ROLLING OF A SYMMETRIC SPHERE ON A HORIZONTAL PLANE WITHOUT SLIDING OR SPINNING

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In this paper we perform a complete study of the dynamics of a symmetric sphere rolling on a horizontal plane without sliding or spinning. Integrals of motion that completely determine the behaviour of this systems in terms of elementary functions are explicitly written. Equilibrium points and closed orbits are systematically described. Our approach is geometric and we find that the system is equivalent to an ODE on the manifold $S^2 \times S^1$.

Keywords: nonholonomic; integrability; symmetric rolling sphere.

1. Introduction

In [1] a detailed study of the behaviour of a heavy homogeneous sphere rolling on a horizontal plane, under the assumption that the area of contact is a small circle and that the force of friction between the sphere and the plane obeys the Coulomb law, has been realized. In particular, it is shown that after a finite time the sphere rolls without sliding or spinning, in which case the dynamics is very simple, namely, the center of the sphere moves along a straight line. Dissipative effects are assumed to appear only during that part of the motion where there is sliding or spinning. See for instance, among several others, [1]–[14], for references on nonholonomic systems and in particular on rolling spheres. According to [10] the no spinning condition would be a simplified model for a rubber ball.

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The no spinning condition is a particular case of the so called *Veselova's constraints* [13]. For a nonhomogeneous sphere subjected to the no sliding or spinning condition the motion is not as simple as in the case of a homogeneous sphere. The integrability of the general case has been established recently by Borisov and Mamaev [14], as part of a general method which includes usage of the Euler–Jacobi theorem, existence of invariant measures and Hamiltonization. However, the detailed description of the dynamics, for instance periodic orbits, equilibrium points, has not been performed yet, to the best of our knowledge.

The purpose of the present work is to give a complete description of the dynamics of a *symmetric* sphere rolling on a horizontal plane without sliding or spinning and with no dissipation of energy. By definition, a symmetric sphere has a distribution of mass such that the center of mass coincides with the center of the sphere, while two of the three principal moments of inertia are equal. Our approach is geometric, integrability follows after we write the equations on the manifold $S^2 \times S^1$, which appears naturally.

2. Equations of motion

Description of the model. We shall model our system as a nonholonomic system on the group $SO(3) \times \mathbb{R}^2$. More precisely, we assume that there is an orthonormal basis fixed in the space, say (e_1, e_2, e_3), $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and also an orthonormal basis moving with the body, (Ae_1, Ae_2, Ae_3), where $A = A(t)$ depends on time. We introduce the variable $z \in S^2$, given by $z = Ae_3$. The spatial angular velocity ω can be written as $\omega = v_0 z + z \times \dot{z}$, so $v_0 = \langle \omega, z \rangle$ is the component of ω along z. The nonholonomic constraint is given by the condition $\omega \times re_3 = \dot{x}$, which is similar to the non sliding condition for a rigid sphere, plus the extra condition (Veselova's constraint) that the vertical component of the spatial angular velocity is 0, that is, $\omega_3 = 0$.

The Lagrangian of the system is given by the kinetic energy

$$\frac{1}{2} I_1 \dot{z}^2 + \frac{1}{2} I_3 v_0^2 + \frac{1}{2} M \dot{x}^2.$$

Using the nonholonomic constraint we can conclude that the kinetic energy of the actual motion of the symmetric sphere is given by

$$E = \frac{1}{2}(I_1 + Mr^2)\dot{z}^2 + \frac{1}{2}(I_3 + Mr^2)v_0^2,$$

which is a preserved quanity.

Derivation of equations. Equations of motion for a general sphere were written in [14], using multipliers. Our derivation is slightly different, for instance equation of balance of momentum (2.2) is written in terms of the *spatial* angular velocity. This leads to a geometric description of the system as an ODE on the manifold $S^2 \times S^1$ which has an expression in terms of elementary functions. One can also derive the same equations using the methods of [2]. We introduce dimensionless quantities $\alpha = I_3/I_1$ and $\beta = Mr^2/I_1$. Moreover, we will assume, without any essential loss of generality, that $I_1 = I_2 = 1$ and $r = 1$, then $\alpha = I_3$ and $\beta = M$.

