

# A posteriori error estimator for exponentially fitted Discontinuous Galerkin approximation of advection dominated problems

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**Abstract** The paper deals with the weakly penalized exponentially fitted Incomplete Interior Penalty (EF-IIPG0) scheme for advection-diffusion problems. In the first part of the paper, the  $M$ -matrix property on conforming weakly-acute meshes is discussed. In the second part, an a posteriori error estimate is derived. The estimator, especially designed for the advection dominated case, controls the energy norm as well as a semi-norm associated with the advective derivative, taking full advantage of the formulation on non-matching grids. The paper is supplemented by numerical experiments, where the estimator is used as local error indicator for marking the triangles to be refined in an adaptive strategy.

**Keywords** Discontinuous Galerkin Methods · Exponentially Fitted Schemes · Advection-Diffusion Equations · a posteriori estimator ·  $M$ -matrix property

**Mathematics Subject Classification (2000)** 65N30

## 1 Introduction

Advection-diffusion problems arise very frequently in applications and it is well known that their numerical discretization requires special care when advection dominates over diffusion. This is the case, for instance, in fluidynamic problems with high Reynolds number, or in semiconductor device simulation under the action of a high electric field. Moreover, for

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an effective numerical resolution, the presence of internal and/or boundary layers requires adaptive strategies able to locally refine the mesh in the vicinity of the layers.

We consider the stationary advection-diffusion model problem

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla u - \beta u) = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_D, \\ (\varepsilon \nabla u - \beta u) \cdot \underline{n} = 0 & \text{on } \Gamma_N, \end{cases} \quad (1)$$

where  $\Omega$  is a convex polygonal domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega = \Gamma_D \cup \Gamma_N$ ,  $\underline{n}$  is the unit outward normal vector, and  $f, g$  are given functions, with  $f \in L^2(\Omega)$ , and  $g \in H^{1/2}(\Gamma_D)$ . Moreover,  $\varepsilon = \varepsilon(x)$  and  $\beta = \beta(x)$  are given regular functions on  $\overline{\Omega}$  such that

$$\exists \varepsilon_0, \varepsilon_M \text{ such that } \varepsilon_M \geq \varepsilon(x) \geq \varepsilon_0 > 0, \quad (2)$$

$$\exists b_0 \text{ such that } \operatorname{div} \beta \geq b_0 > 0. \quad (3)$$

Existence and uniqueness of the solution of (1) then follows by the usual maximum principle. In [9] exponentially fitted Discontinuous Galerkin schemes (namely weak Interior Penalty and weak Local-Discontinuous-Galerkin) for (1) has been designed and studied, showing very good performances in the advection dominated case, as well as in the intermediate cases. In [5] a block solver has been proposed for a weakly penalized exponentially fitted Incomplete-Interior-Penalty (EF-IIPG0) scheme on conforming meshes.

One attractive property when discretizing advection dominated problems (as well as diffusion problems) is the  $M$ -matrix property for the matrix associated to the scheme, since, in particular, it yields positive solutions when positive data are considered and it prevents spurious oscillations. Among the exponentially fitted DG methods mentioned above, the one that provides, on conforming weakly acute meshes, an associated matrix with the  $M$ -matrix property is the EF-IIPG0 scheme. We show this property in Section 3. Although the property does not hold for non-matching grids, it motivated our choice of the discretization scheme for the subsequent a posteriori analysis.

Indeed, the main result of the paper is the design and the analysis of an a posteriori error estimator for the EF-IIPG0 discretization scheme, allowing for non-matching grids. The obtained estimator is used as local error indicator for marking the triangles to be refined (or derefined) in an adaptive strategy. We point out that the use of a Discontinuous Galerkin scheme makes the refining step particularly simple, because non-matching grids are allowed.

The estimator is robust, in the sense that yields an upper bound, independent of the mesh size and the (small) diffusion coefficient, of the error measured in terms of the natural energy norm and a dual semi-norm associated with the convective term. The numerical experiments then confirm that the ratio between the estimator and the energy norm is indeed independent of  $h$  and  $\varepsilon$ . Robust error estimator for advection-diffusion problems have been studied in [12, 15] for SUPG schemes and in [13] for a DG scheme (namely, an Interior Penalty method, with upwind discretization of the convective term). Our analysis approach follows [13], where the error is split into a conforming part and a (discontinuous) remainder. The considered dual semi-norm for the convective term is the one proposed by [15] and used by [13] in the DG framework. Here, an important feature of the upper bound is the presence of a data approximation term which controls the exponential fitting approximation of the flux  $\varepsilon \nabla u - \beta u$ . The numerical tests show that this term is of the same order as the error estimator and it cannot be neglected in the actual implementation of the adaptive strategy. We point out that the presence of this data approximation term (of the same order as the estimator) prevents from obtaining an efficiency result for the estimator.

Robust a posteriori estimation for advection-diffusion equations for DG schemes has been also considered in the  $hp$ -framework in [16], and for the non-stationary case in [2]. Other approaches to energy norm estimate for Discontinuous Galerkin schemes for diffusive problems can be found in [10], e.g., and in the references therein.

The outline of the paper is as follows. Section 2 presents the discretization scheme on non-matching grids. The hypotheses on the allowed meshes are described and the notation used all over the paper are set. In Section 3 the M-matrix property of the EF-IIPG0 scheme on weakly acute conforming meshes is proved. Section 4 contains the presentation of the estimator and the main result on the estimator. In Section 5 the adaptive algorithm is briefly discussed and the numerical experiments are presented.

## 2 Discretization

Let  $\Omega$  be a polygonal domain in  $\mathbb{R}^2$  with boundary  $\partial\Omega = \Gamma_D \cup \Gamma_N$ . We consider the space

$$H_{0,\Gamma_D}^1 = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Let us introduce the bilinear form  $A : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$  as

$$A(w, v) = \int_{\Omega} (\varepsilon \nabla w - \beta w) \cdot \nabla v \, dx. \quad (4)$$

Then we can state the variational formulation of problem (1): Find  $u \in H^1(\Omega)$  such that  $u = g$  on  $\Gamma_D$  and

$$A(u, v) = \int_{\Omega} f v \, dx \quad \forall v \in H_{0,\Gamma_D}^1(\Omega). \quad (5)$$

Let  $\{\mathcal{T}_h\}_h$  be a family of shape regular decompositions (see for example [3]) of the domain  $\Omega$  into triangles  $K$ . A generic edge of one of the triangles will be denoted by the letter  $l$ . The notation  $l_K$  will be used for an edge of the triangle  $K$ . We are going to allow for non-matching meshes (i.e.,  $\mathcal{T}_h$  may contain hanging nodes), with the following restriction. In order to simplify the presentation of the exponential fitting scheme, we assume the very natural condition that, given two neighboring triangles  $K, K' \in \mathcal{T}_h$ , then  $K \cap K'$  is an (entire) edge of either  $K$  or  $K'$ . For the analysis of our estimator we assume local quasi-uniformity of the mesh, that is, if  $K \cap K' \neq \emptyset$  then  $|K| \sim |K'|$ . Moreover, since we will use results from [7, 8] we assume that  $\mathcal{T}_h$  is obtained from a conforming mesh via a (fixed, independent of  $h$ ) finite number of refinement/coarsening steps. Finally, we assume that if a triangle edge  $l$  meets  $\partial\Omega$ , then either  $l \subset \Gamma_D$  or  $l \subset \Gamma_N$  or  $l \cap \partial\Omega$  is a vertex.

