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PERTURBED FRAME SEQUENCES: CANONICAL DUAL SYSTEMS, APPROXIMATE RECONSTRUCTIONS AND APPLICATIONS

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We consider perturbation of frames and frame sequences in a Hilbert space $\mathcal{H}$. It is known that small perturbations of a frame give rise to another frame. We show that the canonical dual of the perturbed sequence is a perturbation of the canonical dual of the original one and estimate the error in the approximation of functions belonging to the perturbed space. We then construct perturbations of irregular translates of a bandlimited function in $L^2(\mathbb{R}^d)$. We give conditions for the perturbed sequence to inherit the property of being Riesz or frame sequence. For this case we again calculate the error in the approximation of functions that belong to the perturbed space and compare it with our previous estimation error for general Hilbert spaces.

Keywords: Frames; Riesz bases; canonical duals; irregular translates; approximate reconstructions; bandlimited functions.

AMS Subject Classification: 42C40, 42C15

1. Introduction

Frames play an important role in signal transmission, sampling and approximation theory.[4,7,16,18,25] Different from bases, the representation of a signal via frames is redundant, i.e. the coefficients in the expansion are not necessarily unique. This is advantageous in many situations, for example when some data is lost. Frequently, for instance in signal processing, the signal might belong to some subspace of the whole Hilbert space $\mathcal{H}$ and then only expansions in this subspace are interesting. Here, it is necessary to work with *frame sequences*, that is a sequence in a Hilbert space which is a frame only for its closed span.

A main question is which properties of frames and Riesz or frame sequences are preserved if we slightly modify the elements of the systems. This gives rise to the so-called perturbation theory. For the first time in Ref. 18, the idea of a specific perturbation of the typical exponential orthonormal basis of $L^2[-\gamma,\gamma]$ appears. Later, frame perturbations have been studied in Refs. 8, 12 and 20, and further on the problem of frame sequence perturbation e.g. in Refs. 15 and 6. There are various results about stability under perturbations for frames and frame sequences, the so-called Paley–Wiener perturbation theorems and also the compact perturbation theorems.[10,12,13] The additional problem that arises when dealing specifically with frame sequences, is that the perturbation might not belong to the subspace that the original sequence spans. For this case the notions of *gap* and *infimum cosine angle*[13,23,24] between the corresponding subspaces have to be involved.

Given a frame $\{f_k\}_{k\in\mathbb{Z}}$ for a Hilbert space $\mathcal{H}$, it is known that there exists a *dual frame* $\{g_k\}_{k\in\mathbb{Z}}$, which is a frame such that

$$f = \sum_{k\in\mathbb{Z}} \langle f, g_k\rangle f_k \ \ \forall\, f \in \mathcal{H} \quad \text{or} \quad f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle g_k\, \forall\, f \in \mathcal{H}.$$

If $\{f_k\}_{k\in\mathbb{Z}}$ is an overcomplete frame, there exist infinite alternatives for dual frames. The choice of dual that gives the classical coefficients in the frame expansion is the *canonical dual frame*.

In this work, we first show that if we do a sufficiently "small" perturbation of a frame, the canonical dual of the new frame is also a "small" perturbation of the canonical dual of the first one. We then obtain a similar result for the case of frame sequences. We exploit this fact in order to obtain different reconstructions of functions that belong to the perturbed space, involving — unlike in the traditionally used frame expansion — the canonical dual of the original frame sequence. We estimate for these cases the deviation from perfect reconstruction.

Once obtained these results for general Hilbert spaces, we consider the concrete case where $\mathcal{H} = L^2(\mathbb{R}^d)$ and the frame sequences are generated by irregular translates of a single function. There is a close connection between frames of translates and frames of complex exponentials via the Fourier transform. Beurling[5] gave sufficient conditions on a sequence $\{\lambda_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{R}^d$ in order to have frames of exponentials when restricted to a ball. These are given in terms of density. Using this, frame properties of irregular translates of a *bandlimited* function were obtained in

Ref. 2. More precisely in Ref. 2 the authors give conditions on the translates of a bandlimited function in order to obtain frames and Riesz or frame sequences.

In the present work, we extend these results for a more general class of generating functions. We obtain conditions on irregular translates of a function in order to be frames and Riesz or frame sequences, weakening the bandlimitedness assumption. For this we study perturbations of irregular translates of a bandlimited function, where the perturbed and the original generating functions differ in a polynomially decaying function. We compute the estimation error for the approximation of a function in the perturbed space in this last particular case and further compare it to the error bound obtained before in the general Hilbert space case.

The paper is organized as follows. In Sec. 2, we give the definitions and state known results we use later. In Sec. 3 we prove that the canonical dual of a perturbed frame (frame sequence) is a perturbation of the canonical dual of the initial frame (frame sequence). We give alternative approximations of functions in the perturbed space and show a bound for the approximation error. Finally, in Sec. 4, we work in $L^2(\mathbb{R}^d)$ perturbing irregular translates of a bandlimited function by adding a polynomially decaying component.

2. Preliminaries

In this section, we set definitions and known results that we will need throughout the paper.

Given a separable Hilbert space $\mathcal{H}$ with norm $\|\cdot\|$, a sequence $\{f_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{k\in\mathbb{Z}} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \tag{2.1}$$

for all $f \in \mathcal{H}$. The constants A and B are called *frame bounds*.

If only the right inequality in (2.1) is satisfied we say that $\{f_k\}_{k\in\mathbb{Z}}$ is a *Bessel sequence* with Bessel bound B. To every Bessel sequence $\{f_k\}_{k\in\mathbb{Z}}$ we associate the *analysis operator* $C_F : \mathcal{H} \to \ell^2(\mathbb{Z})$ defined by $C_F f = \{\langle f, f_k\rangle\}_{k\in\mathbb{Z}}$ for $f \in \mathcal{H}$, and the synthesis operator $U_F : \ell^2(\mathbb{Z}) \to \mathcal{H}$ given by $U_F c = \sum_{k\in\mathbb{Z}} c_k f_k$ for $c \in \ell^2(\mathbb{Z})$.

If $\{f_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{H}$ is a frame the *frame operator* defined by

$$S_F : \mathcal{H} \to \mathcal{H}, \quad S_F f = U_F C_F f = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle f_k,$$

is bounded, positive and invertible. The sequence $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$, where $\tilde{f}_k = S_F^{-1} f_k$, is called the *canonical dual frame* for $\{f_k\}_{k\in\mathbb{Z}}$, and each $f \in \mathcal{H}$ has the following frame decomposition

$$f = \sum_{k\in\mathbb{Z}} \langle f, S_F^{-1} f_k\rangle f_k = \sum_{k\in\mathbb{Z}} \langle f, f_k\rangle S_F^{-1} f_k.$$

It can be equivalently written as $f = U_F C_{\tilde{F}} f = U_{\tilde{F}} C_F f$. Throughout the paper, we will always denote by the subscript to which collection the analysis, synthesis and frame operators are associated.

A sequence $\{f_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ if it is complete in $\mathcal{H}$ and if there exist $0 < A \leq B$ such that for every finite scalar sequence $\{c_k\}_{k\in\mathbb{Z}}$ one has

$$A\|c\|_{\ell^2}^2 \leq \left\|\sum_{k\in\mathbb{Z}} c_k f_k\right\|^2 \leq B\|c\|_{\ell^2}^2.$$

The constants A and B are called *Riesz bounds*.

We say that $\{f_k\}_{k\in\mathbb{Z}} \subseteq \mathcal{H}$ is a *frame sequence* (*Riesz sequence*) if it is a frame (Riesz basis) for the space it spans.

In this paper, we will work with perturbed sequences. More precisely we will use the following notion of perturbation.

Definition 2.1. Let $\{f_k\}_{k\in\mathbb{Z}}$ be a sequence in $\mathcal{H}$ and $\mu > 0$. We say that a sequence $\{g_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ is a *μ-perturbation of* $\{f_k\}_{k\in\mathbb{Z}}$ if for every finite sequence $c \in \ell^2(\mathbb{Z})$,

$$\left\|\sum_{k\in\mathbb{Z}} c_k(f_k - g_k)\right\| \leq \mu\|c\|_{\ell^2}. \tag{2.2}$$

Observe that if condition (2.2) is satisfied for all finite sequence $c \in \ell^2(\mathbb{Z})$, it is valid for all $c \in \ell^2(\mathbb{Z})$.

