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ON ENTROPIES AND COMPLEXITY FOR AMENABLE ACTIONS GROUPS

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Abstract

We consider a definition of entropy for discrete amenable action groups and extend the equality between entropy and dimension for amenable groups. This last result was proved by Simpson for the special case of $\mathbf{Z}^d$ -actions.

1. Introduction

In the area of dynamical systems, the study of dimension of sets constitutes a relevant discipline. One of the main goals in this context is to analyze the complexity of diverse objects. The determination of adequate "characteristic dimensions" of special sets, for instance attractors, was a matter of research by both the mathematicians and physicists. Important invariants to measure the complexity of the systems as well as the average amount of information are the metric and topological entropies.

The use of groups as dynamical objects, which is an interesting issue in ergodic theory, consists in taking groups acting on a measure, or topological, space, obtaining the dynamics from the action. One very developed topic in

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this subject was the extension of the Birkhoff ergodic theorem to measure-preserving action groups as dynamics. The classical ergodic theorem concerns to measure preserving **Z**-actions, and the extension deals with more general group actions, so in [9] and [2] the point-wise convergence of ergodic averages for action groups is considered. One meaningful contribution, in this field, is the work of Lindenstrauss [7] who has generalized the classical Birkhoff ergodic theorem to action amenable groups. In this way dynamical invariants from group actions deserved to be introduced, being countable discrete amenable groups mostly used to this end (see for instance references [11, 5] and [14]). In [14], Zhou introduced the so called tail entropy for countable amenable groups which describes the complexity of the system at small pieces, i.e. counting the elements of refinements of a covering $\mathcal{U}$ relative to subcoverings of $\mathcal{U}$, besides is presented a measure-theoretic entropy and proved a variational principle for refining partitions. The definition in [14] extends that introduced by Misiurewicz for **Z**-actions [8]. In [12], Simpson considered entropies for $\mathbf{Z}^d$ -actions and established equalities among entropy, Hausdorff dimension and Kolmogorov complexity, in the setting of actions on symbolic spaces.

In this article, we extend to amenable groups the equality "entropy = dimension" established by Simpson in [12] for $\mathbf{Z}^d$ -actions, and earlier by Furstenberg [3] for $\mathbf{Z}^+$ -actions. The main tool to do this is the Lindenstrauss covering lemma [7]. This important result states that from translates of sets of a finite Følner sequence contained in a compact *F*, can be randomly extracted an almost-disjoint sequence which covers most part of *F*. This is like a generalization of the Vitali lemma. Finally, we analyze the possibility of extending to amenable actions the relationship between entropy and Kolmogorov complexity, proved in [12] for $\mathbf{Z}^d$ -actions. To study the scaled case for the complexity we consider an extension to amenable action of a version of scaled complexity introduced by Galotolo [4] for the classical **Z**-actions. The Kolmogorov complexity of a sequence in a given description language, is the minimal sequence (program) to produce this sequence to output. The Galatolo version of complexity measures complexity of an orbit.

2. Preliminaries Definitions

Let Γ be a locally compact topological group. Amenability is formulated in different ways, the most convenient for our purpose, which is the one used in Lindenstrauss theory, is the following: Γ is amenable if for any compact $K \subset \Gamma$ and for any $\delta > 0$ there exists a compact $F \subset \Gamma$ such that

$$\frac{|F \Delta KF|}{|F|} < \delta, \tag{1}$$

a set *F* with this condition is called (K, δ)-*invariant*.

A sequence (F_n) of compact subsets of Γ is a *Følner sequence* if for any compact subset *K* of Γ and for any $\delta > 0$,

$$\frac{|F_n \Delta KF_n|}{|F_n|} < \delta \tag{2}$$

holds for large enough *n*. In other word, a (F_n) is a Følner sequence if, for large enough *n*, each set F_n is (K, δ)-invariant. For $\Gamma = \mathbf{Z}$, we can take $F_n = \{1, 2, ..., n\}$, while for $\mathbf{Z}^d$ the natural example is $F_n = \{-n, ..., n\}^d$.

A sequence (F_n) is *tempered* if there is a $C > 0$ such that, the Schulman condition

$$\frac{\left| \bigcup_{k=1}^{n} F_k^{-1} F_n \right|}{|F_n|} < C \tag{3}$$

is satisfied. Any amenable group has a tempered Følner sequence [7]. For a discrete group Γ, the amenability is equivalently formulated as that every continuous action of Γ on a compact space has an invariant measure. With $|\,|$, we denote the left invariant Haar measure on Γ which, for discrete groups, is the usual counting measure.

By a Γ-system is understood a pair (X, Γ), where *X* is a compact metric space and Γ is a group acting on X. A measure μ on X is said Γ-invariant if for any measurable subset E of X and for any $\gamma \in \Gamma$ is $\mu(\gamma E) = \mu(E)$. We denote by $\mathcal{M}(X, \Gamma)$ the set of Γ-invariant probability measures on X. This set is compact with respect to the weak topology.

Symbolic spaces for group actions are defined as follows: let Ω be a non-empty finite set of symbols and let Γ be a discrete amenable group, let us denote

$$\Omega^{\Gamma} = \{x : \Gamma \to \Omega\},$$

i.e. the sequences $(x(\gamma))_{\gamma\in\Gamma}$. Also this space can be expressed as $\Omega^{\Gamma} = \prod_{\gamma\in\Gamma}\Omega$, therefore if Ω has the discrete topology, then Ω^{Γ} can be endowed with the product topology. The shift action on Γ on Ω^{Γ} is given by the map

