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Representation of Cubic Lattices by Symmetric Implication Algebras

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Abstract In this paper a cubic lattice $L(S)$ is endowed with a symmetric implication structure and it is proved that $L(S) \setminus \{0\}$ is a power of the three-element simple symmetric implication algebra. The Metropolis–Rota's symmetries are obtained as partial terms in the language of symmetric implication algebras.

Key words cubic lattices · semilattices · implication algebras

Mathematics Subject Classifications (2000) 06B15 · 08B26

1 Introduction

In this section we reproduce some definitions and results on cubic lattices and the equivalence with the lattice of signed sets of a set S. We also introduce the variety of symmetric implication algebras and we characterize its subvarieties.

In [7], Metropolis and Rota considered the partially ordered set $\mathcal{F}_n$ of all nonempty faces of the n-cube $I_n = [-1, 1]^n$ for each $n = 1, 2, \ldots$, equipped with an operation $\sqcup$ and two *partially defined* operations $\sqcap$ and Δ, where, for $x, y \in \mathcal{F}_n$, $x \sqcup y$ is the least face containing $x \cup y$, $x \sqcap y$ is the set-theoretic intersection of x and y, whenever this intersection exists, and $\Delta_y(x)$ is the unique opposite face of x inside y, whenever x is contained in y. $\mathcal{F}_n$ is a join-semilattice, and $\mathcal{F}_n$ is extended to a lattice by the adjunction of a 0, which corresponds to the empty face.

A *finite cubic lattice* is an algebraic structure isomorphic to the lattice $\mathcal{F}_n$ of faces of the n-cube, for some n. Metropolis and Rota gave in [7] an algebraic characterization of the finite dimensional cubic lattices $\mathcal{F}_n$, which is independent of the dimension n.

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Among other interpretations of the faces of an n-cube, Rota and Metropolis considered the signed subsets of an n-set. A signed set of $\mathbf{n} = \{1, 2, \ldots, n\}$ is an ordered pair $x = (x^+, x^-)$ of disjoints subsets of $\mathbf{n}$. A finite cubic lattice is isomorphic to the lattice of signed subsets of an n-element set, for some n [7]. This can be extended to infinite sets. Indeed, for a set S of arbitrary cardinality, a signed subset of S is an ordered pair $x = (x^+, x^-)$ of disjoints subsets of S. The set $L^+(S)$ of all signed sets of S is an ordered set by the opposite componentwise inclusion:

$$(x^+, x^-) \leq (y^+, y^-) \text{ if and only if } y^+ \subseteq x^+ \text{ and } y^- \subseteq x^-.$$

The poset $L^+(S)$ has a greatest element $1 = (\emptyset, \emptyset)$, and $L^+(S)$ is a join-semilattice, where the operation of supremum is:

$$(x^+, x^-) \vee (y^+, y^-) = (x^+ \cap y^+, x^- \cap y^-).$$

The partial operation of infimum is defined whenever $x^- \cap y^+ = \emptyset$ and $x^+ \cap y^- = \emptyset$, by the stipulation

$$(x^+, x^-) \wedge (y^+, y^-) = (x^+ \cup y^+, x^- \cup y^-).$$

The lattice $L(S)$ obtained from $L^+(S)$ by the adjunction of a least element 0, in notation $L(S) = 0 \oplus L^+(S)$, is complete and it is called the lattice of signed sets on S.

In $L(S)$ we can define a family of symmetries $\{\Delta_x\}_{x \in L(S)}$, such that $\Delta_x : (x] \to (x]$ ($(x] = \{y \in L(S) : y \leq x\}$). The mappings are defined: $\Delta_x(0) = 0$, and for $0 < x = (x^+, x^-)$, $0 < y = (y^+, y^-)$, $y \leq x$, then

$$\Delta_x(y) = (x^+ \cup y^- \backslash x^-, x^- \cup y^+ \backslash x^+).$$

Observe that, in particular, $\Delta_1(y) = (y^-, y^+)$.

Definition 1.1 [9] A cubic lattice L is a lattice satisfying the following six axioms:

1. (1) For $x \in L$, there is an order preserving map $\Delta_x : (x] \to (x]$.
2. (2) $\Delta_x^2 = Id_{(x]}$.
3. (3) For $0 < y, z < x$, $y \vee \Delta_x(z) < x$ if and only if $y \wedge z = 0$.
4. (4) L is complete.
5. (5) L is atomistic: If $x \neq 0$ in L, then there is an atom $a \in L$ such that $a \leq x$.
6. (6) L is coatomistic: If $x \neq 1$ in L, then there is coatom $b \in L$ such that $x \leq b$.

