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Some advances on Lovász–Schrijver semidefinite programming relaxations of the fractional stable set polytope[☆]

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ABSTRACT

We study Lovász and Schrijver's hierarchy of relaxations based on positive semidefiniteness constraints derived from the fractional stable set polytope. We show that there are graphs G for which a single application of the underlying operator, N_+, to the fractional stable set polytope gives a nonpolyhedral convex relaxation of the stable set polytope. We also show that none of the current best combinatorial characterizations of these relaxations obtained by a single application of the N_+ operator is exact.

1. Introduction

Lovász and Schrijver [8] proposed an elegant, general framework to construct the convex hull of 0, 1 points in a given polytope P inside a hypercube, say $[0, 1]^n$. Such methods are called *Lift-and-Project Methods*. Among the methods proposed by Lovász and Schrijver [8], we can mention $N(\cdot)$ and $N_+(\cdot)$; the latter is the focus of this paper.

There are at least two major features of lift-and-project methods. First, by going to the matrix space $\mathbb{R}^{n\times n}$ we possibly square the number of variables; however, in some cases, by enforcing polynomially many linear inequalities in this higher dimensional space, we can generate exponentially many inequalities in the original space $\mathbb{R}^n$. One example of this is the $N(\cdot)$ operator applied to the *fractional stable set polytope* of a graph (which leads to odd-cycle polytope of the graph). A way to impose such valid inequalities in the matrix space is to represent $x \in \mathbb{R}^n$ by xx^T in $\mathbb{R}^{n\times n}$ and note that, for example, if x satisfies some linear inequalities, then every column of every $n \times n$ matrix representing x must also satisfy that inequality. The second feature of lift-and-project methods is that it is natural to add a nonlinear (but convex and tractable) constraint in this "lifted space". As we mentioned above, in this higher dimensional space, instead of only dealing with $x \in \mathbb{R}^n$, we can also deal with $xx^T \in \mathbb{R}^{n\times n}$. If this is the representation we choose, then xx^T is a symmetric, positive semidefinite matrix of rank at most one. Among these three requirements, relaxing the "rank is at most one" condition (which makes the problem nonconvex and intractable), we can derive tractable convex relaxations. The operator $N_+(\cdot)$ enforces such positive semidefiniteness

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constraints in the lifted matrix space. As we will see, $N_+(\cdot)$ operator can be much stronger than $N(\cdot)$ when applied to the relaxations of the stable set polytope.

The behavior of these operators $N(\cdot)$ and $N_+(\cdot)$ has been of particular interest when P is the *fractional stable set polytope* of a graph, given by

$$\mathrm{FRAC}(G) := \left\{x \in [0,1]^{V(G)} : x_u + x_v \le 1,\ \forall \{u,v\} \in E(G)\right\},$$

where $V(G)$, $E(G)$ denote the node set and the edge set of a graph G, respectively. For every graph G, $\mathrm{STAB}(G)$ denotes the convex hull of incidence vectors of stable sets in G. It is elementary to show that $\mathrm{STAB}(G)$ is the convex hull of integer points in $\mathrm{FRAC}(G)$. In general, $\mathrm{FRAC}(G) \neq \mathrm{STAB}(G)$ unless G is bipartite.

Let $\mathbb{S}^n$ denote the space of n-by-n symmetric matrices with real entries. Then, given a graph G,

$$M(G) := \{Y \in \mathbb{S}^{\{0\}\cup V(G)} : Ye_0 = \mathrm{diag}(Y), Ye_v \in \mathrm{cone}(\mathrm{FRAC}(G)), Y(e_0 - e_v) \in \mathrm{cone}(\mathrm{FRAC}(G)), \forall v \in V(G)\}.$$

In the above, 0 is the special homogenizing index, e_i is the ith unit vector, and

$$\mathrm{cone}(\mathrm{FRAC}(G)) := \left\{\begin{pmatrix} x_0 \\ x \end{pmatrix} \in \mathbb{R}^{\{0\}\cup V(G)} : x_u + x_v \le x_0,\ \forall \{u,v\} \in E(G), 0 \le x_v \le x_0,\ \forall v \in V(G)\right\}.$$

Projecting this lifting back to the space of $\mathrm{STAB}(G)$ results in

$$N(G) := \left\{x \in [0,1]^{V(G)} : \begin{pmatrix} x_0 \\ x \end{pmatrix} = Ye_0,\ \text{for some } Y \in M(G)\right\},$$

a relaxation of $\mathrm{STAB}(G)$ satisfying $N(G) \subseteq \mathrm{FRAC}(G)$.

Let $\mathbb{S}^n_+$ denote the space of n-by-n symmetric positive semidefinite (PSD) matrices with real entries. Then

$$M_+(G) := M(G) \cap \mathbb{S}_+^{\{0\}\cup V(G)}$$

yields the tighter relaxation of $\mathrm{STAB}(G)$

$$N_+(G) := \left\{x \in [0,1]^{V(G)} : \begin{pmatrix} x_0 \\ x \end{pmatrix} = Ye_0,\ \text{for some } Y \in M_+(G)\right\}.$$

