Arch. Math. Logic
DOI 10.1007/s00153-006-0023-1

Mathematical Logic

Decomposability of free Łukasiewicz implication algebras

Jose Patricio Díaz Varela · Antoni Torrens Torrell

Received: 15 March 2006 / Accepted: 11 May 2006

Abstract Łukasiewicz implication algebras are $\{\to, 1\}$-subreducts of Wajsberg algebras (MV-algebras). They are the algebraic counterpart of Super-Łukasiewicz Implicational logics investigated in Komori, Nogoya Math J 72:127–133, 1978. The aim of this paper is to study the direct decomposability of free Łukasiewicz implication algebras. We show that freely generated algebras are directly indecomposable. We also study the direct decomposability in free algebras of all its proper subvarieties and show that infinitely freely generated algebras are indecomposable, while finitely free generated algebras can be only decomposed into a direct product of two factors, one of which is the two-element implication algebra.

Keywords Free algebras · Factor congruences · Implicative filters · BCK-algebras · MV-algebras

Mathematics Subject Classification (2000) 06F99 · 08A30 · 08B20

This paper was prepared while the first author was visiting the University of Barcelona, partially supported by Universidad Nacional del Sur, CONICET and Fundación Carolina. The second author was partially supported by grants MTM2004-03101 and TIN2004-07933-C03-02 of M.E.C. of España.

J. P. Díaz Varela (✉)
Departamento de Matemática, Universidad Nacional del Sur,
Avenida Alem 1253, 8000 Bahía Blanca, Argentina
e-mail: usdiavar@criba.edu.ar

A. Torrens Torrell
Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08007 Barcelona, Spain
e-mail: atorrens@ub.edu

1 Introduction and preliminaries

Łukasiewicz implication algebras are the algebraic counterpart of the implicational fragment of Super-Łukasiewicz logic (see [10,11]). In fact they are the class of all $\{\to,1\}$-subreducts of the Wajsberg algebras (Wajsberg algebras are term-wise equivalent to Chang's MV-algebras and bounded commutative BCK-algebras [6,9,12]). They are also called C-algebras in [10,11] and Łukasiewi c z residuation algebras by Berman and Blok in [3].

A *Łukasiewicz implication algebra* is an algebra $\boldsymbol{A} = \langle A, \to, 1\rangle$ of type $\langle 2, 0\rangle$ that satisfies the equations:

$$1 \to x \approx x, \tag{ł1}$$
$$(x \to y) \to ((y \to z) \to (x \to z)) \approx 1, \tag{ł2}$$
$$(x \to y) \to y \approx (y \to x) \to x, \tag{ł3}$$
$$(x \to y) \to (y \to x) \approx y \to x. \tag{ł4}$$

We will denote by $\mathcal{L}$ the variety of all Łukasiewicz implication algebras. The following properties are satisfied in any Łukasiewicz implication algebra:

$$x \to x \approx 1, \tag{ł5}$$
$$x \to 1 \approx 1, \tag{ł6}$$
$$\text{if } x \to y \approx y \to x \approx 1, \quad \text{then } x \approx y, \tag{ł7}$$
$$x \to (y \to x) \approx 1, \tag{ł8}$$
$$x \to (y \to z) \approx y \to (x \to z). \tag{ł9}$$

If $\boldsymbol{A} \in \mathcal{L}$ then the relation $a \leq b$ if and only if $a \to b = 1$ is a partial order on A, called the *natural order of* $\boldsymbol{A}$, with 1 as its greatest element. The join operation $x \vee y$ is given by the term $(x \to y) \to y$ and if $c \in A$, then the polynomial $p(x, y, c) =_{\text{def}} ((x \to c) \vee (y \to c)) \to c$ is such that $p(a, b, c) = a \wedge b = \inf\{a, b\}$ for $a, b \geq c$. The lattice operation satisfies the following properties:

$$(x \vee y) \to z \approx (x \to z) \wedge (y \to z), \tag{ł10}$$
$$z \to (x \vee y) \approx (z \to x) \vee (z \vee y), \tag{ł11}$$

and if for $a, b \in A$ there exists $a \wedge b$, then for any $c \in A$,

$$(a \wedge b) \to c \approx (a \to c) \vee (b \to c), \tag{ł12}$$
$$c \to (a \wedge b) \approx (c \to a) \wedge (c \to b), \tag{ł13}$$

Moreover if $\boldsymbol{A} = \langle A, \to, 1\rangle$ is a Łukasiewicz implication algebra and $c \in A - \{1\}$, then $\boldsymbol{A}_c = \langle [c) = \{a \in A : c \leq a\}, \to, \neg_c, c, 1\rangle$ becomes a Wajsberg algebra defining

(1) $\neg_c x := x \to c$,
(2) $x \to_c y := x \to y$.

