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$\{k\}$-domination for chordal graphs and related graph classes [1]

Gabriela Argiroffo[a] Valeria Leoni[a,b] Pablo Torres[a,b] [2]

[a] *Depto. de Matemática, Universidad Nacional de Rosario, Rosario, Argentina.*

[b] *Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina.*

Abstract

In this work we obtain a new graph class where the $\{k\}$-dominating function problem ($\{k\}$-DOM) is NP-complete: the class of chordal graphs. We also identify some maximal subclasses for which it is polynomial time solvable. Firstly, by relating this problem with the k-dominating function problem (k-DOM), we prove that $\{k\}$-DOM is polynomial time solvable for strongly chordal graphs. Besides, by expressing the property involved in k-DOM in Counting Monadic Second-order Logic, we obtain that both problems are linear time solvable for bounded tree-width graphs. Finally, we show that $\{k\}$-DOM is linear time solvable for spider graphs.

Keywords: $\{k\}$-dominating function, k-dominating function, computational complexity. chordal graph

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[2] Emails:{garua,valeoni,ptorres}@fceia.unr.edu.ar

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1 Introduction

Due to its large range of applications, many variations and extensions of the classical domination problem in graphs have been defined and studied. In general, these problems can be stated as follows: given a graph $G = (V, E)$, $A \subseteq \mathbb{R}$ and $B = \{b_1, \ldots, b_{|V|}\}$, an A, B-*dominating function* of G is a function $f : V \mapsto A$ such that $f(N[v_i]) \geq b_i$ for all $v \in V$, where $f(U) = \sum_{u \in U} f(u)$, for $U \subseteq V$ and $N[v]$ is the *closed neighborhood* of v, i.e. the set of vertices at distance at most 1 from v. The *weigth* of f is given by $w(f) = f(V)$, and $W(G)$ denotes the minimum possible value of $w(f)$.

Clearly, if $A = \{0, 1\}$ and $b_i = 1$ for all $i \in \{1, \ldots, |V|\}$, f is a *dominating funtion* and $W(G)$ is the usual *domination number* of the graph G. This work is focused on two variations of the problem. Let $k \in \mathbb{Z}_+$ and $b_i = k$ for all $i \in \{1, \ldots, |V|\}$. When $A = \{0, 1\}$, f is a k*-tuple dominating function* and $W(G)$ is the k*-tuple domination number* of G denoted by $\gamma_{\times k}(G)$, introduced by Harary and Haynes in [9]. When $A = \{0, 1, \ldots, k\}$, f is a $\{k\}$*-dominating function* and $W(G)$ is the $\{k\}$*-domination number* of G introduced by Bange et al. [1] and denoted by $\gamma_{\{k\}}(G)$. As usual, these definitions induce the study of the following decision problems:

k-TUPLE DOMINATING FUNCTION (k-DOM)
Instance: $G = (V, E)$, $j \in \mathbb{N}$
Question: Does G have a k-tuple dominating function of weight at most j?

$\{k\}$-DOMINATING FUNCTION ($\{k\}$-DOM)
Instance: $G = (V, E)$, $j \in \mathbb{N}$
Question: Does G have a $\{k\}$-dominating function of weight at most j?

Regarding computational complexity results of the above two problems, on the one hand, it is known that if $k = 1$, k-DOM concerns the usual domination function and is linear time solvable for cactus graphs [11], for series-parallel graphs [12] and for the more general class of bounded tree-width graphs [3,6]. For fixed $k \in \mathbb{Z}_+$, k-DOM is linear for trees [8] and polynomial time solvable for strongly chordal graphs [13] and P_4-tidy graphs [5], but it is NP-complete for chordal graphs (even for split graphs) and also for bipartite graphs [13]. On the other hand, Goddard and Henning proved that, for fixed $k \in \mathbb{Z}_+$, $\{k\}$-DOM is NP-complete for planar graphs [8]. Besides, it is NP-complete for bipartite graphs [10].