The quantity $A(I\dot{\Omega} - I\Omega \times \Omega)$, where $\Omega = A^{-1}\omega$ is the body angular velocity, must be compensated by the torque due to the forces of the constraints. This leads to the equation of motion

$$A(I\dot{\Omega} - I\Omega \times \Omega) \times e_3 = -\beta(\dot{\omega} \times e_3). \tag{2.1}$$

For a symmetric sphere we have $I_1 = I_2 = 1$, which leads to a simpler equation. In fact, we have $I\Omega = (\Omega_1, \Omega_2, \alpha\Omega_3) \equiv (\Omega_1, \Omega_2, \alpha v_0) = \Omega - (1-\alpha)v_0 e_3$, from which we obtain $I\dot{\Omega} = \dot{\Omega} - (1-\alpha)\dot{v}_0 e_3$. Then we have $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1-\alpha)\dot{v}_0 z + (1-\alpha)v_0 z \times (v_0 z + z \times \dot{z})$, where we have used the equalities $\omega = A\Omega$, $\dot{\omega} = A\dot{\Omega}$, $\omega = v_0 z + z \times \dot{z}$. Finally, we get $A(I\dot{\Omega} - I\Omega \times \Omega) = \dot{\omega} - (1-\alpha)(v_0 z)^{\cdot}$. Using the equality $\dot{\omega} - (1-\alpha)(v_0 z)^{\cdot} = \alpha\dot{\omega} + (1-\alpha)(z \times \ddot{z})$, the balance of momentum equation (2.1) becomes

$$\alpha(\dot{\omega} \times e_3) + (1-\alpha)(z \times \ddot{z}) \times e_3 = -\beta(\dot{\omega} \times e_3). \tag{2.2}$$

Then the system of dynamical and constraint equations is,

$$(\alpha + \beta)(\dot{\omega} \times e_3) + (1-\alpha)(z \times \ddot{z}) \times e_3 = 0, \tag{2.3}$$

$$\omega = v_0 z + z \times \dot{z}, \tag{2.4}$$

$$w_3 = 0, \tag{2.5}$$

or, equivalently,

$$\lambda(v_0 z)^{\cdot} \times e_3 + (z \times \ddot{z}) \times e_3 = 0, \tag{2.6}$$

$$v_0\langle z, e_3\rangle + \langle z \times \dot{z}, e_3\rangle = 0, \tag{2.7}$$

where $\lambda = (\alpha + \beta)/(1 + \beta)$. Letting $u = \dot{z} \times z$ we have the equations of the system in variables (z, u, v_0) as follows:

$$\lambda(v_0 z)^{\cdot} \times e_3 - \dot{u} \times e_3 = 0, \tag{2.8}$$

$$\dot{z} = z \times u, \tag{2.9}$$

$$u_3 - v_0 z_3 = 0, \tag{2.10}$$

$$z^2 - 1 = 0, \tag{2.11}$$

$$\langle z, u\rangle = 0. \tag{2.12}$$

By taking the inner product of Eq. (2.8) by z we obtain $\lambda v_0\langle \dot{z} \times e_3, z\rangle - \langle \dot{u} \times e_3, z\rangle = 0$. Since $u = \dot{z} \times z$ we obtain $-\lambda v_0\langle e_3, u\rangle - \langle z \times \dot{u}, e_3\rangle = 0$, that is

$$\lambda v_0 u_3 + z_1\dot{u}_2 - z_2\dot{u}_1 = 0. \tag{2.13}$$

Then we have the following system of equations in the space of the variables $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$,

$$\dot{z}_1 = z_2 u_3 - z_3 u_2, \tag{2.14}$$

$$\dot{z}_2 = z_3 u_1 - z_1 u_3, \tag{2.15}$$

$$\dot{z}_3 = z_1 u_2 - z_2 u_1, \tag{2.16}$$

$$z_2 \dot{u}_1 - z_1 \dot{u}_2 = \lambda v_0 u_3, \tag{2.17}$$

$$0 = u_3 - v_0 z_3, \tag{2.18}$$

$$0 = u_1^2 + u_2^2 + u_3^2 + \lambda v_0^2 - \mu, \tag{2.19}$$

$$0 = z_1^2 + z_2^2 + z_3^2 - 1, \tag{2.20}$$

$$0 = z_1 u_1 + z_2 u_2 + z_3 u_3. \tag{2.21}$$

Eq. (2.19) represents conservation of energy.