Łukasiewicz implication algebras are congruence 1-regular. For each congruence relation θ on an algebra $\boldsymbol{A} \in \mathcal{L}$, $1/\theta$ is an implicative filter, i.e., contains 1 and if $a, a \to b \in 1/\theta$, then $b \in 1/\theta$ (modus ponens); in particular, $1/\theta$ is upwardly closed in the natural order. Conversely, for any implicative filter F of $\boldsymbol{A}$ the relation

$$\theta_F = \{\langle a,b\rangle \in A^2 \colon a \to b, b \to a \in F\}$$

is a congruence on $\boldsymbol{A}$ such that $F = 1/\theta_F$. In fact, the correspondence $\theta \mapsto 1/\theta$ gives an order isomorphism from the family of all congruence relations on $\boldsymbol{A}$ onto the family of all implicative filters of $\boldsymbol{A}$, ordered by inclusion. Since any implicative filter F contains 1 and is closed by $\to$, then it is the universe of a subalgebra $\boldsymbol{F}$ of $\boldsymbol{A}$. The bounded distributive lattice of the congruence relations on $\boldsymbol{A}$ is algebraic, and hence it is pseudocomplemented. We represent by Δ and ∇ the diagonal and universal equivalence relations, respectively, and by $\theta^{\perp}$ the pseudocomplement of the congruence θ. It is not difficult to see that for each implicative filter F of $\boldsymbol{A}$,

$$F^{\perp} = \{b \in A \colon b \vee a = 1 \text{ for all } a \in F\}$$

is the pseudocomplement of F in the lattice of all implicative filters. Clearly, for any congruence relation θ we have $(1/\theta)^{\perp} = 1/\theta^{\perp}$.

The subdirectly irreducible algebras in $\mathcal{L}$ are linearly ordered relative to the natural order, or *$\mathcal{L}$-chains* (commutative BCK-chains). Finite $\mathcal{L}$-chains are $\boldsymbol{Ł}_n^{\to}$ the $\{\to, 1\}$-reducts of the finite MV-chains $\boldsymbol{Ł}_n$. The algebra $\boldsymbol{Ł}_n^{\to}$ has as universe the set of rational numbers $Ł_n = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$, and for each $a, b \in Ł_n$, $a \to b = \min(1, 1 - x + y)$. Another important $\mathcal{L}$-chain is the $\{\to, 1\}$-reduct of the Chang's algebra (see [6, p. 474]):

$$\boldsymbol{C}_{\omega}^{\to} = \{(0,y) \colon y \in \mathbb{N}\} \cup \{(1,-y) \colon y \in \mathbb{N}\},$$

where $\mathbb{N}$ is the set of non-negative integers and

$$(x,y) \to (z,u) = \begin{cases} (1,0) & \text{if } z > x, \\ (1, \min(0, u-y)) & \text{if } z = x, \\ (1 - x + z, u - y) & \text{otherwise.} \end{cases}$$

The set $L_{\omega}^{\to} = \{(1,-y) : y \in \mathbb{N}\}$ is the unique maximal (proper) implicative filter of $\boldsymbol{C}_{\omega}^{\to}$ with $\boldsymbol{C}_{\omega}^{\to}/\theta_{L_{\omega}^{\to}} \cong \boldsymbol{Ł}_1^{\to}$. Its associated subalgebra $\boldsymbol{L}_{\omega}^{\to}$ is not finitely generated, and any infinite subalgebra of $\boldsymbol{L}_{\omega}^{\to}$ is isomorphic to a copy of it. Moreover, every non-trivial finite subalgebra of $\boldsymbol{L}_{\omega}^{\to}$ is isomorphic to $\boldsymbol{Ł}_n^{\to}$, for some $n > 0$. In addition, $\boldsymbol{C}_{\omega}^{\to}$ and all $\boldsymbol{Ł}_n^{\to}$, $n > 0$, are two-generated and every non-trivial subalgebra of $\boldsymbol{Ł}_{\omega}^{\to}$ finitely generated is isomorphic to $\boldsymbol{Ł}_n^{\to}$, for some

$n > 0$. In particular, $\boldsymbol{Ł}_n^{\rightarrow}$ is a subalgebra of $\boldsymbol{Ł}_m^{\rightarrow}$ for all $n \leq m$, and every infinite $\mathcal{L}$-chain contains a copy of $\boldsymbol{Ł}_n^{\rightarrow}$ for all $n \geq 0$ (see [11]). Finally, it is easy to see that any simple algebra in $\mathcal{L}$ is isomorphic to $\boldsymbol{Ł}_\alpha^{\rightarrow}$ for some $\alpha \in \omega \cup \{\omega\}$ (see again [11]).