Remark 2.1. Note that condition (2.2) is equivalent to say that $\{f_k - g_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence in $\mathcal{H}$ with Bessel bound μ^2. That is, $\{g_k\}_{k\in\mathbb{Z}}$ is a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ if and only if $\sum_{k\in\mathbb{Z}} |\langle f, f_k - g_k\rangle|^2 \leq \mu^2\|f\|^2$ for all $f \in \mathcal{H}$.

An interesting question about the perturbed sequence is when it inherits the properties of the original one. For instance, it is known that if $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz sequence with lower bound A, the perturbed sequence $\{g_k\}_{k\in\mathbb{Z}}$ is also a Riesz sequence when $\mu < \sqrt{A}$, cf. Theorem 15.3.2 in Ref. 11. For the case that $\{f_k\}_{k\in\mathbb{Z}}$ is a frame sequence the condition $\mu < \sqrt{A}$ is not enough to ensure that $\{g_k\}_{k\in\mathbb{Z}}$ is also a frame sequence. In order to specify when $\{g_k\}_{k\in\mathbb{Z}}$ is a frame sequence we need to consider the gap between the spaces spanned by $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$.

For general nonempty subspaces $V, W \subseteq \mathcal{H}$ the *gap*[23] between V and W is defined as

$$\delta(V, W) := \sup_{f\in V;\|f\|=1} \|f - P_W f\|,$$

where P_W denotes the orthogonal projection of $\mathcal{H}$ onto W.

In Refs. 13 and 24 the *infimum cosine angle* respectively the *supremum cosine angle* are defined by

$$R(V, W) := \inf_{f\in V;\|f\|=1} \|P_W f\| \quad \text{and} \quad S(V, W) := \sup_{f\in V;\|f\|=1} \|P_W f\|.$$

It can be seen that $\delta(V, W)$ and $R(V, W)$ are related by $R(V, W) = \sqrt{1 - \delta(V, W)^2}$.

We denote by I the identity operator on $\mathcal{H}$, and the restriction of an operator T to a subspaces $\mathcal{K}$ by $T|_{\mathcal{K}}$.

The following theorem states conditions for the perturbed sequence to be a frame, a Riesz or a frame sequence. Statement (a) appeared in Refs. 10, 15 and 11, and statements (b) and (c) are from Ref. 13.

Theorem 2.1. *Let* $\{f_k\}_{k\in\mathbb{Z}}$ *be a sequence in* $\mathcal{H}$ *and assume* $\{g_k\}_{k\in\mathbb{Z}}$ *is a* μ*-perturbation of* $\{f_k\}_{k\in\mathbb{Z}}$*. Then the following holds*:

(i) *If* $\{f_k\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{H}$ (*Riesz basis or Riesz sequence in* $\mathcal{H}$) *with frame* (*Riesz*) *bounds* $0 < A \leq B$ *and* $\mu < \sqrt{A}$, *then* $\{g_k\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{H}$ (*Riesz basis or Riesz sequence in* $\mathcal{H}$) *with frame* (*Riesz*) *bounds* $A(1-\mu/\sqrt{A})^2, B(1+\mu/\sqrt{B})^2$.

(ii) *Suppose that* $\{f_k\}_{k\in\mathbb{Z}}$ *is a frame sequence with frame bounds* $0 < A \leq B$ *and that* $\mathcal{H}_F = \overline{\text{span}}\{f_k : k \in \mathbb{Z}\}$*. If* $\mu < \sqrt{A}$, *then* $R(\mathcal{H}_F, \mathcal{H}_G) > 0$, *where* $\mathcal{H}_G = \overline{\text{span}}\{g_k : k \in \mathbb{Z}\}$.

(iii) *If in addition to* (*b*), $R(\mathcal{H}_G, \mathcal{H}_F) > 0$, *then* $\{g_k\}_{k\in\mathbb{Z}}$ *is a frame for* $\mathcal{H}_G$ *with frame bounds* $A(1-\mu/\sqrt{A})^2, B(S(\mathcal{H}_G, \mathcal{H}_F) + \mu/\sqrt{B})^2$ *and moreover,* $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ *is an isomorphism from* $\mathcal{H}_G$ *to* $\mathcal{H}_F$.

Remark 2.2. Since conditions $R(\mathcal{H}_G, \mathcal{H}_F) > 0$ and $\delta(\mathcal{H}_G, \mathcal{H}_F) < 1$ are equivalent, either of them can be used as hypothesis in (c) of the above theorem.

3. Canonical Duals: Perturbations and Reconstruction Errors

We will devote this section to prove a result concerning the canonical duals for frames and frame sequences. Essentially, we prove that the canonical duals of "close" frames (frame sequences respectively) are also "close". In other words, we show that the canonical dual of a perturbed frame is a perturbation of the canonical dual of the original frame.

Throughout this section, we will use the following elementary Banach Algebra's result. Since we could not find a reference for a proof, we provide it here.

Lemma 3.1. *Let* $x, y \in \mathcal{B}$, *where* $\mathcal{B}$ *is a Banach algebra with unit element* e *and norm* $\|\cdot\|_{\mathcal{B}}$. *If* y *is invertible and* $\|x-y\|_{\mathcal{B}} \leq \alpha \|y^{-1}\|_{\mathcal{B}}^{-1}$ *for some* $\alpha < 1$, *then* x *is invertible and* $\|x^{-1}-y^{-1}\|_{\mathcal{B}} \leq \|y^{-1}\|_{\mathcal{B}}\alpha(1-\alpha)^{-1}$.

Proof. From the assumption on x and y it follows that $\|e - xy^{-1}\|_{\mathcal{B}} \leq \|x - y\|_{\mathcal{B}}\|y^{-1}\|_{\mathcal{B}} \leq \alpha$. That means that xy^{-1} is invertible with the inverse given by von Neumann series $(xy^{-1})^{-1} = \sum_{n=0}^{\infty}(e - xy^{-1})^n$. The invertibility of x follows easily from that of xy^{-1}. Now,

$$\begin{aligned}\|x^{-1} - y^{-1}\|_{\mathcal{B}} &= \|y^{-1}(yx^{-1}(e - xy^{-1}))\|_{\mathcal{B}} \leq \|y^{-1}\|_{\mathcal{B}}\|yx^{-1}\|_{\mathcal{B}}\|e - xy^{-1}\|_{\mathcal{B}} \\ &\leq \|y^{-1}\|_{\mathcal{B}}\|e - xy^{-1}\|_{\mathcal{B}}\sum_{n=0}^{\infty}\|e - xy^{-1}\|_{\mathcal{B}}^n = \|y^{-1}\|_{\mathcal{B}}\sum_{n=1}^{\infty}\|e - xy^{-1}\|_{\mathcal{B}}^n \\ &\leq \|y^{-1}\|_{\mathcal{B}}\sum_{n=1}^{\infty}\alpha^n = \|y^{-1}\|_{\mathcal{B}}\alpha(1-\alpha)^{-1}.\end{aligned}$$

□

3.1. *Perturbation of frames*

In this subsection, we will work with frames for the whole Hilbert space $\mathcal{H}$.

Theorem 3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a frame for $\mathcal{H}$ with frame bounds $0 < A_F \le B_F$, and let $\{g_k\}_{k\in\mathbb{Z}}$ be a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ with $0 < \mu < \sqrt{A_F + B_F} - \sqrt{B_F}$. Then $\{g_k\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ and, for every finite sequence $c \in \ell^2(\mathbb{Z})$,*

$$\left\| \sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k) \right\| \le \lambda \|c\|_{\ell^2},$$

where

$$\lambda = \frac{\mu}{A_F}\left(1 + \frac{(\sqrt{B_F} + \mu)(2\sqrt{B_F} + \mu)}{A_F - (2\sqrt{B_F} + \mu)\mu}\right) \tag{3.1}$$

and $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$, $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$ are the canonical dual frames for $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively.