The equations of motion on $S^2 \times S^1$. In this paragraph we show that equations obtained in the previous paragraph can be transformed into an equation, that is, a vector field, on $S^2 \times S^1$. First of all, one can check that Eqs. (2.18)–(2.21) define a submanifold N of the space of the variables $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$, that is, $\mathbb{R}^7$, by using the implicit function theorem at each point of N. Eq. (2.21) tells us that u is a vector tangent to the 2-sphere S^2 given by $z^2 - 1 = 0$. Heuristically, for each $z \in S^2$ we consider the 3-dimensional space $T_z S^2 \times R_z$, where R_z represents a line normal to the sphere at $z \in S^2$, so the collection of all R_z is a trivial real line vector bundle with base S^2. We imagine that the variable v_0 is the coordinate of the axis $R_z \equiv z$ which is normal to $T_z S^2$. Then, for each z, Eq. (2.18) is a plane in $T_z S^2 \times R_z$ containing the origin $0 = 0_z$ since z_3 is fixed once z is fixed. Eq. (2.19) gives an ellipsoid. The intersection of the plane with the ellipsoid is an ellipse. Therefore N must be some fiber bundle with fiber S^1 and base S^2. Using all this and some imagination we can see that it is, in fact, the trivial bundle $S^2 \times S^1$, moreover, we have the following parametrization of N in the variables (θ, φ, ψ):

$$z_1 = \sin\theta \cos\varphi, \tag{2.22}$$

$$z_2 = \sin\theta \sin\varphi, \tag{2.23}$$

$$z_3 = \cos\theta, \tag{2.24}$$

$$u_1 = -a\cos(\varphi - \psi)\cos^2\theta \cos\varphi - b\sin(\varphi - \psi)\sin\varphi, \tag{2.25}$$

$$u_2 = -a\cos(\varphi - \psi)\cos^2\theta \sin\varphi + b\sin(\varphi - \psi)\cos\varphi, \tag{2.26}$$

$$u_3 = a\cos(\varphi - \psi)\cos\theta \sin\theta \tag{2.27}$$

$$v_0 = a\cos(\varphi - \psi)\sin\theta, \tag{2.28}$$

where

$$a = \sqrt{\frac{\mu}{\lambda \sin^2\theta + \cos^2\theta}}, \quad b = \sqrt{\mu}'.$$

In any case, we can check directly that the previous expression of $(z_1, z_2, z_3, u_1, u_2, u_3, v_0)$ in coordinates (θ, φ, ψ) satisfies (2.18)–(2.21). We can also see that Eqs. (2.22)–(2.28) define a diffeomorphism $f : S^2 \times S^1 \to N$, $f(z, (\cos\psi, \sin\psi)) = (z, u, v_0)$.

This is not difficult, but is lengthy, and it involves, in particular, checking that the tangent map of f is injective, for each point of $S^2 \times S^1$.

The differential equation in N, **in variables** (θ, φ, ψ). Considering the parametrization for N given by (2.22)–(2.28) we get equations (2.14)–(2.21) in the coordinates (θ, φ, ψ), as follows

$$\cos\theta\cos\varphi\dot{\theta} - \sin\theta\sin\varphi\dot{\varphi} = a\cos\theta\sin\varphi\cos(\varphi-\psi) - b\cos\theta\cos\varphi\sin(\varphi-\psi), \tag{2.29}$$

$$\cos\theta\sin\varphi\dot{\theta} + \sin\theta\cos\varphi\dot{\varphi} = -a\cos\theta\cos\varphi\cos(\varphi-\psi) - b\cos\theta\sin\varphi\sin(\varphi-\psi), \tag{2.30}$$

$$-\sin\theta\dot{\theta} = b\sin\theta\sin(\varphi-\psi), \tag{2.31}$$

$$a\sin\theta\cos^2\theta\cos(\varphi-\psi)\dot{\varphi} - b\sin\theta\cos(\varphi-\psi)(\dot{\varphi}-\dot{\psi}) = \lambda a^2\cos^2(\varphi-\psi)\sin^2\theta\cos\theta. \tag{2.32}$$