The lattice of all subvarieties of $\mathcal{L}$ was described in [11], and it is a $\omega+1$-chain:

$$V(\boldsymbol{Ł}_0^{\rightarrow}) \subsetneq V(\boldsymbol{Ł}_1^{\rightarrow}) \subsetneq \cdots V(\boldsymbol{Ł}_n^{\rightarrow}) \subsetneq \cdots V(\boldsymbol{Ł}_\omega^{\rightarrow}) = V(\boldsymbol{C}_\omega^{\rightarrow}) = \mathcal{L},$$

where $V(\boldsymbol{A})$ denotes the variety generated by an algebra $\boldsymbol{A}$. Observe that $V(\boldsymbol{Ł}_0^{\rightarrow})$ is the trivial variety and $V(\boldsymbol{Ł}_1^{\rightarrow})$ is the variety of all implication algebras, also known as Tarski algebras (see [1,2,8])

In order to describe equationally the varieties $V(\boldsymbol{Ł}_n^{\rightarrow})$, let us write $x \rightarrow^0 y =_{\text{def}} y$ and for $n \geq 0$, $x \rightarrow^{n+1} y =_{\text{def}} x \rightarrow (x \rightarrow^n y)$. If for any $k \in \omega$, we consider the equation

$$\varepsilon_k\colon\ x \rightarrow^k y \approx x \rightarrow^{k+1} y,$$

then we have

Theorem 1 *$V(\boldsymbol{Ł}_k^{\rightarrow})$ is the variety of implication Łukasiewicz algebras satisfying the equation ε_k.* □

Observe that algebras satisfying ε_1 are implication (Tarski) algebras. Thus the following properties are equivalent for any algebra $\boldsymbol{A} \in \mathcal{L}$:

(i1) $\boldsymbol{A}$ is an implication algebra,
(i2) $\boldsymbol{A}$ satisfies $(x \rightarrow y) \rightarrow x \approx x$,
(i3) $\boldsymbol{A}$ satisfies $(x \rightarrow y) \vee x \approx 1$.

Concerning to filter properties, we give the following known results (see [4] and the references given there).

Lemma 2 *Let $\boldsymbol{A} \in \mathcal{L}$, and let F be an implicative filter of $\boldsymbol{A}$. For each $a \in A$ the filter generated by $F \cup \{a\}$ is the set*

$$\langle F, a\rangle = \{b \in A\colon \textit{ there is } n < \omega \textit{ such that } a \rightarrow^n b \in F\}.$$

Then the filter generated by an element $a \in A$ is the set

$$\langle a\rangle = \{b \in A\colon a \rightarrow^n b = 1, \text{ for some } n < \omega\}.$$

Observe that:

$$\begin{aligned}\langle a\rangle = [a) = \{b \in A\colon b \geq a\} \quad & \text{iff } a \rightarrow b = a \rightarrow (a \rightarrow b), \text{for all } b \in A\\ & \text{iff } (a \rightarrow b) \rightarrow a = a, \text{ for all } b \in A\\ & \text{iff } a \vee (a \rightarrow b) = 1, \text{ for all } b \in A.\end{aligned}$$

In this case a is complemented in the lattice reduct of the Wajsberg algebra $\boldsymbol{A}_c$, for each c in A such that $c \leq a$, and we say that a is *relative boolean*.

2 Factor congruences in $\mathcal{L}$

The following lemmas are based on the results obtained by the authors in [8] for implication algebras.

Lemma 3 *For any congruence relation θ on an algebra $\mathbf{A} \in \mathcal{L}$, the following conditions are equivalent:*

(i) *For every $a \in A$ there are unique $c_a \in 1/\theta$ and $b_a \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$.*
(ii) *$\pi_\theta \restriction_{1/\theta^\perp}: b \mapsto b/\theta$ is an isomorphism from $\mathbf{1/\theta^{\perp}}$ onto $\mathbf{A}/\theta$.*

Moreover, if they hold, then θ is complemented and π_θ is a retraction from $\mathbf{A}$ onto $\mathbf{A}/\theta$, that is, there is an embedding δ: $\mathbf{A}/\theta \to \mathbf{A}$ such that $\pi_\theta \circ \delta = id_{\mathbf{A}/\theta}$.

Proof (i) $\Rightarrow$ (ii): Clearly, $\pi_\theta \restriction_{1/\theta^\perp}$ is a homomorphism from $\mathbf{1/\theta^{\perp}}$ into $\mathbf{A}$. Given $a/\theta \in A/\theta$ consider $c_a \in 1/\theta$ and $b_a \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$. Since the meet is given by a polynomial, we have $a/\theta = b_a/\theta$. This shows that the map is onto. If $d, e \in 1/\theta^{\perp}$ are such that $d/\theta = e/\theta$, then $d \to e, e \to d \in 1/\theta$ and so $d \to e = e \vee (d \to e) = 1$, $e \to d = d \vee (e \to d) = 1$, that is $d = e$. Hence the restriction of π_θ on $1/\theta^{\perp}$ is an isomorphism.
(ii)$\Rightarrow$ (i): Given $a \in A$, consider $b_a = (\pi_\theta \restriction_{1/\theta^\perp})^{-1}(a/\theta) \in 1/\theta^{\perp}$, then $a/\theta = b_a/\theta$. Hence $a \to b_a \in 1/\theta \cap 1/\theta^{\perp} = \{1\}$ and $a \to b_a = 1$, that is $a \leq b_a$. Since $b_a \to a \in 1/\theta$ and $b_a \in 1/\theta^{\perp}$ we have $b_a \vee (b_a \to a) = 1$ and