The main purpose of this work is to provide a new graph class where $\{k\}$-DOM is NP-complete and also to identify some maximal subclasses for which it is polynomial time solvable. More precisely, we prove that it is NP-complete for split graphs, thus, also for chordal graphs (Section 2). In Section 3, we first obtain that $\{k\}$-DOM is polynomial time solvable for strongly chordal graphs. By expressing the property involved in k-DOM in Counting Monadic Second-order Logic, we prove that both problems are linear time solvable for bounded tree-width graphs.

A graph G is *chordal* if it does not contain an induced chordless cycle C_n, for $n \geq 4$. A graph G is a *split graph* if its vertex set admits a partition into a complete set Q and a stable set S. Every split graph is a chordal graph.

2 NP-completeness of $\{k\}$-DOM for split graphs

As already mentioned, $\{k\}$-DOM is NP-complete for general graphs. In this section we state that it is NP-complete even for split graphs. We begin by providing the following property of $\{k\}$-dominating functions.

Lemma 2.1 *Let $G = (V, E)$ be a graph and $k \in \mathbb{Z}_+$. Let $u, v \in V$ such that $N[u] \subseteq N[v]$. There exists a $\{k\}$-dominating function f of G such that $f(u) = 0$ and $w(f)$ is minimum.*

For a graph $G = (V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$, we construct a split graph $H = (V', E')$ with partition (Q, S) defined as follows:

- $Q = \{q_1, \ldots, q_n\}$ is a complete set and $S = \{s_1, \ldots, s_n\}$ is a stable set,
- for all $i, j = 1, \ldots, n$, $q_i s_j \in E'$ if and only if $v_i v_j \in E$ or $i = j$.

From Lemma 2.1, there exists a minimum weight $\{k\}$-dominating function f of H such that $f(S) = 0$. From this fact, we can prove that $\gamma_{\{k\}}(G) = \gamma_{\{k\}}(H)$. As a consequence, we obtain the main result of this section:

Theorem 2.2 *For every fixed $k \in \mathbb{Z}_+$, $\{k\}$-DOM is NP-complete for split graphs.*

As a split graph is also a chordal graph, it is clear that $\{k\}$-DOM is NP-complete for chordal graphs.

3 Polynomial instances of $\{k\}$-DOM

As $\{k\}$-DOM is NP-complete for chordal graphs, it would be worthwhile to determine subclasses of chordal graphs for which it is polynomial time solvable.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs, the *strong product* $G_1 \otimes G_2$ is defined on the vertex set $V_1 \times V_2$, where two vertices (u_1, v_1) and

(u_2, v_2) are adjacent if and only if $u_1 = u_2$ and $v_1v_2 \in E_2$, or $v_1 = v_2$ and $u_1u_2 \in E_1$, or $v_1v_2 \in E_2$ and $u_1u_2 \in E_1$.

In [2] it is proved that for any graph G and $k \geq 1$, $\gamma_{\{k\}}(G) = \gamma_{\times k}(G \otimes K_k)$, where K_k denotes the complete graph on k vertices. As a consequence, we can state the following result.

Corollary 3.1 *Let $k \in \mathbb{N}$ be fixed and let $\mathcal{F}$ and $\mathcal{S}$ be two graph classes such that if $G \in \mathcal{F}$, then $G \otimes K_k \in \mathcal{S}$. If k-DOM is polynomial (linear) time solvable for $\mathcal{S}$, then $\{k\}$-DOM is polynomial (linear) time solvable for $\mathcal{F}$.*

Strongly Chordal Graphs. A graph G is *strongly chordal* if it is chordal and every cycle of even length in G has an odd chord, i.e. an edge that connects two vertices that are at odd distance apart from each other in the cycle. In order to study $\{k\}$-DOM for strongly chordal graphs, we apply their characterization in terms of a simple elimination ordering [7].