### 2.1 Notation

Since we are going to deal with Discontinuous Galerkin schemes, we shall need to write integrals on the skeleton of the mesh. Due to the possible occurrence of hanging nodes, it may happen that given an edge  $l_K$  of a triangle  $K$  there exists an element  $K'$ , neighbor of  $K$  such that  $K \cap K' \subsetneq l_K$ . Therefore, we introduce the following notation for the set of the intersections between triangles, that is

$$\mathcal{E}_0 = \{e = K \cap K' : K, K' \in \mathcal{T}_h, K \neq K'\}.$$

We remark that if  $e = K \cap K' \in \mathcal{E}_0$  then  $e$  is a complete edge of at least one of  $K$  and  $K'$  but not necessarily of both of them. We also set

$$\Gamma_0 = \bigcup_{e \in \mathcal{E}_0} e.$$

On the boundary, the skeleton of the triangulation is made of edges of triangles and  $\mathcal{E}_D$  (resp.  $\mathcal{E}_N$ ) will denote the set of triangle edges on  $\Gamma_D$  (resp.  $\Gamma_N$ ). We write  $\Gamma_D = \bigcup_{e \in \mathcal{E}_D} e$  and  $\Gamma_N = \bigcup_{e \in \mathcal{E}_N} e$ . Finally we denote  $\Gamma = \Gamma_0 \cup \Gamma_D \cup \Gamma_N$  and  $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_D \cup \mathcal{E}_N$ .

The norm of  $L^2(S)$  is denoted by  $\|\cdot\|_{0,S}$  and the norm and seminorm of  $H^k(S)$  by  $\|\cdot\|_{k,S}$  and  $|\cdot|_{k,S}$ , respectively. The subscript  $S$  is omitted when  $S = \Omega$ .

Let  $H^1(\mathcal{T}_h)$  be the space of functions whose restriction to each element  $K$  belongs to the Sobolev space  $H^1(K)$ . It is equipped with the  $h$ -dependent semi-norm

$$|v|_{1,h}^2 = \sum_{K \in \mathcal{T}_h} |v|_{1,K}^2.$$

The space of traces of functions in  $H^1(\mathcal{T}_h)$  is contained in  $Tr(\Gamma)$ , which is defined as  $Tr(\Gamma) := \prod_{K \in \mathcal{T}_h} L^2(\partial K)$ . Thus, functions in  $Tr(\Gamma)$  are double valued on  $\Gamma_0$  and single valued on  $\partial\Omega$ . For scalar functions  $q \in Tr(\Gamma)$  and vector functions  $\phi \in Tr(\Gamma)^2$ , following [1], we introduce the averages  $\{q\}$  and  $\{\phi\}$ , and the jumps  $[[q]]$  and  $[[\phi]]$ , on  $\Gamma$  (using the subscript  $e$  to denote their restriction to  $e$ ). Let  $e = K_1 \cap K_2$ , and let  $n_1$  and  $n_2$  be the outward normals to  $K_1$  and  $K_2$ , respectively. If  $q_i = q|_{\partial K_i}$  then we set

$$\{q\}_e = \frac{1}{2}(q_1 + q_2), \quad [[q]]_e = q_1 n_1 + q_2 n_2.$$

We define  $\phi_1$  and  $\phi_2$  analogously and we set

$$\{\phi\}_e = \frac{1}{2}(\phi_1 + \phi_2), \quad [[\phi]]_e = \phi_1 \cdot n_1 + \phi_2 \cdot n_2.$$

Notice that these definitions do not depend on assigning an ordering to the elements  $K_1$  and  $K_2$ . Also note that the jump of a scalar function is a vector parallel to the normal, and the jump of a vector function is a scalar quantity. On boundary edges we set

$$\{q\}_e = q, \quad [[q]]_e = qn, \quad \{\phi\}_e = \phi, \quad [[\phi]]_e = \phi \cdot n \quad \text{on } e \subset \partial\Omega,$$

where  $n$  is the exterior normal of  $\Omega$ .

Given an element  $K$ , the restriction of a function  $v$  to  $K$  is denoted by  $v_K$ , even when only the value on  $\partial K$  is considered. We shall also use the standard notation for the mean value of a function  $f$  on a triangle  $K \in \mathcal{T}_h$  and on  $l_K$  edge of  $K$ , that is

$$\fint_K f \, dx = \frac{1}{|K|} \int_K f \, dx, \quad \fint_{l_K} f \, ds = \frac{1}{|l_K|} \int_{l_K} f \, ds.$$

## 2.2 Exponentially fitted DG formulation

We define here the primal formulation of exponentially fitted Discontinuous Galerkin schemes in the spirit of [9] in the case of nonconforming meshes. The finite element space  $V_h$  is the set of piece-wise linear functions, that is,

$$V_h = \{v \in L^2(\Omega) : v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h\}. \quad (6)$$

Let  $\varepsilon(x)$  and  $\beta(x)$  be piecewise constant functions. These functions can be viewed as approximations of the actual coefficients in (1). However we prefer to keep the notation  $\varepsilon(x)$  and  $\beta(x)$  for them. We define a piecewise linear  $\psi$  (in general not continuous at the interelements) whose gradient coincides with the constant  $\beta_K$  on the element  $K$ . We notice that such a  $\psi$  always exists, even if  $\beta(x)$  is a piece-wise constant approximation of a not globally irrotational field. Locally on the triangle  $K$ , we can rewrite the flux variable  $\sigma := \varepsilon \nabla u - \beta u$  in terms of the locally defined potential  $\psi$  as

$$\sigma_K = \varepsilon e^{\frac{\psi}{\varepsilon}|_K} \nabla(e^{-\frac{\psi}{\varepsilon}|_K} u). \quad (7)$$

In the subsequent construction of the scheme, we will make precise how to approximate the coefficient  $\varepsilon e^{\frac{\psi}{\varepsilon}|_K}$  and the change of variable  $\rho = e^{-\frac{\psi}{\varepsilon}|_K} u$  appearing in (7).

