EQUIVALENCES BETWEEN TWO MATCHING MODELS: STABILITY

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(Communicated by Jordi Massó)

Abstract. We study the equivalences between two matching models, where the agents in one side of the market, the workers, have *responsive* preferences on the set of agents of the other side, the firms. We modify the firms' preferences on subsets of workers and define a function between the set of many-to-many matchings and the set of *related* many-to-one matchings. We prove that this function restricted to the set of stable matchings is bijective and that preserves the stability of the corresponding matchings in both models. Using this function, we prove that for the many-to-many problem with *substitutable* preferences for the firms and *responsive* preferences for the workers, the set of stable matchings is non-empty and has a lattice structure.

1. **Introduction.** Many-to-many matching models have been useful for studying assignment problems with the distinctive feature that agents can be divided into two disjoint subsets: the set of firms and the set of workers.[1] The nature of the assignment problem consists of matching each agent with a subset of agents from the other side of the market. Thus, each firm may hire a subset of workers while each worker may work for a number of different firms.

Stability has been considered the main property to be satisfied by any matching. A matching is called *stable* if all agents have acceptable partners and there is no unmatched worker-firm pair who both would prefer to be matched to each other rather than staying with their current partners. Unfortunately, the set of stable matchings may be empty. *Substitutability*[2] is the weakest condition that has so far been imposed on agents' preferences under which the existence of stable matchings is guaranteed. An agent has *substitutable* preferences if he wants to continue being

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[1]We will be using as a reference (and as a source of terminology) labor markets with part-time jobs and we will generically refer to these two sets as the two sides of the market.

[2]Hatfield and Kojima [6] in matching models with contracts introduce a weaker condition called *bilateral substitutability* and show that this condition is sufficient for the existence of a stable matching. Also, they consider a strengthening of the bilateral substitutability condition called *unilateral substitutability*. Both conditions reduce to standard *substitutability* in matching problems without contracts. See Definitions 3 and 5 in Hatfield and Kojima [6] for a precise and formal definition of *bilateral* and *unilateral substitutability,* respectively.

matched to an agent on the other side of the market even if other agents become unavailable.[3]

The *college admissions problem* is the name given by Gale and Shapley [4] to a many-to-one matching model. Colleges have *responsive* preferences over students and students have preference over colleges; each college c has a maximum number of positions to be filled (its *quota* q_c), it ranks individual students and orders subsets of students in a *responsive* manner (Roth [12]); namely, to add "good" students to a set leads to a better set, whereas to add "bad" students to a set leads to a worst set. In addition, for any two subsets that differ in only one student, the college prefers the subset containing the most preferred student. In this model the set of stable matchings satisfies the following additional properties: (i) there is a polarization of interests between the two sides of the market along the set of stable matchings, (ii) the set of unmatched agents is the same under every stable matching, (iii) the number of workers assigned to a firm through stable matchings is the same, and (iv) if a firm does not complete its quota under some stable matching then it is matched to the same set of workers at any stable matching.[4]

The case in which all quotas are equal to one is called the *marriage problem*,[5] and is symmetric between the two sides of the market. It was initially thought that the essential features of the college admissions problem could be captured by treating it as a marriage problem in which each of the q_c positions available at a college c would be treated as q_c different colleges, denote by $c_1, c_2, \ldots, c_{q_c}$. There is a natural injective correspondence between matchings in the original college admissions problem and matchings in the *related* marriage problem in this way: a matching μ of the college admissions problem, which matches a college c with the set of students $\mu(c)$, corresponds to the matching μ' in the *related* marriage market in which the students in $\mu(c)$ are matched, in the order that they occur in its preferences, with the ordered positions of c that appear in the *related* marriage market; that is, if s is c's most preferred student in $\mu(c)$, then $\mu'(c_1) = s$, and so forth. This correspondence preserves the stability of the matchings. This construction is due to Gale and Sotomayor [5]. Most of the subsequent theoretical literature concerned with these problems focused on the marriage problem, with the assumption that results established for the marriage problem would carry over to the college admissions problem through this kind of transformation. However, Roth [12] observed that certain results, as those of optimality and incentives, cannot be extended from the case of the marriage problem.

Knuth [8] established that the set of stable matchings for the marriage model has a lattice structure and attributed this result to Conway. Roth [11] showed that the least upper bound and the greatest lower bound used by Knuth [8] did not work in a more general many-to-many matching model. Blair [2] proposed a natural extension of the partial ordering used in Knuth [8]. However, its binary operations were unnatural and complicated since they were obtained as the outcomes of nontrivial

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[3]Kelso and Crawford [7] were the first to use *substitutability* to show the existence of stable matchings in a many-to-one model with money. Roth [10] shows that, if all agents have *substitutable* preferences, the set of many-to-many stable matchings is non-empty.

