Economic MPC for a Changing Economic Criterion for Linear Systems

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***Abstract*—Economic Model Predictive Controllers, consisting of an economic criterion as stage cost for the dynamic regulation problem, have shown to improve the economic performance of the controlled plant, as well as to ensure stability of the economic setpoint. However, throughout the operation of the plant, economic criteria are usually subject to frequent changes, due to variations of prices, costs, production demand, market fluctuations, reconciled data, disturbances, etc. A different economic criterion determines a change of the optimal operation point and this may cause a loss of feasibility and/or stability. In this paper a stabilizing economic MPC for changing economic criterion for linear prediction models is presented. The proposed controller always ensures feasibility for any given economic criterion, thanks to the particular choice of the terminal ingredients. Asymptotic stability is also proved, providing a Lyapunov function.**

***Index Terms*—Asymptotic stability, changing economic criterion, economic cost function, model predictive control (MPC), real time optimizer (RTO).**

I. Introduction

THE main goal of advanced control strategies is to operate the plants as close as possible to the economically optimal operation point, while ensuring stability. In the process industries, this objective is achieved by means of a hierarchical control structure [11], [23], [26], [32]: at the top of this structure, an economic scheduler and planner decides what, when and how much the plant has to produce, taking into account information from the market and from the plant. The output of this layer are production goals, prices, economic cost functions and constraints which are sent to a real time optimizer (RTO). The RTO is a model-based system, operated in closed loop. It implements the economic decision in real time, performing a static optimization, and providing setpoints to the advanced control level. It employs a stationary complex model of the plant and for this reason it works on a time-scale of hours or day. The setpoints calculated by the RTO are sent to the advanced control level, where an advanced control strategy—usually model predictive control (MPC) [9], [22], [31]—calculates the optimal control action to be sent to the plant, in order to regulate it as close as possible to the setpoint, taking into account a dynamic model of the plant, constraints, and stability requirements.

The hierarchical control structure supposes a time-scale separation between the RTO and MPC layers. This separation has two main consequences on the economic performance of the plant.

The first one is that, the economic setpoint calculated by RTO may be inconsistent or unreachable with respect to the dynamic layer [18]. A way to avoid this problem is to add a new optimization level in between of RTO and MPC, referred as the steady state target optimizer (SSTO). The SSTO calculates the steady state to which the system has to be stabilized, solving a linear or quadratic programming and taking into account information from the RTO [21], [24], [28], [36].

In [12], [13], [19] an MPC that integrates the SSTO into the same MPC layer, is presented. This controller ensures that under any change of the economic setpoint, the closed-loop system maintains the feasibility of the controller and ensures local optimality. Similar strategies are also presented in [15].

The second consequence of the above mentioned time-scale separation, is that the MPC control law is designed to ensure asymptotic tracking of the setpoint, without taking into account the issue of transient costs [6]. This way to operate is practically optimal when the setpoint does not change with respect to the dynamic of the system. However, in some industrial applications, the cost in the transient is more significant than the cost at the steady state. This happens when the economic criterion is subject to frequent changes. Hence, it becomes very important to optimize the cost of the entire trajectory, not only at the steady state. All the above motivated in the last years the interest in Economic MPC [29].

A first approach in this direction is represented by the Dynamic Real Time Optimizer (D-RTO) [7], [18], [35], which solves a dynamic economic optimization and delivers target trajectories (instead of target steady state) to the MPC layer. A second approach is represented by the one-layer MPC, which integrates the RTO economic cost function as part of the MPC cost function [1], [2], [37]. This converts the economic objective into a process control objective. Another method is to provide to a setpoint-tracking MPC, an unreachable but economically optimal setpoint [30].

An improved approach is represented by the Economic MPC, which considers the nonlinear economic cost of the RTO, as the stage cost for the dynamic regulation problem. This method has been widely studied in the last few years, and Lyapunov stability has been proved in [7] for cyclic process, in [17] by

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means of a dual-mode MPC, and in [10], under the assumption of strong duality. This strong duality assumption was then relaxed to a dissipativity assumption for the cases of both terminal equality constraint [6] and terminal cost [3], [16]. Stability results, resorting to suboptimal MPC [25], are provided in [5]. A drawback of these formulations might be represented by feedback delays and consequent loss of performance due to on-line optimization of a (strongly) nonlinear cost. However, thanks to novel techniques, like fast-MPC [34], Advanced-Step MPC [38] or simultaneous approach [4], [8], the computational burden seems not to be an issue anymore.

None of the above mentioned strategies takes in account the issue of possible changes in the economic criterion, that is the economic cost to be optimized. This economic criterion might vary during the operation of a plant, due to both: i) market fluctuations, which causes changes in the cost function and in the prices that parameterized this function [27]; ii) variations in disturbances estimation or constraints, due to data reconciliation algorithms. If the economic criterion changes, the economically optimal setpoint, to which the system has to be driven by the controller, may be different. Hence, due to the stability requirements, the feasibility of the controller may be lost.

The main objective of this paper is to present a novel economic MPC formulation, suitable for changing economic criteria. This controller extends the economic MPC [10], following ideas of MPC for tracking [13], [19], by means of a slightly modified cost function and a relaxed terminal constraint, which requieres the terminal state to be *any* admissible equilibrium point. The resulting controller ensures the following properties, interesting from both a theoretical and a practical point of view: i) it guarantees feasibility under any change of the economic criterion; ii) it ensures economic optimality; iii) it provides a larger domain of attraction than [10]. Moreover, asymptotic stability to the economically optimal steady state is proved resorting to a Lyapunov function.

The paper is organized as follows. Section II presents some preliminary notation. In Section III the problem is stated and in Section IV the economic MPC is briefly introduced. In Section V the new economic MPC for a changing economic criterion is proposed. In Section VI the local economic optimality property is described. Finally, illustrative examples and conclusions of this study are provided in Sections VII and VIII.

II. Notation

The symbols $\mathbb{I}$ and $\mathbb{R}$ denote the sets of integers and real numbers, respectively. $\mathbb{I}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of nonnegative integers and reals, respectively, while $\mathbb{I}_{0:N}$ denotes the set $\{0, 1, \ldots, N\}$. Given a vector $x \in \mathbb{R}^n$, x' denotes the transpose vector, and $|x|$ denotes the Euclidean norm. Given a sequence $\mathbf{u}$, $u(j)$, $j \in \mathbb{I}_{\geq 0}$, denotes the j-th element of the sequence. Given two sets, $\mathcal{X}_1 \subseteq \mathbb{R}^n$ and $\mathcal{X}_2 \subseteq \mathbb{R}^n$ containing the origin, define $\mathcal{X}_1 \setminus \mathcal{X}_2 = \{x \mid x \in \mathcal{X}_1 \text{ and } x \notin \mathcal{X}_2\}$, and $\delta\mathcal{X}_1 = \{\delta x \mid x \in \mathcal{X}_1\}$, with $\delta \in (0, 1)$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\vartheta : \mathbb{R}_{\geq 0} \times \mathbb{I}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if it is continuous, and if, for each $k \geq 0$, $\vartheta(\cdot, k)$ is a $\mathcal{K}$-function and for each $r \geq 0$, $\vartheta(r, \cdot)$ is nonincreasing and satisfies $\lim_{k\to\infty} \vartheta(r, k) = 0$. Given an optimization problem $\min_x V(x)$, the optimal value of the cost function is noted as $V^0(x)$, and the minimizer of $V(x)$ as x^0.

III. Problem Statement

Consider a system described by a discrete-time linear time-invariant model

$$x^+ = Ax + Bu \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector and x^+ is the successor state. The solution of this system for a given sequence of control inputs $\mathbf{u}$ and initial state x is denoted as $x(j) = \phi(j; x, \mathbf{u})$, $j \in \mathbb{I}_{\geq 0}$, where $x = \phi(0; x, \mathbf{u})$. The state of the system and the control input applied at sampling time k are denoted as $x(k)$ and $u(k)$ respectively.