$$T : \Gamma \times \Omega^{\Gamma} \to \Omega^{\Gamma}$$

$$T(\gamma', x)(\gamma) = x(\gamma\gamma').$$

The compact dynamical system (Ω^{Γ}, T) is usually called the *full shift*. A *subshift* is a closed subset Z of Ω^{Γ} which is also invariant by the shift action. A *language* of the subshift Z is the set $\mathcal{L}(Z)$ formed by sequences in points of Z. For a Følner sequence (F_n) let $\Omega^{F_n} = \{x|_{F_n} : F_n \to \Omega,\ x \in \Omega^{\Gamma}\}$, i.e. the restrictions of sequences of Ω^{Γ} to F_n. Thus each element of Ω^{F_n} can be identified with a string $\mathbf{L} = (\ell_1, \ell_2, ..., \ell_N)$ with $\ell_i \in \Omega$. Let $\Omega^* = \bigcup_{n\geq 1}\Omega^{F_n}$. Now for a Følner sequence (F_n), set $\mathcal{L}(Z, F_n) := \mathcal{L}(Z) \cap \Omega^{F_n}$ and for any $x^{(n)} = x|_{F_n} \in \mathcal{L}(Z, F_n)$. The *cylinder* $C(x^{(n)})$ is formed by the sequence $y \in Z$ such that $y^{(n)} := y|_{F_n} = x^{(n)}$. The counting of the growing of the cardinality of $\mathcal{L}(Z, F_n)$ is used for the definition of entropy of subshifts. For the full subshift, i.e. $Z = \Omega^{\Gamma}$, $|\mathcal{L}(\Omega^{\Gamma}, F_n)| = |\Omega|^{|F_n|}$.

3. Entropies for Amenable Groups

Let (X, Γ) be a Γ-system with Γ a discrete countable amenable group. For an open cover $\mathcal{U}$ of X, we denote

$$N(\mathcal{U}) = \min\{card\mathcal{V} : \mathcal{V} \text{ is a finite subcovering of } \mathcal{U}\}. \tag{4}$$

If $\mathcal{U}_1, \mathcal{U}_2$ are covers of X, then set $\mathcal{U}_1 \vee \mathcal{U}_2 = \{U_1 \cap U_2 : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2\}$ and for any $\gamma \in \Gamma$ let $\gamma^{-1}(\mathcal{U}) = \{\gamma^{-1}U : U \in \mathcal{U}\}$. If (F_n) is a Følner sequence in Γ, then define $\mathcal{U}^{F_n} = \bigvee_{\gamma \in F_n} \gamma^{-1}(\mathcal{U})$.

Definition. The upper and lower entropies are defined, respectively, as

$$\overline{h}(X, \Gamma) = \sup_{\mathcal{U}} \overline{h}(X, \Gamma, \mathcal{U}),$$

$$\underline{h}(X, \Gamma) = \sup_{\mathcal{U}} \underline{h}(X, \Gamma, \mathcal{U}) \tag{5}$$

with

$$\overline{h}(X, \Gamma, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}),$$

$$\underline{h}(X, \Gamma, \mathcal{U}) = \liminf_{n\to\infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n}). \tag{6}$$

Remark 1. Since $N(\mathcal{U}_1 \vee \mathcal{U}_2) \leq N(\mathcal{U}_1)N(\mathcal{U}_2)$, and hence $\log N(\mathcal{U}^{F_n \cup F_m}) \leq \log N(\mathcal{U}^{F_n}) + \log N(\mathcal{U}^{F_m})$, by the Ornstein and Weiss lemma ([11, 6]), results $\overline{h} = \underline{h}$ and the limit in (5) exists.

Remark 2. If Γ acts isometrically on a compact metric space X, i.e. $d(\gamma x, \gamma y) = d(x, y)$, for any $\gamma \in \Gamma$, then $N(\mathcal{U}^{F_n})$ remains bounded for any n, and therefore $\overline{h}(X, \Gamma) = \underline{h}(X, \Gamma) = 0$.

For closed subshift $Z \subset \Omega^{\Gamma}$, we can set

$$\overline{h}(Z, \Gamma) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log|\mathcal{L}(Z, F_n)|,$$

$$\underline{h}(Z, \Gamma) = \liminf_{n\to\infty} \frac{1}{|F_n|} \log|\mathcal{L}(Z, F_n)|. \tag{7}$$

The explanation of this definition may be adopted for subshifts briefly as follows: a (topological) *generator* is a cover $\mathcal{U}$ with the property that there is an n such that $\mathcal{U}^{F_n}$ is a refinement of any covering $\mathcal{V}$. If $\mathcal{U}$ is a generator, then the supremum in (6) is reached and $\overline{h}(X, \Gamma, \mathcal{U}^{F_n}) = \overline{h}(X, \Gamma, \mathcal{U})$ and $\underline{h}(X, \Gamma, \mathcal{U}^{F_n}) = \underline{h}(X, \Gamma, \mathcal{U})$, for any n. In the case of subshifts, a covering is by cylinders $\mathcal{C} = \{C_1, ..., C_{|\Omega|}\}$, with $C_i = \{x \in \Omega^\Gamma : x(e) = i\}$, where e is the identity in Γ. Thus C can be taken as a generator and the counting $N(\mathcal{C}^{F_n})$ is precisely $|\mathcal{L}(Z, F_n)|$. For the full shift $Z \subset \Omega^\Gamma$, $h(\Omega^\Gamma, \Gamma) = \log|\Omega|$. Since Z is closed, there is an n_0 such that $|\mathcal{L}(Z, F_n)| < |\Omega|^{|F_{n_0}|}$ so that $\frac{1}{|F_{n_0}|}\log|\mathcal{L}(Z, F_{n_0})| < \log|\Omega|$ and then $h(Z, \Gamma) < \log|\Omega|$.

A definition in terms of separated sets, like in the classical case, can be formulated as follows: a subset E of X, is $(\Gamma, F_n, \varepsilon)$-separated if for any $x \neq y \in E$, $d(\gamma x, \gamma x) > \varepsilon$ for some $\gamma \in F_n$.