If $\mathrm{TH}(G)$ denotes the *theta body* of G (see Lovász [4,7]) and $\mathrm{CLQ}(G)$ the polytope defined by the clique constraints that are valid for $\mathrm{STAB}(G)$, it is known that $\mathrm{TH}(G) \subseteq \mathrm{CLQ}(G)$ [4]. In [8], the authors gave a PSD representation for $\mathrm{TH}(G)$ that seems close to the definition of $N_+(G)$:

$$\mathrm{TH}(G) = \left\{x \in [0,1]^{V(G)} : \begin{pmatrix} x_0 \\ x \end{pmatrix} = Ye_0, Y_{uv} = 0, \forall \{u,v\} \in E(G), Ye_0 = \mathrm{diag}(Y),\ \text{for some } Y \in \mathbb{S}_+^{\{0\}\cup V(G)}\right\}.$$

Motivated by that result, we define

$$\hat{\mathrm{TH}}(G) := \left\{x \in [0,1]^{V(G)} : \begin{pmatrix} x_0 \\ x \end{pmatrix} = Ye_0, Ye_v \in \mathrm{cone}(\mathrm{FRAC}(G)), \forall v \in V(G),\right.$$
$$\left. Ye_0 = \mathrm{diag}(Y),\ \text{for some } Y \in \mathbb{S}_+^{\{0\}\cup V(G)}\right\}. \tag{1}$$

Clearly, $N_+(G) \subseteq \hat{\mathrm{TH}}(G) \subseteq \mathrm{TH}(G)$.

Lovász and Schrijver [8] proved that for every graph G, $N(G) = \mathrm{OC}(G)$, where $\mathrm{OC}(G)$ denotes the polytope defined by intersecting $\mathrm{FRAC}(G)$ with all the odd-cycle inequalities that are valid for $\mathrm{STAB}(G)$. To the best of our knowledge, no analogous characterization has been discovered for $N_+(G)$.

Let $\mathrm{ANTI\text{-}HOLE}(G)$ denote the polytope defined by all the anti-hole constraints that are valid for $\mathrm{STAB}(G)$ and let $\mathrm{WHEEL}(G)$ denote the polytope defined by all the wheel constraints that are valid for $\mathrm{STAB}(G)$ (for the underlying inequalities, see for instance [8]).

Given any graph G, let us define

$$\mathrm{LS}(G) := \mathrm{OC}(G) \cap \mathrm{ANTI\text{-}HOLE}(G) \cap \mathrm{WHEEL}(G) \cap \mathrm{CLQ}(G).$$

The following theorem follows from the results in [8]:

Theorem 1.1. *For every graph G,*

$$N_+(G) \subseteq \mathrm{LS}(G) \cap \mathrm{TH}(G).$$

The inclusion in the statement of Theorem 1.1 may be strict. This gives one of the motivations for the current paper: *Find a sharper description of $N_+(G)$ analogous to the partial description in* Theorem 1.1. Full characterizations analogous to Theorem 1.1 may be helpful in analyzing relaxations, approximation ratios and integrality gaps.

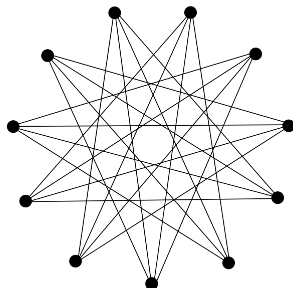

Fig. 1. The antiweb $\overline{W}_{11}^{3}$.

Note that LS(G) may have exponentially many facets and TH(G) may need uncountably many defining linear inequalities. Moreover, it is known that TH(G) is a polyhedron if and only if G is a perfect graph, (see for instance [4]) but for $N_+(G)$, no such characterization has been obtained yet. To the best of our knowledge, no graph with nonpolyhedral $N_+(G)$ is known. The closest existing results in the literature about the nonpolyhedrality of the relaxation obtained by the N_+ operator can be found in Bianchi's Ph.D. Thesis [1]. It was proved there that when the N_+ operator is applied to the relaxation of the matching polytope described by the nonnegativity and degree constraints, the resulting tighter relaxation can be nonpolyhedral. The second motivation of the current work is: *to show that $N_+(G)$ may not be a polyhedron*.

Let us present one of the main technical tools used by Lovász and Schrijver [8] in proving Theorem 1.1. Given a graph $G = (V, E)$ and a node v, we denote by $G \ominus v$ the graph obtained after the *destruction* of node v, that is the subgraph of G obtained after deleting v and its adjacent nodes in G. If $a^T x \leq b$ is a valid inequality for STAB(G), we denote by G_a its *support graph*, that is, the subgraph of G induced by the nodes with positive coefficients in the inequality.

Lemma 1.2 *([8]). Let $G = (V, E)$ and $a^T x \leq b$ be a valid inequality for* STAB(G)*. If, for every $v \in V(G_a)$, the inequality*

$$\sum_{i \in V(G_a \ominus v)} a_i x_i \leq b - a_v \tag{2}$$

is valid for FRAC($G_a \ominus v$)*, then $a^T x \leq b$ is a valid inequality for $N_+(G)$.*

It is well known that all the odd-cycle, anti-hole, wheel and clique constraints that are valid for STAB(G), satisfy the sufficient conditions given in the lemma above. However, there are examples of graphs G for which not every valid inequality of $N_+(G)$ satisfies these conditions. Hence, the third motivation for the current paper is: *to improve (strengthen) the technical tool provided by* Lemma 1.2.

We start towards these goals by considering the following questions:

Q.1. Is there a stronger relaxation of $N_+(G)$ than the one presented in Theorem 1.1? (Here, we are seeking a stronger relaxation which has an elegant, explicit description, analogous to the one given in Theorem 1.1.)

Q.2. Is $N_+(G)$ polyhedral for every G?