The system is subject to hard constraints on state and input

$$x(k) \in X, \quad u(k) \in U \tag{2}$$

for all $k \geq 0$, where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ are compact sets.

It is assumed that the following assumption holds.

Assumption 1: The pair (A,B) is controllable and the state is measured at each sampling time. □

The steady state and input of the plant (x_s, u_s) are such that (1) is fulfilled, i.e., $x_s = Ax_s + Bu_s$.

We define the set of admissible equilibrium states as

$$\mathcal{Z}_s = \{(x, u) \in \delta(X \times U) \mid x = Ax + Bu\} \tag{3}$$

$$\mathcal{X}_s = \{x \in X \mid \exists\, u \in U \text{ such that } (x, u) \in \mathcal{Z}_s\} \tag{4}$$

where $\delta \in (0, 1)$ is a constant arbitrarily close to 1. Notice that $\mathcal{X}_s$ is the projection of $\mathcal{Z}_s$ onto X.

Definition 1: The economic performance measure is given by the function

$$\ell_{eco}(x, u, p) \tag{5}$$

where x and u are the state and the input of the system, and p is a vector of parameters which takes into account prices, costs, production goals, etc. The set of parameters may change throughout the evolution of the plant. □

Function (5) represents the economic criterion to be optimized, and it may change according to the market, the plant scheduling, or the data reconciliation tasks.

The optimal operation point that stabilizes the plant, is the steady state provided by the RTO, which satisfies the following definition:

Definition 2: The optimal steady state and input, (x_s, u_s), satisfy

$$\begin{aligned}(x_s, u_s) = \arg\min_{x,u}\ & \ell_{eco}(x, u, p)\\ s.t.\ & x = Ax + Bu\\ & x \in X, \quad u \in U\end{aligned} \tag{6}$$

and is assumed to be unique. □

Remark 1: Notice that the optimal steady state and input depends on the value of p, that is $(x_s(p), u_s(p))$. However, for the sake of clarity, in the rest of the paper we will use the notation (x_s, u_s). □

Assumption 2: The cost $\ell_{eco}(x,u,p)$ is locally Lipschitz continuous in (x_s, u_s); that is there exists a constant $L > 0$ such that

$$|\ell_{eco}(x,u,p) - \ell_{eco}(x_s,u_s,p)| \leq L\,|(x,u) - (x_s,u_s)|$$

for all p and all $(x,u) \in X \times U$ such that $|x - x_s| \leq \varepsilon$ and $|u - u_s| \leq \varepsilon, \varepsilon > 0$. □

The controller design problem consists in deriving a control law that minimizes the given economic performance cost index

$$\sum_{j=0}^{N-1} \ell_{eco}\left(x(j), u(j), p\right) \tag{7}$$

taking into account that the economic function may change.

A change of the economic function causes also a change of the steady state, defined in Definition 2, to which the system has to be stabilized. The challenge of the proposed control problem is then to design an economic MPC that ensures feasibility, convergence to the optimal steady state and asymptotic stability, under any change of the economic cost function.

IV. Economic MPC

The economic MPC cost function [3], [6], [10] is given by

$$V_N^e(x,p;\mathbf{u}) = \sum_{j=0}^{N-1} \ell_{eco}\left(x(j), u(j), p\right). \tag{8}$$

The economic MPC control law is derived from the solution of the optimization problem $P_N^e(x,p)$

$$\begin{aligned} \min_{\mathbf{u}} \quad & V_N^e(x,p;\mathbf{u}) \\ s.t. \quad & \\ & x(0) = x, \\ & x(j+1) = Ax(j) + Bu(j), \\ & x(j) \in X,\ u(j) \in U, \qquad j \in \mathbb{I}_{0:N-1} \\ & x(N) = x_s \end{aligned} \tag{9}$$

and it is given by the receding horizon application of the optimal solution, $\kappa_N^e(x,p) = u^0(0;x)$. The optimal value of the cost function is noted as $V_N^{e0}(x,p)$.

Define the following set:

$$Z_N(w) = \big\{(x,\mathbf{u}) \in X \times U^N \mid x(j) \in X,\ u(j) \in U,\ j \in \mathbb{I}_{0:N-1},\ x(N) = w\big\}$$

for any $w \in \mathcal{X}_s$, where $x(j) = \phi(j;x,\mathbf{u})$. Then, the feasible region of the optimization problem is given by

$$\mathcal{X}_N^e = \{x \in X \mid \exists (x,\mathbf{u}) \in \mathcal{Z}_N(w),\ \text{for } w = x_s\}.$$

The standard Lyapunov arguments to prove asymptotic stability of MPC cannot be directly used in this case because the optimal cost is not necessarily decreasing along the closed-loop trajectory. In [10], in order to find a suitable Lyapunov function, the following assumption is made:

Assumption 3 (Strong Duality of the Steady-State Problem): Let $L_r(x,u,p)$ be the *rotated* stage cost function given by

$$L_r(x,u,p) = \ell_{eco}(x,u,p) + \lambda'(x - (Ax + Bu)) - \ell_{eco}(x_s,u_s,p) \tag{10}$$

where λ is a multiplier that ensures that the rotated cost exhibits a unique minimum at (x_s,u_s) for all $x \in X,\ u \in U$. Then there exist two $\mathcal{K}$-functions ρ_1 and ρ_2 such that $L_r(x,u,p) \geq \rho_1(|x - x_s|) + \rho_2(|u - u_s|)$. □

If we define the following cost function:

$$\tilde{V}_N^e(x,p;\mathbf{u}) = \sum_{j=0}^{N-1} L_r\left(x(j), u(j), p\right) \tag{11}$$

then the optimization problem consisting of minimizing (11) subject to the same constraints as in (9), delivers the same optimal sequence as (9), and also the optimal cost function is a Lyapunov function [10].

Remark 2 (Convex Problems): As remarked in [10], Assumption 3 is always satisfied in case of linear control systems, if $\ell_{eco}(x,u,p)$ is strictly convex in (x,u), and the steady-state problem (6) is feasible and satisfies a Slater condition (if the constraints are linear, then a Slater condition is not necessary). In general, this assumption is not easy to be satisfied. In [6], strong duality is significantly relaxed by means of a dissipativity assumption. □

V. Economic MPC for a Changing Economic Criterion

When the parameter p in (5) changes, the economic objective of the controller also changes as well as the optimal admissible steady state (x_s,u_s) (to which the system should be steered by the controller). This change may cause a loss of feasibility of the controller.

The loss of feasibility is due to the terminal equality constraint $x(N) = x_s$ imposed to problem (9). Following the same idea as in [13], [19], in this paper we propose a new formulation in which this terminal constraint is relaxed in such a way that $x(N) = x(N-1)$, that is the pair $(x(N-1), u(N-1)) \in \mathcal{Z}_s$. This implies that the terminal constraint can be *any* equilibrium point.

First, a modified cost function is proposed

$$\ell_{eco}\left(x(j) - x(N-1) + x_s, u(j) - u(N-1) + u_s, p\right). \tag{12}$$

Moreover, in order to ensure that $x(N-1)$ converges to x_s, the so-called offset cost function is added to this problem and it is defined as follows:

Definition 3: Let $V_O(x,u)$ be a positive definite convex function such that the unique minimizer of

$$\min_{(x,u)\in\mathcal{Z}_s} V_O(x,u) \tag{13}$$

is (x_s,u_s). □

Assumption 4: There exist a positive constant γ such that

$$V_O(x,u) - V_O(x_s,u_s) \geq \gamma |x - x_s|. \tag{14}$$

□

The economic MPC for a changing economic criterion that we propose in this paper has then the following cost function:

$$V_N(x,p;\mathbf{u}) = \sum_{j=0}^{N-1} \ell_{eco}\left(x(j) - x(N-1) + x_s,\right.$$
$$\left. u(j)-u(N-1)+u_s, p\right) + V_O\left(x(N-1), u(N-1)\right). \tag{15}$$

For any current state x, the optimization problem $P_N(x,p)$ to be solved is given by

$$\begin{aligned}
\min_{\mathbf{u}} \quad & V_N(x,p;\mathbf{u}) \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j), \\
& x(j) \in X,\ u(j) \in U, \quad j \in \mathbb{I}_{0:N-1} \\
& (x(N-1), u(N-1)) \in \mathcal{Z}_s.
\end{aligned} \tag{16}$$

The control law, in the receding horizon fashion, is given by $\kappa_N(x,p) = u^0(0;x)$.