First of all, we approximate the coefficient  $\varepsilon e^{\frac{\psi}{\varepsilon}|_K}$  on  $K$  by means of its harmonic average and thus we introduce the piecewise constant  $a_h$  defined by

$$a_h|_K = \frac{\varepsilon}{\int_K e^{-\psi/\varepsilon} dx}.$$

In order to define the approximated change of variable, we define first the local basis functions  $\varphi_K^i$ ,  $i = 1, 2, 3$  on one element  $K$ . For  $i = 1, 2, 3$ ,  $\varphi_K^i$  is the polynomial of degree 1 which takes the value 1 in the midpoint of the edge  $l_K^i$  and the value 0 in the midpoint of the other two edges (here,  $l_K^i$ ,  $i = 1, 2, 3$  denote the three edges of  $K$ ). For  $v \in \mathcal{P}_1(K)$ , we use the notation

$$v(x) = v^1 \varphi_K^1(x) + v^2 \varphi_K^2(x) + v^3 \varphi_K^3(x).$$

Moreover, for  $K \in \mathcal{T}_h$  and  $l_K$  edge of  $K$  we define

$$E(K, l_K) = \int_{l_K} e^{-\frac{\psi}{\varepsilon}|_K} ds. \quad (8)$$

Now we are ready to introduce the the operator  $T : V_h \rightarrow V_h$  which is the approximation of the change of variable. For all  $v \in V_h$

$$(Tv)_K := E(K, l_K^1) v^1 \varphi_K^1 + E(K, l_K^2) v^2 \varphi_K^2 + E(K, l_K^3) v^3 \varphi_K^3. \quad (9)$$

We point out that the definition of  $E(K, l_K)$ , and consequently of the operator  $T$ , is given locally on the triangle  $K$ .

To set the discrete problem, we introduce the projection operator  $\Pi_h : H^1(\mathcal{T}_h) \rightarrow L^2(\Gamma)^2$ . It is double-valued on the intersection of the triangles, and it is defined for  $e = K \cap K' \in \mathcal{E}_0$  as

$$\Pi_h u = (\Pi_0^e(u|_e), \Pi_0^e(u_{K'}|_e)), \quad (10)$$

where  $\Pi_0^e$  is the  $L^2$ -projection on the space of constant functions on  $e$ . It is similarly defined on boundary edges, where it is single-valued. Analogously, we define the double-valued function  $E(e)$  for  $e = K \cap K' \in \mathcal{E}_0$  as

$$E(e) = (E(K, l_K), E(K', l_{K'})),$$

where  $l_K$  and  $l_{K'}$  are the edges of  $K$  and  $K'$  such that  $l_K \cap l_{K'} = e$ . For a boundary edge,  $E(e)$  is clearly single-valued.

### 2.3 Discrete problem

The discrete problem reads: find  $u_h \in V_h$ , such that, for all  $v \in V_h$

$$\begin{aligned} \int_{\Omega} a_h \nabla_h T u_h \cdot \nabla_h v \, dx - \int_{\Gamma_0 \cup \Gamma_D} \llbracket v \rrbracket \cdot \{a_h \nabla_h T u_h\} \, ds \\ + \int_{\Gamma_0 \cup \Gamma_D} \mu \llbracket \Pi_h u_h \rrbracket \cdot \llbracket \Pi_h v \rrbracket \, ds = \int_{\Omega} f v \, dx + \int_{\Gamma_D} \mu g \Pi_h v \, ds, \end{aligned} \quad (11)$$

where, for  $e = l_K \cap l_{K'}$ , and with the obvious changes for boundary edges, the penalization parameter  $\mu$  is defined by

$$\mu|_e = \frac{\gamma \zeta_e}{h_e}, \quad \text{with} \quad \zeta_e = \{E a_h\}_e, \quad (12)$$

where  $\gamma \geq \gamma_0$  for all  $e$ , with  $\gamma_0$  a constant independent of  $\varepsilon$  and of  $h$ .

*Remark 1* The a priori analysis of the IIPG0 scheme for pure diffusion problems on non-matching grids can be carried out following the guidelines of [4, 9]. In particular, in this case we can take the constant  $\gamma_0$  verifying

$$\gamma_0 \geq \frac{1}{2} + \frac{C_{sr}^2}{2},$$

where the constant  $C_{sr}$ , depending on the shape regularity of the mesh, is given by

$$C_{sr} = \sqrt{2} \max \left\{ \frac{|e|}{|T|^{\frac{1}{2}}} : T \in \mathcal{T}_h, e \text{ edge of } T \right\}.$$

*Remark 2* Taking into account our definitions of the coefficient  $a_h$  and of the operator  $T$ , the quantity  $(a_h \nabla_h T u_h)|_K$  can be seen as an approximation of the flux  $\sigma_K$ , defined in (7).

*Remark 3* We point out that the presence of the projection  $\Pi_h$  in the stabilization term (the last term in the bilinear form in (11)) has the effect of reducing the connectivity of the matrix associated with the discretization (see Remark 4 and [9] for further details). Moreover, the use of the projection is crucial for obtaining the  $M$ -matrix property when conforming meshes are taken, as discussed in the next section.

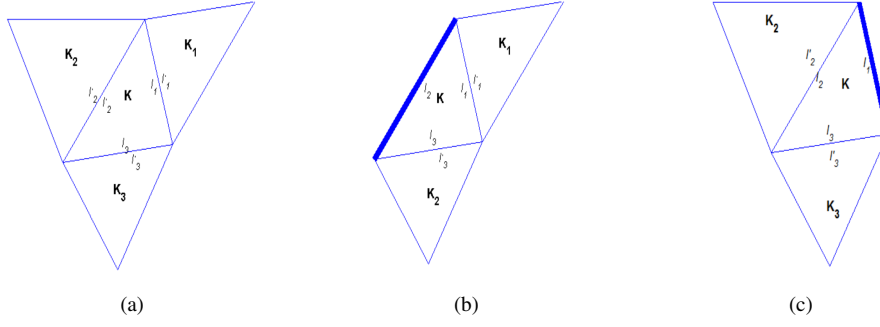


Fig. 1 Different situations of the edge  $l_1$  of the element  $K$ .

### 3 M-matrix property of the EF-IIPG0 method

In this section we assume that  $\mathcal{T}_h$  is a conforming triangulation of  $\Omega$  of weakly acute type, that is, the angles of each triangle in  $\mathcal{T}_h$  do not exceed  $\frac{\pi}{2}$ . We recall that the discretization scheme is defined for piecewise constant (approximation of) coefficients  $\varepsilon(x)$  and  $\beta(x)$ .

As basis functions of  $V_h$  we choose the piecewise linear functions  $\varphi_{K,l}$  which are defined for an element  $K$  and an edge  $l$  of  $K$  as

$$\varphi_{K,l}(m_{l',K'}) = \delta_{l'l} \delta_{KK'} \quad \forall K' \in \mathcal{T}_h, l' \text{ edge of } K',$$

where  $m_{l,K}$  denotes the midpoint of the edge  $l$  of  $K$ . We remark that the functions in  $V_h$  are double-valued on  $\Gamma_0$ . We call  $A_h$  the bilinear form on  $V_h \times V_h$  defined by the left hand side of (11). Then the stiffness matrix of the EF-IIPG0 method can be written as  $M = (M_{K,l;K',l'})$  with

$$M_{K,l;K',l'} = A_h(\varphi_{K,l}, \varphi_{K',l'}), \quad K, K' \in \mathcal{T}_h, l \in \mathcal{L}(K), l' \in \mathcal{L}(K'),$$

where  $\mathcal{L}(K)$  denotes the set of edges of the triangle  $K$ . Finally we use the notation  $n_{K,l}$  for the outer normal to the edge  $l$  of  $K$ .