Let $r(\Gamma, F_n, \varepsilon) = \max\{cardE : E \text{ is } (\Gamma, F_n, \varepsilon)\text{-separated}\}$. Then define

$$\overline{H}(X, \Gamma) = \lim_{\varepsilon\to 0} \overline{H}(X, \Gamma, \varepsilon),$$

$$\underline{H}(X, \Gamma) = \lim_{\varepsilon\to 0} \underline{H}(X, \Gamma, \varepsilon) \tag{8}$$

with

$$\overline{H}(X, \Gamma, \varepsilon) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log r(\Gamma, F_n, \varepsilon),$$

$$\underline{H}(X, \Gamma, \varepsilon) = \liminf_{n\to\infty} \frac{1}{|F_n|} \log r(\Gamma, F_n, \varepsilon). \tag{9}$$

The equivalence between the two definitions can be seen in a rather similar way as in the case of integer actions, we do this in a following section. Also, like in **Z**-actions the definition does not depend on the metric.

3.1. Measure-theoretic entropy for amenable groups

The entropy of a partition $\mathcal{P}$ can be defined like in the classical case of **Z**-actions: let $H(X, \Gamma, \mu, \mathcal{P}) = -\sum_{P \in \mathcal{P}} \mu(P) \log \mu(P)$ and $h_\mu(X, \Gamma, \mathcal{P}) = \frac{1}{|F_n|} \lim_{n \to \infty} H(X, \Gamma, \mu, \mathcal{P}^{F_n})$. The limit does exist ([7, 11]). The entropy $h_\mu(X, \Gamma)$ is defined taking the supremum over all the finite partitions $\mathcal{P}$. For the case of subshifts the canonical partition is taken by cylinders and it may be denoted directly by $h_\mu(Z, \Gamma)$, for $Z \subset \Omega^\Gamma$.

4. Dimension and Entropy

In this section, we extend to amenable action groups the result of Simpson [12], obtained for $\mathbf{Z}^d$ -actions, which relates, for closed subshifts of a symbolic space, Hausdorff dimension with entropy. We firstly recall the definition of Hausdorff dimension of sets, this quantity was widely used in multifractal analysis to describe the structure of level sets in which the phase space is decomposed (multifractal decomposition).

Definition. Let X be a compact metric space and let

$$G(X, \alpha, \varepsilon) = \inf_{\mathcal{G}} \left\{ \sum_{Y \in \mathcal{G}} (diam Y)^\alpha \right\},$$

where the infimum is taken over all the coverings $\mathcal{G}$ of X with $diam \mathcal{G} \leq \varepsilon$. The outer α-Hausdorff measure of X is $\mu_\alpha(X) = \lim_{\varepsilon \to 0} G(X, \alpha, \varepsilon)$ and finally the Hausdorff dimension of X is $\dim_H X = \inf\{\alpha : \mu_\alpha(X) = 0\}$.

The Hausdorff dimension is not a “dynamical quantity”, as it does not depend on the dynamics, to relate it with a dimension given by the dynamics, like the entropy. We work in the setting of symbolic dynamics. Let Z be a closed subset of a full shift Ω^{Γ}, for a Følner sequence (F_n) set $F_n = \{\gamma_1^n, \gamma_2^n, ..., \gamma_{|F_n|}^n\}$ and to any sequence of $\Omega^{F_n} = \{x|_{F_n} : F_n \to \Omega,\ x \in \Omega^{\Gamma}\}$ can be assigned a string $\mathbf{L} = (\ell_1, \ell_2, ..., \ell_{|F_n|})$ with $x(\gamma_1^n) = \ell_i$, $i = 1, 2, ..., |F_n|$, we denote this $x|_{F_n} \leftrightarrow \mathbf{L} = (\ell_1, \ell_2, ..., \ell_{|F_n|})$, and $|\mathbf{L}| = |F_n|$. Let $\mathcal{G}_{|F_n|} = \{x|_{F_n} \leftrightarrow \mathbf{L} = (\ell_{\mathbf{1}}, \ell_{\mathbf{2}}, ..., \ell_{|\mathbf{F_n}|}),\ x \in \Omega^{\Gamma}\}$ and $\mathcal{G}^* = \bigcup \mathcal{G}_{|F_n|}$. The shift Ω^{Γ} can be endowed with a metric as follows: let $x, y \in \Omega^{\Gamma}$, if $|F_n|$ is the maximal value for which $\mathbf{L} = \overline{\mathbf{L}}$, where $x|_{F_n} \leftrightarrow \mathbf{L} = (\ell_1, \ell_2, ..., \ell_{|F_n|}) \cdot y|_{F_n} \leftrightarrow \overline{\mathbf{L}} = (\overline{\ell_1}, \overline{\ell_2}, ..., \overline{\ell_{|F_n|}})$, then set $d(x, y) = 2^{-|F_n|}$. As it was pointed out in [12], the definition of Hausdorff dimension, for the case $\Gamma = \mathbf{Z}^d$, in the setting of symbolic spaces can be reformulated considering cylinders as basic sets and with respect to the above metric, this fact can be easily extended for our more general context. If Z is a closed subshift in Ω^{Γ}, then for any $x|_{F_n} \leftrightarrow \mathbf{L}$, $x \in Z$, is valid $diamC(x|_{F_n}) = diamC(\mathbf{L}) = 2^{-|\mathbf{L}|} = 2^{-|F_n|}$. In this way the Hausdorff measure of a closed subshift $Z \subset \Omega^{\Gamma}$ is defined as

$$\mu_{\alpha}(Z) = \sum_{C(\mathbf{L}),\, \mathbf{L} \in \mathcal{G}} 2^{-|\mathbf{L}|\alpha} \tag{10}$$

and $\dim_H Z = \inf\{\alpha : \mu_{\alpha}(Z) = 0\}$.