Q.3. Which valid inequalities for $N_+(G)$ do not satisfy the sufficient condition in Lemma 1.2?

In Sections 2–4 we provide answers to questions Q.1, Q.2 and Q.3, respectively.

2. A stronger relaxation of $N_+(G)$

A graph is called *near-bipartite* [10] if after the destruction of any node, the resulting graph is bipartite.

Therefore, by Lemma 1.2, every valid inequality for STAB(G) with near-bipartite support graph is also valid for $N_+(G)$. In particular, if G is near-bipartite then $N_+(G) = $ STAB(G). However there are near-bipartite graphs G for which STAB(G) does not coincide with LS(G). Consider as graph G the antiweb $\overline{W}_{11}^{3}$ in Fig. 1.

It is known that the *rank constraint*, $\sum_{v \in V(G)} x_v \leq \alpha(G)$, is needed in the description of STAB(G) [11]; but it is neither one of the inequalities of LS(G), nor implied by them. This motivates the definition of a new polyhedral relaxation of $N_+(G)$. For this purpose, let us recall that if G' is a node-induced subgraph of G ($G' := G[U]$ where $U \subseteq V$), then

$$\mathrm{STAB}(G) \subseteq \mathrm{STAB}(G') \oplus [0, 1]^{V(G) \setminus V(G')},$$

where for $S_1 \subseteq \mathbb{R}^n$ and $S_2 \subseteq \mathbb{R}^m$, $S_1 \oplus S_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} : x \in S_1, y \in S_2 \right\}$. For the sake of simplicity, using the above context, we define the *completion* of STAB(G') as

$$\mathrm{comp}_{V(G)} \left[\mathrm{STAB}(G') \right] := \mathrm{STAB}(G') \oplus [0, 1]^{V(G) \setminus V(G')}.$$

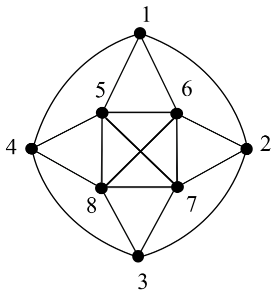

Fig. 2. The graph $\hat{G}$.

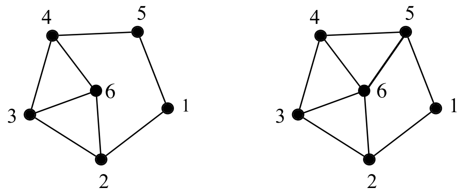

Fig. 3. The graphs G_{LT} and G_{EMN}.

If NB denotes the class of all near-bipartite graphs, given a graph G, we define

$$\mathrm{NB}(G) := \bigcap_{G'=G[U],U\subseteq V;G'\in \mathrm{NB}} \mathrm{comp}_V\left[\mathrm{STAB}(G')\right].$$

It is clear that if G is near-bipartite then $\mathrm{STAB}(G) = \mathrm{NB}(G)$. However, there are other classes of graphs for which this condition holds. For instance, perfect graphs and t-perfect graphs (i.e., a graph G for which $\mathrm{STAB}(G) = \mathrm{OC}(G)$) are examples of this kind.

From the definition of $\mathrm{NB}(G)$ and Lemma 1.2, it is clear that $N_+(G) \subseteq \mathrm{NB}(G)$. Since complete graphs, odd holes, odd antiholes and wheels are near-bipartite graphs, we have

$$\mathrm{NB}(G) \subseteq \mathrm{LS}(G)$$

and the inclusion can be strict (recall the graph $\overrightarrow{W}^3_{11}$). Then, we have a stronger relaxation of $N_+(G)$ analogous to the one given in Theorem 1.1.

Lemma 2.1. *For every graph G, $N_+(G) \subseteq \mathrm{NB}(G) \cap \hat{\mathrm{TH}}(G)$.*

Actually, in the following sections we analyze how tight the above relaxation of $N_+(G)$ is. At the time of this writing, we do not have an example of a graph G for which $N_+(G) \neq \mathrm{NB}(G) \cap \hat{\mathrm{TH}}(G)$.

3. A graph G with nonpolyhedral $N_+(G)$

As we have already mentioned, $\mathrm{TH}(G)$ is polyhedral if and only if G is a perfect graph, and in this case $\mathrm{TH}(G) = \mathrm{STAB}(G)$. In addition, if G is perfect then $N_+(G)$ is polyhedral, but it is known that it is not the only class of graphs for which this condition (polyhedrality of $N_+(G)$) holds. For instance, near-bipartite, t-perfect and all graphs for which $\mathrm{STAB}(G)$ coincides with $\mathrm{NB}(G)$ are graphs for which $N_+(G)$ is a polyhedron. It remains intriguing to characterize when $N_+(G)$ is polyhedral. In what follows we will identify a small, symmetric obstruction to the polyhedrality of $N_+(G)$, namely the 8-node graph in Fig. 2.

Let us denote by $\hat{G}$ that 8-node graph. Utilizing similar techniques as the ones used in [1] we will prove that $N_+(\hat{G})$ is nonpolyhedral. It will follow from our results that every graph G which contains $\hat{G}$ as an induced subgraph has $N_+(G)$ nonpolyhedral. It seems likely that there are smaller obstructions to nonpolyhedrality of $N_+(G)$. Any such obstruction must have at least 6 nodes and there are exactly two possibilities of such obstructions with 6 nodes, namely, G_{LT} and G_{EMN}, depicted in Fig. 3 (see [3,6]). In fact, $\hat{G}$ can be considered as a "symmetrization" of G_{EMN}. Moreover, G_{EMN} is an induced subgraph of $\hat{G}$ (for example, with node set $\{1, 2, 3, 4, 5, 6\}$).