The following theorem was proved by Oliveira in [9].

Theorem 1.2 *Given a cubic lattice L, there exists a set S unique up to bijections such that $L \cong L(S)$.*

The notion of *implication algebra* was introduced and developed by Abbot in [1, 2]. An implication algebra is an algebra $\langle A, \to \rangle$ of type (2) that satisfies the equations:

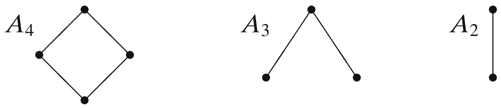
(T1) $(x \to y) \to x \approx x$.
(T2) $(x \to y) \to y \approx (y \to x) \to x$.
(T3) $x \to (y \to z) \approx y \to (x \to z)$.

These algebras were named Tarski algebras by others authors (see for example, [8, p. 35], [6]).

In an implication algebra A, the term $x \to x$ is an algebraic constant which we represent by 1. The relation $a \leq b$ if and only if $a \to b = 1$ is a partial order on A, with 1 as its greatest element, and the join operation is given by $x \vee y = (x \to y) \to y$.

An *implicative filter* in an implication algebra A is a non-empty increasing subset F of A such that if $x, y \in F$ and there exists $x \wedge y$ in A, then $x \wedge y \in F$. It is known [2] that every congruence on an implication algebra A is determined by an implicative filter F, where the relation is $a \equiv b \pmod{F}$ if and only if $a \to b$ and $b \to a \in F$. Thus the lattice of implicative filters of A is isomorphic to its lattice of congruences. From this it is clear that the 2-element implication algebra $\mathbf{2} = \{0, 1\}$ is the only simple implication algebra.

By a *symmetric implication algebra* we understand a pair $\langle A, T \rangle$ such that A is an implication algebra and T is a distinguished automorphism of A of period two, i.e., $T^2(x) = x$ for every $x \in A$, considered as a new unary operation. Examples of symmetric implication algebras are the symmetric Boolean algebras [8, p. 212]. An implicative filter F of an implication algebra A such that $T(x) \in F$ whenever $x \in F$ is called an *implicative T-filter* of A. It is not difficult to see that the lattice of congruences on a symmetric implication algebra A is isomorphic to the lattice of implicative T-filters of A. Consequently, the algebra $\langle A_4, T \rangle = \langle \mathbf{2} \times \mathbf{2}, T \rangle$, where the automorphism T is defined by $T((x, y)) = (y, x)$, and its subalgebras $\langle A_2 = \{(0, 0), (1, 1)\}, T \rangle$ and $\langle A_3 = \{(0, 1), (1, 0), (1, 1)\}, T \rangle$ are simple implication algebras. It can be seen that they are the only subdirectly irreducible (and simple) algebras in the variety of all symmetric implication algebras.

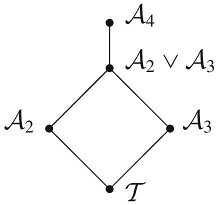


If $\mathcal{A}_i = V(A_i)$ is the variety generated by A_i, for $i = 2, 3, 4$, then $\mathcal{A}_4$ is the variety of all symmetric implication algebras and we have the following characterization. The proof is straightforward.

Theorem 1.3

(1) *The identity* $x \to T(x) \approx 1$ *is an equational basis for* $\mathcal{A}_2$.
(2) *The identity* $x \vee T(x) \approx 1$, *is an equational basis for* $\mathcal{A}_3$.
(3) *The identity* $(x \to T(x)) \vee (T(y) \vee y) \approx 1$ *is an equational basis for* $\mathcal{A}_2 \vee \mathcal{A}_3$.

The lattice of subvarieties of $\mathcal{A}_4$ looks like the following figure, where $\mathcal{T}$ is the trivial variety:



In $L^+(S)$ we can define a structure of implication algebra, by defining the implication

$$x \to y := y \vee \Delta_1(\Delta_{x\vee y}(y)).$$

Chen and Oliveira [5] proved that $\langle L^+(S), \to\rangle$ is an implication algebra. Observe that $L^+(S)$ is a symmetric implication algebra, where $T(x) = T((x^+, x^-)) = \Delta_1(x) = \Delta_1(x^+, x^-) = (x^-, x^+)$.

2 Representation of Cubic Lattices

In this section we give a decomposition of $L^+(S)$ as a direct product of certain simple symmetric implication algebras. We also give an expression of the symmetries Δ_x as partial terms in the language of symmetric implication algebras.