We recall now the main grounds of the Lindenstrauss theory for amenable groups. Let us consider a Γ-system (X, Γ), let (F_n) be a tempered sequence in Γ and let F be a compact subset of Γ. Lindenstrauss introduced collections of right translates of sets $F_1, F_2, ..., F_N$, which "almost" cover F. The collection $\overline{\mathcal{F}}$ is specified by subsets A_j of Γ with $F_j A_j \subset F$, $j = 1, 2, ..., N$, and

$$\overline{\mathcal{F}} = \{F_j a : a \in A_j,\ j = 1, 2, ..., N\}. \tag{11}$$

From the collection $\overline{\mathcal{F}}$, we can randomly extract an adequate finite subcollection. More formally this is done in the following way: let $(\Omega, \mathbf{P})$ be a probability space, $\Omega = \{\varpi : \{1, 2, ..., N\} \times \Gamma \to \{0, 1\}\}$ and let

$$\mathcal{F} : \Omega \to \mathcal{P}(\overline{\mathcal{F}}),$$
$$\varpi \mapsto \overline{\mathcal{F}}(\varpi). \tag{12}$$

Define a counting function $\Lambda^{\varpi} : F \to \mathbf{N}$, by

$$\Lambda^{\varpi}(\gamma) = \sum_{B \in \mathcal{F}(\varpi)} I_B(\gamma), \tag{13}$$

with I_B the characteristic function of B. For any $B \in \mathcal{P}(\overline{\mathcal{F}})$, let $\| S \| = \sum_{B \in \mathcal{S}} | B |$.

The covering lemma of Lindenstrauss [7] says that for a given $\delta > 0$ the map $\mathcal{F}$ can be chosen such that

(i) $\mathbf{E}(\Lambda^{\varpi}(\gamma) | \Lambda^{\varpi}(\gamma) \geq 1 |) \leq 1 + \delta,$

(ii) $\mathbf{E}(\| \mathcal{F}(\varpi) \|) \leq h(\delta, C) \left| \bigcup_{j=1}^{N} A_j \right|,$

where $h(\delta, C) = \dfrac{\delta}{1 + C\delta}$ with the constant C for the tempered sequence (F_N) and the expectation value $\mathbf{E}$.

These conditions mean that, on average, the random subcollections are almost disjoint and the sets in $\overline{\mathcal{F}}(\varpi)$ cover "the most" of F. A collection $\overline{\mathcal{F}} \subset \mathcal{P}(\Gamma)$ is δ-disjoint if for any $A \in \overline{\mathcal{F}}$, there is a $A_0 \subset A$ such that $| A_0 | \geq (1 - \delta) | A |$ and $A_0 \cap B_0$ for every $B \neq A \in \overline{\mathcal{F}}$. The covering lemma is proved in [7] mainly by Lemma 2.6 and Corollary 2.7 which express that: if $\overline{\mathcal{F}}$ is a collection of subsets of Γ specified by the translates of a given sequence of subsets of Γ and $F \subset \Gamma$, then there is a subcollection $\mathcal{F}$ of $\overline{\mathcal{F}}$ such that $| \bigcup \mathcal{F} | \geq (1 - \delta) | F |$, where $\bigcup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$.

If **L** is a string associated to a sequence $x|_{F_n}$, then the translate of the string **L** by an element γ is understood as $\gamma\mathbf{L} = \gamma F_n$.

The following fact is used in the next theorem: if Z is a closed subshift in Ω^{Γ}, then Z can be covered by cylinders associated to strings I_ℓ in $\mathcal{G}^*$ [12], $Z \subset \bigcup_{\mathbf{L}\in I_\ell} C(\mathbf{L})$. Let $\mathcal{I} = \bigcup_{\ell=1}^{\infty} I_\ell$, in a similar form as is done in [12]. Then the sets I_ℓ can be chosen such that $\sum_{C(\mathbf{L}),\, \mathbf{L}\in\mathcal{I}} 2^{-|\mathbf{L}|\alpha} < 1$ and

$$\sum_{k=1}^{\infty}\left(\sum_{C(\mathbf{L}),\, \mathbf{L}\in\mathcal{I}} 2^{-|\mathbf{L}|\alpha}\right)^{k} = S < \infty.$$

Theorem 1. *Let Z be a closed subshift in Ω^{Γ}, with Γ a discrete countable amenable group, and let (F_n) be a Følner tempered sequence. Then $h(Z, \Gamma) = \dim_H Z$.*

Proof. The inequality $h(Z, \Gamma) \geq \dim_H Z$ is proved in a similar way as in [12] (we display the proof for completeness). For converse, lines from [12] are followed using the above results from Lindenstrauss theory. In the definition of entropy log with base 2 is used.

Let $\mathcal{C}$ be the partition by cylinders of Z. Then, it can be proved that if $\alpha > h(Z, \Gamma)$, then $\lim_{n\to\infty} \frac{N(\mathcal{C}^{F_n})}{2^{\| F_n \|\alpha}} = 0$. To see this, we have, by the definition of entropy that for any $\varepsilon > 0$, $\alpha > h(Z, \Gamma)$ and for n large enough, $N(\mathcal{C}^{F_n}) < 2^{| F_n |(\alpha-\varepsilon)}$. Therefore $2^{-| F_n |\alpha} N(\mathcal{C}^{F_n}) < 2^{| -F_n |\varepsilon}$ and thus $\lim_{n\to\infty} \frac{N(\mathcal{C}^{F_n})}{2^{-| F_n |\alpha}} = 0$ if $\alpha > h(Z, \Gamma)$. As is pointed out earlier $Z \subset \bigcup_{x\in Z} C(x^{(n)}) = \bigcup_{\substack{\mathbf{L}\in I_\ell \\ x\leftrightarrow\mathbf{L}}} C(\mathbf{L})$.

Recall that $diamC(x|_{F_n}) = diamC(\mathbf{L}) = 2^{-|\mathbf{L}|} = 2^{-|F_n|}$. Since $\lim_{n\to\infty} \frac{N(\mathcal{C}^{F_n})}{2^{-|F_n|\alpha}} = 0$, if $\alpha > h(Z, \Gamma)$, we get $\mu_\alpha(Z) = 0$ for $\alpha > h(Z, \Gamma)$ and then $h(Z, \Gamma) \geq \dim_H Z$.