Remark 3: Notice that the pair $(x(N-1), u(N-1))$ defines an admissible equilibrium point, such that $x(N) = x(N-1) = Ax(N-1) + Bu(N-1) \in \mathcal{X}_s$. □

Remark 4: The constraint $(x(N-1), u(N-1)) \in \mathcal{Z}_s$ is equivalent to imposing $x(N) = x(N-1)$ and is actually removing a degree of freedom to the controller. If we add the constraint $x(N) = x_s$ to (16), the optimization problem is the same as (9). Also notice that, since the cost function (15) depends on the optimal steady state (x_s, u_s), we need to solve first the steady state optimization problem given in Definition 2 whenever p changes, and then the economic MPC for a changing economic criterion problem, (16). □

The feasible region of problem (16) is a compact set given by

$$\mathcal{X}_N = \left\{ x \in X \mid \exists (x,\mathbf{u}) \in \mathcal{Z}_N(w), \text{ for } w \in \mathcal{X}_s \right\}.$$

As we remarked before, since by imposing $x(N) = x(N-1)$ we lose a degree of freedom, the set $\mathcal{X}_N$ is given by $N-1$ admissible inputs. So it can be compared with $\mathcal{X}^e_{N-1}$.

Since $\{x_s\} \subset \mathcal{X}_s$, we have that $\mathcal{X}^e_{N-1} \subset \mathcal{X}_N$.

The set $\mathcal{X}_N$ is a feasible set of initial x such that one can reach *any* feasible steady state with $N-1$ admissible inputs.

The set $\mathcal{X}^e_{N-1}$ is a feasible set of initial x such that one can reach *the optimal* steady state with $N-1$ admissible inputs.

Then, the set $\mathcal{X}_N$ is larger, and, in some applications much larger, than $\mathcal{X}^e_{N-1}$.

Remark 5: The result presented in [3], that is the use of a terminal cost function and a terminal inequality constraint, provides a larger feasible set than $\mathcal{X}^e_{N-1}$. However, the set $\mathcal{X}_N$ may still be larger, since the proposed terminal constraint spans the entire steady state manifold. Moreover, this set may be enlarged by extending the proposed formulation, considering a terminal cost as in [3] and a terminal inequality constraint as in [19]. □

A) Asymptotic Stability of the Proposed Controller: Let us rewrite (12) in this way

$$\ell_t(z,v) = \ell_{eco}(z + x_s, v + u_s, p). \tag{17}$$

In order to prove stability of the proposed controller, following [10] we introduce a *rotated* cost function $L_t(z,v,p) = L_r(z + x_s, v + u_s, p)$, where $L_r(x,u,p)$ is defined in (10). This function satisfies the following properties:

Property 1:

1) $L_t(x - x_s, u - u_s, p) = L_r(x,u,p)$
2) $L_t(0,0,p) = L_r(x_s,u_s,p) = 0$
3) $L_t(z,v,p) \geq \alpha_1(|z|) + \alpha_2(|v|)$ for certain $\mathcal{K}$-functions α_1 and α_2. □

Define also, the *rotated* offset cost function $\tilde{V}_O(x,u)$ as follows:

Definition 4: The *rotated* offset cost function is given by

$$\tilde{V}_O(x,u) = V_O(x,u) + \lambda'(x - x_s) - V_O(x_s,u_s) \tag{18}$$

where λ is the same multiplier as in (10). □

Notice that, since $V_O(x,u)$ and $\lambda'(x - x_s)$ are convex, then $\tilde{V}_O(x,u)$ is also convex.

Hence, we can define an *auxiliary* optimization problem given the *auxiliary* cost function

$$\tilde{V}_N(x,p;\mathbf{u}) = \sum_{j=0}^{N-1} L_t\left(x(j) - x(N-1), u(j) - u(N-1), p\right)$$
$$+ \tilde{V}_O\left(x(N-1), u(N-1)\right). \tag{19}$$

Lemma 1: The *auxiliary* optimization problem $\tilde{P}_N(x,p)$ given by

$$\begin{aligned}
\min_{\mathbf{u}} \quad & \tilde{V}_N(x,p;\mathbf{u}) \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j), \\
& x(j) \in X,\ u(j) \in U, \quad j \in \mathbb{I}_{0:N-1} \\
& (x(N-1), u(N-1)) \in \mathcal{Z}_s.
\end{aligned} \tag{20}$$

delivers the same optimal control sequence as problem (16).

The proof of this lemma is given in the Appendix A.

Assumption 5: The prediction horizon N is such that

$$rank(Co_N) \geq n$$

where $Co_N = [A^{N-1}B \ldots AB\ B]$ is the N-controllability matrix of system (A,B). Moreover, there exists a control gain K_{db}, such that $A_K = A + BK_{db}$ has null eigenvalues. □

Let us define the following constant:

$$\gamma_0 = L\left(1 + |K_{db}|\right) \frac{1 - |A_K|^N}{1 - |A_K|}. \tag{21}$$

The following theorem is the main result of this paper. It establishes asymptotic stability of the closed-loop system under the proposed economic MPC for a changing economic criterion.

Theorem 1: Consider system (1) subject to (2). Consider that Assumptions 1–5 hold. Assume that (x_s, u_s) lies in the interior of $\mathcal{Z}_s$ and $\mathcal{X}_N$ is compact. Then, for all $\gamma > \gamma_0$, x_s is an asymptotically stable equilibrium point for the closed-loop system $x^+ = Ax + B\kappa_N(x,p)$ and its domain of attraction is $\mathcal{X}_N$.

Proof: Consider $x \in \mathcal{X}_N$ at time k, then the optimal cost function is given by $\tilde{V}_N^0(x,p) = \tilde{V}_N(x,p;\mathbf{u}^0(x))$, where $\mathbf{u}^0(x) = \{u^0(0), u^0(1), \ldots, u^0(N-1)\}$ defines the optimal solution to problem (20), and $\kappa_N(x,p) = u^0(0)$.

Define the function $J(x,p) = \tilde{V}_N^0(x,p) - \tilde{V}_O(x^0(N-1), u^0(N-1))$. Define also $e(x) = x - x^0(N-1)$. Notice that $J(x,p)$ is defined on $\mathcal{X}_N$, and moreover: due to Property 1, $J(x,p) \geq \alpha_1(|e(x)|)$, for all $x \in \mathcal{X}_N$; due to Lemma 2 (Appendix B), given the successor state $x^+ = \phi(k+1; x, \mathbf{u}^0(x)) = Ax + B\kappa_N(x,p)$, we have that $J(x^+,p) - J(x,p) \leq -\alpha_1(|e(x)|)$, for all $x \in \mathcal{X}_N$.

From Lemma 6 (Appendix B), it follows that:

$$\alpha_1\left(|e(x)|\right) \geq \alpha_1\left(\alpha_e\left(|x - x_s|\right)\right) = \alpha_J\left(|x - x_s|\right)$$

where a α_J is $\mathcal{K}$-function. Then, we can conclude that:

i) $J(x,p) \geq \alpha_J(|x - x_s|)$, for all $x \in \mathcal{X}_N$.
ii) $J(x^+,p) - J(x,p) \leq -\alpha_J(|x - x_s|)$, for all $x \in \mathcal{X}_N$.
iii) Since $\mathcal{X}_N$ is compact, $J(x_s,p) = 0$, and $J(x,p)$ is continuous in $x = x_s$, then there exists a $\mathcal{K}$-function β_J such that $J(x,p) \leq \beta_J(|x - x_s|)$, for all $x \in \mathcal{X}_N$, [31].