Proof. Since $\sqrt{A_F + B_F} - \sqrt{B_F} < \sqrt{A_F}$, by Theorem 2.1(a) we have that $\{g_k\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ with frame bounds $A_G = A_F(1 - \mu/\sqrt{A_F})^2$ and $B_G = B_F(1 + \mu/\sqrt{B_F})^2$. We denote by S_F and S_G the frame operators associated to $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively.

Now, we set

$$\begin{aligned}\left\| \sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k) \right\| &= \|S_F^{-1}U_F c - S_G^{-1}U_G c\| \le \|S_F^{-1}(U_F c - U_G c)\| \\ &\quad + \|(S_F^{-1} - S_G^{-1})U_G c\| \\ &\le \mu\|c\|_{\ell^2}\|S_F^{-1}\|_{\mathrm{op}} + \sqrt{B_G}\|c\|_{\ell^2}\|S_F^{-1} - S_G^{-1}\|_{\mathrm{op}}.\end{aligned}$$

In order to estimate $\|S_F^{-1} - S_G^{-1}\|_{\mathrm{op}}$ we utilize Lemma 3.1. By the perturbation conditions of $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ together with Remark 2.1, we obtain

$$\begin{aligned}\|S_F - S_G\|_{\mathrm{op}} &= \|U_F C_F - U_G C_G\|_{\mathrm{op}} \le \|U_F(C_F - C_G)\|_{\mathrm{op}} + \|(U_F - U_G)C_G\|_{\mathrm{op}} \\ &\le \sqrt{B_F}\mu + \sqrt{B_G}\mu \le \|S_F^{-1}\|_{\mathrm{op}}^{-1}\frac{\sqrt{B_F} + \sqrt{B_G}}{A_F}\mu.\end{aligned}$$

Let $\alpha = \frac{\sqrt{B_F} + \sqrt{B_G}}{A_F}$. The perturbation condition implies that $\sqrt{B_G} \le \sqrt{B_F} + \mu$, hence $\alpha\mu \le A_F^{-1}(2\sqrt{B_F} + \mu)\mu$. By assumption on μ, we obtain that $\alpha\mu < 1$. Therefore, by Lemma 3.1, it follows that $\|S_F^{-1} - S_G^{-1}\|_{\mathrm{op}} \le \frac{\alpha\mu}{1-\alpha\mu}\|S_F^{-1}\|_{\mathrm{op}}$.

Finally, collecting all the estimates,

$$\left\| \sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k) \right\| \le \left(\frac{\mu}{A_F} + \frac{\sqrt{B_G}}{A_F}\frac{\alpha\mu}{1 - \alpha\mu}\right)\|c\|_{\ell^2}.$$

Therefore, setting

$$\lambda = \frac{\mu}{A_F} + \frac{\sqrt{B_G}}{A_F}\frac{\alpha\mu}{1 - \alpha\mu} = \frac{\mu}{A_F}\left(1 + \frac{(\sqrt{B_F} + \mu)(2\sqrt{B_F} + \mu)}{A_F - (2\sqrt{B_F} + \mu)\mu}\right)$$

we complete the proof. □

Remark 3.1. Note that the constant λ is small provided μ is small.

For the special case of perturbations of a Riesz basis Theorem 3.1 is also true asking only $0 < \mu < \sqrt{A_F}$. Riesz bases are in particular frames. Different from frames, each Riesz bases has a unique dual which is the canonical dual.

Proposition 3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ be a Riesz basis for $\mathcal{H}$ with Riesz bounds $0 < A_F \leq B_F$, and let $\{g_k\}_{k\in\mathbb{Z}}$ be a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ with $0 < \mu < \sqrt{A_F}$. Then $\{g_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$ and, for each finite sequence $c \in \ell^2(\mathbb{Z})$,*

$$\left\|\sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k)\right\| \leq \lambda\|c\|_{\ell^2},$$

where $\lambda = \mu(A_F - \mu\sqrt{A_F})^{-1}$ and $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$, $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$ are the duals for $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively.

Proof. Since $\{g_k\}_{k\in\mathbb{Z}}$ is a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ with $\mu < \sqrt{A_F}$, by Theorem 2.1(a) $\{g_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$. Denote by A_G its lower Riesz bound. Let $c \in \ell^2(\mathbb{Z})$ be a finite sequence. Since $\{f_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$, the analysis operator $C_F : \mathcal{H} \to \ell^2(\mathbb{Z})$ is onto, hence there exists $h \in \mathcal{H}$ such that $c = C_F h$. Now, using that $I = U_{\tilde{G}}C_G = U_{\tilde{F}}C_F$ and Remark 2.1, we have

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k)\right\| &= \|U_{\tilde{F}}c - U_{\tilde{G}}c\| = \|U_{\tilde{F}}C_F h - U_{\tilde{G}}C_F h\| = \|(I - U_{\tilde{G}}C_F)h\| \\ &= \|(U_{\tilde{G}}C_G - U_{\tilde{G}}C_F)h\| \leq \|U_{\tilde{G}}\|_{\mathrm{op}}\|C_G - C_F\|_{\mathrm{op}}\|h\| \\ &\leq \frac{\mu}{\sqrt{A_G}}\|h\| \leq \frac{\mu}{\sqrt{A_G}\sqrt{A_F}}\|c\|_{\ell^2},\end{aligned}$$

where the last inequality follows from a frame property of $\{f_k\}_{k\in\mathbb{Z}}$. Replacing A_G with the estimate in Theorem 2.1(a) completes the proof. □

We want to point out that duals of perturbed frames are also studied in Ref. 14. Different from our approach, in Ref. 14 the authors work with the notion of *approximately dual frames*. Two Bessel sequences $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are *approximately dual frames* if $\|I - U_F C_G\|_{\mathrm{op}} < 1$, or equivalently $\|I - U_G C_F\|_{\mathrm{op}} < 1$. They show that if $\{f_k\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}$ and $\{g_k\}_{k\in\mathbb{Z}}$ is a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ with $\mu < \sqrt{A_F}$, then $\|I - U_{\widetilde{F}}C_G\|_{\mathrm{op}} < \mu/\sqrt{A_F} < 1$, meaning the canonical dual $\{\widetilde{f}_k\}_{k\in\mathbb{Z}}$ of $\{f_k\}_{k\in\mathbb{Z}}$, is *approximately dual* to $\{g_k\}_{k\in\mathbb{Z}}$. Inexplicitly, this notion also measures the closeness of two canonical dual systems $\{\widetilde{f}_k\}_{k\in\mathbb{Z}}$ and $\{\widetilde{g}_k\}_{k\in\mathbb{Z}}$, since

$$\|I - U_{\widetilde{F}}C_G\|_{\mathrm{op}} = \|(U_{\widetilde{G}} - U_{\widetilde{F}})C_G\|_{\mathrm{op}} < \frac{\mu}{\sqrt{A_F}}.$$

Theorem 3.1 above instead measures the similarity between the two canonical systems explicitly in terms of μ and frame bounds of $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$.

3.2. *Perturbation of frame sequences*

In this section, we show a similar result to Theorem 3.1 for frame sequences. The main difficulty in the case of frame sequences is that the frame operators do not have the same domain.

We begin by setting our Standing Assumptions which will be in force for the remainder of this section.

Standing Assumption:

- $\{f_k\}_{k\in\mathbb{Z}}$ is a frame sequence in $\mathcal{H}$ with frame bounds $A_F \leq B_F$.
- $\{g_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ is a μ-perturbation of $\{f_k\}_{k\in\mathbb{Z}}$ such that $0 < \mu < \sqrt{A_F}$.
- $\mathcal{H}_F = \overline{\operatorname{span}}\{f_k : k \in \mathbb{Z}\}$ and $\mathcal{H}_G = \overline{\operatorname{span}}\{g_k : k \in \mathbb{Z}\}$.
- $\delta(\mathcal{H}_G, \mathcal{H}_F) < 1$.