Using the results in [9] (e.g., Theorem 4.11 of [9]) we know that

$$\mathrm{STAB}(\hat{G}) = \mathrm{CLQ}(\hat{G}) \cap \left\{ x \in \mathbb{R}^{V(G)} : \sum_{i=1}^{8} x_i \leq 2 \right\}. \tag{3}$$

We prove that a two-dimensional cross-section (defined by the linear subspace determined by the constraints $x_1 = x_2 = x_3 = x_4$ and $x_5 = x_6 = x_7 = x_8$) of the compact convex relaxation $N_+(\hat{G})$ has a nonlinear piece on its boundary. In order to do so, for every pair of nonnegative numbers α and β, let $z \in \mathbb{R}^8$ be such that

$$z_i := \begin{cases} \alpha & \text{if } i \in \{1,2,3,4\}, \\ \beta & \text{if } i \in \{5,6,7,8\}. \end{cases} \tag{4}$$

Since $N_+(\hat{G}) \subseteq \mathrm{CLQ}(\hat{G})$, every z defined by (4) which belongs to $N_+(\hat{G})$ must satisfy the nonnegativity and clique constraints, i.e.,

$$2\alpha + \beta \le 1, \qquad \alpha + 2\beta \le 1, \qquad 4\beta \le 1, \quad \alpha \ge 0,\ \beta \ge 0.$$

It is easy to check that the inequality $\alpha + 2\beta \le 1$ can be deduced from the other inequalities, leading us to the following definition:

Definition 3.1. Given nonnegative numbers α and β, we say that $z \in \mathbb{R}^8$ defined in (4) is an $\alpha\beta$-*point* and we write $z(\alpha, \beta)$, if α and β satisfy

$$2\alpha + \beta \le 1 \quad \text{and} \quad 4\beta \le 1. \tag{5}$$

The main result of this section is that the convex set of $\alpha\beta$-points in $N_+(\hat{G})$ is not a polyhedron. In order to prove it, we characterize the set of $\alpha\beta$-points in $N_+(\hat{G})$. Let us begin by considering an appropriate matrix $Y \in M_+(\hat{G})$ for such an $\alpha\beta$-point.

Definition 3.2. For nonnegative numbers α and β, let z be as defined in (4). If λ_α and λ_β are nonnegative numbers, we define $Y(z; \lambda_\alpha, \lambda_\beta)$ the symmetric matrix satisfying:

1. the diagonal and the zeroth column are equal to $(1, z)^T \in \mathbb{R}^9$,
2. for each $\{i, j\} \in E(\hat{G})$, $\left[Y(z; \lambda_\alpha, \lambda_\beta)\right]_{ij} := 0$,
3. for each $\{i, j\} \notin E(\hat{G})$ and $i \neq 0, j \neq 0$,

$$\left[Y(z; \lambda_\alpha, \lambda_\beta)\right]_{ij} := \begin{cases} \lambda_\alpha & \text{if } z_i = z_j = \alpha,\ i \neq j, \\ \lambda_\beta & \text{if } z_i = \alpha,\ z_j = \beta. \end{cases}$$

Then we have,

Lemma 3.3. *Let $z(\alpha, \beta)$ be an $\alpha\beta$-point. Then, $z \in N_+(\hat{G})$ if and only if there exist nonnegative numbers $\lambda_\alpha, \lambda_\beta$ such that*

$$\lambda_\alpha + \lambda_\beta \le \alpha, \qquad 2\lambda_\beta \le \beta, \qquad 2\lambda_\beta \le \alpha, \tag{6}$$

$$3\alpha - 1 \le \lambda_\alpha, \tag{7}$$

$$Y(z; \lambda_\alpha, \lambda_\beta) \textit{ is PSD}. \tag{8}$$

Proof. Trivially, if z is an $\alpha\beta$-point and there is a PSD matrix $Y(z; \lambda_\alpha, \lambda_\beta)$ for which $\lambda_\alpha, \lambda_\beta$ satisfy (6) and (7), then $Y(z; \lambda_\alpha, \lambda_\beta) \in M_+(\hat{G})$ and $z(\alpha, \beta) \in N_+(\hat{G})$.

Let $z \in N_+(\hat{G})$ and let $\mathcal{S}$ be the set of automorphisms of $\hat{G}$. Given $Y \in M_+(\hat{G})$ and $\sigma \in \mathcal{S}$, let $\sigma(Y)$ be the matrix such that, for every $i, j \in \{0, 1, \dots, 8\}$, $[\sigma(Y)]_{ij} := Y_{\sigma(i)\sigma(j)}$ where $\sigma(0) = 0$. It is not hard to see that $\sigma(Y) \in M_+(\hat{G})$. Moreover, as $M_+(\hat{G})$ is a convex set, defining

$$\overline{Y} := \frac{1}{|\mathcal{S}|} \sum_{\sigma \in \mathcal{S}} \sigma(Y),$$

we have that $\overline{Y} \in M_+(\hat{G})$.