Hence $J(x,p)$ is a Lyapunov function and x_s is an asymptotically stable equilibrium point for the closed-loop system $x^+ = Ax + B\kappa_N(x,p)$, that is, there exists a $\mathcal{KL}$-function ϑ such that

$$|x(k) - x_s| \leq \vartheta\left(|x(0) - x_s|, k\right)$$

for all $x(0) \in \mathcal{X}_N$. ■

Remark 6: Notice that the stability of the closed-loop system is uniform with respect to p. In fact, the existence of functions α_J and β_J can be ensured for any value of p. □

Property 2 (Changing Economic Criterion): Since the set of constraints of $P_N(x,p)$ does not depend on (x_s, u_s) or p, the proposed controller is able to guarantee recursive feasibility and constraints satisfaction for any $p(k)$, $k > 0$. Moreover, if $p(k)$ converges to a constant value, the controller ensures asymptotic stability of (x_s, u_s). In fact, since the domain of attraction $\mathcal{X}_N$ does not depend on the optimal steady state, for all $x \in \mathcal{X}_N$ every admissible steady state is reachable. Moreover, since the trajectory remains in $\mathcal{X}_N$, if the economic criterion (and hence the optimal steady state) changes, problem $P_N(x,p)$ does not lose feasibility and the system is driven to the new optimal steady state in an admissible way.

Corollary 1: If the gradient of $\ell_{eco}(x,u,p)$ in (x_s, u_s) is null, that is if constraints are not active at (x_s, u_s), then convergence is ensured for any $V_O(x,u)$, even if it does not fulfill Assumption 4 nor the condition $\gamma > \gamma_0$.

Proof: The proof of this corollary follows same arguments as the proof of Theorem 1, but considering Lemma 7 (Appendix C), instead of Lemma 5 (Appendix B). ■

VI. Local Economic Optimality

Property 2 is the main advantage of the proposed controller. Indeed, one of the consequence of the effort to maintain feasibility is that, during the transient phase, that is while $x(N-1) \not\approx x_s$, the economic MPC for a changing economic criterion may be suboptimal, in the sense that its performance may differ from the economic MPC controller [10]. This suboptimality is due to the particular cost to minimize and to the terminal constraint imposed to the control problem.

However, in the following it is proved that under mild conditions on the offset cost function, the proposed controller ensures the economic optimality property as in [10].

Since the economic optimality is provided by the stage cost, there is no restriction on the form of the offset cost function $V_O(\cdot,\cdot)$. Hence, this function can be chosen in any form that fulfills Definition 3 and Assumption 4.

Property 3 (Local Optimality): Consider that Assumptions 1–5 hold. Then there exists a $\alpha^0 > 0$ such that for all $\gamma \geq \alpha^0$ and for all $x \in \mathcal{X}_N^e$ the proposed economic MPC for a changing economic criterion is equal to the economic MPC, i.e. $\kappa_N(x,p) = \kappa_N^e(x,p)$.

Proof: First, define problem $\hat{P}_{N-1}^e(x,p)$, which is equivalent to problem (9), but rewritten as follows:

$$\begin{aligned}
\min_{\mathbf{u}} \quad & \sum_{j=0}^{N-1} \ell_t\left(x(j) - x(N-1), u(j) - u(N-1), p\right) \\
& + V_O\left(x(N-1), u(N-1)\right) \\
s.t. \quad & \\
& x(0) = x, \\
& x(j+1) = Ax(j) + Bu(j) \\
& x(j) \in X,\ u(j) \in U, \quad j \in \mathbb{I}_{0:N-1} \\
& (x(N-1), u(N-1)) \in \mathcal{Z}_s \\
& |x(N-1) - x_s|_q = 0.
\end{aligned} \tag{22}$$

Let $\nu(x)$ be the Lagrange multiplier of the equality constraint $|x(N-1) - x_s|_q = 0$ of the optimization problem $\hat{P}_{N-1}^e(x,p)$. We define the following constant α^0:

$$\alpha^0 = \max_{x \in \mathcal{X}_{N-1}^e} |\nu(x)|.$$

Define the optimization problem $P_{N,\gamma}(x,p)$ as a particular case of (16) with $V_O(x,u) \triangleq \gamma|x - x_s|_p$, where $|.|_p$ is the dual norm of $|.|_q$.[1] This optimization problem results from (22) with the last constraint posed as an exact penalty function. Therefore, in virtue of the well-known result on the exact penalty functions [20], taking any $\gamma \geq \alpha^0$ we have that $V_{N,\gamma}(x,p) = V_{N-1}^{e0}(x,p)$, and hence $\kappa_N(x,p) = \kappa_N^e(x,p)$, for all $x \in \mathcal{X}_{N-1}^e$. ■

Remark 7: As it has been mentioned before, for all $x(0) \in \mathcal{X}_N$, while $x(k) \in \mathcal{X}_N \setminus \mathcal{X}_{N-1}^e$, the controller provides a suboptimal solution. But once $x(k) \in \mathcal{X}_{N-1}^e$, Property 3 ensures that the solution provided by the controller is the economically optimal one.

[1] The dual $|.|_p$ of a given norm $|.|_q$ is defined as $|u|_p \triangleq \max_{|v|_q \leq 1} u'v$. For instance, $p = 1$ if $q = \infty$ and vice versa, or $p = 2$ if $q = 2$ [20].

The suboptimality in $\mathcal{X}_N \setminus \mathcal{X}^e_{N-1}$ is the price one has to pay for always ensuring feasibility under any change of p. □

VII. ILLUSTRATIVE EXAMPLE

Consider an isothermal stirred-tank reactor, in which reagent A form product B. The linearized dynamics of the system are given by

$$A = \begin{bmatrix} -\frac{Q_o}{V_R} - k_r - 2k_r c_{Ao} & 0 \\ k_r & -\frac{Q_o}{V_R} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{c_{Af}-c_{Ao}}{V_R} \\ \frac{c_{Bf}-c_{Bo}}{V_R} \end{bmatrix}$$

where $x = (c_A, c_B)$ are the molar concentrations of A and B, $c_{Af} = 1\ \mathrm{mol/L}$ and $c_{Bf} = 0\ \mathrm{mol/L}$ are the feed concentrations of A and B, and $u = Q$ is the flow through the reactor. $(c_{Ao}, c_{Bo}, Q_o) = (0.4142, 0.4142, 12)$ defines the point around which the system has been linearized. $k_r = 1.2\ \mathrm{L/(mol\ min)}$ is the rate constant. The volume of the reactor is fixed to $V_R = 10\ \mathrm{L}$. The constraint on the state is taken as $0 \leq x \leq 1\ \mathrm{mol/L}$, while an upper-bound of 20 L/min is imposed to the flow rate. The system has been discretized with a sample time $T_s = 0.5\ \mathrm{sec}$.

The economic cost function is based on the price of product B and a separation cost (assumed to be directly proportional to the flow rate [10]) plus a regularization cost

$$\ell_{eco}(x,u,p) = -(p(1)ux(2) - p(2)u) + |x - x_s|^2_{\bar{Q}} + |u - u_s|^2_{\bar{R}} \tag{23}$$

where $p = (p(1), p(2))$ are prices, $\bar{Q} = diag(0.5, 0.5)$, $\bar{R} = 0.5$, and (x_s, u_s) is the economically optimal steady state corresponding to the prices p.

In this example, the performance of the following three controllers are assessed: economic MPC for a changing economic criterion (E-MPCT) proposed in this paper, economic MPC (E-MPC, [10]), MPC for tracking (MPCT, [13], [19]).