Remark 3.2. Note that under our Standing Assumptions, $\{g_k\}_{k\in\mathbb{Z}}$ is a frame sequence — as a consequence of Theorem 2.1 — which frame bounds we call $A_G \leq B_G$. We will use later that $\delta(\mathcal{H}_G, \mathcal{H}_F) < 1$ is equivalent to say that $P_{\mathcal{H}_F}|_{\mathcal{H}_G} : \mathcal{H}_G \to \mathcal{H}_F$ is an isomorphism, cf. Refs. 15 and 13. Moreover, $R(\mathcal{H}_G, \mathcal{H}_F) = \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}}^{-1}$, cf. Proposition 2.1 in Ref. 22.

We now present the main result of this section.

Theorem 3.2. *Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be sequences in $\mathcal{H}$ satisfying the Standing Assumptions. If for $\alpha = \frac{(\sqrt{B_G}+\sqrt{B_F})}{A_F R(\mathcal{H}_G,\mathcal{H}_F)}$, $\mu\alpha < 1$, then for all finite sequences $c \in \ell^2(\mathbb{Z})$,*

$$\left\|\sum_{k\in\mathbb{Z}} c_k(\tilde{f}_k - \tilde{g}_k)\right\| \leq \lambda\|c\|_{\ell^2},$$

where

$$\lambda = \frac{\sqrt{B_F}}{A_F}\frac{\mu\alpha}{1-\mu\alpha} + \frac{\sqrt{B_F}\delta(\mathcal{H}_G,\mathcal{H}_F) + \mu}{A_G R(\mathcal{H}_G,\mathcal{H}_F)} \tag{3.2}$$

and $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$, $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$ are the canonical dual frames for $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively.

Remark 3.3. Using explicit expression for the upper frame bound B_G, it can be easily computed that $\mu\alpha < 1$ only for

$$\mu < 2^{-1}\left(\sqrt{B_F(1 + S(\mathcal{H}_G,\mathcal{H}_F))^2 + 4A_F R(\mathcal{H}_G,\mathcal{H}_F)} - \sqrt{B_F}(1 + S(\mathcal{H}_G,\mathcal{H}_F))\right).$$

This bound on μ is much smaller than $\sqrt{A_F}$, which is sufficient, together with $R(\mathcal{H}_G,\mathcal{H}_F) > 0$, for establishing frame property of $\{g_k\}_{k\in\mathbb{Z}}$.

For the proof of Theorem 3.2 we need following lemmas.

Lemma 3.2. *Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be sequences in $\mathcal{H}$ satisfying the Standing Assumptions. Then $\|U_F c - P_{\mathcal{H}_F} U_G c\| \leq \mu\|c\|_{\ell^2}$ for all finite sequences $c \in \ell^2(\mathbb{Z})$.*

Proof. Let $c \in \ell^2(\mathbb{Z})$ be a finite sequence. Since $P_{\mathcal{H}_F} f_k = f_k$ we have that

$$\|U_F c - P_{\mathcal{H}_F} U_G c\| = \|P_{\mathcal{H}_F}(U_F c - U_G c)\| \leq \|U_F c - U_G c\| \leq \mu \|c\|_{\ell^2}. \qquad \square$$

The following lemma is a consequence of Lemma 3.1.

Lemma 3.3. *Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be sequences in $\mathcal{H}$ satisfying the hypotheses of Theorem* 3.2. *Then,*

$$\|S_F^{-1} - P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}} \leq \|S_F^{-1}\|_{\mathrm{op}} \frac{\mu\alpha}{1-\mu\alpha}.$$

Proof. Let $f \in \mathcal{H}_F$. Since $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ is an isomorphism, there exists $h \in \mathcal{H}_G$ such that $h = P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} f$ and $P_{\mathcal{H}_F} h = f$. Moreover, $C_F P_{\mathcal{H}_F} h = C_F h$. Therefore, by the triangle inequality

$$\begin{aligned}\|P_{\mathcal{H}_F} S_G P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} f - S_F f\| &= \|P_{\mathcal{H}_F} U_G C_G h - U_F C_F P_{\mathcal{H}_F} h\| \\ &\leq \|(P_{\mathcal{H}_F} U_G - U_F) C_G h\| + \|U_F(C_G h - C_F h)\|.\end{aligned}$$

Using that $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are frame sequences and Lemma 3.2, we obtain

$$\begin{aligned}\|P_{\mathcal{H}_F} S_G P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} f - S_F f\| &\leq \mu(\sqrt{B_G} + \sqrt{B_F})\|h\| \\ &\leq \mu(\sqrt{B_G} + \sqrt{B_F})\|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}}\|f\| \\ &\leq \|S_F^{-1}\|_{\mathrm{op}}^{-1} \mu \underbrace{\frac{\sqrt{B_G} + \sqrt{B_F}}{A_F R(\mathcal{H}_G, \mathcal{H}_F)}}_{\alpha} \|f\|,\end{aligned}$$

where the last inequality is due to $\|S_F^{-1}\|_{\mathrm{op}} \leq A_F^{-1}$ and $R(\mathcal{H}_G, \mathcal{H}_F) = \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}}^{-1}$. By assumption, $\mu\alpha < 1$, hence, directly by Lemma 3.1, it follows that

$$\|S_F^{-1} - P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}} \leq \|S_F^{-1}\|_{\mathrm{op}} \frac{\mu\alpha}{1-\mu\alpha}. \qquad \square$$

We are now in the position to prove Theorem 3.2.

Proof of Theorem 3.2. Let $c \in \ell^2(\mathbb{Z})$ be a finite sequence. In order to estimate

$$\left\|\sum_{k\in\mathbb{Z}} c_k (S_F^{-1} f_k - S_G^{-1} g_k)\right\| = \|S_F^{-1} U_F c - S_G^{-1} U_G c\| \leq \|S_F^{-1} U_F - S_G^{-1} U_G\|_{\mathrm{op}} \|c\|_{\ell^2},$$

we use the triangle inequality,

$$\begin{aligned}\|S_F^{-1} U_F - S_G^{-1} U_G\|_{\mathrm{op}} &\leq \underbrace{\|(S_F^{-1} - P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}) U_F\|_{\mathrm{op}}}_{T_I} \\ &\quad + \underbrace{\|P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F - S_G^{-1} U_G\|_{\mathrm{op}}}_{T_{II}}.\end{aligned}$$

The first term, T_I, can be estimated using Lemma 3.3 and the frame property of $\{f_k\}_{k\in\mathbb{Z}}$, thus becoming

$$T_I \le \|S_F^{-1} - P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}} \|U_F\|_{\mathrm{op}} \le \|S_F^{-1}\|_{\mathrm{op}} \frac{\mu\alpha}{1-\mu\alpha}\sqrt{B_F} \le \frac{\sqrt{B_F}}{A_F}\frac{\mu\alpha}{1-\mu\alpha}.$$

For the second term T_{II}, we compute

$$\begin{aligned} T_{II} &\le \|P_{\mathcal{H}_F} S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F - S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F\|_{\mathrm{op}} + \|S_G^{-1} P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F - S_G^{-1} U_G\|_{\mathrm{op}} \\ &\le \|P_{\mathcal{H}_F} S_G^{-1} - S_G^{-1}\|_{\mathrm{op}} \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F\|_{\mathrm{op}} + \|S_G^{-1}\|_{\mathrm{op}} \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F - U_G\|_{\mathrm{op}}. \end{aligned}$$

Since $P_{\mathcal{H}_F} S_G^{-1} - S_G^{-1} = -P_{\mathcal{H}_F^\perp} S_G^{-1}$ and $\|S_G\|_{\mathrm{op}}^{-1} \le A_G^{-1}$,

$$\begin{aligned} \|P_{\mathcal{H}_F} S_G^{-1} - S_G^{-1}\|_{\mathrm{op}} = \|P_{\mathcal{H}_F^\perp} S_G^{-1}\|_{\mathrm{op}} &= \sup_{y\in\mathcal{H}_G\setminus\{0\}} \frac{\|P_{\mathcal{H}_F^\perp} y\|}{\|S_G y\|} \\ &\le A_G^{-1} \sup_{y\in\mathcal{H}_G, \|y\|=1} \|P_{\mathcal{H}_F^\perp} y\| \\ &= A_G^{-1}(1 - R(\mathcal{H}_G, \mathcal{H}_F)^2)^{\frac{1}{2}} \\ &= A_G^{-1}\delta(\mathcal{H}_G, \mathcal{H}_F). \end{aligned} \tag{3.3}$$

