is dominated by the rank constraint. Also, if n' is a multiple of k' then it is valid for $Q(C_n^k)$.

In summary, the relevant minor inequalities correspond to minors M isomorphic to $C_{n'}^{k'}$ such that $n' \neq 0 \pmod{k'}$ and $\left\lceil \frac{n'}{k'} \right\rceil > \left\lceil \frac{n}{k} \right\rceil$. In this case, we will say that M is a *relevant minor*.

The following result identifies relevant minors:

Lemma 4. *Let M be a circulant minor of C_n^k isomorphic to $C_{n'}^{k'}$ with parameters d, n_1, n_2 and n_3 and let r be such that $1 \leq r \leq k' - 1$ and $n' = r \pmod{k'}$. Then, M is a relevant minor if and only if $dn_3 \geq kr$.*

Proof. We know that $nn_1 = n_2k + n_3(k + 1)$, $n' = n - d(n_2 + n_3)$ and $k' = k - dn_1$.

Let s be such that $n - d(n_2 + n_3) = s(k - dn_1) + r$ then $\left\lceil \frac{n'}{k'} \right\rceil = s + 1$.

It follows that M is a relevant minor if and only if $\left\lceil \frac{n}{k} \right\rceil \leq s$. Since

$$n = sk - (sdn_1 - d(n_2 + n_3) - r),$$

we have that $\left\lceil \frac{n}{k} \right\rceil \leq s$ if and only if

$$sdn_1 - d(n_2 + n_3) - r \geq 0.$$

It is not hard to see that

$$dn_3 - kr = (k - dn_1)(sdn_1 - d(n_2 + n_3) - r).$$

Since $k - dn_1 > 0$, the proof is complete. □

Taking advantage of the same ideas in proving Theorem 6.10 in [3], we can prove the following generalization:

Theorem 5. *Let $W \subset \mathbb{Z}_n$ be a subset defining a relevant minor isomorphic to $C_{n'}^{k'}$. Then, if $n' = 1(\text{mod } k')$ the inequality*

$$\sum_{i\in W} 2x_i + \sum_{i\notin W} x_i \geq \left\lceil \frac{n'}{k'} \right\rceil \tag{3}$$

defines a facet of $Q^*(C_n^k)$.

Proof. We will show that there are n linearly independent roots of inequality (3), i.e. n linearly independent covers of C_n^k that satisfy (3) at equality.

Let $N = \cup_{j\in\mathbb{Z}_d} N^j \subset \mathbb{Z}_n$ be the subset defining the minor, i.e., $C_n^k/N \approx C_{n'}^{k'}$ and let us denote the elements of $\mathbb{Z}_n - N$ as $\{v_0, \ldots, v_{n'-1}\}$ with $0 \leq v_0 \leq v_1 \leq \cdots \leq v_{n'-1} \leq n-1$.

Recall that the subsets $\tilde{x}^l = \left\{ v_{l+sk'} : 0 \leq s \leq \left\lfloor \frac{n'}{k'} \right\rfloor \right\}$ with $l \in \mathbb{Z}_{n'}$ are n' linearly independent minimum covers of C_n^k/N and then they are n' linearly independent roots of (3).

For the remaining $|N|$ roots, we will construct a root z^i for every $i \in N$.

Observe that as $n' = 1\,(\text{mod } k')$, if $l \in \mathbb{Z}_{n'}$ then $l + \lfloor \frac{n'}{k'} \rfloor k' = l - 1\,(\text{mod } n')$. Hence, $v_{l-1} \in \tilde{x}^l$ for every $l \in \mathbb{Z}_{n'}$. Therefore, for every $l \in \mathbb{Z}_{n'}$ there are two consecutive elements of $\mathbb{Z}_n - N$ that belong to $\tilde{x}^l$, i.e. $\{v_{l-1}, v_l\} \subset \tilde{x}^l$ for every $l \in \mathbb{Z}_{n'}$. Moreover, by Lemma 2(ii), we know that for every $i \in \mathbb{Z}_n$, $k' \leq |C^i - N| \leq k' + 1 < n'$ and then, there exists $l \in \mathbb{Z}_{n'}$ such that $v_l \notin C^i$ and $v_{l+1} \in C^i$.

Let us start with $i \in N - W$. Let $l \in \mathbb{Z}_{n'}$ such that $v_l \notin C^i$ and $v_{l+1} \in C^i$. Observe that, by Lemma 2(ii) we have that $|C^{i-k} - N| = k'$ and since $k' \geq 2$ it follows that $v_{l-1} \in C^{i-k}$. Then, the vector $z^i = (\tilde{x}^l - \{v_l\}) \cup \{i\}$ satisfies the inequality (3) at equality and by the condition (iii) in Remark 1 it is a cover. Also observe that $z^i \cap N = \{i\}$, and then $\{\tilde{x}^l : l \in \mathbb{Z}_{n'}\} \cup \{z^i : i \in N - W\}$ is a set of linearly independent covers of C_n^k.

Let us now obtain z^i for $i \in W$. Let $i \in W$ and w.l.o.g. assume that $i \in N^0$. First, consider the minimum cover of C_n^k

$$x^i = \left\{ i + tk : 0 \leq t \leq \left\lceil \frac{n}{k} \right\rceil - 1 \right\}.$$

If $x^i \subset N^0$ then $x^i \cap W = \{i\}$, and since x^i satisfies (3) we have

$$\sum_{j\in W} 2x_j^i + \sum_{j\notin W} x_j^i = 2 + \left\lceil \frac{n}{k} \right\rceil - 1 \geq \left\lceil \frac{n'}{k'} \right\rceil \geq \left\lceil \frac{n}{k} \right\rceil + 1.$$

Hence, x^i is a root of (3) and then we set $z^i = x^i$.