If $\sin\theta \neq 0$ the system (2.29)–(2.32) becomes

$$\dot{\theta} = -b\sin(\varphi-\psi), \tag{2.33}$$

$$\dot{\varphi} = -a\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi), \tag{2.34}$$

$$\dot{\psi} = a\cos(\varphi-\psi)\frac{\cos\theta}{\sin\theta}\left(\frac{b}{a}-1\right), \tag{2.35}$$

or equivalently,

$$\dot{\theta} = -b\sin(\varphi-\psi), \tag{2.36}$$

$$\dot{\varphi} = -a\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi), \tag{2.37}$$

$$\dot{\psi} = (b-a)\frac{\cos\theta}{\sin\theta}\cos(\varphi-\psi). \tag{2.38}$$

It is not difficult to see that the only solution with some initial condition compatible with the system and involving the condition $\sin\theta = 0$, that is, an initial condition of the type $(z_{10}, z_{20}, z_{30}, u_{10}, u_{20}, u_{30}, v_{00}) = (0, 0, \pm 1, u_{10}, u_{20}, 0, 0)$, consists of a uniform circular motion of z on a vertical plane perpendicular to the constant vector $(u_1(t), u_2(t), u_3(t)) = (u_{10}, u_{20}, 0)$, while $v_0(t) = 0$. This is also consistent with physical reasoning. Then we have a smooth vector field on $S^2 \times S^1$ which represents equations of motion of the system.

It can be easily seen that this system can be integrated by quadratures. For instance, if we call $w = \varphi - \psi$, from (2.36)–(2.38) we obtain a planar system in the coordinates (θ, w),

$$\dot{\theta} = -b\sin w, \tag{2.39}$$

$$\dot{w} = -b\frac{\cos\theta}{\sin\theta}\cos w, \tag{2.40}$$

which, in turn, leads to the separable equation

$$\frac{d\theta}{dw} = \tan\theta \tan w. \tag{2.41}$$

3. Dynamics

By introducing a reparametrization of time, if necessary we can normalize the system (2.39)-(2.40) so that $b = 1$, therefore $\mu = 1$.

Equilibrium points. Using (2.39)–(2.40) we conclude that equilibrium points are given by $(\theta, w) = ((2k+1)\pi/2, l\pi)$, $k, l \in \mathbb{Z}$. Linearization of the system at any equilibrium point is given by the matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

so the classification and stability of each equilibrium point must be studied at the nonlinear level.

The solutions. By integrating (2.39)–(2.40) we obtain the solutions

$$\theta(t) = \arctan\left[\frac{\sqrt{1 + C_1(t)^2}}{C - 2(t)}\right], \tag{3.1}$$

$$w(t) = -\arctan(C_1(t)), \tag{3.2}$$

where $C_1(t) = \bar{c}_1 \sin t - \bar{c}_2 \cos t$, $C_2(t) = \bar{c}_1 \cos t + \bar{c}_2 \sin t$. Eq. (2.41) can be easily integrated by separation of variables, so the solutions of the system must be in the level surface

$$\sin\theta \, \cos w = c. \tag{3.3}$$

Now we are going to obtain another level surface containing the solutions and also an explicit expression of $\varphi(t)$. Using (3.1) and (3.2) we can obtain expressions for $\tan\theta$, $\cos^2 w$, $\sin^2\theta$, $\cos^2\theta$ and a in terms of $C_1(t)$ and $C_2(t)$. Then using (2.37) and taking into account that $\tan^2 w = C_1^2(t)$, we obtain the expression