The cost functions have been taken as:

Economic MPC (E-MPC):

$$V^e_N(x, p; \mathbf{u}) = \sum_{j=0}^{N-1} \ell_{eco}\left(x(j), u(j), p\right). \tag{24}$$

MPC for tracking (MPCT):

$$V^t_N(x; \mathbf{u}) = \sum_{j=0}^{N-1} |x(j) - x(N-1)|^2_{\bar{Q}} + |u(j) - u(N-1)|^2_{\bar{R}} + V_O\left(x(N-1), u(N-1)\right) \tag{25}$$

where $V_O(x(N-1), u(N-1)) = \alpha|x(N-1) - x_s|_1$ is the offset cost function.

Economic MPC for a changing economic criterion (E-MPCT):

$$V_N(x,p;\mathbf{u}) = \sum_{j=0}^{N-1} \ell_{eco}(x(j) - x(N-1) + x_s, u(j) - u(N-1) + u_s) + V_O\left(x(N-1), u(N-1)\right) \tag{26}$$

where $V_O(x(N), u(N-1)) = \alpha|x(N-1) - x_s|_1$ is the offset cost function.

Remark 8: As we already mentioned in Section VI, there are no restrictions on the form of the offset cost function, since the economic optimality depends on the stage cost. In this example, we chose $V_O(.,.)$ as a 1-norm of the distance to the optimal steady state. It is clear that this function can assume different forms—like distances in the state space, distance in both state and input space, distance to a set (see [14])—with the only requirement that Definition 3 and Assumption 4 are fulfilled. □

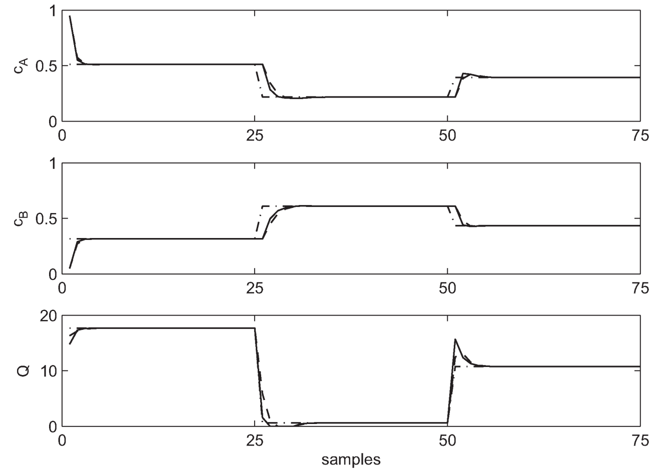

Fig. 1. E-MPCT versus MPCT. Time evolution of states and input: E-MPCT in solid line, MPCT in dashed line, economically optimal steady state in dashed-dotted line.

TABLE I
ECONOMIC PERFORMANCE: E-MPCT VS MPCT

	$\Phi(p_1)$	$\Phi(p_2)$	$\Phi(p_3)$
E-MPCT	54.3903	0.4766	−8.0388
MPCT	58.3299	1.8571	−7.0908

A) Feasibility Guarantee When the Economic Criterion Changes: First of all, the economic MPC for a changing economic criterion (E-MPCT) has been compared to the MPC for tracking (MPCT). For this test, three cost changes have been considered given, respectively by $p_1 = (10, 0.1)$, $p_2 = (1, 0.6)$, and $p_1 = (2, 0.5)$. The economically optimal steady conditions obtained from these prices are, respectively, $(x_{s,1}, u_{s,1}) = (0.5128, 0.3156, 17.7111)$, $(x_{s,2}, u_{s,2}) = (0.2177, 0.6106, 0.6168)$, and $(x_{s,3}, u_{s,3}) = (0.3928, 0.4356, 10.7574)$. The prediction horizon has been taken as $N = 4$, while $\alpha = 200$. The initial condition is $x_0 = (0.95, 0.05)$.

Fig. 1 shows the time evolution of the states and the input. Thanks to the relaxed terminal constraint, the E-MPCT is capable to guarantee feasibility for all three economic criteria, as the MPCT does. Notice how the two controllers approaches the economic setpoint in different ways.

The controllers performances have been assessed using the following closed-loop control performance measure:

$$\Phi(p) = \sum_{k=0}^{T} \ell_{eco}\left(x(k), u(k), p\right) - \ell_{eco}(x_s, u_s, p) \tag{27}$$

where T is the simulation time. The results are shown in Table I. As it was expected, the E-MPCT shows better economic performance than the MPCT.

B) Domain of Attraction: The aim of this test is to compare the domain of attraction provided by the E-MPCT ($\mathcal{X}_N$) to

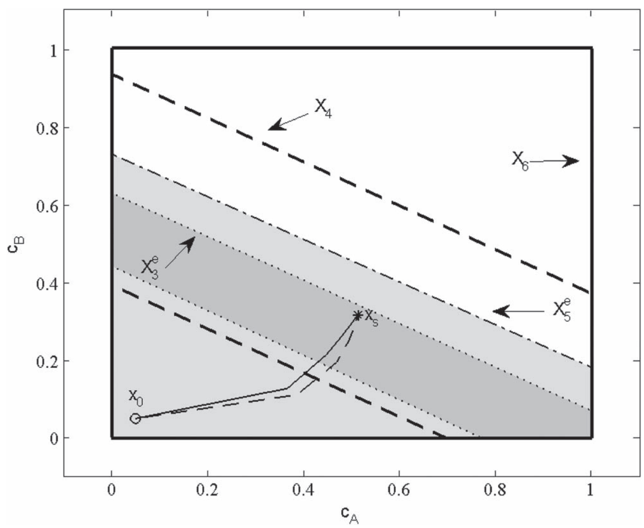

Fig. 2. Comparison of the domains of attraction of the E-MPC and the E-MPCT: $\mathcal{X}_6$ in solid line, $\mathcal{X}_4$ in dashed line, $\mathcal{X}_5^e$ in dashed-dotted line, and $\mathcal{X}_3^e$ in dotted line. The economic optimal point x_s is plotted as a star.

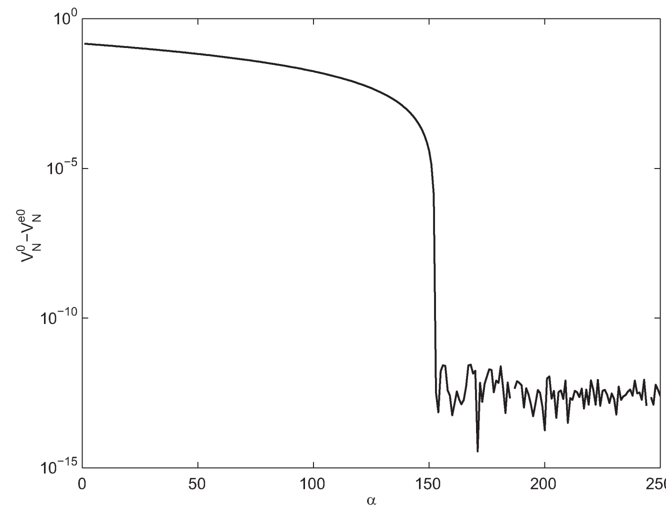

Fig. 3. Local optimality: the optimal costs of the two controllers become the same when the weighting of the offset cost function α starts to be greater than $\alpha^* = 152.4546$.

the one provided by the E-MPC proposed in [10] ($\mathcal{X}^e_{N-1}$). Recall that, due to the terminal constraint $x(N) = x(N-1)$, the E-MPCT domain of attraction is actually given by $N-1$ admissible inputs, so it has to be compared to $\mathcal{X}^e_{N-1}$ (see Section V for more details).

Fig. 2 shows the domains of attraction of the E-MPCT, for $N = 4$ and $N = 6$ ($\mathcal{X}_4$ in dashed line and $\mathcal{X}_6$ in solid line), compared to the corresponding domains of attraction of the E-MCP ($\mathcal{X}_3^e$ in dotted line and $\mathcal{X}_5^e$ in dashed-dotted line), calculated with respect to the economic steady state x_s provided by the price p_1. This point is plotted in Fig. 2 as a star.