It only remains to observe that if $Y \in M_+(\hat{G})$ and $Ye_0 = \binom{1}{z}$ then $\overline{Y} = Y(z; \lambda_\alpha, \lambda_\beta)$ for some nonnegative values $\lambda_\alpha, \lambda_\beta$. The conditions (6) and (7) follow from the facts that $Y(z; \lambda_\alpha, \lambda_\beta)e_i \in \mathrm{FRAC}(\hat{G})$ and $Y(z; \lambda_\alpha, \lambda_\beta)(e_0 - e_i) \in \mathrm{FRAC}(\hat{G})$, respectively. □

Observe that, by using (1), the same arguments in the above proof can be applied to $\hat{\mathrm{TH}}(\hat{G})$. Actually,

Remark 3.4. Let $z(\alpha, \beta)$ be an $\alpha\beta$-point. Then, $z \in \hat{\mathrm{TH}}(\hat{G})$ if and only if there exist nonnegative numbers $\lambda_\alpha, \lambda_\beta$ such that conditions (6) and (8) hold.

In order to characterize the points $z(\alpha,\beta)$ in $N_+(\hat{G})$ we must handle the PSD restriction of a matrix $Y(z;\lambda_\alpha,\lambda_\beta)$ with $\lambda_\alpha,\lambda_\beta$ satisfying (6) and (7). Indeed, we have

Lemma 3.5. *For nonnegative numbers $\alpha,\beta,\lambda_\alpha,\lambda_\beta$ satisfying (6), let*

$$\begin{aligned} q(\gamma) :=\; & (4\lambda_\beta^2 - \alpha\,\beta + 4\,\alpha^2\beta - \lambda_\alpha\,\beta - 16\,\alpha\,\lambda_\beta\,\beta + 4\,\alpha\,\beta^2 + 4\,\lambda_\alpha\beta^2) \\ & + (\alpha - 4\,\alpha^2 + \lambda_\alpha - 4\,\lambda_\beta^2 + \beta + \alpha\,\beta + \lambda_\alpha\,\beta - 4\,\beta^2)\gamma - (1+\alpha+\lambda_\alpha+\beta)\gamma^2 + \gamma^3. \end{aligned} \tag{9}$$

Let z be as in (4). Then, $Y(z;\lambda_\alpha,\lambda_\beta)$ is PSD if and only if the roots of the polynomial q are nonnegative.

Proof. The characteristic polynomial of the matrix $Y(z;\lambda_\alpha,\lambda_\beta)$ is

$$p(\gamma) := -(\alpha+\lambda_\alpha-\gamma)(\beta-\gamma)\left((-2\lambda_\beta^2+\alpha\,\beta-\lambda_\alpha\,\beta)-(\alpha-\lambda_\alpha+\beta)\gamma+\gamma^2\right)^2 q(\gamma).$$

Since the matrix $Y(z;\lambda_\alpha,\lambda_\beta)$ is symmetric, all the roots of $p(\gamma)$ are real. Clearly, from conditions (6), the roots of $(\alpha+\lambda_\alpha-\gamma)$ and $(\beta-\gamma)$ are nonnegative. The roots given by the factor

$$(-2\lambda_\beta^2+\alpha\,\beta-\lambda_\alpha\,\beta)-(\alpha-\lambda_\alpha+\beta)\gamma+\gamma^2$$

are

$$\gamma_3 = \frac{1}{2}\left(\alpha-\lambda_\alpha+\beta-\sqrt{\alpha^2-2\,\alpha\,\lambda_\alpha+\lambda_\alpha^2+8\lambda_\beta^2-2\,\alpha\,\beta+2\,\lambda_\alpha\,\beta+\beta^2}\right),$$

and

$$\gamma_4 = \frac{1}{2}\left(\alpha-\lambda_\alpha+\beta+\sqrt{\alpha^2-2\,\alpha\,\lambda_\alpha+\lambda_\alpha^2+8\lambda_\beta^2-2\,\alpha\,\beta+2\,\lambda_\alpha\,\beta+\beta^2}\right).$$

From (6), we have $\alpha-\lambda_\alpha+\beta\geq 0$. Then, proving that $\gamma_3\geq 0$ is equivalent to proving that:

$$(\alpha-\lambda_\alpha+\beta)^2-\left(\alpha^2-2\,\alpha\,\lambda_\alpha+\lambda_\alpha^2+8\lambda_\beta^2-2\,\alpha\,\beta+2\,\lambda_\alpha\,\beta+\beta^2\right)\geq 0.$$

Or equivalently:

$$\left(\alpha-\lambda_\alpha-\lambda_\beta\right)\beta+\left(\beta-2\lambda_\beta\right)\lambda_\beta\geq 0.$$

The last inequality holds by (6). Finally, observing that $\gamma_3\leq\gamma_4$, the claim of the lemma follows. □

We analyze the roots of the polynomial q defined in the previous lemma by using the same techniques as in [1] and based on the following results:

Theorem 3.6 (*Hurwitz [5]*). *Let $q(x)=q_0+q_1x+q_2x^2+\cdots+q_nx^n$ with $q_i\in\mathbb{R}$ for every $i\in\{0,\ldots,n\}$ and $q_n>0$. Then, all the roots of q have negative real part if and only if the determinants:*

$$\det[q_1],\ \det\begin{bmatrix} q_1 & q_0 \\ q_3 & q_2\end{bmatrix},\ \det\begin{bmatrix} q_1 & q_0 & 0 \\ q_3 & q_2 & q_1 \\ q_5 & q_4 & q_3\end{bmatrix},\ldots,\ \det\begin{bmatrix} q_1 & q_0 & 0 & \cdots & 0 \\ q_3 & q_2 & q_1 & \cdots & 0 \\ q_5 & q_4 & q_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ q_{2n-1} & q_{2n-2} & q_{2n-3} & \cdots & q_n \end{bmatrix}$$

are all positive. In the matrices above we let $q_r:=0$ if $r>n$.