$$\begin{aligned} a = 1 \to a &= (b_a \vee (b_a \to a)) \to a = (b_a \to a) \wedge ((b_a \to a) \to a) \\ &= (b_a \to a) \wedge (b_a \vee a) = (b_a \to a) \wedge b_a; \end{aligned}$$

hence we can take $c_a = b_a \to a$. Assume that there are $c_1, c_2 \in 1/\theta$, and $b_1, b_2 \in 1/\theta^{\perp}$, such that $a = c_1 \wedge b_1 = c_2 \wedge b_2$. Since π_θ is isomorphism and $a/\theta = c_1/\theta = c_2/\theta$, we have $c_1 = c_2$. Then $c_1 \vee b_1 = 1$ say us that c_1 is a boolean in $\mathbf{A}_a$, hence $c_1 \vee b_2 = 1$ implies $b_1 = b_2$.
It is clear that if (i) holds, then $1/\theta \bigvee 1/\theta^{\perp} = A$ and hence $\theta \bigvee \theta^{\perp} = \nabla$, and if (ii) holds then $\delta = (\pi_\theta \restriction_{1/\theta^\perp})^{-1}$ is the section of π_θ. □

We recall that a complemented congruence relation θ on an algebra $\mathbf{A}$ that permutes with a complement θ' is called *factor*, and $\{\theta, \theta'\}$ is called a pair of factor congruences. This is equivalent to that $\mathbf{A} \cong \mathbf{A}/\theta \times \mathbf{A}/\theta'$ (see [5] for details). Thus an algebra is directly indecomposable if and only if the unique pair of factor congruences is the trivial $\{\Delta, \nabla\}$. Observe that in a congruence distributive algebra, a congruence relation θ is factor if and only if $\theta \circ \theta^{\perp} = \nabla$.

Lemma 4 *Let θ be a congruence relation on an algebra $\mathbf{A}$ in $\mathcal{L}$. Then θ is factor if and only if the following conditions hold:*

(a) *For every $a \in A$ there are unique $c_a \in 1/\theta$ and $b_a \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$.*
(b) *For every $c \in 1/\theta$ and $b \in 1/\theta^{\perp}$ the element $c \wedge b$ is in A.*

Proof Assume that $\theta \circ \theta^{\perp} = \theta^{\perp} \circ \theta = \nabla$. Let $a \in A$, since $\langle a, 1\rangle \in \nabla$, then there are $c_a, b_a \in A$ such that $a \overset{\theta}{\equiv} b_a \overset{\theta^{\perp}}{\equiv} 1$ and $a \overset{\theta^{\perp}}{\equiv} c_a \overset{\theta}{\equiv} 1$. From $a \overset{\theta}{\equiv} b_a$ we get $a \to b_a \in 1/\theta$, and from $b_a \in 1/\theta^{\perp}$ we get $a \to b_a \in 1/\theta^{\perp}$. But $1/\theta \cap 1/\theta^{\perp} = \{1\}$, so $a \to b_a = 1$. This means that $a \leq b_a$. Likewise $a \leq c_a$. Therefore $c_a \wedge b_a \in A$. From $a \overset{\theta}{\equiv} b_a$ and $a \overset{\theta^{\perp}}{\equiv} c_a$ we can meet with $c_a \wedge b_a$ to obtain $a \overset{\theta^{\perp} \cap \theta}{\equiv} c_a \wedge b_a$. Since $\theta^{\perp} \cap \theta = \Delta$ we have $a = c_a \wedge b_a$. The uniqueness can be obtained as in Lemma 3. And we have (a). To show (b) let $c \in 1/\theta$ and $b \in 1/\theta^{\perp}$, so $c \overset{\theta}{\equiv} 1 \overset{\theta^{\perp}}{\equiv} b$. Then there exists $a \in A$ such that $b \overset{\theta^{\perp}}{\equiv} a \overset{\theta}{\equiv} c$. Thus $a \to b \in 1/\theta$, and since $b \in 1/\theta^{\perp}$, we have $a \to b = b \vee (a \to b) = 1$, so $a \leq b$. Likewise $a \leq c$, then $c \wedge b = p_{\wedge}(c, b, a) \in A$.
Conversely, assume that (a) and (b) are satisfied. Let $a, d \in A$, then by (a) there are $c_a, c_d \in 1/\theta$ and $b_a, b_d \in 1/\theta^{\perp}$ such that $a = c_a \wedge b_a$ and $d = c_d \wedge b_d$. By (b) $c_d \wedge b_a \in A$, and it is easy to see that $a \overset{\theta}{\equiv} c_d \wedge b_a \overset{\theta^{\perp}}{\equiv} d$. This shows that $\theta \circ \theta^{\perp} = \nabla$, i.e., θ is factor congruence. □

Remark 5 Given an algebra $\boldsymbol{A}$ in $\mathcal{L}$, for each factor congruence θ of $\boldsymbol{A}$, it follows from Lemmas 4 and 3 that $\boldsymbol{A} \cong \boldsymbol{1/\theta} \times \boldsymbol{1/\theta^{\perp}}$.
Observe also that given $a \in A$, if c_a and b_a are as in Lemma 4, since for any $c \in 1/\theta$ and any $b \in 1/\theta^{\perp}$ we have $c \vee b = 1$ and $c \wedge b \in A$, then using distributivity it is easy to argue that a is minimal in A if and only if c_a is minimal (relative to the natural order) in $1/\theta$ and b_a is minimal in $1/\theta^{\perp}$.