Otherwise, let s be the smallest nonnegative integer such that $s \leq \lceil \frac{n}{k} \rceil - 1$ and $i + sk \notin N^0$. It holds that $i + sk + 1 \in W \cap N^0$ and for all $1 \leq t \leq s - 1$, $i + tk \in N^0 - W$.

Now, let $l \in \mathbb{Z}_{n'}$ such that $v_{l-1} \notin C^i$ but $v_l \in C^i$.

Hence, by Lemma 2(ii) we have $|C^{i+(t-1)k} - N| = k'$, for $1 \leq t < s$ and $|C^{i+(s-1)k+1} - N| = k' + 1$. Then,

$$C^{i+(t-1)k} - N = \{v_{l+(t-1)k'}, \ldots, v_{l+tk'-1}\}$$

for all $1 \leq t \leq s - 1$ and

$$C^{i+(s-1)k+1} - N = \{v_{l+(s-1)k'}, \ldots, v_{l+sk'}\}.$$

We define

$$z^i = \tilde{x}^l - (\{v_{l-1}\} \cup \{v_{l+tk'} : 0 \leq t \leq s - 1\}) \cup \{i + tk : 0 \leq t \leq s - 1\}.$$

We have seen that $v_{l+sk'} \in C^{i+(s-1)k+1}$. By Remark 1(ii), we only need to prove that $v_{l+sk'} \in z^i$. For this, we need to verify that $v_{l+sk'} \neq v_{l-1}$.

But $v_{l+sk'} = v_{l-1}$ if and only if $s = \left\lfloor \frac{n'}{k'} \right\rfloor$ and this cannot happen since we consider $s \leq \lceil \frac{n}{k} \rceil - 1$ and $\lceil \frac{n}{k} \rceil < \left\lceil \frac{n'}{k'} \right\rceil$. Then z^i is a cover and it is easy to check that it is also a root of (3). In addition $z^i \cap W = \{i\}$. Hence, it is not hard to see that $\{\tilde{x}^l : l \in \mathbb{Z}_{n'}\} \cup \{z^i : i \in N\}$ is a set of linearly independent covers of C_n^k. □

Computational experiences lead us to conjecture that the converse of Theorem 5 always holds, i.e., a minor inequality defines a facet only when it corresponds to a relevant minor isomorphic to $C_{n'}^{k'}$ with $n' = 1(\text{mod } k')$.

Moreover, we conjecture the following

Conjecture 6. *If $W \subset \mathbb{Z}_n$ defines a relevant minor of C_n^k isomorphic to $C_{n'}^{k'}$ then there exists $W' \subset W$ that defines a relevant minor isomorphic to $C_{n''}^{k'}$ such that $n'' = 1(\text{mod } k')$ and $\left\lceil \frac{n''}{k'} \right\rceil \geq \left\lceil \frac{n'}{k'} \right\rceil$.*

Clearly, if the conjecture holds, the converse of Theorem 5 is true. Nevertheless, we have a weaker result than the previous conjecture:

Lemma 7. *If $W \subset \mathbb{Z}_n$ defines a relevant minor of C_n^k isomorphic to $C_{n'}^{k'}$ then there exists $W' \subset \mathbb{Z}_n$ with $|W'| \le |W|$ that defines a relevant minor isomorphic to $C_{n''}^{k'}$ such that $n'' = 1 (\text{mod } k')$ and $\left\lceil \frac{n''}{k'} \right\rceil \ge \left\lceil \frac{n'}{k'} \right\rceil$.*

Proof. Let s be such that $n' = sk' + r$. If $r = 1$ the result clearly holds. Let r be such that $2 \le r \le k' - 1$.

By Theorem 1 there exist non negative integers d, n_1, n_2 and n_3 such that $n_1 n = (n_2 + n_3)k + n_3$, $n' = n - d(n_2 + n_3)$, $k' = n - dn_1$ and $|W| = dn_3$.

In addition, by Lemma 4, we have that $dn_3 \ge kr$.

Then if we set $\tilde{n}_3 = dn_3 - k(r-1)$ and $\tilde{n}_2 = dn_2 + (k+1)(r-1)$, we have that $n_1 n = (\tilde{n}_2 + \tilde{n}_3)k + \tilde{n}_3$ with $0 < \tilde{n}_3 < dn_3$.

Considering $\tilde{d} = \gcd(n_1, \tilde{n}_2, \tilde{n}_3)$, Theorem 1 states that there exists a minor of C_n^k isomorphic to $C_{n''}^{k'}$ with subset W' such that $|W'| = \tilde{n}_3 < dn_3 = |W|$ and $n'' = n - (\tilde{n}_2 + \tilde{n}_3)$.

Moreover, $n'' = 1 (\text{mod } k')$ and $\left\lceil \frac{n''}{k'} \right\rceil = \left\lceil \frac{n'}{k'} \right\rceil = s + 1$. □

In addition, we can state:

Lemma 8. *Conjecture 6 holds for relevant minors with parameters $d = n_1 = 1$.*

Proof. Let $W \subset \mathbb{Z}_n$ be a subset defining a relevant minor of C_n^k isomorphic to $C_{n'}^{k-1}$ and $n' = s(k-1) + r$ with $2 \le r \le k-1$. Assume that $W = \{w_i : i \in \mathbb{Z}_{|W|}\}$ with $0 \le w_0 < w_1 < \cdots < w_{|W|-1} \le n - 1$.