It is clear how the proposed controller provides a larger domain of attraction than the E-MPC.

C) Local Economic Optimality: The local optimality property has been checked in the following test. The difference of the optimal costs of the E-MPC and E-MPCT, V_N^{e0} and V_N^0, respectively, has been compared while the weighting of the offset cost function—α—has been varied. Theses costs have been calculated starting from the initial condition $x_0 = (0.05, 0.05)$ and considering price p_1. The prediction horizon considered is $N = 5$ for the E-MPC and $N = 6$ for the E-MPCT.

The result of this test is presented in Fig. 3. It is clear how the optimal costs of this two controllers become practically the same when α starts to be greater than a certain value, which is the value of the Lagrange multiplier of the equality constraint of the E-MPC problem, $\alpha^* = 152.4546$. See Section VI for more details.

Moreover, in Table II, the closed-loop performance of these two controllers are compared to the one of the MPCT, by means of the performance index (27).

The economic MPC and the economic MPC for a changing economic criterion have a better performance than the MPC for tracking. Moreover, the proposed controller shows the same performance index as the economic MPC, which means that the economic optimality is guaranteed.

The state space evolutions for this simulation are also drawn in Fig. 2. The E-MPC evolution is plotted in dotted line, the E-MPCT in solid line and the MPCT in dashed line. The initial condition x_0 is plotted as a dot. Notice how the evolutions of E-MPC and E-MPCT coincide.

TABLE II
COMPARISON OF CONTROLLER PERFORMANCE

	E-MPC	MPCT	E-MPCT
$\Phi(p_1)$	129.7553	136.4140	129.7553

VIII. Conclusion

In this paper, an economic MPC that handles a changing economic criterion has been presented. Following the main idea of the MPC for tracking, the proposed controller considers a slightly modified economic cost function, and a terminal constraint that accounts for *any* admissible steady state.

This paper proves that this formulation ensures: i) feasibility under any change of the economic criterion; ii) optimality with respect to the economic setpoint; iii) larger domain of attraction than standard economic MPC.

Asymptotic stability of the controller has been established, by means of a Lyapunov function.

Appendix

A. Proof of Lemma 1

Proof: From the definition of the rotated cost

$$\begin{aligned} &L_t\left(x - x(N-1), u - u(N-1), p\right) \\ &= \ell_t\left(x - x(N-1), u - u(N-1), p\right) \\ &\quad + \lambda'\left(x - x(N-1) - (Ax + Bu - Ax(N-1) - Bu(N-1))\right) \\ &\quad - \ell_{eco}(x_s, u_s, p) \\ &= \ell_t\left(x - x(N-1), u - u(N-1), p\right) \\ &\quad + \lambda'\left(x - (Ax + Bu)\right) - \ell_{eco}(x_s, u_s, p) \end{aligned}$$

where the last equality comes from the fact that $x(N-1)$ is an equilibrium point, such that $x(N) = Ax(N-1) + Bu(N-1)$.

Therefore, operating with the cost function we have that

$$\begin{aligned}\tilde{V}_N(x,p;\mathbf{u}) &= \sum_{j=0}^{N-1} L_t\left(x(j)-x(N-1),u(j)-u(N-1),p\right)\\ &\quad+\tilde{V}_O\left(x(N-1),u(N-1)\right)\\ &= \sum_{j=0}^{N-1}\left(\ell_t\left(x(j)-x(N-1),u(j)-u(N-1),p\right)\right.\\ &\quad+\lambda'(x(j)-x(j+1))-\ell_{eco}(x_s,u_s,p))-V_O(x_s,u_s)\\ &\quad+V_O\left(x(N-1),u(N-1)\right)+\lambda'\left(x(N-1)-x_s\right).\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{j=0}^{N-1}\lambda'\left(x(j)-x(j+1)\right) &= \lambda'\left(x(0)-x(1)+x(1)-x(2)+\dots\right.\\ &\qquad\left.+x(N-1)-x(N)\right)\\ &= \lambda'\left(x(0)-x(N)\right)=\lambda'(x-x(N-1))\end{aligned}$$

where the last equality comes from the constraints of (20). Hence

$$\begin{aligned}\tilde{V}_N(x,p;\mathbf{u}) &= \sum_{j=0}^{N-1}\ell_t\left(x(j)-x(N-1),u(j)-u(N-1),p\right)\\ &\quad+\lambda'(x-x(N-1))-N\ell_{eco}(x_s,u_s,p)-V_O(x_s,u_s)\\ &\quad+V_O\left(x(N-1),u(N-1)\right)+\lambda'\left(x(N-1)-x_s\right)\\ &= \sum_{j=0}^{N-1}\ell_t\left(x(j)-x(N-1),u(j)-u(N-1),p\right)\\ &\quad+\lambda'(x-x_s)-N\ell_{eco}(x_s,u_s,p)\\ &\quad+V_O\left(x(N-1),u(N-1)\right)-V_O(x_s,u_s).\end{aligned}$$

This is equivalent to optimize, at any instant, the cost function

$$V_N(x,p;\mathbf{u})=\sum_{j=0}^{N-1}\ell_t\left(x(j)-x(N-1),u(j)-u(N-1),p\right)+V_O\left(x(N-1),u(N-1)\right)$$

which is the cost function of problem (16). ■

B. Stability Proof: Technical Lemmas

Lemma 2 (Decrease of the Cost Function): Consider system (1) subject to constraints (2). Let Assumptions 1–5 hold. Consider an initial condition $x\in\mathcal{X}_N$. Consider the successor state $x^+=Ax+B\kappa(x,p)$. Then $x^+\in\mathcal{X}_N$. Moreover, there exists a $\mathcal{K}$-function α_1 such that

$$\tilde{V}_N^0(x^+,p)-\tilde{V}_N^0(x,p)\le-\alpha_1\left(\left|x-x^0(N-1)\right|\right).\tag{28}$$

Proof: Consider that $x\in\mathcal{X}_N$ at time k, then the optimal cost function is given by $\tilde{V}_N^0(x,p)=\tilde{V}_N(x,p;\mathbf{u}^0(x))$, where $\mathbf{u}^0(x)=\{u^0(0),u^0(1),\dots,u^0(N-1)\}$ defines the optimal solution to (20). The resultant state sequence is given by $\mathbf{x}^0(x)=\{x^0(0),x^0(1),\dots,x^0(N-1),x^0(N)\}$, where $x^0(0)=x$, $x^0(1)=x^+$ and $x^0(N)=x^0(N-1)\in\mathcal{X}_s$.

Define $\kappa(x,p)=u^0(0)$ and let the successor state at time $k+1$ be $x^+=Ax+B\kappa(x,p)$. Since $\mathbf{u}^0(x)$ is the optimal solution to (20), then x^+ is feasible, and moreover $x^+\in\mathcal{X}_N$.