Let us consider the set F_n of the Følner sequence with n large enough. Let $\overline{\mathcal{F}} = \{\overline{F_1}, \overline{F_2}, ..., \overline{F_N}\}$ be a tempered sequence in Γ, and let $F = F_n$. Then, by the Lindenstrauss theory F can be "almost covered" by the translates of the sets $\overline{F_i}$. Let $\mathcal{F}$ be the subcollection randomly obtained form the translates with almost cover F, i.e. for $\delta > 0$ holds $\bigcup \mathcal{F} \subset F$ and $|\bigcup \mathcal{F}| \geq (1 - \delta)|F|$. Let $U = F - \bigcup \mathcal{F}$, so $|U| = |F| - |\bigcup \mathcal{F}|$ and $|U| <$ $\delta|F|$. Let $G_i = \overline{F_i} a_i$, $i = 1, 2, ..., M$ be translates of the elements of $\overline{\mathcal{F}}$ which form the sequence $\mathcal{F}$ and let $G_i = \{\gamma_1^i, ..., \gamma_{|G_i|}^i\}$. The set F can be described by these translated sets. Let us select one element from each G_i, namely $H = \{\gamma_1, ..., \gamma_M\}$, and let $V = U - H$. The set F, or equivalently, the string $x|_F$, can be characterized by sequences $a_1, ..., a_{[2\delta|F|]} \in \Omega$, $x(\gamma) = a_j$, for any $\gamma \in V$ as well as the translates of elements of $\overline{\mathcal{F}}$. Let α be such that $\mu_\alpha(Z) = 0$. Then, we have $|\mathcal{L}(Z, F)| < 2^{|F|\alpha}|\Omega|^{2\delta|F|}S$ or $|\mathcal{L}(Z, F)|2^{|F|(\alpha+2\delta\log|\Omega|)} < S$. Therefore, since F is a member of the sequence with n large enough, then it is proved that $h(Z, \Gamma) \leq \alpha$ with α such that $\mu_\alpha(Z) = 0$. Hence, $h(Z, \Gamma) \leq \dim_H Z$. □

5. On the Relationship Between Dimension and Kolmogorov Complexity for Amenable Groups

In [12], Simpson proved the equality between Hausdorff dimension of a shift and the Kolmogorov complexity and the growing rate of the Kolmogorov complexity of sequences $x|_{F_n}$, $x \in Z$ for $\mathbf{Z}^d$-actions. By

Z-action groups the equality between complexity and measure-theoretic entropy was established by Brudno [1]. The extension to general amenable discrete groups can be done in a more or less direct way with some modifications that we shall point out. Let us begin with a brief account of Kolmogorov complexity, which is a very important subject in information theory. Informally speaking the Kolmogorov complexity of a sequence in a given description language, is the minimal program that produces this sequence as the output. More formally by a description language, $\mathcal{L}$ may be understood as a map $\mathcal{L} : S \to \mathcal{H},$ where S consists of a set of sequences in, say, two symbols (binary sequence), and S is formed by sequences in n-symbols, for any n. Thus $\mathcal{L}$ makes a correspondence between binary strings and strings of any length. If the description language $\mathcal{L}$ can be described by a Turing machine M says that it is *computable*, i.e. a sequence s is in the domain of $\mathcal{L}$ whenever $M(s)$ is an accepted state by M. A description of a string $t \in \mathcal{H}$ by the language $\mathcal{L}$ is a string $s \in S$ with $\mathcal{L}(s) = t.$ By $|s|,$ we denote the length of the sequence s.

Definition. The *Kolmogorov complexity*, with respect to a language $\mathcal{L}$ of a string s is defined as the minimal length of a description of s, namely

$$K_{\mathcal{L}}(t) = \min\{|s| : \mathcal{L}(s) = t\} \cup \{\infty\},$$

where ∞ is the output for the language when t is not possible.

The invariance theorem establishes the existence of a “universal” language in the following sense, if a string has a description in a given language this description may be used in the universal language except for a constant. More precisely the invariance theorem says that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two description languages, then there is a constant C such that $K_{\mathcal{L}_1}(t) = K_{\mathcal{L}_2}(t) + C,$ for any s and where the constant C depends only on the languages. Thus a universal optimal language can be considered and the complexity can be defined with respect to this language, and hence the language can be omitted in the definition of Kolmogorov complexity.

In the case of subshifts, take $\mathcal{S} = \mathcal{G}^* = \bigcup \mathcal{G}_{|F_n|}$ with $\mathcal{G}_{|F_n|} = \{x|_{F_n} \leftrightarrow \mathbf{L},$

$x \in \Omega^{\Gamma}\}$ and the languages are the allowed sequences of the subshifts. Then in this setting the definition of complexity of a sequence x belonging to a subshift Z can be formulated by

$$K(x) = \limsup_{n\to\infty} \frac{K(x|_{F_n})}{|F_n|}. \tag{14}$$

In [12], it is proved that $K(x) \le h(Z, \Gamma)$ for any $x \in Z$ and $K(x) = h_{\mu}(Z, \Gamma)$ for any μ-a.e. $x \in Z$. Here we display the first inequality and $K(x) \le h_{\mu}(Z, \Gamma)$ for any ergodic Γ-invariant measure μ. The following variational theorem holds.

Theorem [10, 13]**.** *For an amenable action group on a probability space* (X, μ),

$$h(X, \Gamma) = \sup_{\mathcal{M}(X,\Gamma)} \{h_{\mu}(X, \Gamma)\} = \sup_{\mathcal{M}_E(X,\Gamma)} \{h_{\mu}(X, \Gamma)\},$$

where $\mathcal{M}_E(X,\Gamma)$ *is the set of* Γ-*invariant ergodic probability measures on X.*

Therefore, if the supremum is attained at an ergodic measure $\overline{\mu}$, then $K(x) = h(Z, \Gamma) = h_{\overline{\mu}}(Z, \Gamma)$.