Take $W' = \{w_i : 0 \le i \le |W| - k(r-1) - 1\}$. It is not hard to see that, by Remark 2, W' defines a relevant minor with parameters $d = n_1 = 1$ and by using the same arguments as in the previous lemma, the minor is isomorphic to $C_{n''}^{k-1}$ with $n'' = 1 (\text{mod } (k-1))$. □

As a consequence we have:

Corollary 9. *Let $k \le 4$. If $W \subset \mathbb{Z}_n$ defines a relevant minor of C_n^k isomorphic to $C_{n'}^{k'}$ then the corresponding minor inequality defines a facet of $Q^*(C_n^k)$ if and only if $n' = 1 (\text{mod } k')$.*

Proof. If $k \le 4$, every minor inequality valid for $Q^*(C_n^k)$ corresponds to a relevant minor isomorphic to $C_{n'}^{k'}$ with $k' = 2$ or $k' = 3$. If $k' = 3$, the minor has parameters $d = n_1 = 1$ and then the corollary follows from Lemma 8. It only remains to observe that when $k' = 2$ and the minor inequality defines a facet of $Q^*(C_n^k)$, then n' has to be odd. □

4. The separation problem for minor inequalities

In the context of the study of the dominating set problem on cycles, the authors in [10] give a polynomial time algorithm to separate minor inequalities valid for $Q^*(C_n^3)$. Let us observe that every circulant minor of C_n^3 has parameters $d = n_1 = 1$.

In this section we study the separation problem for inequalities associated with circulant minors of any circulant matrix with parameter $n_1 = 1$ and any $d \ge 1$.

In order to do so, let us first present a technical lemma for these inequalities.

Lemma 10. *Let $d, n_1 = 1, n_2, n_3$ be the parameters associated with a circulant minor of C_n^k such that $n_3 = r(\text{mod } (k-d))$ with $1 \le r < k - d$. Then*

$$\left\lceil \frac{n - d(n_2 + n_3)}{k - d} \right\rceil = \left(\frac{n}{k} - \frac{r}{k-d} + 1 \right) + \frac{1}{k(k-d)} dn_3.$$

Proof. Let s be the nonnegative integer such that $n_3 = s(k-d) + r$. Since $n = k(n_2 + n_3) + n_3$ we have that

$$\left\lceil \frac{n - d(n_2 + n_3)}{k - d} \right\rceil = \left\lceil \frac{(k-d)(n_2 + n_3) + n_3}{k - d} \right\rceil = n_2 + n_3 + s + 1.$$

Since $s = \frac{n_3 - r}{k-d}$ and $n_2 + n_3 = \frac{n - n_3}{k}$ it follows that

$$n_2 + n_3 + s + 1 = \frac{n - n_3}{k} + \frac{n_3 - r}{k - d} + 1 = \left(\frac{n}{k} - \frac{r}{k-d} + 1 \right) + \frac{1}{k(k-d)} dn_3$$

and the proof is complete. □

From the previous lemma, if $W \subset \mathbb{Z}_n$ defines a relevant minor of C_n^k with parameters $d, n_1 = 1, n_2, n_3$ and $n_3 = r(\text{mod } (k-d))$ with $1 \le r < k - d$, then the corresponding minor inequality can be written as

$$\sum_{i \in W} x_i + \sum_{i=1}^{n} x_i \ge \alpha(d, r) + \beta(d) \, |W|$$

where

$$\alpha(d,r)=\frac{n}{k}-\frac{r}{k-d}+1,\qquad \beta(d)=\frac{1}{k(k-d)}$$

or equivalently

$$\sum_{i\in W}(x_i-\beta(d))\geq\alpha(d,r)-\sum_{i=1}^{n}x_i. \tag{4}$$

Given C_n^k and two integer numbers d, r with $1\leq d\leq k-2$ and $1\leq r<k-d$, we define the function c^d on $\mathbb{R}$ such that $c^d(t)=t-\beta(d)$ and the function $L^{d,r}$ on $\mathbb{R}^n$ such that $L^{d,r}(x)=\alpha(d,r)-\sum_{i=1}^{n}x_i$.

Then, the inequality (4) can be written as

$$\sum_{i\in W}c^d(x_i)\geq L^{d,r}(x). \tag{5}$$

We will first extend the techniques used in [10] for matrices C_n^3 to any matrix C_n^k, in order to separate inequalities corresponding to relevant minors with parameters $d=n_1=1$.

Let us denote by $\mathcal{W}(d,r)$ the set of subsets $W\subset\mathbb{Z}_n$ defining relevant minors with parameters $d, n_1=1, n_2, n_3=r(\text{mod }(k-d))$. Observe that, from Lemma 8, when $d=n_1=1$ every relevant minor inequality corresponds to the case $r=1$, that is why we only consider subsets $W\in\mathcal{W}(1,1)$.

To this end, given n, k let $K_n^k=(V,A)$ be the digraph with set of nodes $V=\{v_i^j: i\in\mathbb{Z}_n,\ j\in\mathbb{Z}_{k-1}\}\cup\{t\}$ and set of arcs defined as follows: first consider in A the arcs

- (v_0^0, v_l^1) for all l such that $k+1\leq l\leq n-1$ and $l=1$ (mod k),

then consider in a recursive way:

- for each $(v, v_i^j)\in A$, add (v_i^j, v_l^{j+1}) whenever l is such that $i+k+1\leq l\leq n-1$ and $l-i=1$ (mod k),
- for each $(v, v_i^0)\in A$, add (v_i^0, t) whenever i is such that $i\leq n-(k+1)$ and $n-i=1$ (mod k).