On the other hand, Lemma 3.2 yields

$$\begin{aligned} \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F - U_G\|_{\mathrm{op}} &= \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} (U_F - P_{\mathcal{H}_F} U_G)\|_{\mathrm{op}} \\ &\le \|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1}\|_{\mathrm{op}} \|U_F - P_{\mathcal{H}_F} U_G\|_{\mathrm{op}} \\ &\le R(\mathcal{H}_G, \mathcal{H}_F)^{-1}\mu. \end{aligned} \tag{3.4}$$

Thus, by (3.3), (3.4) and $\|P_{\mathcal{H}_F}|_{\mathcal{H}_G}^{-1} U_F\|_{\mathrm{op}} \le \sqrt{B_F} R(\mathcal{H}_G, \mathcal{H}_F)^{-1}$,

$$T_{II} \le \frac{\sqrt{B_F}\delta(\mathcal{H}_G, \mathcal{H}_F) + \mu}{A_G R(\mathcal{H}_G, \mathcal{H}_F)}.$$

Finally, collecting all the estimates, we obtain

$$\|S_F^{-1} U_F - S_G^{-1} U_G\|_{\mathrm{op}} \le T_I + T_{II} \le \frac{\sqrt{B_F}}{A_F}\frac{\mu\alpha}{1-\mu\alpha} + \frac{\sqrt{B_F}\delta(\mathcal{H}_G, \mathcal{H}_F) + \mu}{A_G R(\mathcal{H}_G, \mathcal{H}_F)}. \qquad \square$$

Remark 3.4. Since both $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ are frame sequence, we have that for $f \in \mathcal{H}_G$, $\|f\| \ge \sqrt{A_G}\|C_{\widetilde{G}} f\|$, and by Lemma 3.2,

$$\|f - P_{\mathcal{H}_F} f\| \le \|(U_G - U_F) C_{\widetilde{G}} f\| + \|(U_F - P_{\mathcal{H}_F} U_G) C_{\widetilde{G}} f\| \le 2\mu \|C_{\widetilde{G}} f\|.$$

Therefore, $\delta(\mathcal{H}_G, \mathcal{H}_F) \le \frac{2\mu}{\sqrt{A_G}}$. Consequently, $\lambda \le \mu(\frac{\alpha}{A_F(1-\alpha\mu)}\sqrt{B_F} + \frac{2\sqrt{B_F}+\sqrt{A_G}}{R(\mathcal{H}_G,\mathcal{H}_F)A_G^{3/2}})$ and from this it follows that λ tends to zero when μ tends to zero.

Theorem 3.2 states that if two frame sequences are "close", then their canonical duals are also "close". This leads to consider the following approximations of a function belonging to $\mathcal{H}_G$, where we replace in the classical frame decomposition the dual of the perturbed sequence $\{g_k\}_{k\in\mathbb{Z}}$ by the dual of the original frame sequence $\{f_k\}_{k\in\mathbb{Z}}$.

Corollary 3.1. *Let $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$ be sequences in $\mathcal{H}$ satisfying assumptions of Theorem* 3.2, *and let $f \in \mathcal{H}_G$. Then the following hold*:

(i) $\|f - f_1\| \le \sqrt{B_G}\lambda\|f\|$, *where* $f_1 = \sum_{k\in\mathbb{Z}}\langle f, \tilde{f}_k\rangle g_k \in \mathcal{H}_G$;
(ii) $\|f - f_2\| \le \sqrt{B_G}\lambda\|f\|$, *where* $f_2 = \sum_{k\in\mathbb{Z}}\langle f, g_k\rangle \tilde{f}_k \in \mathcal{H}_F$;

where λ is defined in (3.2), *B_G denotes the upper frame bound of $\{g_k\}_{k\in\mathbb{Z}}$ and $\{\tilde{f}_k\}_{k\in\mathbb{Z}}$, $\{\tilde{g}_k\}_{k\in\mathbb{Z}}$ are canonical dual frames for $\{f_k\}_{k\in\mathbb{Z}}$ and $\{g_k\}_{k\in\mathbb{Z}}$, respectively.*

Remark 3.5. For perturbed frame sequences another kind of dual windows can be considered, namely oblique dual frames. Such frames can be constructed when a direct sum condition $\mathcal{H} = \mathcal{H}_F \oplus \mathcal{H}_G^{\perp}$ is satisfied. This condition is equivalent to having $R(\mathcal{H}_F, \mathcal{H}_G) > 0$ and $R(\mathcal{H}_G, \mathcal{H}_F) > 0$, which is met for perturbation of frame sequences, as in Theorem 2.1. For a detailed exposition of oblique dual frames we refer the reader to Ref. 19.

4. Perturbation of the Generator of Frames of Translates

In order to use approximate reconstructions, the previous results all assumed that a pair of frames, one being a perturbation of another, is given. When considering frame sequences for closed subspaces it is a nontrivial task to verify when a given sequence $\{g_k\}_{k\in\mathbb{Z}}$ is a perturbation of $\{f_k\}_{k\in\mathbb{Z}}$. A condition that causes difficulty is that on the gap between spaces, $\delta(\mathcal{H}_G, \mathcal{H}_F)$. However, in particular situations the gap can be computed. One instance of such a case are frames of translates.

In order to apply results developed in the previous section, we restrict ourselves to a space $\mathcal{H} = L^2(\mathbb{R}^d)$ with the usual norm $\|\cdot\|_2$. We consider frame sequences of irregular translates i.e. of the form $\{f(\cdot - \lambda_k)\}_{k\in\mathbb{Z}}$, where $\{\lambda_k\}_{k\in\mathbb{Z}}$ is an arbitrary sequence in $\mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$. Frames of irregular translates appear in numerous applications, for example when dealing with jittered samples in sound analysis. Frames of irregular translates were studied for example in Ref. 1, 21 and 2.

We denote the translation of f by λ_k as $f(\cdot - \lambda_k) = f_{\lambda_k}$. We consider translations along a set of points $\Lambda = \{\lambda_k : k \in \mathbb{Z}\} \subseteq \mathbb{R}^d$ that form a γ-separated set. That is, there exists $\gamma > 0$ such that $\|\lambda_k - \lambda_m\| > \gamma$ for $\lambda_k, \lambda_m \in \Lambda$, whenever $k \ne m$ and $\|\cdot\|$ denotes a standard Euclidean distance in $\mathbb{R}^d$.

Let E be a bounded subset of $\mathbb{R}^d$. When we say that the exponentials $\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in\mathbb{Z}}$ are a frame for $L^2(E)$ we mean that the set $\{e^{-2\pi\langle\lambda_k,\cdot\rangle}\chi_E\}_{k\in\mathbb{Z}}$ has the property, where χ_E denotes the characteristic function of E.

We denote by $\mathcal{P}_E$ the space defined by

$$\mathcal{P}_E = \{h \in L^2(\mathbb{R}^d) : \text{supp } \hat{h} \subseteq E\}. \tag{4.1}$$

Let $f \in L^2(\mathbb{R}^d)$ be a bandlimited function i.e. $\text{supp}(\widehat{f})$ is compact. We consider a perturbation $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ of $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ such that the difference between f and g is a function with polynomial decay.

When dealing with polynomially decaying functions, we will repeatedly use the following lemma which is the version in $\mathbb{R}^d$ of Lemma 2.2 in Ref. 17. Its proof is analogous to the one-dimensional case.

Lemma 4.1. *Let $\Lambda = \{\lambda_k : k \in \mathbb{Z}\} \subseteq \mathbb{R}^d$ be a γ-separated set and $p > 1$. Then*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \sum_{k\in\mathbb{Z}} (1 + \|x - \lambda_k\|)^{-p} \le 2 \left[2^{d-1} + \frac{(2^d - 1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1 + \frac{\gamma}{\sqrt{d}}\right)^p (p-1)} \right].$$

Now, we state the first result about a perturbation of a bandlimited function.