Lindenstrauss proved the Shannon-McMillan theorem for amenable groups.

Theorem [7]**.** *Let* Γ *be a discrete amenable group, which acts ergodically on probability space* (X, μ), *and let* $\mathcal{P}$ *be a finite partition of X. If* (F_n) *be a Følner tempered sequence with the property* $\frac{|F_n|}{\log n} \to \infty$ (*increasing property*), *then*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \log \mu(\mathcal{P}^{Fn}(x)) = h_{\mu}(X, \Gamma, \mathcal{P}), \tag{15}$$

for μ-*almost every x, where* $\mathcal{P}^{Fn}(x)$ *is the member of* $\mathcal{P}^{Fn}$ *which contains x* (*the* $\mathcal{P}$-*name of x*).

Now we state the theorem which claims the equality between Kolmogorov complexity and entropy. As we have mentioned the proof follows steps closely to those of [12], with some modifications. For instance, the increasing condition be added.

Theorem 2. *Let* Γ *be a discrete amenable group, and let Z be a closed subshift of* Ω^{Γ}, *let* μ *be a shift invariant ergodic measure on Z. If* (F_n) *is a Følner tempered with the increasing property* $\frac{|F_n|}{\log n} \to \infty$, *then*

$$K(x) = \limsup_{n\to\infty} \frac{K(x|_{F_n})}{|F_n|} \leq h_{\mu}(Z, \Gamma),$$

for μ*-almost every* $x \in Z$, *and*

$$K(x) \geq h_{\mu}(Z, \Gamma),$$

for any $x \in Z$ (*we use in the entropy log in base* 2). *So that* $K(x) = h_{\mu}(Z, \Gamma)$, *for* μ*-almost every* $x \in Z$.

Proof. Let $x \in Z$. Let $\alpha < h_{\mu}(Z, \Gamma)$. Choose $\varepsilon > 0$ be such that $\alpha + \varepsilon < h_{\mu}(Z, \Gamma)$. Using the Shannon-McMillan theorem, we can consider the sets:

$$S_n = \{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\} \cap \{x|_{F_n} : \mu(C(x|_{F_n})) < 2^{-|F_n|(\alpha+\varepsilon)}\}.$$

We have $card\{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\} \leq 2^{|F_n|\alpha}$.

For a set $Y \subset \Omega^{F_n} = \{x|_{F_n} : F_n \to \Omega\}$, let us denote $C(Y) = \{C(y|_{F_n}) : y \in Y\}$. Thus

$$\begin{aligned}\mu(C(S_n)) &\leq \mu(C(\{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\}) \\ &\quad \cap C\{x|_{F_n} : \mu(C(x|_{F_n})) < 2^{-|F_n|(\alpha+\varepsilon)}\}) \\ &\leq 2^{|F_n|\alpha} 2^{-|F_n|(\alpha+\varepsilon)} = 2^{-|F_n|\varepsilon}.\end{aligned}$$

By the increasing property, we have that $\frac{|F_n|}{\log n} > \frac{2}{\varepsilon}$, for n large enough. Thus $2^{-|F_n|\varepsilon} < \frac{1}{n^2}$, and hence $\sum \mu(C(S_n)) < \infty$. Then by the Borel-Cantelli lemma, all μ-a.e. $x \in Z$ do not belong to all the sets $C(\{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\}) \cap C\{x|_{F_n} : \mu(C(x|_{F_n})) < 2^{-|F_n|(\alpha+\varepsilon)}\}$, for large enough n. In particular, $x \notin C(\{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\})$, μ-a.e., for n large. But if $x \notin C(\{x|_{F_n} : K(x|_{F_n}) < |F_n|2\alpha\})$, then $x|_{F_n} \notin \{x|_{F_n} : K(x|_{F_n}) < |F_n|\alpha\}$. Therefore, $K(x|_{F_n}) \geq |F_n|\alpha$ for n large enough and for μ-almost every $x \in Z$. Finally,

$$K(x) \leq h_\mu(Z, \Gamma)$$

for $x \in Z$, μ-a.e.

For the second inequality, let $m \geq n$ and $k \in \mathbf{N}$ such that $km \leq n \leq km + m$. Let us consider a partition of each F_n in r-blocks of measure $|F_m|$, hence $|F_n| \geq r|F_m|$, and $|\mathcal{L}(Z, F_n)| \leq r|\mathcal{L}(Z, F_m)|$. Thus, there is a constant C such that $K(x|_{F_n}) \leq r \log|\mathcal{L}(Z, F_m)| + \log n^2 + C$, and so for any $x \in Z$,

$$\limsup_{n\to\infty} \frac{K(x|_{F_n})}{|F_n|} \leq \limsup_{m\to\infty} \frac{r \log|\mathcal{L}(Z, F_m)|}{r|F_m|} + \frac{\log|F_n|^2 + C}{r|F_m|}$$

$$= \limsup_{m\to\infty} \frac{\log|\mathcal{L}(Z, F_m)|}{|F_m|} = h_\mu(Z, \Gamma). \qquad \square$$

This result can be extended to an action on a compact topological space X, defining the complexity of a point x via the coding of the orbit of x by its name with respect to a given partition and a Følner sequence. This was done in [4] for $\mathbf{Z}$-actions. Let $\mathcal{P} = \{P_1, ..., P_k\}$ be finite partition of X and let (F_n) be a Følner sequence. Recall that $\mathcal{P}$-name of x of length $|F_n|$ is the member of $\mathcal{P}^{Fn}$ which contains x, i.e. the $\mathcal{P}^{Fn}$-name of x will be a string $\mathbf{L}_{\mathcal{P}, F_n}(x) = (\ell_1, \ldots, \ell_{|F_n|})$ such that $\gamma x \in P_{\ell_i}$, when γ varies in F_n. In this way, we set

$$K(x, \mathcal{P}, F_n) = K(\mathbf{L}_{\mathcal{P}, F_n}(x)) \tag{16}$$

and

$$K(x, \mathcal{P}) = \limsup_{n \to \infty} \frac{K(\mathbf{L}_{\mathcal{P}, F_n}(x))}{|F_n|}. \tag{17}$$

Now we can state the above theorem for more general case.