Let $A(S)$ be the set of ordered pairs (x_1, x_2) of subsets of S with the order given by the opposite componentwise inclusion

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } y_1 \subseteq x_1 \text{ and } y_2 \subseteq x_2.$$

$A(S)$ is a Boolean algebra, and if we put $T((x_1, x_2)) = (x_2, x_1)$, then $\langle A(S), T\rangle$ is a symmetric implication algebra (in fact, it is a symmetric Boolean algebra).

It is clear that $L^+(S)$ is a symmetric implication subalgebra of $A(S)$. Let $\big[\big(x, \complement x\big)\big)$ denote the filter of $A(S)$ generated by $\big(x, \complement x\big)$, that is, $\big[\big(x, \complement x\big)\big) = \{(x_1, x_2) \in A(S) : \big(x, \complement x\big) \leq (x_1, x_2)\}$.

Lemma 2.1 $L^+(S) = \bigcup_{x\subseteq S}[(x, \complement x))$

Proof If $(x_1, x_2) \in L^+(S)$, then $x_1 \cap x_2 = \emptyset$, then $x_2 \subseteq \complement x_1$ and consequently $(x_1, \complement x_1) \leq (x_1, x_2)$. The other inclusion is trivial. □

Lemma 2.2 Let $x \in A(S)$. Then $x \in L^+(S)$ if and only if $x \vee T(x) = 1$. In particular, $L^+(S) \in \mathcal{A}_3$.

Proof For $(x_1, x_2) \in L^+(S)$, $(x_1, x_2) \vee T((x_1, x_2)) = (x_1, x_2) \vee (x_2, x_1) = (x_1 \cap x_2, x_1 \cap x_2) = (\emptyset, \emptyset) = 1$. Conversely, if $(x_1, x_2) \vee T((x_1, x_2)) = 1$, then $x_1 \cap x_2 = \emptyset$, and thus $(x_1, x_2) \in L^+(S)$. □

Observe that $\mathcal{A}_3 \cong L^+(\{s\})$ and consequently $L^+(S)$ generates $\mathcal{A}_3$ for any S. $\mathcal{A}_3$ is the variety of cubic algebras studied by Bailey and Oliveira in [3].

The next theorem proves that the symmetries Δ_x are partial terms in the language of symmetric implication algebras.

Theorem 2.3 *For $x, y \in L^+(S)$ and $y \leq x$,*

$$\Delta_x(y) = x \wedge T(x \to y).$$

Proof Let $x, y \in L^+(S)$, $x = (x_1, x_2) \geq y = (y_1, y_2)$. Then $x \wedge T(x \rightarrow y) = (x_1, x_2) \wedge T((x_1, x_2) \rightarrow (y_1, y_2)) = (x_1, x_2) \wedge T(\neg(x_1, x_2) \vee (y_1, y_2)) = (x_1, x_2) \wedge T\left((\complement x_1 \cap y_1, \complement x_2 \cap y_2)\right) = (x_1, x_2) \wedge (\complement x_2 \cap y_2, \complement x_1 \cap y_1) = (x_1 \cup (\complement x_2 \cap y_2), x_2 \cup (\complement x_1 \cap y_1)) = (x_1 \cup y_2 \backslash x_2, x_2 \cup y_1 \backslash x_1) = \Delta_x(y)$. □

By Lemma 2.2 and Theorem 1.3, we have that $L^+(S)$ is a subdirect product of copies of A_3. In what follows we prove that this subdirect product is in fact a direct product.

Lemma 2.3 For $s \in S$, let $F_s = \{(x, y) \in L^+(S) : s \notin x \cup y\}$. Then

(1) F_s is T-filter of $L^+(S)$.
(2) $(x_1, x_2)\theta(F_s)(y_1, y_2)$ if and only if $\{s\} \cap x_1 = \{s\} \cap y_1$ and $\{s\} \cap x_2 = \{s\} \cap y_2$.
(3) $L^+(S)/F_s \cong A_3$ and F_s is maximal.
(4) $\bigcap_{s \in S} F_s = \{(\emptyset, \emptyset)\} = 1$.
(5) Let $A_3 \cong A_{3,s} = \{1_s, a_s, b_s\}$ be the three-element simple symmetric implication algebra with top 1_s. Then $f_s : L^+(S) \rightarrow A_{3,s}$ defined by

$$f_s(x_1, x_2) = \begin{cases} 1_s & \text{if } s \notin x_1 \cup x_2 \\ a_s & \text{if } s \notin x_1 \text{ and } s \in x_2 \\ b_s & \text{if } s \notin x_2 \text{ and } s \in x_1 \end{cases}$$

is an epimorphism such that $Ker(f_s) = \theta(F_s)$.