Theorem 4.1. *Let $\{\lambda_k : k \in \mathbb{Z}\} \subseteq \mathbb{R}^d$ be a γ-separated set and $f \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp}(\widehat{f}) = E$ is compact. Consider $g = f + r$ with $r \in L^2(\mathbb{R}^d)$ satisfying*

$$|r(x)| \le C(1 + \|x\|)^{-p} \tag{4.2}$$

for some $p > d$ and C a positive constant. Set

$$\mu = \sqrt{\frac{2C^2 C_d}{p-d} \left[2^{d-1} + \frac{(2^d - 1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1 + \frac{\gamma}{\sqrt{d}}\right)^p (p-1)} \right]}, \tag{4.3}$$

where $C_d = \frac{\pi^{\frac{d}{2}} d!}{\Gamma(\frac{d}{2}+1)}$ and Γ is the Gamma function. Then:

(a) *If $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R}^d)$ with lower Riesz bound A_F and $\mu^2 < A_F$, then $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a Riesz sequence.*

(b) *Let $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ be a frame sequence in $L^2(\mathbb{R}^d)$ with lower frame bound A_F and $\mu^2 < A_F$. Assume that there exist $\alpha, \beta > 0$ such that $\alpha \le |\hat{f}(\omega)| \le \beta$ a.e. in E and that $\mathcal{H}_F = \overline{\operatorname{span}}\{f_{\lambda_k} : k \in \mathbb{Z}\} = \mathcal{P}_E$. Denote $\mathcal{H}_G = \overline{\operatorname{span}}\{g_{\lambda_k} : k \in \mathbb{Z}\}$. If there exists $0 < \eta$ such that $|\hat{g}| > \eta$ a.e. on E, then $\delta(\mathcal{H}_G, \mathcal{H}_F) < 1$ and so $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame sequence.*

Proof. (a) Let $c \in \ell^2(\mathbb{Z})$ be a finite sequence indexed by $F \subseteq \mathbb{Z}$. Then

$$\begin{aligned} \left\| \sum_{k\in F} c_k (g_{\lambda_k} - f_{\lambda_k}) \right\|_2^2 &= \left\| \sum_{k\in F} c_k r_{\lambda_k} \right\|_2^2 = \left\langle \sum_{k\in F} c_k r_{\lambda_k}, \sum_{m\in F} c_m r_{\lambda_m} \right\rangle \\ &\le \sum_{k,m\in F} |c_k||c_m||\langle r_{\lambda_k}, r_{\lambda_m}\rangle| \end{aligned}$$

$$\leq \frac{1}{2} \sum_{k,m \in F} (|c_k|^2 + |c_m|^2) |\langle r_{\lambda_k}, r_{\lambda_m} \rangle|$$

$$\leq \frac{1}{2} \left[\sum_{k \in F} |c_k|^2 \sum_{m \in F} |\langle r_{\lambda_k}, r_{\lambda_m} \rangle| + \sum_{m \in F} |c_m|^2 \sum_{k \in F} |\langle r_{\lambda_k}, r_{\lambda_m} \rangle| \right]$$

$$\leq \|c\|_{\ell^2}^2 \sup_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle r_{\lambda_k}, r_{\lambda_m} \rangle|.$$

Using the assumption (4.2) on r, we find that

$$\sum_{k \in \mathbb{Z}} |\langle r_{\lambda_k}, r_{\lambda_m} \rangle| \leq C^2 \int_{\mathbb{R}^d} (1 + \|x\|)^{-p} \sum_{k \in \mathbb{Z}} (1 + \|x + \lambda_k - \lambda_m\|)^{-p} dx$$

$$\leq C^2 \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}} (1 + \|x + \lambda_k - \lambda_m\|)^{-p} \int_{\mathbb{R}^d} (1 + \|x\|)^{-p} dx$$

$$\leq \frac{C^2 C_d}{p - d} \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \sum_{k \in \mathbb{Z}} (1 + \|x + \lambda_k - \lambda_m\|)^{-p},$$

where $\int_{\mathbb{R}^d} (1 + \|x\|)^{-p} dx = \frac{C_d}{p-d}$ with $C_d = \pi^{d/2} d! \Gamma(d/2 + 1)^{-1}$ and Γ the Gamma function. By Lemma 4.1, we have that

$$\sup_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} |\langle r_{\lambda_k}, r_{\lambda_m} \rangle| \leq \frac{2C^2 C_d}{p - d} \left[2^{d-1} + \frac{(2^d - 1)\left(\frac{\sqrt{d}}{\gamma} + p \right)}{\left(1 + \frac{\gamma}{\sqrt{d}} \right)^p (p - 1)} \right] = \mu^2.$$

Since $\mu < \sqrt{A_F}$, by Theorem 2.1(a) $\{g_{\lambda_k}\}_{k \in \mathbb{Z}}$ is a Riesz sequence.

(b) We already showed in part (a) that

$$\left\| \sum_{k \in F} c_k (g_{\lambda_k} - f_{\lambda_k}) \right\|_2 \leq \mu \|c\|_{\ell^2}$$

for every finite sequence $\{c_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$, where $\mu < \sqrt{A_F}$. Hence, by Theorem 2.1(b), it follows that $R(\mathcal{H}_F, \mathcal{H}_G) > 0$. We want to verify that also $R(\mathcal{H}_G, \mathcal{H}_F) > 0$. By Proposition 3.3 in Ref. 6, $R(\mathcal{H}_F, \mathcal{H}_G) > 0$ and $R(\mathcal{H}_G, \mathcal{H}_F) > 0$ if and only if $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ is invertible. From Lemma 3.1 in Ref. 6 we also know that $R(\mathcal{H}_F, \mathcal{H}_G) > 0$ implies that $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ is onto. Hence it is only left to check that $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ is injective. For this, let $\phi \in \mathcal{H}_G$ such that $P_{\mathcal{H}_F}\phi = 0$. Since the Fourier transform is an isometry, $\widehat{P_{\mathcal{H}_F}\phi} = P_{\widehat{\mathcal{H}_F}}\widehat{\phi}$ where $\widehat{\mathcal{H}_F} = \{\widehat{h} : h \in \mathcal{H}_F\}$. Now, using that $\mathcal{H}_F = \mathcal{P}_E$ it follows that $P_{\widehat{\mathcal{H}_F}}\widehat{\phi} = \chi_E \widehat{\phi}$. Hence, $\widehat{\phi} = 0$ a.e. in E. We have to see now that $\widehat{\phi} = 0$ a.e. also in E^c.

Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a sequence such that $\phi_n \to \phi$ in $L^2(\mathbb{R}^d)$, where $\phi_n = \sum_{k \in F_n} c_k^n g_{\lambda_k}$ with $F_n \subseteq \mathbb{Z}$ a finite set. We write $\widehat{\phi_n}(\omega) = \theta_n(\omega)\widehat{g}(\omega)$, where $\theta_n(\omega) = \sum_{k \in F_n} c_k^n e^{-2\pi i \langle \lambda_k, \omega \rangle}$. Then $\widehat{P_{\mathcal{H}_F}\phi_n} = \chi_E \widehat{\phi_n} \to \widehat{P_{\mathcal{H}_F}\phi} = 0$.

For $\omega \in E^c$ we have $\widehat{\phi_n}(\omega) = \theta_n(\omega)\widehat{g}(\omega) = \theta_n(\omega)\widehat{r}(\omega)$. Then, analogously as in part (a) we obtain that

$$\int_{E^c} |\widehat{\phi_n}(\omega)|^2 d\omega = \int_{E^c} |\theta_n(\omega)\widehat{r}(\omega)|^2 d\omega \le \int_{\mathbb{R}^d} |\theta_n(\omega)\widehat{r}(\omega)|^2 d\omega$$

$$= \Big\| \sum_{k\in F_n} c_k^n r_{\lambda_k} \Big\|_2^2 \le \|c^n\|_{\ell^2}^2 \frac{2C^2 C_d}{p-d} \left[2^{d-1} + \frac{(2^d-1)\left(\frac{\sqrt{d}}{\gamma}+p\right)}{\left(1+\frac{\gamma}{\sqrt{d}}\right)^p (p-1)} \right],$$

where $c^n = \{c_k^n\}_{k\in F_n}$. We will show that $\|c^n\|_{\ell^2}^2 \to 0$ if $n \to +\infty$.