We have the next result.

**Theorem 1** *Suppose that  $\mathcal{T}_h$  is a conforming triangulation of  $\Omega$  of weakly acute type and that, for a given  $K$ , for every edge  $l \in \mathcal{L}(K)$  it holds*

$$\mu_l \geq \frac{|l|}{2|K|} a_h |_{\kappa} E(K, l),$$

where  $\mu_l$  is the penalization parameter over the edge  $l$  (see (12)). Then the stiffness matrix  $M$  is an M-matrix.

*Proof* Suppose that  $l_1$  is an edge of  $K$  and that  $K$  has not any side on the boundary of  $\Omega$ . We use the notation introduced in Figure 1(a), where  $l_2$  and  $l_3$  are the other edges of  $K$ ,  $K_1, K_2$  and  $K_3$  are neighbors of  $K$  sharing the edges  $l_1, l_2$  and  $l_3$  respectively. We denote  $l'_i$  the edge  $l_i$  viewed as an edge of  $K_i$ . It follows that  $M_{K,l_1;K',l'} \neq 0$  only if

$$(K', l') \in \{(K, l_1), (K, l_2), (K, l_3), (K_1, l'_1), (K_2, l'_2), (K_3, l'_3)\}. \quad (13)$$

Taking into account that

$$\nabla \varphi_{K,l} = \begin{cases} \frac{|l|}{|K|} n_{K,l} & \text{on } K, \\ 0 & \text{elsewhere,} \end{cases}$$

an explicit computation of the corresponding entries of  $M$  gives

$$\begin{aligned} M_{K,l_1;K,l_1} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1|^2}{|K|} + \mu_{l_1} |l_1|, \\ M_{K,l_1;K,l_2} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1||l_2|}{|K|} n_{K,l_1} \cdot n_{K,l_2}, \\ M_{K,l_1;K,l_3} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1||l_3|}{|K|} n_{K,l_1} \cdot n_{K,l_3}, \\ M_{K,l_1;K_1,l'_1} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1|^2}{|K|} - \mu_{l_1} |l_1|, \\ M_{K,l_1;K_2,l'_2} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1||l_2|}{|K|} n_{K,l_1} \cdot n_{K,l_2}, \\ M_{K,l_1;K_3,l'_3} &= \frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1||l_3|}{|K|} n_{K,l_1} \cdot n_{K,l_3}. \end{aligned} \quad (14)$$

Since  $\mathcal{T}_h$  is weakly acute, we have that

$$M_{K,l_1;K,l_2} = M_{K,l_1;K_2,l'_2} \leq 0, \quad \text{and} \quad M_{K,l_1;K,l_3} = M_{K,l_1;K_3,l'_3} \leq 0.$$

Moreover, if

$$\mu_{l_1} \geq \frac{|l_1|}{2|K|} a_h |_K E(K, l_1), \quad (15)$$

then we also have  $M_{K,l_1;K_1,l'_1} \leq 0$ .

Therefore, for all off-diagonal entries, we have checked

$$M_{K,l_1;K',l'} \leq 0 \quad \text{if } (K, l_1) \neq (K', l'). \quad (16)$$

We also have

$$\sum_{(K',l'): K' \in \mathcal{T}_h, l' \in \mathcal{L}(K')} M_{K,l_1;K',l'} = a_h |_K E(K, l_1) \frac{|l_1|}{|K|} (|l_1| n_{K,l_1} + |l_2| n_{K,l_2} + |l_3| n_{K,l_3}) \cdot n_{K,l_1} = 0.$$

Now we suppose that  $l_1$  is an internal edge of  $K \in \mathcal{T}_h$  and that  $K$  has the edge  $l_2 \neq l_1$  on the boundary of  $\Omega$ , as depicted in Figure 1(b). Then, we have

$$M_{K,l_1;K,l_2} = 0,$$

while the other entries remain unchanged. It follows that property (16) holds true and

$$\sum_{(K',l'): K' \in \mathcal{T}_h, l' \in \mathcal{L}(K')} M_{K,l_1;K',l'} = -\frac{1}{2} a_h |_K E(K, l_1) \frac{|l_1||l_2|}{|K|} n_{K,l_1} \cdot n_{K,l_2} > 0.$$

Finally, we consider the case in which  $l_1$  is a boundary edge of  $K \in \mathcal{T}_h$  while  $l_2$  and  $l_3$  are the other edges  $K$ , see Figure 1(c). In this case we have

$$M_{K,l_1;K,l_1} = \mu_{l_1} |l_1|,$$



while the other entries remain as in the first considered case apart from  $M_{K,l_1;K'_1,l'_1}$  which is not defined because  $l_1$  is a boundary edge. Then (16) again holds true and if (15) holds we have

$$\sum_{(K',l'):K' \in \mathcal{T}_h, l' \in \mathcal{L}(K')} M_{K,l_1;K',l'} = -\frac{1}{2} a_h |_{\kappa} E(K, l_1) \frac{|l_1|^2}{|K|} + \mu_{l_1} |l_1| > 0.$$

Then we can apply, for instance [11, Part III Theorem 3.1, page 279] to conclude that  $M$  is a  $M$ -matrix.  $\square$

*Remark 4* As already observed, the projection reduces the connectivity of  $M$ . Indeed, without the projection in the penalization term (as in the case of the standard IIP method) the set in (13) would be larger, that is, there will be more non-zero entries in the matrix  $M$ . This fact spoils the  $M$ -matrix property, since some of the additional entries have the wrong sign.

*Remark 5* In the case of nonconforming meshes the  $M$ -matrix property does not hold, in general. Nevertheless, the numerical experiments presented in Section 5 show that the quality of the solution is not spoiled by the lack of this property (see remark 6).

#### 4 The a posteriori estimation

In this section we introduce the estimator and prove the main result of the paper.

For the sake of simplicity, from now on we are going to assume that  $\varepsilon(x) \equiv \varepsilon$  constant on  $\Omega$  and  $g$  is the restriction to  $\Gamma_D$  of a function in  $V_h \cap H^1(\Omega)$ . Furthermore, we assume that all the integrals involving the right hand side  $f$  are exactly computed (as, for instance, in the case that  $f$  is a polynomial).