3 Decomposability in free algebras

In what follows $\boldsymbol{T}(X) = \langle T(X), \to, 1\rangle$ will represent the $\{\to, 1\}$-term algebra over the set of variables X, and $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ will represent the $|X|$-free algebra in the subvariety $\mathcal{V}$ of $\mathcal{L}$. If $s(x_1, \ldots, x_n) \in T(X)$, we write $\bar{s}$ for $s^{\boldsymbol{F}_{\mathcal{V}}(\overline{X})}(\bar{x}_1, \ldots, \bar{x}_n)$, if there is no confusion.

There is no a description of the $|X|$-free Łukasiewicz implication algebras $\boldsymbol{F}_{\mathcal{L}}(\overline{X})$. In [3], Berman and Blok described the finitely generated free algebras for the subvarieties $V(\boldsymbol{Ł}_n^{\to})$ and they give their cardinality.

For a given class $\mathcal{K}$ of $\{\to, 1\}$-algebras, we say that two terms s, t in $T(X)$ are $\mathcal{K}$*-equivalent* if the equation $s \approx t$ holds in $\mathcal{K}$. Since the class $\mathcal{L}$ is a variety of BCK-algebras, it follows from [4, Fact 0] (cf. [13, Lemma 4]) that any $\{\to, 1\}$-term $s(x_1, \ldots, x_n)$ is $\mathcal{L}$-equivalent to a $\{\to, 1\}$-term

$$s'(x_1, \ldots, x_n) = s_1 \to (s_2 \to (\cdots \to (s_r \to x_i) \cdots)),$$

where $x_i \in \{x_1, \ldots, x_n\}$ and s_i, $1 \leq i \leq r$, are terms in the variables $x_1, \ldots, x_n$ in which 1 does not appear. Thus, for any $\{\to, 1\}$-term s there is a variable x which appear in s, such that the equation $x \to s \approx 1$ hold in $\mathcal{L}$. Therefore for every subvariety $\mathcal{V}$ of $\mathcal{L}$, every element of $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ is greater than or equal to some generator. Hence:

Lemma 6 *If $\mathcal{V}$ is a subvariety of $\mathcal{L}$, then* $F_{\mathcal{V}}(\overline{X}) = \bigcup_{x \in X} [\overline{x})$. □

Lemma 7 *Let $\mathcal{V}$ be a nontrivial subvariety of $\mathcal{L}$. If X is infinite, then the unique upper bound of $\overline{X}$ relative to natural order in $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ is* 1.

Proof Let $\overline{s}(\overline{x}_1, \ldots, \overline{x}_n) \neq 1$ be an upper bound of $\overline{X}$ in $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$, then there is a $\mathcal{L}$-chain $\boldsymbol{C}$ and a homomorphism f from $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ onto $\boldsymbol{C}$ such that $s^{\boldsymbol{C}}(f(\overline{x}_1), \ldots, f(\overline{x}_n)) = f(\overline{s}) \neq 1$. Let $f' : \overline{X} \to C$ such that

$$f'(\overline{x}) = \begin{cases} f(\overline{x}), & \text{if } x \in \{x_1, \ldots, x_n\} \\ 1, & \text{if } x \in X - \{x_1, \ldots, x_n\} \end{cases}$$

Thus f' extends to a homomorphism $\widehat{f'}$: $\boldsymbol{F}_{\mathcal{V}}(\overline{X}) \to \boldsymbol{C}$, such that

$$\begin{aligned} \widehat{f'}(\overline{s}(\overline{x}_1, \ldots, \overline{x}_n)) &= s^{\boldsymbol{C}}(f'(\overline{x}_1), \ldots, f'(\overline{x}_n)) \\ &= s^{\boldsymbol{C}}(f(\overline{x}_1), \ldots, f(\overline{x}_n)) = f(\overline{s}) < 1, \end{aligned}$$

Since X is infinite, there is $y \in X - \{x_1, \ldots, x_n\} \neq \emptyset$, and $\widehat{f'}(y) = 1 > \widehat{f'}(\overline{s})$, but this contradicts the fact that $\overline{y} \leq \overline{s}$. □

Theorem 8 *Let $\mathcal{V}$ be a subvariety of $\mathcal{L}$, and X a set of variables. If θ is a factor congruence of $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$, then there is a relative boolean element α of $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ such that $\overline{x} \leq \alpha$ for all $x \in X$, and $\theta = \theta_{[\alpha)}$ or $\theta^{\perp} = \theta_{[\alpha)}$.*