As a consequence of this theorem, we have

Corollary 3.7 (*[1]*). *Let $q(x)=q_0+q_1x+q_2x^2+x^3$ be a polynomial with real coefficients. Then, the roots of q have nonnegative real part if and only if:*

$$q_0\leq 0,\qquad q_1\geq 0,\qquad q_2\leq 0\quad\text{and}\quad q_1q_2-q_0\leq 0.$$

Observe that after applying the above result to the polynomial q in Lemma 3.5, it yields that

$$\begin{aligned} -q_0 &= c_1(\lambda_\alpha,\lambda_\beta),\\ q_1 &= c_2(\lambda_\alpha,\lambda_\beta),\\ -q_2 &= 1+\alpha+\lambda_\alpha+\beta,\\ -q_1q_2+q_0 &= c_3(\lambda_\alpha,\lambda_\beta), \end{aligned}$$

where

$$c_1\left(\lambda_\alpha,\lambda_\beta\right) := -4\lambda_\beta^2 + 16\,\alpha\,\beta\,\lambda_\beta + (\beta-4\,\beta^2)\lambda_\alpha + \alpha\,\beta - 4\,(\alpha^2\beta+\alpha\,\beta^2),$$
$$c_2\left(\lambda_\alpha,\lambda_\beta\right) := -4\lambda_\beta^2 + \lambda_\alpha\,(1+\beta) + \alpha + \beta + \alpha\,\beta - 4\,(\alpha^2+\beta^2),$$
$$c_3\left(\lambda_\alpha,\lambda_\beta\right) := (\lambda_\alpha+\alpha+\beta+1)\,c_2\left(\lambda_\alpha,\lambda_\beta\right) - c_1\left(\lambda_\alpha,\lambda_\beta\right).$$

Hence, we can state,

Theorem 3.8. *Let $z(\alpha,\beta)$ and $\lambda_\alpha,\lambda_\beta$ satisfying (6). Then, $Y(z;\lambda_\alpha,\lambda_\beta)$ is PSD if and only if*

$$c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0,\qquad c_2\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0 \quad \textit{and} \quad c_3\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0.$$

Proof. Since $1+\alpha+\lambda_\alpha+\beta\geq 0$, $Y(z;\lambda_\alpha,\lambda_\beta)$ is PSD if and only if

$$c_1\left(\lambda_\alpha,\lambda_\beta\right)\geq 0,\qquad c_2\left(\lambda_\alpha,\lambda_\beta\right)\geq 0,\quad \text{and} \quad c_3\left(\lambda_\alpha,\lambda_\beta\right)\geq 0.$$

If $\lambda_\alpha,\lambda_\beta$ satisfy (6) and we define $\lambda'_\alpha=\alpha-\lambda_\beta$, then $(\lambda'_\alpha,\lambda_\beta)$ also satisfy (6) and $\lambda_\alpha\leq\lambda'_\alpha$. Then, it only remains to prove that the functions c_1, c_2 and c_3 are nondecreasing with respect to λ_α. If we differentiate them with respect to λ_α, we obtain:

- $\frac{\partial c_1}{\partial\lambda_\alpha}=\beta-4\beta^2=\beta\,(1-4\beta)$,
- $\frac{\partial c_2}{\partial\lambda_\alpha}=1+\beta$,
- $\frac{\partial c_2}{\partial\lambda_\alpha}=1+2\,\alpha\,(1-2\,\alpha+\beta)+2\,\lambda_\alpha+2\,\beta+2\,\lambda_\alpha\,\beta+\left(\beta-2\,\lambda_\beta\right)\left(\beta+2\,\lambda_\beta\right)$.

Using the facts that z is an $\alpha\beta$-point and inequalities in (6) hold, the functions above are nonnegative and the proof is complete. □

We summarize all the results obtained so far in the following:

Corollary 3.9. *Let $z(\alpha,\beta)$ be an $\alpha\beta$-point. Then, the following statements are equivalent:*

1. $z\in N_+(\hat{G})$*;*
2. $z\in \hat{\text{TH}}(\hat{G})$*;*
3. *there exists* $0\leq\lambda_\beta\leq\min\left\{\frac{\alpha}{2},\frac{\beta}{2}\right\}$ *such that*

$$c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0,\qquad c_2\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0 \quad \textit{and} \quad c_3\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0.$$

Proof. Observe that (i) trivially implies (ii). Now, let $z\in\hat{\text{TH}}(\hat{G})$; then by Remark 3.4, there exist nonnegative numbers λ_α and λ_β satisfying inequalities (6) and condition (8). Then (iii) follows from the previous theorem.