Since $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame sequence in $L^2(\mathbb{R}^d)$ and there exist $\alpha, \beta > 0$ such that $\alpha \le |\hat{f}(\omega)| \le \beta$ a.e. in E, then $\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in\mathbb{Z}}$ is a frame for $L^2(E)$, cf. Proposition 3.6 in Ref. 2. Denote its frame bounds A_1 and B_1.

We can assume without loss of generality that $F_1 \subseteq F_2 \ldots \subseteq F_n \subseteq \ldots$ and $\bigcup_{n\in\mathbb{N}} F_n = \mathbb{Z}$. Let $\mathcal{H}_n = \operatorname{span}\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in F_n}$ and $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0, A_1)$ be a decreasing sequence converging to zero. As a consequence of Lemma 3.4 in Ref. 9, there exists a finite set J_n containing F_n such that

$$\sum_{k\notin J_n} |\langle h, e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle_{L^2(E)}|^2 \le \frac{\varepsilon_n^2}{B_1}\|h\|_{L^2(E)}^2, \quad \text{for every } h \in \mathcal{H}_n.$$

Let S_{J_n} denote the frame operator of $\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in J_n}$. Then $S_{J_n}|_{\mathcal{H}_n}$ is an isomorphism from $\mathcal{H}_n$ onto $S_{J_n}(\mathcal{H}_n)$. It can be seen analogously as in the proof of Theorem 3.5 in Ref. 9 that $\|(S_{J_n}|_{\mathcal{H}_n})^{-1}\|_{\text{op}} \le \frac{1}{A_1-\varepsilon_n}$. Now, since $\theta_n \in \mathcal{H}_n \subseteq \operatorname{span}\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in J_n}$

$$\theta_n = \sum_{k\in J_n} \langle \theta_n, S_{J_n}^{-1} e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle e^{-2\pi i\langle\lambda_k,\cdot\rangle}.$$

Using that $\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in J_n}$ is a linearly independent system and that $F_n \subseteq J_n$, we find

$$\langle \theta_n, S_{J_n}^{-1} e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle = \begin{cases} c_k^n & \text{if } k \in F_n, \\ 0 & \text{if } k \in J_n\backslash F_n. \end{cases}$$

Now, $S_{J_n}^{-1}\theta_n \in \operatorname{span}\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in J_n}$ and $\langle S_{J_n}^{-1}\theta_n, e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle = \langle\theta_n, S_{J_n}^{-1} e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle = 0$ for $k \in J_n\backslash F_n$. Hence $S_{J_n}^{-1}\theta_n \in \mathcal{H}_n$ and so $\theta_n = S_{J_n}(S_{J_n}^{-1}\theta_n) \in S_{J_n}(\mathcal{H}_n)$.

Then we have

$$\|c^n\|_{\ell^2}^2 = \sum_{k\in J_n} |\langle\theta_n, S_{J_n}^{-1} e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle|^2 = \sum_{k\in J_n} |\langle S_{J_n}^{-1}\theta_n, e^{-2\pi i\langle\lambda_k,\cdot\rangle}\rangle|^2$$

$$\leq B_1 \|S_{J_n}^{-1}\theta_n\|^2_{L^2(E)} \leq B_1 \|(S_{J_n}|_{\mathcal{H}_n})^{-1}\|^2_{\text{op}} \|\theta_n\|^2_{L^2(E)}$$
$$\leq B_1 \left(\frac{1}{A_1 - \varepsilon_n}\right)^2 \|\theta_n\|^2_{L^2(E)}.$$

Note that since $\chi_E \widehat{\phi_n} = \chi_E \theta_n \widehat{g} \to 0$ and $|\hat{g}| > \eta$ a.e. on E, it follows that $\theta_n \to 0$ in $L^2(E)$. From this, we obtain that $\|c^n\|^2_{\ell^2} \to 0$. Thus, $\chi_{E^c}\widehat{\phi_n} \to 0$ in $L^2(\mathbb{R}^d)$ and finally $\widehat{\phi} = 0$ a.e $\mathbb{R}^d$. Therefore, $P_{\mathcal{H}_F}|_{\mathcal{H}_G}$ is injective and so $R(\mathcal{H}_G, \mathcal{H}_F) > 0$. Then, by Theorem 2.1(c) $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame sequence. □

Remark 4.1. When $d = 1$, that is for $L^2(\mathbb{R})$, μ in Theorem 4.1 is given by

$$\mu = \sqrt{\frac{4C^2}{p-1}\left(1 + \frac{\gamma^{-1} + p}{(1+\gamma)^p (p-1)}\right)}.$$

This follows from $\int_{\mathbb{R}} (1 + |x|)^{-p} dx = 2(p-1)^{-1}$.

Remark 4.2. Theorem 4.1 can be extended for γ-relatively separated sets. We say that $\Lambda \subseteq \mathbb{R}^d$ is a γ-relatively separated set if $\text{rel}(\Lambda) := \max_{x\in\mathbb{R}^d} \#\{\Lambda \cap ([0,\gamma]^d + x)\} < +\infty$, cf. Ref. 11. In this case, the estimate in Lemma 4.1 becomes

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \sum_{k\in\mathbb{Z}} (1 + \|x - \lambda_k\|)^{-p} \leq \text{rel}(\Lambda) 2 \left[2^{d-1} + \frac{(2^d - 1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1 + \frac{\gamma}{\sqrt{d}}\right)^p (p-1)}\right].$$

Roughly speaking, Theorem 4.1 states that $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame sequence provided r has sufficient decay. We are interested now in approximating a function $h \in \mathcal{H}_G$ by $h_1 = \sum_{m\in\mathbb{Z}} \langle h, \widetilde{f_{\lambda_m}}\rangle g_{\lambda_m}$, and to estimate the error $\|h_1 - h\|_2$. When in addition to the hypotheses of Theorem 4.1 we ask $\mu < \min\{1, \frac{A_F R(\mathcal{H}_G, \mathcal{H}_F)}{2\sqrt{B_F}+1}\}$ where $0 < A_F \leq B_F$ are the frame bounds of $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$, then we are under hypotheses of Theorem 3.2 and, in particular, Corollary 3.1 can be used to estimate the error $\|h_1 - h\|_2$. Since in this case, an expression for the canonical dual of $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ is known, cf. Ref. 3, we are able to compute the error as Theorem 4.2 shows.

Theorem 4.2. *Under the same hypothesis of Theorem* 4.1 *part* (*b*), *let* $h \in \mathcal{H}_G$ *and* $h_1 = \sum_{m\in\mathbb{Z}} \langle h, \widetilde{f_{\lambda_m}}\rangle g_{\lambda_m}$. *Then, we have*

$$\|h - h_1\|_2 \leq \left(\delta(\mathcal{H}_G, \mathcal{H}_F) + \frac{\sqrt{B_G}}{\sqrt{A_F} A_G} \frac{2C^2 C_d}{p-d} \left[2^{d-1} + \frac{(2^d - 1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1 + \frac{\gamma}{\sqrt{d}}\right)^p (p-1)}\right]\right) \|h\|_2,$$

where $C_d = \frac{\pi^{\frac{d}{2}} d!}{\Gamma(\frac{d}{2}+1)}$ *and* Γ *is the Gamma function.*

Proof. Since $h \in \mathcal{H}_G$ and $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}_G$, $h = \sum_{k\in\mathbb{Z}} c_k g_{\lambda_k}$, where $c_k = \langle h, S_G^{-1} g_{\lambda_k}\rangle$ and S_G is the frame operator associated to $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$. Therefore,

$$\widehat{h}(\omega) = \left(\sum_{k\in\mathbb{Z}} c_k e^{-2\pi i\langle\lambda_k,\omega\rangle}\right)\widehat{g}(\omega) = \theta(\omega)\widehat{g}(\omega).$$