Proof Each $\overline{x} \in \overline{X}$ is minimal in $F_{\mathcal{V}}(\overline{X})$ so, by Remark 5, if $c_{\overline{x}}$ and $b_{\overline{x}}$ are as in Lemma 4, then $c_{\overline{x}}$ and $b_{\overline{x}}$ are minimal in $1/\theta$ and $1/\theta^{\perp}$, respectively. Suppose $c \neq c'$ are two minimal elements of $1/\theta$ and $b \neq b'$ are two minimal elements of $1/\theta^{\perp}$. Then each of the four elements $c \wedge b, c' \wedge b, c \wedge b'$ and $c' \wedge b'$ are minimal in $F_{\mathcal{V}}(\overline{X})$, so, by Remark 5 and Lemma 6, they belong to $\overline{X}$ and by condition (a) of Lemma 4 they are pairwise distinct. Let h be a homomorphism from $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ onto $\boldsymbol{Ł}_1^{\to}$, for which $h(c \wedge b) = h(c' \wedge b') = 0$ and $h(c' \wedge b) = h(c \wedge b') = 1$. By freeness such homomorphism exists. By monotonicity $h(c) = h(c') = h(b) = h(b') = 1$, but $0 = h(c \wedge b') = h(c) \wedge h(b') = 1$ that is a contradiction. So at least one of $1/\theta$ and $1/\theta^{\perp}$ has at most one minimal element, say $1/\theta$. For each $x \in X$ we then have that $c_{\overline{x}}$ is minimal in $1/\theta$. So for all $x, y \in X$ we have $c_{\overline{x}} = c_{\overline{y}}$. Thus $\alpha = c_{\overline{x}}$ is the least element of $1/\theta$. Since for any $\overline{s} \in 1/\theta$ there exists $y \in X$ such that $\overline{y} \leq \overline{s}$, we have $\overline{s} \geq c_{\overline{y}} = c_{\overline{x}} = \alpha \geq \overline{x}$ for all $x \in X$. Thus $1/\theta = [\alpha)$, i.e., $\theta = \theta_{[\alpha)}$ and then α is relatively boolean. □

By the remarks given in precedent section, if $\mathcal{V}$ is a subvariety of $\mathcal{L}$, then there is $\beta \in \omega \cup \{\omega\}$ such that $\mathcal{V} = V(\boldsymbol{Ł}_\beta)$, and hence $\boldsymbol{F}_{\mathcal{V}}(\overline{X})$ is isomorphic to a subalgebra of the power $\boldsymbol{Ł}_\beta^{Ł_\beta^{\overline{X}}}$ (see [7, Chapter IV Theorem 3.13]). To simplify, we consider $\boldsymbol{F}_{V(\boldsymbol{Ł}_\beta)}(\overline{X}) \subseteq \boldsymbol{Ł}_\beta^I$ for a suitable set I. Then the free $|X|$-algebra is semisimple and it can be looked as subdirect product of $\prod_{i\in I}\boldsymbol{S}_i$, where each $\boldsymbol{S}_i$ is isomorphic to $\boldsymbol{Ł}_\gamma$, for some $\gamma \le \beta$. For any $i \in I$, let π_i be the projection homomorphism from $\prod_{i\in I}\boldsymbol{S}_i$ onto $\boldsymbol{S}_i$, then the algebra image $\pi_i(\boldsymbol{F}_{V(\boldsymbol{Ł}_\beta)}(\overline{X})) = \boldsymbol{S}_i$ is generated by $\pi_i(\overline{X})$. It is clear that there is an unique $i_0 \in I$ such that $\pi_{i_0}(\overline{x}) = \pi_{i_0}(\overline{y}) \neq \pi_{i_0}(1)$ for any $\overline{x}, \overline{y} \in \overline{X}$. Moreover, $\boldsymbol{S}_{i_0} \cong \boldsymbol{Ł}_1$ and $\pi_{i_0}[\overline{X}] = \{0\}$.

Theorem 9 *If X is an infinite set of variables, then for any subvariety $\mathcal{V}$ of $\mathcal{L}$, $\boldsymbol{F}_\mathcal{V}(\overline{X})$ is directly indecomposable.*

Proof Assume that X is infinite. Let θ be a factor congruence of $\boldsymbol{F}_\mathcal{V}(\overline{X})$, then, by Theorem 8, we can assume that every element of $1/\theta$ is an upper bound of $\overline{X}$. Therefore by Lemma 7, we have that $\{\alpha = 1\} = 1/\theta$ and hence $\theta = \Delta$. That is, θ and $\theta^\perp$ form a trivial pair of factor congruences, and so $\boldsymbol{F}_\mathcal{V}(\overline{X})$ is directly indecomposable. □