##### 4.1 The result

Let us introduce the following mesh dependent norm. For  $v \in H^1(\mathcal{T}_h)$  we define

$$\|v\|^2 = |\varepsilon^{1/2} v|_{1,h}^2 + \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_D} \frac{\gamma \varepsilon}{h_e} \|[[v]]\|_{0,e}^2, \quad (17)$$

with  $\gamma > \gamma_0 > 0$  independent of  $h$  and of  $\varepsilon$ .

Moreover, as in [13], for  $q \in (L^2(\Omega))^2$  we take

$$|q|_* = \sup_{v \in H_{0,\Gamma_D}^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} q \cdot \nabla v \, dx}{\|v\|}. \quad (18)$$

We then define

$$|v|_{\beta}^2 = |\beta v|_*^2 + \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_D} \frac{h_e}{\varepsilon} \|[[v]]\|_{0,e}^2. \quad (19)$$

The bilinear form  $A$  defined in (4) satisfies the inf-sup condition: there exists  $C > 0$  independent of  $\varepsilon$  such that

$$\inf_{u \in H_{0,\Gamma_D}^1(\Omega) \setminus \{0\}} \sup_{v \in H_{0,\Gamma_D}^1(\Omega) \setminus \{0\}} \frac{A(u, v)}{(\|u\| + |\beta u|_*) \|v\|} \geq C. \quad (20)$$

This property is proved in [13, Lemma 4.4] for the case  $\Gamma_D = \partial\Omega$ . It extends without changes to our case.

For each element  $K$ , we define the local error indicator  $\eta_K$ , as given by the sum of three contributions

$$\eta_K^2 := \eta_{R_K}^2 + \eta_{J_K^g}^2 + \eta_{J_K^u}^2, \quad (21)$$

where the three terms are defined as follows

$$\eta_{R_K}^2 := \frac{h_K^2}{\varepsilon} \|f - \operatorname{div}(a_h \nabla T u_h)\|_{0,K}^2 = \frac{h_K^2}{\varepsilon} \|f\|_{0,K}^2, \quad (22)$$

which is the interior residual term,

$$\eta_{J_K^g}^2 := \frac{1}{2} \sum_{e \in \partial K \setminus \Gamma_D} \frac{h_e}{\varepsilon} \| [ [ a_h \nabla T u_h ] ] \|_{0,e}^2, \quad (23)$$

which controls the jump of the approximated flux, and

$$\eta_{J_K^u}^2 := \frac{1}{2} \sum_{e \in \partial K \setminus \partial\Omega} \left( \frac{\gamma \varepsilon}{h_e} + \frac{h_e}{\varepsilon} \right) \| [ [ u_h ] ] \|_{0,e}^2 + \sum_{e \in \partial K \cap \Gamma_D} \left( \frac{\gamma \varepsilon}{h_e} + \frac{h_e}{\varepsilon} \right) \| u_h - g \|_{0,e}^2, \quad (24)$$

which is the jump of the discrete solution.

The a posteriori error estimator is then defined as

$$\eta := \left( \sum_{K \in \mathcal{T}_h} \eta_{R_K}^2 + \eta_{J_K^g}^2 + \eta_{J_K^u}^2 \right)^{1/2}. \quad (25)$$

We further define the exponentially fitted data approximation term

$$\theta_K^2 := \frac{1}{\varepsilon} \| \varepsilon \nabla u_h - \beta u_h - a_h \nabla T u_h \|_{0,K}^2, \quad (26)$$

and the exponential fitting approximation error by

$$\theta := \left( \sum_K \theta_K^2 \right)^{\frac{1}{2}}. \quad (27)$$

We shall prove the following result.

**Theorem 2** *Let  $u$  be the solution of (1) and let  $u_h$  be the solution of (11). Let the error estimator  $\eta$  be defined by (25) and the exponential fitting approximation error  $\theta$  defined by (27). Then, we have the a posteriori error bound*

$$\| |u - u_h| \| + |u - u_h|_\beta \leq C(\eta + \theta), \quad (28)$$

with  $C$  independent of  $\varepsilon$  and  $h$ .

## 4.2 Approximation Operators

We need to define a conforming approximation for functions in  $V_h$ . We recall that we assume that the family of meshes  $\mathcal{T}_h$  satisfies the restrictions described in section 2.

We denote by  $V_h^c$  the conforming subspace of  $V_h$  given by  $V_h^c = V_h \cap H_{0,\Gamma_D}^1(\Omega)$ . Next result is taken from [8, Theorem 2.1].

Suppose that  $\chi$  is the restriction to  $\Gamma_D$  of a function in  $V_h \cap H^1(\Omega)$ . Then, there exists an operator  $\Xi_h : V_h \rightarrow H^1(\Omega)$  such that for all  $v \in V_h$  we have  $\Xi_h v|_{\Gamma_D} = \chi$  and for a constant  $C$

$$\sum_{K \in \mathcal{T}_h} \|v - \Xi_h v\|_{L^2(K)}^2 \leq C \left( \sum_{e \in \mathcal{E}_0} h_e \|\llbracket v \rrbracket\|_{0,e}^2 + \sum_{e \in \mathcal{E}_D} h_e |v - \chi|_{0,e}^2 \right), \quad (29)$$

$$\sum_{K \in \mathcal{T}_h} \|\nabla(v - \Xi_h v)\|_{L^2(K)}^2 \leq C \left( \sum_{e \in \mathcal{E}_0} h_e^{-1} \|\llbracket v \rrbracket\|_{0,e}^2 + \sum_{e \in \mathcal{E}_D} h_e^{-1} |v - \chi|_{0,e}^2 \right). \quad (30)$$

We further need the existence of an interpolation operator  $\mathcal{I}_h : H_{0,\Gamma_D}^1(\Omega) \rightarrow V_h^c$  with the following error estimates: for all  $v \in H_{0,\Gamma_D}^1(\Omega)$

$$\|v - \mathcal{I}_h v\|_{L^2(\Omega)} \leq C \left( \sum_{K \in \mathcal{T}_h} h_K^2 |v|_{H^1(K)}^2 \right)^{\frac{1}{2}} \quad (31)$$

$$\|v - \mathcal{I}_h v\|_{H^1(\Omega)} \leq C |v|_{H^1(\Omega)}. \quad (32)$$

Since we are assuming that the mesh  $\mathcal{T}_h$  is obtained from a conforming shape regular mesh by means of a few number of refinements/derefinements together with the local quasi-uniformity property, we can take as  $\mathcal{I}_h$  the Scott-Zhang interpolation operator [14] on the mentioned conforming mesh.

## 4.3 Estimate

Now we are ready for the proof of our main result.