Note that, by construction, K_n^k is acyclic. For illustration, digraph K_{20}^4 is depicted in Fig. 1.

We have the following result:

Lemma 11. *There is a one-to-one correspondence between v_0^0t-paths in K_n^k and subsets $W\in\mathcal{W}(1,1)$ with $0\in W$.*

Proof. Let $W\in\mathcal{W}(1,1)$ and assume that $W=\{i_j: j\in\mathbb{Z}_{n_3}\}\subset\mathbb{Z}_n$ with $0=i_0<i_1<\cdots<i_{n_3-1}\leq n-1$. Let α be the positive integer such that $|W|=n_3=\alpha(k-1)+1$.

Then, by Remark 2(i), $i_{j+1}-i_j=1$ (mod k) and $i_{j+1}-i_j\geq k+1$ for all $j\in\mathbb{Z}_{n_3}$. Then,

$$\left\{v_{i_j}^s\in V(K_n^k): i_j\in W,\ s=j(\text{mod }(k-1))\right\}\cup\{t\}$$

induces a v_0^0t-path in K_n^k.

Conversely, let P be a v_0^0t-path in K_n^k. By construction, there exists a positive integer α such that $|V(P)\cap V^j|=\alpha$ for all $j\neq 0$ and $|V(P)\cap V^0|=\alpha+1$. Then, $|V(P)-\{t\}|=\alpha(k-1)+1$.

Now, if we define

$$W=\{i\in\mathbb{Z}_n: v_i^j\in V(P)\text{ for some }j\in\mathbb{Z}_{k-1}\}$$

then $|W|=\alpha(k-1)+1$ and from Remark 2(i) and Lemma 4, it follows that $W\in\mathcal{W}(1,1)$. □

Theorem 12. *Given C_n^k, the separation problem for inequalities corresponding to minors with parameters $d=n_1=1$ can be polynomially reduced to at most n minimum weight path problems in an acyclic digraph.*

Proof. Let $\hat{x}\in\mathbb{R}^n$. We will show that the problem of deciding if, given $j\in\mathbb{Z}_n$, there exists $W\in\mathcal{W}(1,1)$ with $j\in W$ and such that $\hat{x}$ violates the inequality (5) can be reduced to a shortest path problem. W.l.o.g we set $j=0$.

Consider the digraph K_n^k and associate the weight $c^1(\hat{x}_i)$ with every arc $(v_l^j, v_i^{j+1})\in A$ and the weight $c^1(\hat{x}_0)$ with every arc $(v_l^0, t)\in A$.

Clearly, if W is the subset corresponding to a v_0^0t-path P in K_n^k, the weight of P is equal to $\sum_{i\in W}c^1(\hat{x}_i)$.

Then, there exists $W\in\mathcal{W}(1,1)$ with $0\in W$ and such that $\hat{x}$ violates the inequality (5) if and only if the minimum weight on all v_0^0t-paths in K_n^k is less than $L^{1,1}(\hat{x})$. Since K_n^k is acyclic, computing this minimum weight path can be done in polynomial time using for instance Bellman's algorithm [5]. □

In what follows we consider inequalities corresponding to minors with parameters $n_1=1$ and $d\geq 2$. More precisely, we will focus on a particular family of minors that we call *alternated minors*.

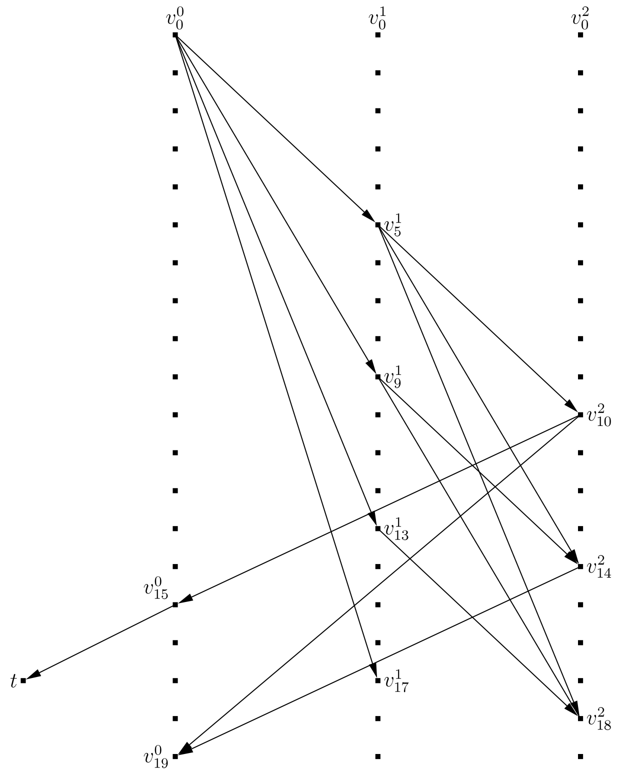

Fig. 1. The digraph K_{20}^4.

Definition 1. Let $W = \{i_s : s \in \mathbb{Z}_{dn_3}\} \subset \mathbb{Z}_n$ with $0 \leq i_0 < i_1 < \cdots < i_{dn_3-1} \leq n-1$ be a subset defining a relevant minor of C_n^k with parameters $d \geq 2$, $n_1 = 1$, n_2, n_3. Then, the minor defined by W is a *d-alternated minor* if, for every $j \in \mathbb{Z}_d$, $W^j = \{i_{j+td} : t \in \mathbb{Z}_{n_3}\}$.