Lemma 3.2 *If G is a strongly chordal graph, then $G \otimes K_k$ also is.*

Finally, as mentioned before, in [13] it is proved that k-DOM is polynomial time solvable for strongly chordal graphs. Then, from the lemma above and Corollary 3.1, we can state the following result.

Theorem 3.3 $\{k\}$-DOM *is polynomial time solvable for strongly chordal graphs.*

Bounded tree-width Graphs. We introduce some concepts: a graph property P is expressible in *counting monadic second-order logic*, $CMSOL$ for short (see [4] for further reference), if P can be defined using vertices, edges, sets of vertices and sets of edges of a graph, the logical operators OR, AND, NOT, the logical quantifiers $\forall$ and $\exists$ over vertices, edges, sets of vertices or sets of edges, the membership relation $\in$, the equality operator $=$ for vertices and edges, the unary cardinality operator *card* for sets of vertices, the binary adjacency relation *adj* (where $adj(u, v)$ holds if and only if u and v are adjacent vertices) and the binary incidence relation *inc* (where $inc(v, e)$ holds if and only if edge e is incident to vertex v). $CMSOL$ is useful when combined with graph tree-width. Moreover, if P is a graph property expressible in $CMSOL$ and c is a constant, then, for any graph G with tree-width at most c, it can be checked in linear time whether G has property P [4].

We can prove that given a tree decomposition of width t of a graph G, it can be found a tree decomposition of width $k(t + 1) - 1$ of $G \otimes K_k$. We have:

Lemma 3.4 *If a graph G has bounded tree-width, then $G \otimes K_k$ also has.*

Given integers k and t and a graph G, we first prove that the property $\gamma_{\times k}(G) \leq t$ can be expressed in CMSOL. As a consequence, we obtain that k-DOM can be solved in linear time for all graph classes with tree-width bounded by a constant. Afterwards, from these facts and from Corollary 3.1 and Lemma 3.4, we obtain the following result.

Theorem 3.5 $\{k\}$-DOM *can be solved in linear time for all graph classes with tree-width bounded by a constant.*

In particular, the theorem above states that for fixed k, $\{k\}$-DOM can be solved in linear time for the subclass of chordal graphs given by k-trees.

Spider Graphs. A *spider graph* is a graph whose vertex set can be partitioned into S, C and R, where $S = \{s_1, \ldots, s_r\}$ is a stable set, $C = \{c_1, \ldots, c_r\}$ is a complete set, $r \geq 2$ and all the vertices in R are adjacent to all the vertices in C and non-adjacent to all the vertices in S. In a *thin spider graph*, s_i is adjacent to c_j if and only if $i = j$, and in a *thick spider graph*, s_i is adjacent to c_j if and only if $i \neq j$. The triple (S, C, R) can be found in linear time [14].

Theorem 3.6 *Let G be a spider graph with partition (S, C, R).*

(i) *If G is a thin spider graph, then $\gamma_{\{k\}}(G) = k|C|$.*

(ii) *If G is a thick spider graph, then $\gamma_{\{k\}}(G) = k + 1$ if $k + 1 \leq |C|$ and $\gamma_{\{k\}}(G) = k + \left\lceil \frac{k}{|C|-1} \right\rceil$ if $k \geq |C|$.*

Corollary 3.7 $\{k\}$-DOM *is linear time solvable for spider graphs.*

In this work we proved that $\{k\}$-DOM is NP-complete for chordal graphs, even for split graphs. Nevertheless, we proved that it is polynomial time solvable for some subfamilies of chordal graphs (namely, strongly chordal graphs, k-trees for fixed k, and any spider for which the graph induced by its head is chordal). Moreover, we obtained that $\{k\}$-DOM is polynomial time solvable for general spider graphs, which play an important role in the modular decomposition of graph classes with a bounded number of P_4's (see [15]). To combine modular decomposition and $\{k\}$-dominating functions could be a promising line of future work.

In addition, from Theorem 3.5 and due to the relationship between $\{k\}$-DOM and k-DOM, we enlarged the family of graph classes for which k-DOM is linear time solvable.

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