Theorem 3. *Let Γ be a discrete amenable group acting on a topological compact space X, and let μ be a Γ-invariant ergodic measure on X. Let (F_n) be a Følner tempered with the increasing property $\frac{|F_n|}{\log n} \to \infty$. Then*

$$K(x, \mathcal{P}) \leq h_\mu(X, \Gamma, \mathcal{P}),$$

for any μ-a.e. $x \in X$, and

$$K(x, \mathcal{P}) \geq h_\mu(X, \Gamma, \mathcal{P}),$$

for any $x \in X$.

The proof is similar to that of the theorem for subshifts with some modifications, for instance the sets S_n must be defined as

$$S_n = \{x|_{F_n} : K((\mathbf{L}_{\mathcal{P}, F_n}(x))) < |F_n| \alpha\}$$

$$\cap \{x|_{F_n} : \mu(\mathcal{P}^{F_n}(x)) < 2^{-|F_n|(\alpha + \varepsilon)}\},$$

with $\alpha < h_\mu(X, \Gamma, \mathcal{P})$.

5.1. Scaled orbit complexity

A version of complexity was introduced by Galotolo in [4], for the classical case of **Z**-actions. This quantity is computed with respect to so called computable structure and measures the amount of information needed to follow the orbit of a point in a separable space X with determined precision. Thus this version gives the length of the smallest sequence (program) to approximate the orbit of a point. Recall that the Kolmogorov complexity of a point in an abstract space is the minimal program to produce the name of a point, with respect to a given partition, to output. Finite strings can be interpreted as points in an abstract metric space X by means of an "interpretation function" C which assigns to a finite string a point in the space X and whose image is dense in X. A point which is the dense image of interpretation map is called *ideal*, and an interpretation is computable when the distance between two ideal points can be approximated by a recursive algorithm. This concept will be useful to analyze the scaled case for a complexity indicator.

Let Γ be an amenable discrete group acting on a separable space (X, d), and let S be the set of finite binary strings. As we said a *computable interpretation* on (X, d) is a map $C : S \to X$ such that $C(S)$ is dense in X and if $x_1 = C(s_1)$, $x_2 = C(s_2)$ are two ideal points, then there is a recursive algorithm function F defined on $S \times S \times \mathbf{N}$ such that, for any s_1, s_2, n, $|d(x_1, x_2) - F(s_1, s_2, n)| < \delta(n)$, with $\delta(n) \to_n 0$. Two interpretations C_1, C_2 are equivalent when $d(C_1(s_1), C_2(s_2))$ can be approximated by a recursive algorithm $F = F(s_1, s_2, n)$, for any n. A *computable structure* is an equivalence class $\mathcal{C}$ of computable structures. Now, to define the orbit complexity a separable space (X, d) endowed with a computable structure $\mathcal{C}$ is considered which allows to model sequences as points of X.

If (X, Γ) is a Γ-system where X is endowed with a computable structure $\mathcal{C}$, then a Γ-action is a *morphism of computable structures* if for any $C \in \mathcal{C}$, $s_1, s_2 \in S$, there is a recursive sequence $\{F_n(s_1, s_2)\}$ such that

$$|\, d(\gamma C(s_1), C(s_2)) - F_n(s_1, s_2) < 1/2^n \,|, \text{ for any } \gamma \in \Gamma.$$

An interesting property of morphisms is that for any ideal point $x = C(s) \in X$, there exists an algorithm which allows to find an ideal point $y \in X$ as close to γx as we wish [4]. The description of the process to check this fact is as follows: let $x = C(s)$ be a given ideal point, then the set $\{C(z) : z \in S\}$ is dense in *X*, and therefore if we calculate $F_{n+1}(s, t)$ fixing *s* and varying *t*, a string t_0 can be obtained such that $F_{n+1}(s, t_0) < 1/2^{n+1}$. Setting $y = C(t_0)$, we obtain the desired point.

If (F_n) is a Følner sequence, then the (F_n)-orbit of a point *x* is the $|F_n|$-sequence $o_{F_n}(x) = \{\gamma x\}_{\gamma \in F_n}$. A consequence of the above fact is that the orbit of ideal points can be approximated within a given accuracy. For an ideal point $x = C(s)$, $n \in \mathbf{N}$ and $\varepsilon > 0$, if $F_n = \{\gamma_1^n, \gamma_2^n, ..., \gamma_{|F_n|}^n\}$, then there are strings, $s_i = s_i(s, F_n, \varepsilon)$, $i = 1, 2, ..., |F_n|$ such that $d(\gamma_1^n x, C(s_i)) < \varepsilon$, $i = 1, 2, ..., |F_n|$.