Example 1. In the following two cases, let us consider the circulant minors of C_{33}^6 induced by $W = W^0 \cup W^1$:

(i) $W^0 = \{7, 14, 21\}$ and $W^1 = \{8, 15, 22\}$,
(ii) $W^0 = \{7, 14, 21\}$ and $W^1 = \{12, 25, 32\}$.

Clearly, the case (i) corresponds to a 2-alternated minor, while the case (ii) does not.

We are interested in characterizing the subsets $W \subset \mathbb{Z}_n$ that define alternated minors of C_n^k.

From now on, whenever $W \subset \mathbb{Z}_n$ and $|W| = m$ we assume that $W = \{i_s : s \in \mathbb{Z}_m\}$ with $0 \leq i_0 < i_1 < \cdots < i_{m-1} \leq n-1$. In addition, according to W we let $\delta_s = i_{s+1} - i_s$, for all $s \in \mathbb{Z}_m$.

Remark 3. If $W \subset \mathbb{Z}_n$ defines a d-alternated minor it can be checked that

$$\sum_{s=0}^{d-1} \delta_s = i_d - i_0 = 1 (\text{mod } k) \geq k+1.$$

If in addition to the necessary condition above, we have that $\delta_{s+d} = \delta_s$ for all $s \in \mathbb{Z}_{|W|}$, then W clearly defines a d-alternated minor. However, not every alternated minor verifies this condition as the following example shows:

Example 2. Consider the minor of C_{47}^9 defined by $W = W^0 \cup W^1$ where $W^0 = \{0, 10\}$ and $W^1 = \{3, 22\}$. It is easy to see that W defines a 2-alternated minor but $\delta_0 = 3$ and $\delta_2 = 12$.

However, we have:

Lemma 13. *Let $W \subset \mathbb{Z}_n$ be a subset defining a d-alternated minor. Then, $\delta_{s+d} = \delta_s (\text{mod } k)$ for all $s \in \mathbb{Z}_{|W|}$.*

Proof. We know that, since W defines a d-alternated minor, for any $s \in \mathbb{Z}_{|W|}$, i_s and i_{s+d} belong to W^j for some $j \in \mathbb{Z}_d$. Then, by Remark 2(ii), $i_{s+d} - i_s = 1$ (mod k).

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Hence, for all $s \in \mathbb{Z}_{|W|}$, we have that

$$\delta_{s+d} - \delta_s = (i_{s+1+d} - i_{s+d}) - (i_{s+1} - i_s) = (i_{s+1+d} - i_{s+1}) - (i_{s+d} - i_s),$$

proving that $\delta_{s+d} = \delta_s (\bmod\ k)$ for all $s \in \mathbb{Z}_{|W|}$. □

We also have the following result:

Lemma 14. *Let $W \subset \mathbb{Z}_n$ be a subset defining a d-alternated minor with $|W| = dn_3$. Then,*

(i) *if $\sum_{s=j}^{r} \delta_s = 0(\bmod\ k)$ for some $j \le r < j+d$, then $r = j+d-2$ and $\delta_{r+1} = 1$,*
(ii) *if $\delta_s = 1(\bmod\ k)$ for some $s \in \mathbb{Z}_d$ then $\delta_{s+td} = 1$ for all $t \in \mathbb{Z}_{n_3}$.*

Proof. In order to prove item (i), let $j, r \in \mathbb{Z}_{dn_3}$ with $j \le r < j+d$ and $\sum_{s=j}^{r} \delta_s = 0(\bmod\ k)$. Considering that

$$\sum_{s=j}^{r} \delta_s = i_{r+1} - i_j,$$

we have that $i_{r+1} - i_j = 0(\bmod\ k)$.

Since $r+1 \le j+d$ and $i_{j+d} - i_j = 1(\bmod\ k)$, then $r+1 < j+d$ and $i_j < i_{r+1} < i_{j+d}$.

W.l.o.g. let us assume that $j \in W^j$. Since $i_{j+d} \in W^j$, $i_{j+d} = i_j + tk + 1$ for some positive integer t and $i_j + t'k \in N^j$ for all $1 \le t' \le t-1$. Since $i_j < i_{r+1} < i_{j+d}$, $i_{r+1} - i_j = 0(\bmod\ k)$ and $i_{r+1} \notin N^j$, then $i_{r+1} = i_j + tk$.

Equivalently, $i_{r+1} = i_{j+d} - 1$, $r = j+d-2$ and $\delta_{r+1} = 1$.

To prove item (ii), we only need to observe that if $\delta_s = 1(\bmod\ k)$ for some s, then using the previous lemma for all $t \in \mathbb{Z}_{n_3}$ we have,

$$\sum_{j=s+(t-1)d+1}^{s+td-1} \delta_j = 0(\bmod\ k),$$

and by item (i), $\delta_{s+td} = 1$ for all $t \in \mathbb{Z}_{n_3}$. □

The previous results describe necessary conditions that the values in $\{\delta_s : s \in \mathbb{Z}_d\}$ associated with a subset $W \subset \mathbb{Z}_n$ must satisfy in order to define a d-alternated minor. Actually, we will see that these conditions characterize these subsets. For this purpose, let us define the following:

Definition 2. Given $k \ge 4$ and $2 \le d \le k-2$, let $R_{d,k} \subset \mathbb{Z}_k^d$ such that $(a_0, a_1, \ldots, a_{d-1}) \in R_{d,k}$ if and only if

(i) $\sum_{s=0}^{d-1} a_s = 1(\bmod\ k) \ge k+1$,
(ii) if $\sum_{s=j}^{r} a_s = 0(\bmod\ k)$ for some $0 \le j \le r \le d-1$ then $r = j+d-2$ and $j \in \{0, 1\}$.