*Proof (Proof of Theorem 2)* As in [6], the key starting point is the decomposition of the solution  $u_h$  into a conforming part plus a remainder. More precisely, given  $u_h$  solution of the discrete problem (11), we can split it as  $\Xi_h u_h$  plus a remainder. Denoting  $u_h^c = \Xi_h u_h$ , and denoting the remainder by  $u_h^r$ , we have

$$u_h = u_h^c + u_h^r.$$

Therefore, we have

$$\| \|u - u_h\| \| + |\beta(u - u_h)|_* \leq \| \|u - u_h^c\| \| + |\beta(u - u_h^c)|_* + \| \|u_h^r\| \| + |\beta u_h^r|_*.$$

First, we notice that  $\| \|u_h^r\| \| + |\beta u_h^r|_*$  can be controlled by means of (29) and (30), since  $u_h^r = u_h - \Xi_h u_h$  by definition, and by the fact that  $\llbracket u_h^r \rrbracket = \llbracket u_h \rrbracket$ , obtaining

$$\begin{aligned} \| \|u_h^r\| \|^2 + |\beta u_h^r|_*^2 &\leq \sum_{e \in \mathcal{E}_0} \left[ (1 + \gamma) \frac{\varepsilon}{h_e} + \|\beta\|_{L^\infty(\Omega)} \frac{h_e}{\varepsilon} \right] \|\llbracket u_h \rrbracket\|_{0,e}^2 + \\ &\quad \sum_{e \in \mathcal{E}_D} \left[ (1 + \gamma) \frac{\varepsilon}{h_e} + \|\beta\|_{L^\infty(\Omega)} \frac{h_e}{\varepsilon} \right] \|u_h - g\|_{0,e}^2. \end{aligned} \quad (33)$$

Since  $g$  is the restriction to  $\Gamma_D$  of a function in  $V_h \cap H^1(\Omega)$  and because of the construction of  $\mathfrak{E}_h$ , we have  $u = u_h^c = g$  on  $\Gamma_D$  and, therefore,  $u - u_h^c \in H_{0,\Gamma_D}^1(\Omega)$ . So we use the *inf-sup* condition (20) to obtain for all  $v \in H_{0,\Gamma_D}^1(\Omega)$

$$\| \|u - u_h^c\| \| + |\beta(u - u_h^c)|_* \leq C \sup_{v \in H_{0,\Gamma_D}^1(\Omega) \setminus \{0\}} \frac{A(u - u_h^c, v)}{\| \|v\| \|}. \quad (34)$$

Using that  $u$  is solution of the continuous problem and  $u_h^c = u_h - u_h^r$ , for  $v \in H_{0,\Gamma_D}^1(\Omega)$  we obtain

$$\begin{aligned} A(u - u_h^c, v) &= \int_{\Omega} f v \, dx - \sum_K \int_K (\varepsilon \nabla u_h^c - \beta u_h^c) \cdot \nabla v \, dx \\ &= \int_{\Omega} f v \, dx - \sum_K \int_K (\varepsilon \nabla u_h - \beta u_h) \cdot \nabla v \, dx + \sum_K \int_K (\varepsilon \nabla u_h^r - \beta u_h^r) \cdot \nabla v \, dx \\ &= \int_{\Omega} f(v - \mathcal{I}_h v) \, dx - \sum_K \int_K (\varepsilon \nabla u_h - \beta u_h - a_h \nabla T u_h) \cdot \nabla v \, dx \\ &\quad + \int_{\Omega} f \mathcal{I}_h v \, dx - \sum_K \int_K a_h \nabla T u_h \cdot \nabla v \, dx + \sum_K \int_K (\varepsilon \nabla u_h^r - \beta u_h^r) \cdot \nabla v \, dx. \end{aligned}$$

Using now the discrete equation (11), tested against  $\mathcal{I}_h v \in H_{0,\Gamma_D}^1(\Omega)$ , and taking into account that the jumps of  $\mathcal{I}_h v$  on internal and Dirichlet edges vanish, we have

$$\int_{\Omega} f \mathcal{I}_h v \, dx = \sum_K \int_K a_h \nabla T u_h \cdot \nabla \mathcal{I}_h v \, dx.$$

Hence

$$\begin{aligned} A(u - u_h^c, v) &= \int_{\Omega} f(v - \mathcal{I}_h v) \, dx - \sum_K \int_K (\varepsilon \nabla u_h - \beta u_h - a_h \nabla T u_h) \cdot \nabla v \, dx \\ &\quad - \sum_K \int_K a_h \nabla T u_h \cdot \nabla(v - \mathcal{I}_h v) \, dx + \sum_K \int_K (\varepsilon \nabla u_h^r - \beta u_h^r) \cdot \nabla v \, dx. \end{aligned} \quad (35)$$

Let us consider separately each term of the last equation. Using the  $L^2$ -approximation estimate (31) for the Scott-Zhang interpolation operator, we can estimate the first term of (35) as

$$\left| \int_{\Omega} f(v - \mathcal{I}_h v) \, dx \right| \leq \left( \sum_K \frac{h_K^2}{\varepsilon} \|f\|^2 \right)^{1/2} \| \|v\| \|. \quad (36)$$

From the definition (27) of the exponentially fitted data approximation error, for the second term of (35) we have

$$\left| \sum_K \int_K (\varepsilon \nabla u_h - \beta u_h - a_h \nabla T u_h) \cdot \nabla v \, dx \right| \leq \theta \| \|v\| \|. \quad (37)$$

For the third term of (35), integrating by parts, taking into account that  $a_h \nabla_h T u_h$  is piecewise constant, using that  $H^1$ -functions have vanishing jumps, using a trace inequality to bound

$\|v - \mathcal{I}_h v\|_{L^2(e)}$  on edges, and finally using the approximation estimates (31) and (32) for the Scott-Zhang interpolation operator, we have

$$\begin{aligned} \left| \sum_K \int_K a_h \nabla T u_h \cdot \nabla (v - \mathcal{I}_h v) dx \right| &= \left| \int_{\Gamma_N} (v - \mathcal{I}_h v) n \cdot a_h \nabla T u_h ds + \int_{\Gamma^0} \llbracket a_h \nabla T u_h \rrbracket \{v - \mathcal{I}_h v\} ds \right| \\ &\leq \left( \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_N} \frac{h_e}{\varepsilon} \|\llbracket a_h \nabla T u_h \rrbracket\|_{0,e}^2 \right)^{1/2} \|v\|. \end{aligned} \quad (38)$$

The fourth term of (35) is controlled thanks to (29) (since  $u_h^r = u_h - \Xi_h u_h$ , by definition)

$$\begin{aligned} &\left| \sum_K \int_K (\varepsilon \nabla u_h^r - \beta u_h^r) \cdot \nabla v dx \right| \\ &\leq \left[ \left( \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_D} \frac{\varepsilon}{h_e} \|\llbracket u_h \rrbracket\|_{0,e}^2 \right)^{1/2} + \|\beta\|_{L^\infty(\Omega)} \left( \sum_{e \in \mathcal{E}_0 \cup \mathcal{E}_D} \frac{h_e}{\varepsilon} \|\llbracket u_h \rrbracket\|_{0,e}^2 \right)^{1/2} \right] \|v\|. \end{aligned} \quad (39)$$

Collecting (36), (37), (38) and (39), and using the definition of (25) of  $\eta$  we obtain from (35)

$$|A(u - u_h^c, v)| \leq C(\eta + \theta) \|v\|.$$

Therefore, inserting this inequality in (34), we have

$$\|u - u_h^c\| + |\beta(u - u_h^c)|_* \leq C(\eta + \theta)$$

and, which together with (33) gives

$$\|u - u_h\| + |\beta(u - u_h)|_* \leq C(\eta + \theta).$$

The proof is then concluded since  $|u - u_h|_\beta \leq |\beta(u - u_h)|_* + \eta$ .