Finally, let $0\leq\lambda_\beta\leq\min\left\{\frac{\alpha}{2},\frac{\beta}{2}\right\}$ and set $\lambda'_\alpha:=\alpha-\lambda_\beta$. By assumption, $z(\alpha,\beta)$ satisfies (5). Then, $\beta+2\alpha\leq 1$. Since $\lambda_\beta\leq\beta/2$, it follows that $\lambda_\beta\leq 1-2\alpha$ or equivalently $\alpha-\lambda_\beta\geq 3\alpha-1$. Hence, $\lambda_{\alpha'}$ and λ_β satisfy (6) and (7). Applying Theorem 3.8 and Lemma 3.3, we conclude that $z\in N_+(\hat{G})$. □

In order to show that $N_+(\hat{G})$ is not a polyhedron, we identify a nonlinear piece on its boundary, by restricting to $\alpha\beta$-points in $N_+(\hat{G})\setminus\text{STAB}(\hat{G})$. Recall that the only facet of $\text{STAB}(\hat{G})$ that is not a clique inequality is the full rank inequality (3). This allows us to consider only $\alpha\beta$-points in the set A (see Fig. 4) given by

$$A:=\left\{z\,(\alpha,\beta):0\leq\beta\leq\frac{1}{4},\ \beta+2\alpha\leq 1,\ 2\alpha+2\beta\geq 1\right\}.$$

Now, for $\alpha\beta$-points in $A\cap N_+(\hat{G})$, conditions in Corollary 3.9 can be simplified as follows:

Lemma 3.10. *Let $z(\alpha,\beta)\in A$. Then, $z\in N_+(\hat{G})$ if and only if there exists $0\leq\lambda_\beta\leq\frac{\beta}{2}$ satisfying $c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0$.*

Proof. Observe that, for every $z(\alpha,\beta)\in A$, we have $\beta\leq\alpha$. By Corollary 3.9, $z\in N_+(\hat{G})$ if and only if $0\leq\lambda_\beta\leq\frac{\beta}{2}$, $c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0$, $c_2\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0$, and $c_3\left(\alpha-\lambda_\beta,\lambda_\beta\right)\geq 0$.

For $\lambda_\beta\in[0,\frac{\beta}{2}]$ we define

$$g(\lambda_\beta):=c_2\left(\alpha-\lambda_\beta,\lambda_\beta\right)-c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right)$$

and

$$h(\lambda_\beta)=c_3\left(\alpha-\lambda_\beta,\lambda_\beta\right)-c_1\left(\alpha-\lambda_\beta,\lambda_\beta\right).$$

To prove the result we only need to show that $g(\lambda_\beta)\geq 0$ and $h(\lambda_\beta)\geq 0$ for every $\lambda_\beta\in[0,\frac{\beta}{2}]$. It is not difficult to see that they are decreasing functions for $\lambda_\beta\in[0,\frac{\beta}{2}]$ and then, it is enough to prove that $g\left(\frac{\beta}{2}\right)\geq 0$ and $h\left(\frac{\beta}{2}\right)\geq 0$. Taking

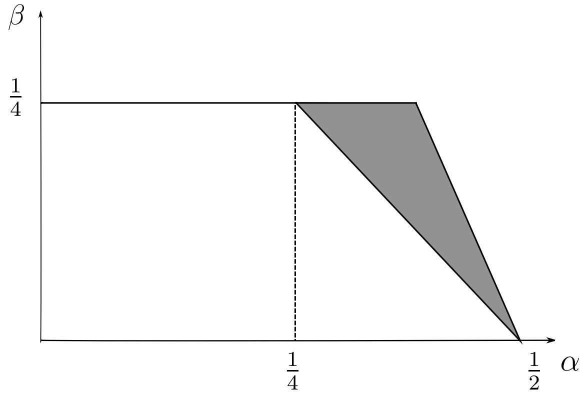

Fig. 4. The set A corresponds to the shaded region.

into account that

$$g\left(\frac{\beta}{2}\right) = 2\alpha - 4\alpha^2 + \frac{\beta}{2} + 4\alpha^2\beta - 4\beta^2 - 2\beta^3$$

and

$$h\left(\frac{\beta}{2}\right) = \frac{1}{4}\left[8\alpha - 32\alpha^3 + 2\beta + 40\alpha^2\beta - 9\beta^2 - 40\alpha\beta^2 - 27\beta^3\right],$$

it can be easily checked that the minimum values of $g\left(\frac{\beta}{2}\right)$ and $h\left(\frac{\beta}{2}\right)$ are both achieved at $\beta = 1/4$.

Finally, it is not hard to see that, for every $\alpha \in [1/4, 1/2]$, we have $g\left(\frac{1}{8}\right) \geq 0$ and $h\left(\frac{1}{8}\right) \geq 0$. $\square$

We are now ready to present the main result of this section.

Theorem 3.11. *Let α and β be nonnegative numbers satisfying $2\alpha + \beta \leq 1$ and $4\beta \leq 1$. An $\alpha\beta$-point belongs to $N_+(\hat{G}) \cap A$ if and only if*

$$\beta \leq \frac{3 - \sqrt{1 + 8(-1 + 4\alpha)^2}}{8}.$$

Proof. For every $\lambda_\beta \in [0, \frac{\beta}{2}]$ we define $f(\lambda_\beta) := c_1\left(\alpha - \lambda_\beta, \lambda_\beta\right)$. Let $z(\alpha, \beta)$. By Lemma 3.10, $z \in N_+(\hat{G}) \cap A$ if and only if there exists $\lambda_\beta \in [0, \frac{\beta}{2}]$ such that $f(\lambda_\beta) \geq 0$. We will prove that f is a nondecreasing function in $[0, \frac{\beta}{2}]$ and then $f(\lambda_\beta) \geq 0$ if and only if $f(\frac{\beta}{2}) \geq 0$.