$$\dot{\varphi} = -\bar{c}\sqrt{\frac{1}{(\lambda - 1)(1 + C_1^2(t)) + \bar{c}^2}}\frac{C_1'(t)}{1 + C_1^2(t)} \tag{3.4}$$

$$= -\bar{c}\sqrt{\frac{1}{(\lambda - 1)(1 + \tan^2 w) + \bar{c}^2}}dw, \tag{3.5}$$

where $\bar{c} = \sqrt{1 + \bar{c}_1^2 + \bar{c}_2^2}$. We can integrate this expression of φ and we obtain

$$-\sin(\varphi - d) = \frac{\bar{c}\sin w}{m}, \tag{3.6}$$

where $m^2 = \lambda - 1 + \bar{c}^2$.

Using the expressions of $\sin^2\theta$ and $\cos^2 w$ in terms of $C_1(t)$ and $C_2(t)$ we can show that

$$\sin\theta\cos w = \frac{1}{\bar{c}} = c,$$

which gives a relation between c and $\bar{c}$, then from (3.6) we obtain

$$\sqrt{(\lambda-1)c^2+1}\,\sin(\varphi-d)+\sin w = 0. \tag{3.7}$$

We have shown that the solutions of (2.39)–(2.40) must satisfy (3.3) and (3.7), in other words, c and d are constants of motion. The following theorem describes completely the dynamics of the elastic rolling sphere.

THEOREM 3.1. *Let us consider the system*

$$\dot{\theta} = -\sin w, \tag{3.8}$$

$$\sin\theta\dot{\varphi} = -a\,\cos\theta\,\cos(w), \tag{3.9}$$

$$(\sin\theta)\dot{w} = -\cos\theta\,\cos w. \tag{3.10}$$

(a) *The solution of the planar system* (3.8), (3.10) *for a given initial condition* (θ_0, w_0) *is unique and is described as follows*

(i) *If* $(\theta_0, w_0) = ((2k+1)\pi/2, l\pi)$, *then the only solution is* $(\theta(t), w(t)) = (\theta_0, w_0)$. *These are the only equilibrium points for the system.*

(ii) *If* $(\theta_0, w_0) = (\theta_0, (2l+1)\pi/2)$, *then the only solution is* $(\theta(t), w(t)) = ((-1)^{l+1}t + \theta_0, (2l+1)\pi/2)$.

(iii) *If* (θ_0, w_0) *is such that* $\theta_0 \in (k\pi, (k+1)\pi)$, $w_0 \in ((2l-1)\pi/2, (2l+1)\pi/2)$, $(\theta_0, w_0) \neq ((2k+1)\pi/2, l\pi)$; *the only solution is a closed curve in* $(\theta(t), w(t)) \in (k\pi, (k+1)\pi) \times ((2l-1)\pi/2, (2l+1)\pi/2)$.

(iv) *If* (θ_0, w_0) *is such that* $\theta_0 = k\pi$, $w_0 \neq (2l+1)\pi/2$, *there is no solution.*

(b) *The solution of the system* (3.8)–(3.10) *for each initial condition* $(\theta_0, \varphi_0, w_0)$ *is unique and is given by the solution to the planar system, described in part* (a), *and* (3.7). *Each solution, including equilibrium points, is stable.*

Proof: We first prove (a). (i) and (iv) are easily verified. (ii) is also easily verified using (3.3). The statement (iii) can be proved using the fact that for each $c \in (0,1)$ the subset

$$S_c = \{(\theta, w) \in (k\pi, (k+1)\pi) \times ((2l-1)\pi/2, (2l+1)\pi/2) \mid \sin\theta\cos w = c\},$$

is a simple closed, therefore compact, curve. For each $(\theta_0, w_0) \in S_c$ there is a unique solution $(\theta(t), w(t))$ such that $(\theta(0), w(0)) = (\theta_0, w_0)$, moreover $(\theta(t), w(t)) \in S_c$, for all t.

The proof of part (b) is straightforward, taking into account part (a). □

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