## 5 Numerical experiments

In this section we present some numerical experiments using the a posteriori error estimates of Theorem 2 as error indicator for an adaptive refinement strategy.

The starting mesh is a non structured shape regular mesh of weakly acute type with 28 elements and 84 degrees of freedom. In the experiments, we construct adaptively refined sequences of meshes by marking elements for refinement according to the size of the local indicator  $\zeta_K$  defined as  $\zeta_K^2 := \eta_K^2 + \theta_K^2$ , where  $\eta_K$  and  $\theta_K$  are defined in (21) and (26), respectively. The refinement fraction is set to 50% and the derefinement fraction is set to 10%. In each refinement step, we split the triangles which are marked for refinement into four congruent triangles. In this way the weakly acute property of the initial mesh is conserved in all the refinement steps.

### 5.1 Test 1

In this first test we deal with an example for which we know the exact solution, so that we can study numerically the errors. Consider the equation (1) in the domain  $\Omega = (0, 1)^2$  with  $\beta = [1, 1]$ , homogeneous Dirichlet boundary condition on  $\partial\Omega$  and the right hand side  $f$  given by

$$f(x, y) = 2 \left( \frac{1 + e^{-\frac{1}{\varepsilon}} - 2e^{\frac{x-1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + \frac{1 + e^{-\frac{1}{\varepsilon}} - 2e^{\frac{y-1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + 2x + 2y - 2 \right)$$

The exact solution is then

$$u(x, y) = \left( \frac{1 + e^{-\frac{1}{\varepsilon}} - 2e^{\frac{x-1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + 2x - 1 \right) \left( \frac{1 + e^{-\frac{1}{\varepsilon}} - 2e^{\frac{y-1}{\varepsilon}}}{1 - e^{-\frac{1}{\varepsilon}}} + 2y - 1 \right).$$

Since the solution, otherwise smooth, presents boundary layers at  $x = 1$  and  $y = 1$ , this test is designed to study the relationship between the estimator  $\zeta := \eta + \theta$  and the true error in dependence on the diffusion parameter  $\varepsilon$ . Figure 2 plots (in log-log scale) the value of the estimator (labelled as EST), of the energy norm (17) (labelled as T-ERR) against the squared root of the number of elements, for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ . As in [13] we also plot the  $\varepsilon$ -weighted  $L^2$ -norm  $\varepsilon^{-1/2} \|u - u_h\|_{0, \Omega}$  (labelled as L-ERR), which is a bound of  $|\beta(u - u_h)|_*$ .

We observe that the estimator curve is always above the true error curve, according to Theorem 2. Moreover, in all cases, the energy norm reaches an order one convergence, while the norm  $\varepsilon^{-1/2} \|u - u_h\|_0$  reaches an order 2 convergence (orders 1 and 2 are displayed by the small triangles). In figure 2(d), we further plot, for  $\varepsilon = 10^{-3}$ , the energy and  $L^2$  norms for an uniformly refined sequence of meshes. As expected, with a comparable number of elements as in figure 2(c), optimal order of convergence is far to be attained.

Figure 3 shows the ratio between the estimator  $\zeta$  and the energy norm, for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ . The asymptotic value of the ratio seems not to depend significantly on  $\varepsilon$ , showing the robustness of the presented estimator with respect to  $\varepsilon$ .

### 5.2 Test 2

We study now a propagation of a boundary discontinuity for  $\varepsilon = 10^{-4}$ . In the domain  $\Omega = (-1, 1)^2$  we take  $\beta = [1, 1]$ ,  $f = 0$  and  $\Gamma_D = \partial\Omega$ . The boundary condition is

$$g(x, y) = \begin{cases} 1 & \text{if } x = -1 \text{ and } y \geq -\frac{1}{3}, \text{ or if } y = 1 \text{ and } x \leq \frac{1}{3} \\ 0 & \text{elsewhere} \end{cases}$$

Figure 4 shows the mesh and the solution after 17 refinement steps. We observe that in a large portion of the domain, far from internal and boundary layers, several triangles have not been refined. The estimator provides marked elements for the refinement iteration steps only where needed.

Figure 5 plots the total estimator  $\zeta$ , introduced above, and the different terms that contribute to it. In this example, we do not know the exact solution, so we cannot plot the true

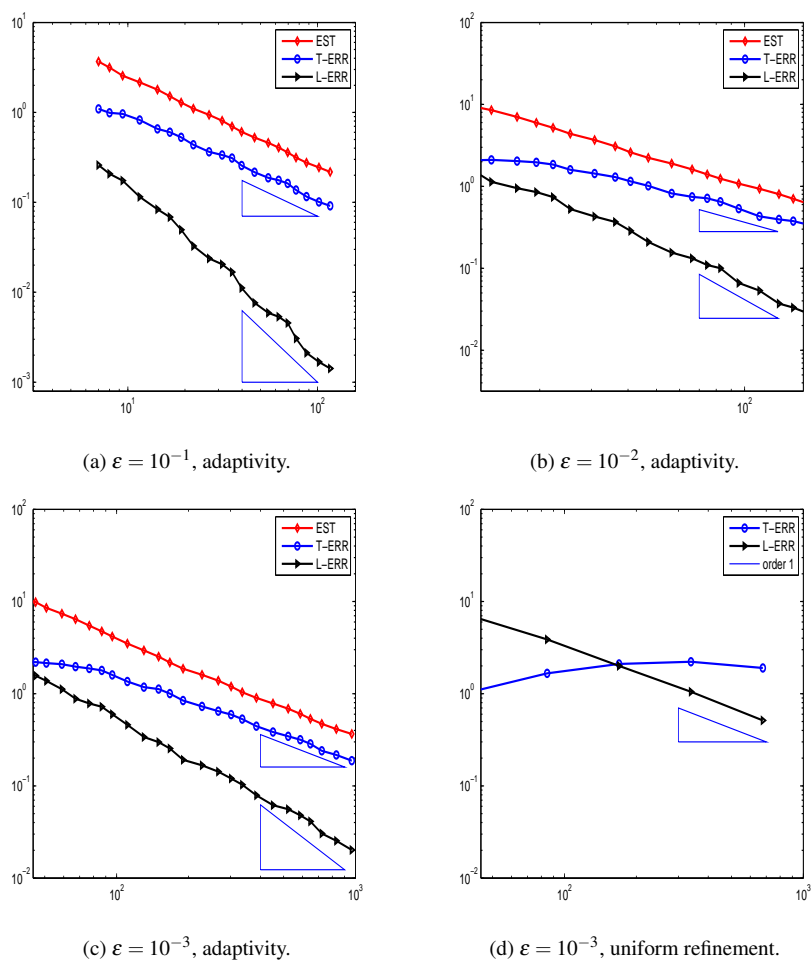


Fig. 2 Test 1. The estimator and the true error.