Recall that

$$f\left(\lambda_\beta\right) = -4\,\lambda_\beta^2 + 2\,\alpha\,\beta - 4\,\alpha^2\beta + \lambda_\beta\,\beta + 16\,\alpha\,\lambda_\beta\,\beta - 8\,\alpha\,\beta^2 + 4\,\lambda_\beta\,\beta^2,$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda_\beta} f\left(\lambda_\beta\right) &= 4\left(-2\lambda_\beta + \beta\right) + \beta\left(-5 + 16\alpha + 4\beta\right)\\ &= 4\left(-2\lambda_\beta + \beta\right) + \beta\left[(-5 + 12\alpha) + 4(\alpha + \beta)\right].\end{aligned}$$

Observe that for $z \in A$, we have $\alpha \geq \frac{1}{4}$ and $4(\alpha + \beta) \geq 2$. Hence,

$$\frac{\partial}{\partial\lambda_\beta} f\left(\lambda_\beta\right) \geq 4\left(-2\lambda_\beta + \beta\right) + \beta(-3 + 12\alpha).$$

Thus, $\frac{\partial}{\partial\lambda_\beta} f\left(\lambda_\beta\right) \geq 0$.

It only remains to verify that condition $f(\frac{\beta}{2}) \geq 0$ is equivalent to

$$\beta \leq \frac{1}{8}\left(3 - \sqrt{1 + 8\left(-1 + 4\alpha\right)^2}\right).$$

By definition,

$$f\left(\frac{\beta}{2}\right) = 2\alpha\,\beta - 4\alpha^2\beta - \frac{3}{2}\beta^2 + 2\beta^3$$
$$= \frac{\beta}{32}\left((-3+8\beta)^2 - 1 - 8\,(-1+4\alpha)^2\right);$$

then,

$$f\left(\frac{\beta}{2}\right) \geq 0 \quad \text{if and only if} \quad (-3+8\beta)^2 \geq 1 + 8\,(-1+4\alpha)^2 .$$

Since $z \in A$, we have $-3+8\beta \leq 0$ yielding

$$f\left(\frac{\beta}{2}\right) \geq 0 \quad \text{if and only if} \quad 3 - 8\beta \geq \sqrt{1+8\,(-1+4\alpha)^2},$$

or equivalently,

$$f\left(\frac{\beta}{2}\right) \geq 0 \quad \text{if and only if} \quad \beta \leq \frac{1}{8}\left(3 - \sqrt{1+8\,(-1+4\alpha)^2}\right). \quad \square$$

The result in Theorem 3.11 allows us to establish the following:

Corollary 3.12. *For any graph G having $\hat{G}$ as an induced subgraph, $N_+(G)$ is not a polyhedron.*

4. More valid inequalities for $N_+(\hat{G})$

Let us observe that Theorem 3.11 provides an infinite family of valid inequalities for $N_+(\hat{G})$ that do not satisfy the conditions of Lemma 1.2. Actually, we can state,

Theorem 4.1. *For $\alpha_0 \in \left(\frac{1}{4}, \frac{1}{2}\right]$, let*

$$a(\alpha_0) := 4(4\alpha_0 - 1), \qquad b(\alpha_0) := \sqrt{1+8(4\alpha_0-1)^2} \quad \text{and}$$
$$c(\alpha_0) := \frac{3}{2}\sqrt{1+8(4\alpha_0-1)^2} + 16\alpha_0 - \frac{9}{2}.$$

Then,

$$a(\alpha_0)(x_1+x_2+x_3+x_4) + b(\alpha_0)(x_5+x_6+x_7+x_8) \leq c(\alpha_0) \tag{10}$$

is a valid inequality for $N_+(\hat{G})$ that does not satisfy the conditions of Lemma 1.2.

Proof. Let us observe that if there exists a point $x \in N_+(G)$ violating an inequality of the form (10), then by convexity there exists an $\alpha\beta$-point violating this same inequality. Thus, it is enough to prove that

$$4a(\alpha_0)\alpha + 4b(\alpha_0)\beta \leq c(\alpha_0)$$

is valid for any $\alpha\beta$-point in $N_+(\hat{G})$. Indeed, this fact follows after computing the tangent line to the function

$$g(\alpha) := \frac{3 - \sqrt{1+8(4\alpha_0-1)^2}}{8}$$

at the point $(\alpha_0, g(\alpha_0))$ for $\alpha_0 \in \left(\frac{1}{4}, \frac{1}{2}\right]$ and observing that this tangent line is exactly $4a(\alpha_0)\alpha + 4b(\alpha_0)\beta = c(\alpha_0)$.

Finally, it is easy to check that the point $x = \left(1, 0, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2}\right)^T \in \text{FRAC}(\hat{G}) \cap \{x : x_1 = 1\}$ and violates the inequality (10) for every $\alpha_0 \in \left(\frac{1}{4}, \frac{1}{2}\right]$. In other words, inequality (10) does not satisfy the conditions of Lemma 1.2 when $G = \hat{G}$ in Fig. 2 and $v = 1$. $\square$

Also observe that by Corollary 3.9, the inequalities in (10) are valid inequalities for $\hat{\text{TH}}(\hat{G})$, for every $\alpha_0 \in \left(\frac{1}{4}, \frac{1}{2}\right]$.

The results presented herein lead us to wonder if every valid inequality for $N_+(G)$ which is not valid for $\hat{\text{NB}}(G)$ is valid for $\hat{\text{TH}}(G)$ or, equivalently, whether $N_+(G) = \text{NB}(G) \cap \hat{\text{TH}}(G)$.

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