As standard in MPC ([22], [31]), choose a sequence of future control inputs $\tilde{\mathbf{u}}=\{u^0(1),\dots,u^0(N-1),u^0(N-1)\}$, feasible solution to problem (20). The state sequence due to $\tilde{\mathbf{u}}$ is $\tilde{\mathbf{x}}=\{x^0(1),x^0(2),\dots,x^0(N),x^0(N)\}$, which is clearly feasible. Compare now the optimal cost $\tilde{V}_N^0(x,p)$, with the cost given by $\tilde{\mathbf{u}}$, $\tilde{V}_N(x^+,p;\tilde{\mathbf{u}})$. We have that

$$\begin{aligned}&\tilde{V}_N(x^+,p;\tilde{\mathbf{u}})\\ &\quad=\tilde{V}_N^0(x,p)-L_t\left(x-x^0(N-1),u^0(0)-u^0(N-1),p\right)\\ &\qquad-\tilde{V}_O\left(x^0(N-1),u^0(N-1)\right)\\ &\qquad+L_t\left(x^0(N-1)-x^0(N-1),u^0(N-1)-u^0(N-1),p\right)\\ &\qquad+\tilde{V}_O\left(x^0(N-1),u^0(N-1)\right)\\ &\quad=\tilde{V}_N^0(x,p)-L_t\left(x-x^0(N-1),u^0(0)-u^0(N-1),p\right).\end{aligned}$$

Since, by optimality, $\tilde{V}_N^0(x^+,p)\le\tilde{V}_N(x^+,p;\tilde{\mathbf{u}})$, then

$$\begin{aligned}&\tilde{V}_N^0(x^+,p)-\tilde{V}_N^0(x,p)\\ &\qquad\le-L_t\left(x-x^0(N-1),u^0(0)-u^0(N-1),p\right)\end{aligned}$$

$\forall x\in\mathcal{X}_N$.

Taking into account that $L_t(\cdot,\cdot,p)$ is positive definite (Property 1), then there exists a $\mathcal{K}$-function α_1 such that

$$\tilde{V}_N^0(x^+,p)-\tilde{V}_N^0(x,p)\le-\alpha_1\left(\left|x-x^0(N-1)\right|\right)\tag{29}$$

$\forall x\in\mathcal{X}_N$. ■

Lemma 3: Consider system (1) subject to constraints (2). Let Assumptions 1–5 hold. Let the system be controlled by the control law $u(k)=K_{db}(x(k)-x_s)+u_s$. Assume that $|x(0)-x_s|\le\omega(\varepsilon)$, $\omega>0$, in such a way that $|x(j)-x_s|\le\varepsilon$, for $j\in\mathbb{I}_{0:N-1}$. Then

$$\sum_{j=0}^{N-1}\left(\ell_{eco}\left(x(j),u(j),p\right)-\ell_{eco}(x_s,u_s,p)\right)\le\gamma_0\left|x(0)-x_s\right|$$

where γ_0 is defined in (21).

Proof: We have that

$$\begin{aligned}&\sum_{k=0}^{N-1}\left(\ell_{eco}\left(x(j),u(j),p\right)-\ell_{eco}(x_s,u_s,p)\right)\\ &\quad\le\sum_{j=0}^{N-1}\left|\ell_{eco}\left(x(j),u(j),p\right)-\ell_{eco}(x_s,u_s,p)\right|\\ &\quad\le L\sum_{j=0}^{N-1}\left(|x(j)-x_s|+\left|K_{db}\left(x(j)-x_s\right)\right|\right)\\ &\quad\le L\sum_{j=0}^{N-1}\left((1+|K_{db}|)\left|x(j)-x_s\right|\right)\end{aligned}$$

$$\leq L\left(1+|K_{db}|\right)\sum_{j=0}^{N-1}\left(\left|A_K^j\right||x(0)-x_s|\right)$$
$$\leq L\left(1+|K_{db}|\right)|x(0)-x_s|\sum_{j=0}^{N-1}|A_K|^j$$
$$= L\left(1+|K_{db}|\right)\frac{1-|A_K|^N}{1-|A_K|}|x(0)-x_s|$$
$$= \gamma_0|x(0)-x_s|.$$

■

Lemma 4: Consider system (1) subject to constraints (2). Let Assumptions 1–5 hold. Let $(\hat{x}_s,\hat{u}_s)\in\mathcal{Z}_s$. Assume that $|x(0)-\hat{x}_s|\leq\omega(\varepsilon)$, $\omega>0$, in such a way that $|x(j)-x_s|\leq\varepsilon$, for $j\in\mathbb{I}_{0:N-1}$. Let $\hat{\mathbf{u}}$ be a sequence of control actions such that $\hat{u}(j)=K_{db}(x(j)-\hat{x}_s)+\hat{u}_s$, with $x(j+1)=Ax(j)+B\hat{u}(j)$. Then $\hat{\mathbf{u}}$ is feasible and

$$\tilde{V}_N\left(x(0),p;\hat{\mathbf{u}}\right)\leq\gamma_0|x(0)-\hat{x}_s|+V_O(\hat{x}_s,\hat{u}_s)+d$$

where $d=\lambda'(x(0)-x_s)+V_O(x_s,u_s)$ and γ_0 is given in (21).

Proof: From (19) we have that

$$\begin{aligned}\tilde{V}_N\left(x(0),p;\hat{\mathbf{u}}\right)&=\sum_{j=0}^{N-1}L_t\left(x(j)-\hat{x}_s,\hat{u}(j)-\hat{u}_s\right)+\tilde{V}_O(\hat{x}_s,\hat{u}_s)\\&=\sum_{j=0}^{N-1}\left(\ell_{eco}\left(x(j)-\hat{x}_s+x_s,\hat{u}(j)-\hat{u}_s+u_s,p\right)\right.\\&\quad\left.-\ell_{eco}(x_s,u_s,p)\right)+\lambda'\left(x(0)-\hat{x}_s\right)\\&\quad+V_O(\hat{x}_s,\hat{u}_s)+\lambda'(\hat{x}_s-x_s)+V_O(x_s,u_s)\\&=\sum_{j=0}^{N-1}\left(\ell_{eco}(x(j)-\hat{x}_s+x_s,\hat{u}(j)-\hat{u}_s+u_s,p)\right.\\&\quad\left.-\ell_{eco}(x_s,u_s,p)\right)\\&\quad+V_O(\hat{x}_s,\hat{u}_s)+\lambda'\left(x(0)-x_s\right)+V_O(x_s,u_s)\\&\leq\gamma_0\left|(x(0)-\hat{x}_s+x_s)-x_s\right|+V_O(\hat{x}_s,\hat{u}_s)+d\\&=\gamma_0|x(0)-\hat{x}_s|+V_O(\hat{x}_s,\hat{u}_s)+d\end{aligned}$$

where $d=\lambda'(x(0)-x_s)+V_O(x_s,u_s)$ and the last inequality comes from Lemma 3. ■

Lemma 5: Consider system (1) subject to constraints (2). Let Assumptions 1–5 hold. Consider the constant $\gamma>\gamma_0$, where γ_0 is defined in (21). Assume that for an initial state x the optimal solution to problem (20) is such that $x^0(N-1)=x$ and $u^0(N-1)=\kappa_N(x,p)$. Then

$$x^0(N-1)=x_s,\quad u^0(N-1)=u_s. \tag{30}$$

Proof: Consider that the optimal solution to (20) is $(x^0(N-1),u^0(N-1))$ and consider that $x=x^0(N-1)$ and $u=u^0(N-1)$. Then, $(x,u)\in\mathcal{Z}_s$ and the optimal cost function is $\tilde{V}_N^0(x,p)=\tilde{V}_O(x^0(N-1),u^0(N-1))$.

The lemma will be proved by contradiction. Assume that $(x^0(N-1),u^0(N-1))\neq(x_s,u_s)$. Hence there exists a $\hat{\beta}\in(0,1)$ such that for any $\beta\in[\hat{\beta},1)$

1) $(\hat{x}_s,\hat{u}_s)=\beta(x^0(N-1),u^0(N-1))+(1-\beta)(x_s,u_s)$
2) $(\hat{x}_s,\hat{u}_s)\in\mathcal{Z}_s$
3) the dead-beat control law $u=K_{db}(x-\hat{x}_s)+\hat{u}_s$ drives the system from $x^0(N-1)$ to $\hat{x}_s$ in an admissible way.