Theorem 10 *If X is a finite set of variables with at least two elements, then*

(1) *For any $0 < n \in \omega$, $\boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\rightarrow})}(\overline{X})$ has a unique non-trivial pair of factor congruences $\{\theta, \theta^\perp\}$ given by*

$$1/\theta = \{1, \alpha = \bigvee_{x,y\in X} (x \to y)^{n-1} \to x\}.$$

(2) *$\boldsymbol{F}_\mathcal{L}(\overline{X})$ is directly indecomposable.*

Proof of (1). Given $0 < n \in \omega$, we can assume that, up isomorphism, $\boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\rightarrow})}(\overline{X})$ is a subdirect product of $\prod_{i\in I}\boldsymbol{Ł}_{r_i}$, for some I finite and $r_i \le n$ for all $i \in I$. Take i_0 the unique element of I such that $\pi_{i_0}(\overline{x}) = \pi_{i_0}(\overline{y}) \neq 1$ for all $x, y \in X$, then for any $i \in I - \{i_0\}$ there is $z \in X$ such that $0 < \pi_i(\overline{z}) \le 1$. Let $\alpha \in \boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\rightarrow})}(\overline{X})$ such that it is relative boolean and an upper bound, relative to natural order, of $\overline{X}$. Then since to be relative boolean in preserved by homomorphic images, we have that for any $i \in I$, $\pi_i(\alpha) \in \{0, 1\}$, in particular for $i \neq i_0$, $\pi_i(\alpha) = 1$. Hence the unique possibility in order that $\alpha \neq 1$ is that $\pi_i(\alpha) = \begin{cases} 0, \text{ if } i = i_0 \\ 1, \text{ if } i \neq i_0 \end{cases}$. Our next task is to see that such element α exists in $\boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\rightarrow})}(\overline{X})$. For this we **claim**:

$$\alpha = \bigvee_{x,y\in X} (\overline{x} \to \overline{y})^{n-1} \to \overline{x}.$$

Indeed, it is clear that for all $x, y \in X$,

$$\pi_{i_0}((\overline{x} \to \overline{y})^{n-1} \to \overline{x}) = (0 \to 0)^{n-1} \to 0 = 0.$$

If $i \in I - \{i_0\}$, then there are at least two elements $x_i, y_i \in X$ such that $\pi_i(\overline{y}_i) = 0$, and $\pi_i(\overline{x}_i) \neq 0$. It is a simple verification on $\boldsymbol{Ł}_n$ that $\pi_i(\overline{x}_i) \to 0 \neq 1$ implies

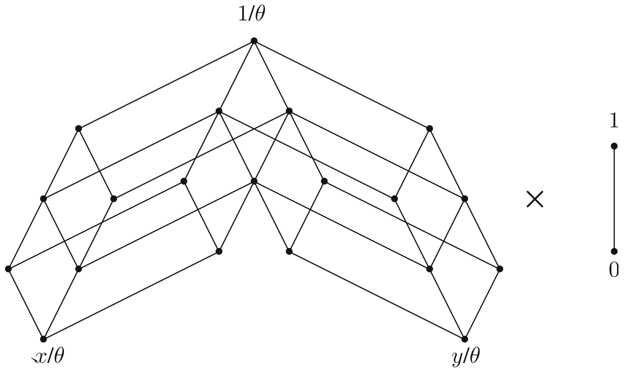
$$\begin{aligned}\pi_i((\overline{x}_i \to \overline{y}_i)^{n-1} \to \overline{x}_i)) &= (\pi_i(\overline{x}_i) \to \pi_i(\overline{y}_i))^{n-1} \to \pi_i(\overline{x}_i)) \\ &= (\pi_i(\overline{x}_i) \to 0)^{n-1} \to \pi_i(\overline{x}) = 1,\end{aligned}$$