Remark 4. Observe that:

(i) $R_{2,k} = \{(a_0, a_1) \in \mathbb{Z}_k^2 : a_0 + a_1 = 1(\bmod\ k)\}$ and
(ii) in general, $|R_{d,k}| = O(k^d)$.

So, we have the following characterization for d-alternated minors.

Theorem 15. *Let $d \ge 2$, $W = \{i_s : s \in \mathbb{Z}_{dn_3}\} \subset \mathbb{Z}_n$ and $W^j = \{i_{j+td} : t \in \mathbb{Z}_{n_3}\}$, for every $j \in \mathbb{Z}_d$. Then, W defines a d-alternated minor of C_n^k if and only if there exists $a \in R_{d,k}$ such that:*

(i) *$a_j = \delta_{j+td}(\bmod\ k)$ for all $j \in \mathbb{Z}_d$, $t \in \mathbb{Z}_{n_3}$ and*
(ii) *if $a_j = 1$ for some $j \in \mathbb{Z}_d$ then $\delta_{j+td} = 1$ for all $t \in \mathbb{Z}_{n_3}$.*

Proof. Let W be a subset defining a d-alternated minor of C_n^k. For every $j \in \mathbb{Z}_d$, let $a_j \in \mathbb{Z}_k$ such that $a_j = \delta_j(\bmod\ k)$.

We first prove that $a = (a_j)_{j \in \mathbb{Z}_d} \in R_{d,k}$. If $d = 2$, it is clear that $a = (a_0, a_1) \in R_{2,k}$.

Let $d \ge 3$. By definition of a and Remark 3,

$$\sum_{s=0}^{d-1} a_s = 1(\bmod\ k) \ge k+1$$

and condition (i) in Definition 2 is verified. Moreover, by Lemma 14(i), if $\sum_{s=j}^{r} a_s = 0(\bmod\ k)$ for some $0 \le j \le r \le d-1$ then $j+d = r+2$. Since $r+2 \le d+1$ then $j \in \{0, 1\}$ and condition (ii) in Definition 2 holds. Therefore, $a \in R_{d,k}$.

Moreover, from the definition and Lemma 14, a satisfies assumptions (i) and (ii).

Conversely, let $a \in R_{d,k}$ satisfy assumptions (i) and (ii). Since, for any $s \in \mathbb{Z}_{dn_3}$, $i_{s+d} - i_s = \sum_{j=s}^{s+d-1} \delta_j$, by Definition 2(i) it holds that $i_{s+d} - i_s = \sum_{j=0}^{d-1} a_j = 1(\bmod\ k)$ and then $i_{s+d} - i_s = 1(\bmod\ k) \ge k+1$. Then, from Remark 2(i), each W^j

induces a circulant minor with parameters $d = n_1 = 1$. Again from Remark 2, we only need to prove that subsets $N^j,\ j \in \mathbb{Z}_d$ are mutually disjoint.

Let us start with the case $d = 2$. Suppose that there exists $v \in N^0 \cap N^1$. W.l.o.g. we set $i_1 < v \le i_2$. Then, since $v \in N^0$, $v - i_0 = 0 \pmod k$ and since $v \in N^1$, $v - i_1 = 0 \pmod k$. Moreover, as $i_2 - i_0 = 1 \pmod k$, then $i_2 - v = 1 \pmod k$ and $\delta_1 = i_2 - i_1 = (i_2 - v) + (v - i_1) = 1 \pmod k$. Hence, $a_1 = 1$. By assumption (ii), $\delta_{1+t2} = 1$ for all $t \in \mathbb{Z}_{n_3}$ and it is not hard to check that, in this case, $N^0 \cap N^1 = \emptyset$, which is a contradiction.

Let $d \ge 3$. W.l.o.g., it is enough to prove that $N^0 \cap N^r = \emptyset$, for any $r \in \mathbb{Z}_d$, $r \ne 0$. To this end let $\widetilde{W} = W^0 \cup W^r$. We will see that $\widetilde{W}$ defines 2-alternated minor of C_n^k. By using the same arguments as in the case $d = 2$, we only need to find $\tilde{a} = (\tilde{a}_0, \tilde{a}_1) \in R_{2,k}$ satisfying assumptions (i) and (ii) for $\widetilde{W}$ with $\tilde{\delta}_{2t} = i_{r+td} - i_{td}$ and $\tilde{\delta}_{1+2t} = i_{(t+1)d} - i_{r+td}$ for all $t \in \mathbb{Z}_{n_3}$.

Let $\tilde{a}_0, \tilde{a}_1 \in \mathbb{Z}_k$ be such that $\tilde{a}_0 = i_r - i_0 \pmod k$ and $\tilde{a}_1 = i_d - i_r \pmod k$. Clearly, $\tilde{a} = (\tilde{a}_0, \tilde{a}_1) \in R_{2,k}$ and verifies assumption (i).

If $\tilde{a}_0 = 1, \tilde{a}_1 = 0$ i.e. $\sum_{i=r}^{d-1} \delta_i = 0 \pmod k$. Then, $\sum_{i=r}^{d-1} a_i = 0 \pmod k$. Hence, since $a \in R_{d,k}$ we have that $r = 1$ and $\tilde{a}_0 = a_0 = 1$. By hypothesis, $\tilde{\delta}_{2t} = \delta_{2t} = 1$ for all $t \in \mathbb{Z}_{n_3}$.