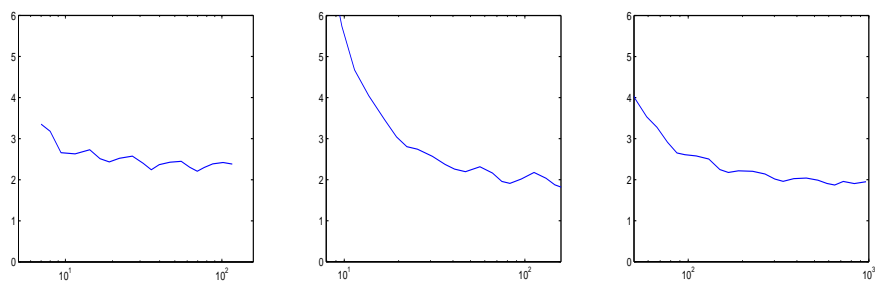


Fig. 3 Test 1. The ratio  $\zeta / \|u - u_h\|$ , for  $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$ .

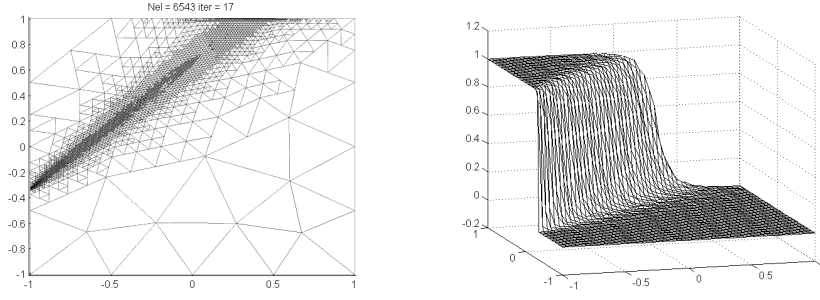


Fig. 4 Test 2, mesh and solution after 17 refinements.

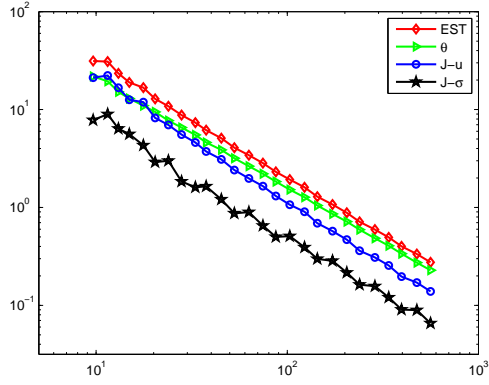


Fig. 5 Test 2. The total error estimator and its different terms.

error. Observing the different contributions to the estimator, we notice that the term  $\theta$  (defined by (27)) that in principle could be considered as a data approximation error term, due to the exponential fitting, cannot be neglected in the indicator in the actual implementation of the adaptive strategy. Indeed, it is of the same order of the total estimator.

### 5.3 Test 3

Now we explore the case of a rotating flow transporting a boundary sharp profile. In the domain  $\Omega = (-1, 1) \times (0, 1)$  we consider

$$\beta = (2y(1-x^2), -2x(1-y^2)), \quad f = 0.$$

The Dirichlet boundary condition is taken as

$$g(x,y) = \begin{cases} 1 + \tanh(10(2x+1)) & \text{if } x \leq 0 \text{ and } y = 0 \\ 0 & \text{elsewhere} \end{cases}$$



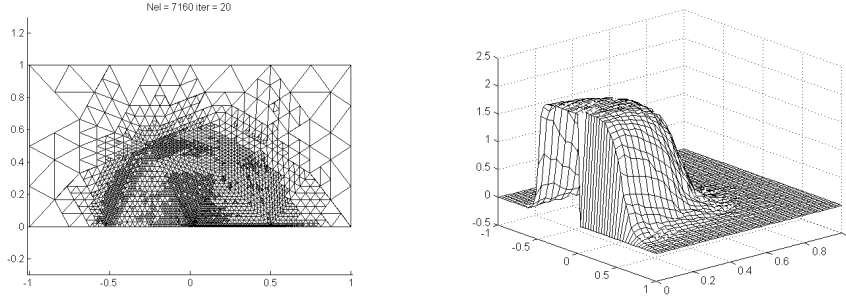


Fig. 6 Test 3. Mesh and solution after 20 refinements.

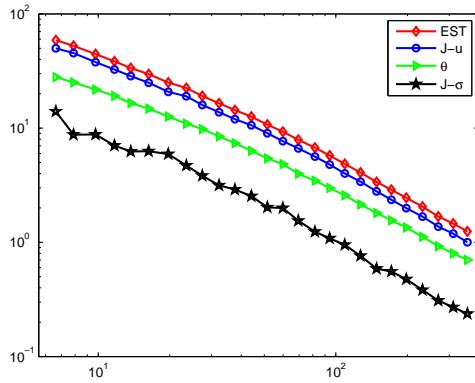


Fig. 7 Test 3. The total error estimator and its different terms.

In this case, we approximate the vector  $\beta$  on each element  $K$  by its value in the barycenter, which we denote by  $(\tilde{\beta}_K^1, \tilde{\beta}_K^2)$ , and we define the piecewise linear function  $\psi$  on  $K$  as

$$\psi_K(x, y) = \tilde{\beta}_K^1 x + \tilde{\beta}_K^2 y.$$

Figure 6 shows the mesh and the solution after 20 refinement steps. We observe that most elements are within the boundary layer, close to the singularity in the origin and where the solution exhibits the steepest behavior.

Figure 7 plots the total estimator  $\zeta$ , and the different terms that contribute to it. We point out that here the field  $\beta$  is approximated and this fact is taken care of in the exponential fitting data approximation error term  $\theta$ . However, its effect on the overall indicator is not predominant, as shown in figure 7. Moreover, this is also confirmed by the quality of the mesh in figure 6, where the stronger refinement takes place in correspondence to the worse behavior of the solution.

*Remark 6* The adaptive algorithm is initialized on a weakly acute conforming mesh, therefore, thanks to the  $M$ -matrix property the initial solution is non negative and it does not

exhibits spurious oscillations. Instead, in the successive refinement steps, the  $M$ -matrix property does not hold, since the inverse of the matrix is not positive. Nevertheless, the quality of the solution is not spoiled. For test 2, in Figure 4 we observe that the solution conserves the positivity and does not have spurious oscillations. For test 3, the solution in Figure 6 presents only very small wiggles.

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