and the claim is proved. Now, since α is a coatom in $\prod_{i\in I} \boldsymbol{Ł}_{r_i}$, then it is a coatom in $\boldsymbol{F}_{V(\boldsymbol{Ł})}(\overline{X})$, and so $[\alpha) = \{1, \alpha\}$. Observe that α is the unique relative boolean in $\boldsymbol{F}_{V(\boldsymbol{Ł})}(\overline{X})$ that is an upper bound of $\overline{X}$. It is clear that α is also a relative boolean element of $\prod_{i\in I} \boldsymbol{Ł}_{r_i}$.
To close the proof of (1) it suffices to see that the congruence relation $\theta_{[\alpha)}$ satisfies (a) and (b) of Lemma 4. To show the condition (a), consider a in $\boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\to})}(\overline{X})$, then $\alpha \vee a \in \{1, \alpha\}$. If $\alpha \vee a = 1$, we have $a \in [\alpha)^{\perp}$ and $c_a = \alpha$ and $b_a = a$. If $a \vee \alpha = \alpha$, then $a \leq \alpha$ and α is boolean in the Wajsberg algebra $\boldsymbol{F}_{V(\boldsymbol{Ł})}(\overline{X})_a$, hence $\alpha \to a \in [\alpha)^{\perp}$ and $a = \alpha \wedge (\alpha \to a)$ and $c_a = \alpha$ and $b_a = \alpha \to a$. Uniqueness easily follows from complementation. For condition (b), as for every $b \in [\alpha)^{\perp}$ there is $x \in X$ such that $\overline{x} \leq b$, and $\alpha \geq \overline{x}$, then we have $b \wedge \alpha = p(b, \alpha, \overline{x}) \in \boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\to})}(\overline{X})$.
Proof of (2). Assume that $\boldsymbol{F}_{\pounds}(\overline{X})$ is not directly indecomposable, then by Theorem 8 there is a relative boolean $\alpha < 1$, that is an upper bound of $\overline{X}$. It follows from the remarks given in item (1), that for each homomorphism h from $\boldsymbol{F}_{\pounds}(\overline{X})$ onto a $\pounds$-chain $\boldsymbol{C}$, $h(\alpha)$ is the lower bound 0 of $\boldsymbol{C}$, if and only if, $\boldsymbol{C} \cong \boldsymbol{Ł}_1$ and $h[\overline{X}] \subseteq \{0\}$. Otherwise $h(\alpha) = 1$. To find a contradiction, let g: $\boldsymbol{F}_{\pounds}(\overline{X}) \to \boldsymbol{C}_{\omega}^{\to}$, such that for a fixed $x_0 \in X$, $g(\overline{x}_0) = (0, 1)$ and $g[\overline{X} - \{x_0\}] = \{(0, 0)\}$. Since $\{(0, 1), (0, 0)\}$ is a set of generators of $\boldsymbol{C}_{\omega}^{\to}$, then g is onto and $g(\alpha) = (1, 0)$ which is the greatest element in $\boldsymbol{C}_{\omega}^{\to}$. Let κ be the natural projection from $\boldsymbol{C}_{\omega}^{\to}$ onto $\boldsymbol{C}_{\omega}^{\to}/L_{\omega}^{\to} \cong \boldsymbol{Ł}_1^{\to}$, then $\kappa \circ g$ can be seen as a homomorphism from $\boldsymbol{F}_{\pounds}(\overline{X})$ onto $\boldsymbol{Ł}_1^{\to}$ such that $\kappa \circ g[\overline{X}] \subseteq \{0\}$, $\kappa \circ g(\alpha) = 1$. That is a contradiction. Hence $\boldsymbol{F}_{\pounds}(\overline{X})$ is directly indecomposable. □

Finally we want to remark that:

- If $|X| = 1$ the $|X|$-free algebra in any subvariety is the two- element implication algebra
- If $|X|$ is finite then for all $0 < n \in \omega, \boldsymbol{F}_{V(\boldsymbol{Ł}_n^{\to})}(\overline{X})$ is the direct product of two-element boolean algebra and a suitable subalgebra given by Lemma 3.

In particular the Hasse diagram, relative to natural order, of $\boldsymbol{F}_{V(\boldsymbol{Ł}_2^{\to})}(2)$ is given by:



where $\theta = \theta_{[\alpha)}$ and $\alpha = ((\overline{x} \to \overline{y}) \to \overline{x}) \vee ((\overline{y} \to \overline{x}) \to \overline{y})$.

References

1. Abbott, J.C.: Semi-boolean algebras. Mat. Vesnik **4**, 177–198 (1967)
2. Abbott, J.C.: Implicational algebras. Bull. Math R. S. Roumaine **11**, 3–23 (1967)
3. Berman, J., Blok, W.J.: Free Lucasiewicz hoop residuation algebras. Studia Logica **77**, 153–180 (2004)
4. Blok, W.J., Raftery, J.G.: On the quasivariety of BCK-algebras and its subquasivarieties. Algebra Universalis **33**, 68–90 (1995)
5. Burris, S., Sankappanavar, H.P.: A course in universal algebra. In: Graduate Texts in Mathematics, vol. 78. Springer, Berlin Heidelberg New York (1981)
6. Chang, C.C.: Algebraic analysis of many-valued logics. Trans. Am. Math. Soc. **88**, 467–490 (1958)
7. Cohn, P.M.: Universal Algebra, Revised edn. Reidel, Dordrecht (1981)
8. Díaz Varela, J.P., Torrens, A.: Decomposability of free Tarski algebras. Algebra Universalis **50**, 1–5, (2003)
9. Font, J.M., Rodriguez, A.J., Torrens, A.: Wajsberg algebras. Stochastica **8**, 5–31 (1984)
10. Komori, Y.: The separation theorem on of the $\aleph_0$-valued propositional logic. Rep. Fac. Sci. Shizouka Univ. **12**, 1–5. (1978) **72**, 127–133 (1978)
11. Komori, Y.: Super-Łucasiewicz implicational logics. Nogoya Math. J. **72**, 127–133 (1978)
12. Mundici, D.: MV-algebras are categorically equivalent to bounded commutative BCK-algebras. Math. Japonica **31**, 889–894 (1986)
13. Torrens, A.: On the role of the polynomial $(X \to y) \to y$ in some implicative algebras. In: Zeitsch. F. Math. Logik Grundl. Math. **34**, 117–122 (1988)