Therefore, defining as $\hat{\mathbf{u}}$ the sequence of control actions derived from the control law $u=K_{db}(x-\hat{x}_s)+\hat{u}_s$, it is inferred that $\hat{\mathbf{u}}$ is a feasible solution to problem (20). Then considering Lemma 3 and Lemma 4, we have that

$$\begin{aligned}\tilde{V}_N^0(x^0(N-1),p)&=\tilde{V}_O\left(x^0(N-1),u^0(N-1)\right)\\&\leq\tilde{V}_N\left(x^0(N-1),p;\hat{\mathbf{u}}\right)\\&=\sum_{j=0}^{N-1}L_t\left(\left(x(j)-\hat{x}_s\right),\left(\hat{u}(j)-\hat{u}_s\right),p\right)\\&\quad+\tilde{V}_O(\hat{x}_s,\hat{u}_s)\\&\leq\gamma_0\left|x^0(N-1)-\hat{x}_s\right|+V_O(\hat{x}_s,\hat{u}_s)+d\\&=\gamma_0(1-\beta)\left|x^0(N-1)-x_s\right|+V_O(\hat{x}_s,\hat{u}_s)+d\end{aligned}$$

where $d=\lambda'(x^0(N-1)-x_s)+V_O(x_s,u_s)$.

Define $W(x^0(N-1),\beta)\stackrel{\Delta}{=}\gamma_0(1-\beta)|x^0(N-1)-x_s|+V_O(\hat{x}_s,\hat{u}_s)+d$ and notice that

$$\begin{aligned}W\left(x^0(N-1),1\right)&=V_O\left(x^0(N-1),u^0(N-1)\right)+d\\&=V_O\left(x^0(N-1),u^0(N-1)\right)\\&\quad+\lambda'\left(x^0(N-1)-x_s\right)+V_O(x_s,u_s)\\&=\tilde{V}_O\left(x^0(N-1),u^0(N-1)\right)\\&=\tilde{V}_N^0\left(x^0(N-1),p\right).\end{aligned}$$

Taking the partial of W about β we have that

$$\frac{\partial W}{\partial\beta}=-\gamma_0\left|x^0(N-1)-x_s\right|+g'\left(x^0(N-1)-x_s,u^0(N-1)-u_s\right)$$

where $g'\in\partial V_O(\hat{x}_s,\hat{u}_s)$, defining $\partial V_O(\hat{x}_s,\hat{u}_s)$ as the subdifferential of $V_O(\hat{x}_s,\hat{u}_s)$. Evaluating this partial for $\beta=1$ we obtain

$$\left.\frac{\partial W}{\partial\beta}\right|_{\beta=1}=-\gamma_0\left|x^0(N-1)-x_s\right|+\bar{g}'\left(x^0(N-1)-x_s,u^0(N-1)-u_s\right)$$

where $\bar{g}'\in\partial V_O(x^0(N-1),u^0(N-1))$, defining $\partial V_O(x^0(N-1),u^0(N-1))$ as the subdifferential of $V_O(x^0(N-1),u^0(N-1))$.

Then, from convexity, we can state that, for every $x^0(N-1)$ and x_s

$$\begin{aligned}\bar{g}'\left(x^0(N-1)-x_s,u^0(N-1)-u_s\right)&\geq V_O\left(x^0(N-1),u^0(N-1)\right)\\&\quad-V_O(x_s,u_s)\\&\geq\gamma\left|x^0(N-1)-x_s\right|\end{aligned}$$

where the last inequality comes from Assumption 4. Therefore

$$\begin{aligned}\left.\frac{\partial W}{\partial\beta}\right|_{\beta=1}&\geq-\gamma_0\left|x^0(N-1)-x_s\right|+\gamma\left|x^0(N-1)-x_s\right|\\&=(\gamma-\gamma_0)\left|x^0(N-1)-x_s\right|.\end{aligned}$$

Since $\gamma>\gamma_0$ and $|x^0(N-1)-x_s|>0$, hence $\partial W/\partial\beta|_{\beta=1}>0$.

This means that there exists a $\beta \in [\hat{\beta}, 1)$ such that the cost to move the system from $x^0(N-1)$ to $\hat{x}_s$, $W(x^0(N-1), \beta)$, is smaller than the cost to remain in $x^0(N-1)$, that is $W(x^0(N-1), 1) = \tilde{V}_O(x^0(N-1), u^0(N-1)) = \tilde{V}_N^0(x^0(N-1), p)$.

This contradicts the optimality of the solution to problem (20) and hence $x^0(N-1) = x_s$, which proves the Lemma. ■

Remark 9: Lemma 5 plays an important role in the proof of Theorem 1. It proves that, if the system converges to an equilibrium point $x^0(N-1)$, then this point can be only the economically optimal steady state, x_s. □

Lemma 6 (Positive Definiteness of the Error Function): Consider system (1) subject to constraints (2). Consider that Assumptions 1–5 hold. For all $x \in \mathcal{X}_N$ and $x^0(N-1) \in \mathcal{X}_s$, define the function $e(x) = x - x^0(N-1)$. Then, there exists a $\mathcal{K}$-function α_e such that

$$|e(x)| \geq \alpha_e \left(|x - x_s|\right). \tag{31}$$

Proof: Notice that, due to convexity, $e(x)$ is a continuous function [31]. Moreover, let us consider these two cases.

1) $|e(x)| = 0$ iff $x = x_s$. In fact, (i) if $e(x) = 0$, then $x = x^0(N-1)$, and from Lemma 5, this implies that $x^0(N-1) = x_s$; (ii) if $x = x_s$, then by optimality $x^0(N-1) = x_s$, and then $x = x^0(N-1)$.
2) $|e(x)| > 0$ for all $|x - x_s| > 0$. In fact, for any $x \neq x_s$, $|e(x)| \neq 0$ and moreover $|x - x_s| > 0$. Then, $|e(x)| > 0$.

Then, since $\mathcal{X}_N$ is compact, in virtue of [33, Ch. 5, Lemma 6, pag. 148], there exists a $\mathcal{K}$-function α_e such that $|e(x)| \geq \alpha_e(|x - x_s|)$. ■

C. Proof of Corollary 1: Technical Lemma

Lemma 7: Consider system (1) subject to constraints (2). Let Assumptions 1–3 and 5 hold. Assume that the gradient of $\ell_{eco}(x, u, p)$ in (x_s, u_s) is null. Assume that for an initial state x the optimal solution to problem (20) is such that $x^0(N-1) = x$ and $u^0(N-1) = \kappa_N(x, p)$. Then

$$x^0(N-1) = x_s, \quad u^0(N-1) = u_s. \tag{32}$$

Proof: The proof of this Lemma follows similar arguments as Lemma 5. In this case

$$\begin{aligned}\tilde{V}_N^0(x^0(N-1), p) &\leq \left|x^0(N-1) - \hat{x}_s\right|_P^2 + V_O(\hat{x}_s, \hat{u}_s) + d \\ &= (1-\beta)^2 \left|x^0(N-1) - x_s\right|_P^2 + V_O(\hat{x}_s, \hat{u}_s) + d.\end{aligned}$$

where $d = \lambda'(x^0(N-1) - x_s) + V_O(x_s, u_s)$, and the inequality comes from [3, Equation (22)], taking into account that the gradient of $\ell_{eco}(x, u, p)$ in (x_s, u_s) is null.

Define $W(x^0(N-1), \beta) \stackrel{\Delta}{=} (1-\beta)^2 |x^0(N-1) - x_s|_P^2 + V_O(\hat{x}_s, \hat{u}_s) + d$ and take the partial of W about β

$$\begin{aligned}\frac{\partial W}{\partial \beta} = &-2(1-\beta) \left|x^0(N-1) - x_s\right|_P^2 \\ &+ g'\left(x^0(N-1) - x_s, u^0(N-1) - u_s\right).\end{aligned}$$

where $g' \in \partial V_O(\hat{x}_s, \hat{u}_s)$, defining $\partial V_O(\hat{x}_s, \hat{u}_s)$ as the subdifferential of $V_O(\hat{x}_s, \hat{u}_s)$. This partial for $\beta = 1$, is strictly positive.

This contradicts the optimality of the solution and hence the result is proved. ■

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