Proof It is immediate that $1 \in F_s$ and $T(F_s) \subseteq F_s$. Let $(x_1, x_2), (x_1, x_2) \rightarrow (y_1, y_2) \in F_s$. Then $s \notin x_1 \cup x_2$, and $s \notin (\complement x_1 \cap y_1) \cup (\complement x_2 \cap y_2)$. Then $s \in x_1 \cup \complement y_1$ and $s \in x_2 \cup \complement y_2$. Since $s \notin x_1$ and $s \notin x_2$, we have that $s \in \complement y_1$ and $s \in \complement y_2$. Thus $s \notin y_1 \cup y_2$. So $(y_1, y_2) \in F_s$, and we have (1).

For (2), let $(x_1, x_2), (y_1, y_2) \in L^+(S)$ such that $(x_1, x_2)\theta(F_s)(y_1, y_2)$. Then $(x_1, x_2) \rightarrow (y_1, y_2)$ and $(y_1, y_2) \rightarrow (x_1, x_2)$ belong to F_s. Thus $(\complement x_1 \cap y_1, \complement x_2 \cap y_2)$ and $(\complement y_1 \cap x_1, \complement y_2 \cap x_2) \in F_s$. Hence, $s \notin (\complement x_1 \cap y_1) \cup (\complement x_2 \cap y_2)$ and $s \notin (\complement y_1 \cap x_1) \cup (\complement y_2 \cap x_2)$. Then $s \in (x_1 \cup \complement y_1) \cap (x_2 \cup \complement y_2)$ and $s \in (y_1 \cup \complement x_1) \cap (y_2 \cup \complement x_2)$. From this it follows that $s \in x_1$ if and only if $s \in y_1$, and $s \in x_2$ if and only if $s \in y_2$.

Conversely, let $(x_1, x_2), (y_1, y_2) \in L^+(S)$ such that $\{s\} \cap x_1 = \{s\} \cap y_1$ and $\{s\} \cap x_2 = \{s\} \cap y_2$. We have the following three cases: (1) $s \notin x_1 \cup x_2$, (2) $s \notin x_1$ but $s \in x_2$, and (3) $s \notin x_2$ but $s \in x_1$. It is easy to prove that in all these three cases $(x_1, x_2) \rightarrow (y_1, y_2) = (\complement x_1 \cap y_1, \complement x_2 \cap y_2) \in F_s$, and similarly $(y_1, y_2) \rightarrow (x_1, x_2) \in F_s$, so $(x_1, x_2)\theta(F_s)(y_1, y_2)$.

In order to prove (3), observe that F_s is a proper filter in $L^+(S)$, so there are three congruence classes of $\theta(F_s)$, namely, $1_s = \{(x_1, x_2) \in L^+(S) : s \notin x_1 \cup x_2\}$, $a_s = \{(x_1, x_2) \in L^+(S) : s \notin x_1, s \in x_2\}$ and $b_s = \{(x_1, x_2) \in L^+(S) : s \notin x_2, s \in x_1\}$. We have that $T(a_s) = b_s$.

Condition (4) is clear and (5) is a consequence of (2) and (3). □

Theorem 2.5 *$L^+(S)$ is isomorphic to A_3^S, as symmetric implication algebras.*

Proof Let $f : L^+(S) \to \prod_{s\in S} A_{3,s}$, defined by $f((x_1, x_2))(s) = f_s((x_1, x_2))$. By (4) in the previous lemma we have that f is one-to-one. Let $z \in \prod_{s\in S} A_{3,s}$, and let $x_1 = \{s \in S : z(s) = 1_s\}$, $x_2 = \{s \in S : z(s) = a_s\}$ and $x_3 = \{s \in S : z(s) = b_s\}$.

$$f(x_3, x_2)(s) = f_s(x_3, x_2) = \begin{cases} 1_s & \text{if } s \notin x_2 \cup x_3 & \text{if } s \in x_1 \\ a_s & \text{if } s \notin x_3 \text{ and } s \in x_2 & \text{if } s \in x_2 \\ b_s & \text{if } s \in x_3 \text{ and } s \notin x_2 & \text{if } s \in x_3 \end{cases}$$

Thus $f(x_3, x_2)(s) = z(s)$. Then f is onto. □

Corollary 2.6 The cubic lattice $L(S)$ is isomorphic to $0 \oplus \prod_{s\in S} A_{3,s}$ and then isomorphic to $0 \oplus A_3^S$.

Some properties for the face lattice of an n-dimensional cube, can be deduced from Corollary 2.6. In that case, $S = \mathbf{n}$, and, for instance, $|L(\mathbf{n})| = |0 \oplus A_3{}^n| = 1 + 3^n$, and $L(\mathbf{n})$ has $2n$ coatoms and $2n$ atoms [4].

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