On the other hand, the canonical dual frame for $\{f_{\lambda_k}\}_{k\in\mathbb{Z}}$ has the form

$$\widehat{\widetilde{f_{\lambda_k}}} = \begin{cases} \dfrac{\widehat{f}}{|\widehat{f}|^2}\psi_{\lambda_k} & \text{on } E, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{\psi_{\lambda_k}\}_{k\in\mathbb{Z}}$ is the canonical dual frame for $\{e^{-2\pi i\langle\lambda_k,\cdot\rangle}\}_{k\in\mathbb{Z}}$, cf. Ref. 3. Then, using that $\widehat{g} = \widehat{f} + \widehat{r}$ we have

$$\begin{aligned}
\widehat{h_1} &= \sum_{m\in\mathbb{Z}} \left\langle \widehat{h}, \frac{\widehat{f}}{|\widehat{f}|^2}\psi_{\lambda_m}\chi_E \right\rangle e^{-2\pi i\langle\lambda_m,\cdot\rangle}\widehat{g} \\
&= \sum_{m\in\mathbb{Z}} \left(\int_E \theta(\omega)\widehat{g}(\omega)\overline{\frac{\widehat{f}(\omega)}{|\widehat{f}(\omega)|^2}}\,\overline{\psi_{\lambda_m}}d\omega\right) e^{-2\pi i\langle\lambda_m,\cdot\rangle}\widehat{g} \\
&= \sum_{m\in\mathbb{Z}} \left(\int_E \theta(\omega)\overline{\psi_{\lambda_m}}d\omega + \int_E \theta(\omega)\widehat{r}(\omega)\overline{\widehat{\widetilde{f_{\lambda_m}}}(\omega)}d\omega\right) e^{-2\pi i\langle\lambda_m,\cdot\rangle}\widehat{g} \\
&= \sum_{m\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} c_k\langle e^{-2\pi i\langle\lambda_k,\cdot\rangle}, \psi_{\lambda_m}\rangle_{L^2(E)} + \sum_{k\in\mathbb{Z}} c_k\langle e^{-2\pi i\langle\lambda_k,\cdot\rangle}\widehat{r}, \widehat{\widetilde{f_{\lambda_m}}}\rangle\right) e^{-2\pi i\langle\lambda_m,\cdot\rangle}\widehat{g} \\
&= \sum_{k\in\mathbb{Z}} c_k \left(\sum_{m\in\mathbb{Z}} \langle e^{-2\pi i\langle\lambda_k,\cdot\rangle}, \psi_{\lambda_m}\rangle_{L^2(E)} e^{-2\pi i\langle\lambda_m,\cdot\rangle}\right)\widehat{g} \\
&\quad + \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_k\langle e^{-2\pi i\langle\lambda_k,\cdot\rangle}\widehat{r}, \widehat{\widetilde{f_{\lambda_m}}}\rangle e^{-2\pi i\langle\lambda_m,\cdot\rangle}\widehat{g}.
\end{aligned}$$

Now, since $\sum_{m\in\mathbb{Z}}\langle e^{-2\pi i\langle\lambda_k,\cdot\rangle}, \psi_{\lambda_m}\rangle_{L^2(E)} e^{-2\pi i\langle\lambda_m,\cdot\rangle} = \chi_E e^{-2\pi i\langle\lambda_k,\cdot\rangle}$,

$$\widehat{h_1} = \chi_E\widehat{h} + \widehat{R}(\omega) = \widehat{P_{\mathcal{H}_F}h} + \widehat{R}(\omega),$$

where $R = \sum_{m\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} c_k\langle r_{\lambda_k}, \widetilde{f_{\lambda_m}}\rangle g_{\lambda_m}$. Therefore,

$$\|h - h_1\|_2 \le \|h - P_{\mathcal{H}_F}h\|_2 + \|R\|_2 \le \delta(\mathcal{H}_G, \mathcal{H}_F)\|h\|_2 + \|R\|_2. \tag{4.4}$$

If we call $d_m = \sum_{k\in\mathbb{Z}} c_k\langle r_{\lambda_k}, \widetilde{f_{\lambda_m}}\rangle$ for each $m \in \mathbb{Z}$, then

$$\|R\|_2 = \left\|\sum_{m\in\mathbb{Z}} d_m g_{\lambda_m}\right\|_2 \le \sqrt{B_G}\|\{d_m\}_{m\in\mathbb{Z}}\|_{\ell^2}. \tag{4.5}$$

In order to estimate $\|\{d_m\}_{m\in\mathbb{Z}}\|_{\ell^2(\mathbb{Z})}$, we compute

$$\sum_{m\in\mathbb{Z}} |d_m|^2 = \sum_{m\in\mathbb{Z}} \left| \left\langle \sum_{k\in\mathbb{Z}} c_k r_{\lambda_k}, \widetilde{f_{\lambda_m}} \right\rangle \right|^2$$

and if $\phi := \sum_{k\in\mathbb{Z}} c_k r_{\lambda_k}$, using that $\{\widetilde{f_{\lambda_m}}\}_{k\in\mathbb{Z}}$ is a frame for $\mathcal{H}_F$, we obtain

$$\sum_{m\in\mathbb{Z}} |d_m|^2 = \sum_{m\in\mathbb{Z}} |\langle \phi, \widetilde{f_{\lambda_m}}\rangle|^2 = \sum_{m\in\mathbb{Z}} |\langle P_{\mathcal{H}_F}\phi, \widetilde{f_{\lambda_m}}\rangle|^2 \leq A_F^{-1}\|\phi\|_2^2. \qquad (4.6)$$

Therefore, replacing (4.6) in (4.5) it follows that

$$\|R\|_2 \leq \frac{\sqrt{B_G}}{\sqrt{A_F}} \left\| \sum_{k\in\mathbb{Z}} c_k r_{\lambda_k} \right\|_2 .$$

Now as in the proof of Theorem 4.1

$$\|R\|_2 \leq \frac{\sqrt{B_G}}{\sqrt{A_F}} \frac{2C^2 C_d}{p-d} \left[2^{d-1} + \frac{(2^d-1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1+\frac{\gamma}{\sqrt{d}}\right)^p (p-1)} \right] \|c\|_{\ell^2}$$

and using the frame condition of $\{S_G^{-1} g_{\lambda_k}\}_{k\in\mathbb{Z}}$ to estimate the norm of c in terms of the norm of f, we finally obtain that

$$\|R\|_2 \leq \frac{\sqrt{B_G}}{\sqrt{A_F} A_G} \frac{2C^2 C_d}{p-d} \left[2^{d-1} + \frac{(2^d-1)\left(\frac{\sqrt{d}}{\gamma} + p\right)}{\left(1+\frac{\gamma}{\sqrt{d}}\right)^p (p-1)} \right] \|f\|_2. \qquad \square$$

Remark 4.3. For the space $L^2(\mathbb{R})$, that is for $d = 1$, the estimate of Theorem 4.2 in terms of γ, p, the gap $\delta(\mathcal{H}_G, \mathcal{H}_F)$ and frame bounds of $\{g_{\lambda_k}\}_{k\in\mathbb{Z}}$ and $\{\widetilde{f_{\lambda_k}}\}_{k\in\mathbb{Z}}$ is given by

$$\|h - h_1\|_2 \leq \left(\delta(\mathcal{H}_G, \mathcal{H}_F) + \frac{\sqrt{B_G}}{\sqrt{A_F} A_G} \frac{4C^2}{p-1} \left[1 + \frac{\gamma^{-1} + p}{(1+\gamma)^p (p-1)} \right] \right) \|h\|_2.$$

We mentioned before Theorem 4.2 that if in addition to hypothesis of Theorem 4.1, μ satisfies $\mu < \min\{1, \frac{A_F R(\mathcal{H}_G, \mathcal{H}_F)}{2\sqrt{B_F}+1}\}$, the error $\|h_1 - h\|_2$ can be estimated using Corollary 3.1. As it is expected, the error that Corollary 3.1 gives is, in general, greater than the error obtained in Theorem 4.2. For instance, this can be easily seen in case when $A_F = B_F = 1$. However, we emphasize that both errors are small when μ is small and μ is sufficiently small if we choose p big enough.

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