To define the orbit complexity with respect to computable structure approximations of orbits must be considered, i.e. strings of strings be modelled in space *X* as orbits of points of *X*. To do this let *S* be the set of finite binary strings and let $Q : S \to S^*$ be a recursive function, where S^* is the set of finite sequences in elements of *S*. To assign a string in *S* to a sequence of points of *X*, consider a computable interpretation *C* and its natural extension to $C : S^* \to X^*$ (with X^* the set of finite sequences in points of *X*). Then define the map

$$H : S \to X^*$$

$$H(s) = C(Q(s)).$$

Definition. If *s* is a $|F_n|$-sequence in *X*, then write $H(s) = (H_\gamma(s))_{\gamma \in F_n}$. The *scaled orbit complexity* of the (F_n)-orbit of a point *x* with respect to a computable structure *C* and a scale sequence **a** is defined as

$$\mathcal{K}(x, C, \varepsilon, \mathbf{a}) = \limsup_{n\to\infty} \frac{\mathcal{K}(x, F_n, C, \varepsilon)}{\mathbf{a}(|F_n|)}, \tag{18}$$

where

$$\mathcal{K}(x, F_n, C, \varepsilon) = \min\{|s| : \min_{\gamma\in F_n}\{d(H_\gamma(s), \gamma x) < \varepsilon\}\}, \tag{19}$$

and set

$$\mathcal{K}(x, C, \mathbf{a}) = \sup_{\varepsilon>0} \mathcal{K}(x, C, \varepsilon, \mathbf{a}). \tag{20}$$

Recall that the upper and lower local scaled entropies, in the style of Brin-Katok, are defined by the rate of the measure of the dynamical balls $B(x, F_n, \varepsilon) = \{y : d(\gamma x, \gamma y) < \varepsilon$, for any $\gamma \in F_n\}$. Following [4], we propose indicators based in the rate of the biggest and smallest balls containing $B(x, F_n, \varepsilon)$ and be contained in $B(x, F_n, \varepsilon)$. By $B_r(x)$ is denoted the ball of centre x and radius r.

Definition. Let $r(x, F_n, \varepsilon) = \sup\{r : B_r(x) \subset B(x, F_n, \varepsilon)\}$ and $R(x, F_n, \varepsilon) = \inf\{r : B_r(x) \supset B(x, F_n, \varepsilon)\}$. Consider

$$r(x, \varepsilon, \mathbf{a}) = \limsup_{n\to\infty} \frac{-\log r(x, F_n, \varepsilon)}{\mathbf{a}(|F_n|)}$$

and

$$R(x, \varepsilon, \mathbf{a}) = \liminf_{n\to\infty} \frac{-\log R(x, F_n, \varepsilon)}{\mathbf{a}(|F_n|)}.$$

Then the *maximal and minimal* initial condition sensitivity at the point x are defined by $r(x, \mathbf{a}) = \sup_{\varepsilon>0} r(x, \varepsilon, \mathbf{a})$ and $R(x, \mathbf{a}) = \sup_{\varepsilon>0} R(x, \varepsilon, \mathbf{a})$, respectively.

Definition. Let (X, Γ) be a Γ-system where X is equipped with a computable structure $\mathcal{C}$, and let $C \in \mathcal{C}$, $x \in X$, $\varepsilon > 0$. Then the *information contained* in x, with accuracy ε, is given by

$$S(C, x, \varepsilon) = \min\{|s| : d(C(s), x) < \varepsilon\}.$$

Let

$$\overline{S}(C, x, \varepsilon) = \limsup_{n\to\infty} \frac{S(C, x, r(x, F_n, \varepsilon))}{\log r(x, F_n, \varepsilon)},$$

$$\underline{S}(C, x, \varepsilon) = \limsup_{n\to n} \frac{S(C, x, R(x, F_n, \varepsilon))}{\log R(x, F_n, \varepsilon)}$$

and

$$\overline{S}(C, x) = \lim_{\varepsilon\to 0} \overline{S}(C, x, \varepsilon),$$

$$\underline{S}(C, x) = \lim_{\varepsilon\to 0} \underline{S}(C, x).$$

Then, we have

Proposition 4. *Let* (X, Γ) *be a* Γ*-system an* $\mathcal{C}$ *be a computable structure on X. If a is a scale sequence such that* $\mathbf{a}(|F_n|)$ *has the same asymptotic behavior than* $\frac{\log|F_n|}{\mathbf{a}(|F_n|)} \to 0$, *then for any* $x \in X$ *and* $C \in \mathcal{C}$,

$$\mathcal{K}(x, C, \mathbf{a}) \le \overline{S}(C, x) r(x, \mathbf{a}), \tag{21}$$

and if $\mathbf{a}(|F_n|) = \log|F_n|$, *then*

$$\mathcal{K}(x, C, \mathbf{a}) \le \overline{S}(C, x) r(x, \mathbf{a}) + 1. \tag{22}$$

Proof. Let s_0 be a string in *S*. Then, we saw that the orbit of $C(s_0)$ can be approximated with accuracy ε, by the computable values of strings $s_i = s_i(s, F_n, \varepsilon)$, $i = 1, 2, ..., |F_n|$. Let $s \in S$ such that $Q(s) = (s_0, s_1, ..., s_{|F_n|})$. Then $|s| \le C + \log|F_n| + |s_0|$. If s_0 is considered as minimal for $d(C(s_0), x) < r(x, F_n, \varepsilon)$, then $|s_0| = S(C, x, r(x, F_n, \varepsilon))$. Therefore, for $\varepsilon > 0$, the string s satisfies $d(H_\gamma(s), \gamma x) < 2\varepsilon$ and $|s| \le D + \log|F_n| +$

$S(C, x, r(x, F_n, \varepsilon))$, for some constant D. Thus $\mathcal{K}(x, F_n, C, \varepsilon) \le D + \log|F_n| + S(C, x, r(x, F_n, \varepsilon))$ and then $\frac{\mathcal{K}(x, F_n, C, \varepsilon)}{\mathbf{a}(|F_n|)} \le \frac{D}{\mathbf{a}(|F_n|)} + \frac{\log|F_n|}{\mathbf{a}(|F_n|)}$ $+ \frac{S(C, x, r(x, F_n, \varepsilon))(-\log r(x, F_n, \varepsilon))}{-\log r(x, F_n, \varepsilon)\mathbf{a}(|F_n|)}$.

Letting $n \to \infty$, $\varepsilon \to 0$, with $\mathbf{a}$ such that $\frac{\log|F_n|}{\mathbf{a}(|F_n|)} \to 0$, we get equation (21) and for $\mathbf{a}(|F_n|) = \log|F_n|$ is obtained equation (22). □

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