If $\tilde{a}_1 = 1, \tilde{a}_0 = 0$, i.e. $\sum_{i=0}^{r-1} \delta_i = 0 \pmod k$ and $\sum_{i=0}^{r-1} a_i = 0 \pmod k$. Hence, since $a \in R_{d,k}$, $r-1 = d-2$ and $\tilde{a}_1 = a_{d-1} = 1$. By hypothesis, $\tilde{\delta}_{1+2t} = \delta_{d-1+td} = 1$ for all $t \in \mathbb{Z}_{n_3}$. Therefore, $\tilde{a}$ satisfies assumption (ii) and the proof is complete. □

Given C_n^k and d, r integer numbers such that $2 \le d \le k-2$, $1 \le r \le k-d$ we define $\mathcal{A}(d, r)$ as the set of all subsets $W \subset \mathbb{Z}_n$ defining a d-alternated minor of C_n^k such that $|W| = dn_3$ with $n_3 = r \pmod{(k-d)}$. Moreover, if $a \in R_{d,k}$, we define the separation problem $C_n^k - SP(d, r, a)$ as follows:

INSTANCE: $\hat{x} \in \mathbb{R}^n$
QUESTION: Is there $W \in \mathcal{A}(d, r)$ such that $0 \in W$,
$\delta_s = a_s \pmod k$ for all $s \in \mathbb{Z}_d$ and
$\sum_{i \in W} c^d(\hat{x}_i) < L^{d,r}(\hat{x})$?

We will reduce $C_n^k - SP(d, r, a)$ to a shortest path problem in the digraph $K_n^k(d, r, a)$ with node set

$$V = \left(\bigcup_{i \in \mathbb{Z}_d,\, j \in \mathbb{Z}_{k-d+r}} V_j^i \right) \cup \{t\}$$

where $V_j^i = \{v_j^i(p) : p \in \mathbb{Z}_n\}$ for all $i \in \mathbb{Z}_d, j \in \mathbb{Z}_{k-d+r}$.

The set of arcs A of $K_n^k(d, r, a)$ is defined as follows: first consider in A the arcs $(v_0^0(0), v_0^1(p))$ for all $p = a_0 \pmod k$ and $1 \le p \le n-1$ when $a_0 \ne 1$, and $(v_0^0(0), v_0^1(1))$ when $a_0 = 1$.

Then consider in a recursive way:

- for each $(v, v_j^i(p)) \in A$ with $1 \le i \le d-2$
  if $a_i \ne 1$ then add $(v_j^i(p), v_j^{i+1}(q))$, for all q such that $p + a_i \le q \le n-1$ and $q - p = a_i \pmod k$, else add $(v_j^i(p), v_j^{i+1}(p+1))$;
- for each $(v, v_j^{d-1}(p)) \in A$ with $j \le k-d+r-2$
  if $a_{d-1} \ne 1$ then add $(v_j^{d-1}(p), v_{j+1}^0(q))$, for all $p + a_{d-1} \le q \le n-1$ and $q - p = a_{d-1} \pmod k$, else add $(v_j^{d-1}(p), v_{j+1}^0(p+1))$;
- for each $(v, v_{k-d-1}^{d-1}(p)) \in A$
  if $a_{d-1} \ne 1$ then add $(v_{k-d-1}^{d-1}(p), v_0^0(q))$, for all $p + a_{d-1} \le q \le n-1$ and $q - p = a_{d-1} \pmod k$, else add $(v_{k-d-1}^{d-1}(p), v_{k-d-1}^0(p+1))$.

Finally, consider the following arcs: for each $(v, v_{k-d+r-1}^{d-1}(p)) \in A$, if $a_{d-1} \ne 1$ then add $(v_{k-d+r-1}^{d-1}(p), t)$, for all $p \le n-1$ and $n - p = a_{d-1} \pmod k$, else add $(v_{k-d+r-1}^{d-1}(p), t)$ only when $p = n-1$.

In Fig. 2 we sketch the digraph $K_{29}^4(2, 1, (3, 2))$ where only the arcs corresponding to two $v_0^0(0)t$-paths are drawn.

Note that, by construction, if $(v_j^i(p), v_l^s(q)) \in A$ then$q > p$. Hence, $K_n^k(d, r, a)$ is acyclic.

We have the following result:

Lemma 16. *There is a one-to-one correspondence between $v_0^0(0)t$-paths in $K_n^k(d, r, a)$ and subsets $W \in \mathcal{A}(d, r)$ with $0 \in W$.*

Proof. Let $W \in \mathcal{A}(d, r)$. Then, for all $j \in \mathbb{Z}_d$, $W^j = \{i_{j+hd} : h \in \mathbb{Z}_{n_3}\}$ and $n_3 = \alpha(k-d) + r$ for some positive integer α.

For each $h \in \mathbb{Z}_{n_3}$ we define $t(h)$ such that $t(h) = h \pmod{(k-d)}$ and

- if $0 \le h \le \alpha(k-d) - 1$ then $t(h) \in \mathbb{Z}_{k-d}$
- if $\alpha(k-d) \le h \le \alpha(k-d) + r - 1$ then $k - d \le t(h) \le k - d + r - 1$.

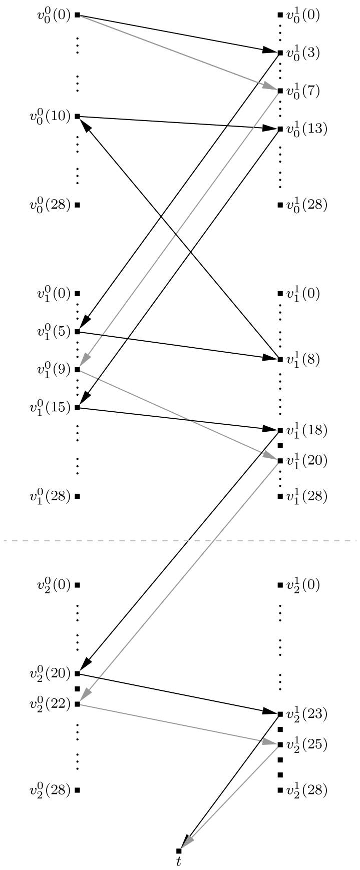

Fig. 2. Two $v_0^0(0)t$-paths in the digraph $K_{29}^4(2, 1, (3, 2))$.

Then, we associate with every $i_{j+hd} \in W^j$, the node $v_{t(h)}^j(i_{j+hd})$ and

$$\left\{v_{t(h)}^j(i_{j+hd}) : j \in \mathbb{Z}_d, h \in \mathbb{Z}_{n_3}\right\} \cup \{t\}$$

induces a $v_0^0(0)t$-path in $K_n^k(d, r, a)$.

Conversely, let P be a $v_0^0(0)t$-path in $K_n^k(d, r, a)$. By construction, there exists a positive integer α such that $|V(P) \cap V^j| = \alpha(k-d) + r$ for all $j \in \mathbb{Z}_d$. Hence, if we define

$$W^j = \{p \in \mathbb{Z}_n : v_i^j(p) \in V(P) \cap V^j \text{ for some } j \in \mathbb{Z}_{k-d},\ i \in \mathbb{Z}_{k-d+r}\},$$

then $|W^j| = \alpha(k-d) + r$. Clearly $W = \cup_{j \in \mathbb{Z}_d} W^j \in \mathcal{A}(d, r)$. □

Theorem 17. *The $C_n^k - SP(d, r, a)$ can be polynomially reduced to a shortest path problem in a weighted acyclic digraph.*

Proof. Let us consider the digraph $K_n^k(d, r, a)$ and assign the weight $c^d(\hat{x}_p)$ to every arc $(v_i^j(q), v_l^m(p)) \in A$ and the weight $c^d(\hat{x}_0)$ to every arc $(v_l^{d-1}(q), t) \in A$.

Clearly, if W is the subset corresponding to a $v_0^0(0)t$-path P in $K_n^k(d, r, a)$, the weight of P is equal to $\sum_{i \in W} c^d(\hat{x}_i)$.

Then, $\hat{x}$ violates an inequality corresponding to a circulant minor of C_n^k with parameters d and $n_1 = 1$ and subset W with $0 \in W$ if and only if the minimum weight of all $v_0^0(0)t$-paths P in $K_n^k(d, r, a)$ is less than $L^{d,r}(\hat{x})$.

Since $K_n^k(d, r, a)$ is acyclic, computing this minimum path can be done in polynomial time using for instance Bellman's algorithm [5]. □

Finally, the separation problem for inequalities corresponding to alternated minors can be formally stated as:

INSTANCE: $\hat{x} \in \mathbb{R}^n$
QUESTION: Is there an alternated minor whose corresponding inequality is violated by $\hat{x}$?

Hence, from Theorems 12 and 17 and Remark 4(ii), we have:

Theorem 18. *For a fixed k, the separation problem for inequalities corresponding to alternated minors of C_n^k can be solved in polynomial time.*

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5. Conclusions

In this paper we study the description of the set covering a polyhedron of circulant matrices. We associate a valid inequality with each circulant minor and we show (Theorem 5) that minor inequalities include an important family of facet defining inequalities. We also give a polynomial time algorithm to separate a subfamily of them.

As it was mentioned at the beginning of the paper, the dominating set polyhedron of web graphs is the set covering polyhedron of certain circulant matrices. Then, the results obtained so far have direct consequences on the dominating set problem on web graphs.

We also state some interesting open questions. Theorem 5 gives sufficient conditions for a minor inequality in order to define a facet. In this way, we give a conjecture (Conjecture 6) whose validity would imply that these conditions are also necessary. Moreover, the computational complexity of the separation problem for general minor inequalities remains open.

Finally, we would like to sketch some new research lines on this topic.

Given $N \subset \mathbb{Z}_n$ such that $C_n^k/N \approx C_{n'}^{k'}$ and following the proof of Theorem 3, if before adding up all the rows of $A = (C_n^k)_{\mathbb{Z}_n - R(N), \mathbb{Z}_n}$ we multiply them by a positive integer r and then we divide all the coefficients by k' and round up, we obtain the following Chvátal–Gomory inequality:

$$(r+1)\sum_{i \in W} x_i + r \sum_{i \notin W} x_i \geq \left\lceil \frac{rn'}{k'} \right\rceil .$$

Let us call r-*minor inequalities* those ones obtained by the previous procedure. Clearly, r-minor inequalities are valid for the set covering polyhedron of circulant matrices for every $r \in \mathbb{Z}_+$. The inequalities considered in this paper correspond to the case $r = 1$.

Initially, we strongly believed that the only r-minor inequalities defining a facet would be the 1-minor inequalities. The results in [10] for matrices C_n^3 for any $n \geq 5$ together with those in [9] for matrices C_{sk}^k, with $s = 1, 2$ and any $k \geq 2$ strengthened this conjecture. But recently in [22], the authors have found a 2-minor inequality defining a facet of $Q^*(C_{59}^4)$, contradicting our early beliefs.

This last result determines the lines of our current research work. Indeed, we focus on obtaining sufficient conditions and necessary conditions for an r-minor inequality to define a facet of the set covering polyhedron of a circulant matrix as well as studying the underlying separation problem. In addition, we are interested in characterizing classes of matrices for which these inequalities completely describe the polyhedron.

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