[4]Property (i) is a consequence of the decomposition lemma proved by Gale and Sotomayor [5]. Properties (ii) and (iii) were proved independently by Gale and Sotomayor [5] and Roth [10]. Property (iv) was proved by Roth [13].

[5]It is the name given to the one-to-one matching model. See Roth and Sotomayor [15] for a precise and formal definition of such model.

sequences of matchings. Roth and Sotomayor [15] extended the result of the marriage problem to the college admission problem and this work was further extended by Martínez et al. [9] by proposing, for a many-to-one model with *substitutable* and *separable with quota* preferences, two very natural binary operations that endow to the set of stable matchings with a lattice structure.

The *college admissions problem with substitutable preferences* is the name given by Roth and Sotomayor [15] to the most general many-to-one model with ordinal preferences. Firms are restricted to have *substitutable* preferences over subsets of workers, while workers may have all possible preferences over the set of firms. Under this hypothesis Roth and Sotomayor [15] showed that the deferred-acceptance algorithms produce either the firm-optimal stable matching or the worker-optimal stable matching, depending on whether the firms or the workers make the offers. The firm (worker)-optimal stable matching is unanimously considered by all firms (respectively, workers) to be the best among all stable matchings.

It is natural to think that there exists a natural bijective function between the set of stable matchings in the original many-to-many model with *responsive* preferences for the workers and the set of stable matchings in the *related* many-to-one model. However, we show that the natural extension presented by Gale and Sotomayor [5] does not preserve the equivalence between the set of stable matchings in the many-to-many matching model and the corresponding set of stable matchings of the *related* many-to-one model.[6] We study a many-to-many matching model with *responsive* preferences for the workers, by investigating how to define agents' preferences in its *related* many-to-one model to preserve the stability between matchings in the original many-to-many model and matchings in the *related* many-to-one model. For this reason, over the subsets of copies of workers in the *related* many-to-one model, we modify the firms' preferences and define a function between the set of many-to-many matchings and the set of *related* many-to-one matchings. We show that this function, restricted to the set of stable matchings, is bijective. Hence, it preserves the stability between both sets of stable matchings, obtaining our main theorem.

Moreover, we prove that the modified firms' preferences, defined over the subsets of copies of students in the *related* many-to-one market, inherit the restriction of *substitutability* when the firms have *substitutable* preferences in the original many-to-many model. Since the function defined preserves the stability between both sets of stable matchings, we give an alternative proof that the set of stable matchings is non-empty for the many-to-many model with *substitutable* preferences for the firms, and *responsive* preferences for the workers. Also, we prove that this function preserves the Blair partial order of the agents on each side of the market over the set of stable matchings, and that this set has a lattice structure for the many-to-many model with *substitutable* preferences for the firms, and *responsive* preferences for the workers. From here, it follows that both sets of stable matchings have equivalent lattice structures.

Our paper contributes to this literature by proposing a new tool to prove results in the many-to-many model that follow from results in the many-to-one model. In this paper, we present novel proofs of two already well-known results in the many-to-many model. However, it is important to mention that this new technique of demonstration introduced in this paper, could be used to prove results not known yet in the many-to-many model.

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[6]See Example 1, Section 3.

The paper is organized as follows. In Section 2, we present the preliminary notations and definitions. In Section 3, we introduce a many-to-many matching model and its *related* many-to-one matching model, and we present the principal result of this paper. In Section 4, we prove that the set of stable matchings in the many-to-many model has a lattice structure. Finally, in Section 5, we conclude with some final remarks.

2. **Preliminaries.** There are two finite and disjoint sets of agents, the set of n *firms* $F = \{f_1, ..., f_n\}$ and the set of m *workers* $W = \{w_1, ..., w_m\}$. Each worker $w_j \in W$ with $j = 1, \ldots, m$ has a maximum number of positions to be filled: its *quota*,[7] denoted by s_j. Observe that, the quota of worker w imposes only a restriction on the maximal number of firms to which w can be assigned. Let $\mathbf{s} = (s_1, \ldots, s_m)$ be the list of *quotas*, one for each worker $w_j \in W$. To simplify the notation, sometimes, we denote a generic firm by f (instead of f_i) and a generic worker by w (instead of w_j), and its quota by s_w. Each firm $f \in F$ has an antisymmetric, transitive and complete preference relation $\succeq_f$ over the set of all subsets of W, and each worker $w \in W$ has an antisymmetric, transitive and complete preference relation $\succeq_w$ over the set of all subsets of F. Given $A, B \subseteq W$, we write $A \succeq_f B$ to indicate that firm f likes A at least as well as B. Given the preference relation $\succeq_f$, we say that $A \succ_f B$ when $A \succeq_f B$ and $A \neq B$. Analogously, for each worker $w \in W$ and any two sets of firms $C, D \subseteq F$, we write $C \succeq_w D$ and $C \succ_w D$. *Preferences profiles* are $(n+m)$-tuples of preference relations and they are represented by $\succ = (\succ_{f_1}, \ldots, \succ_{f_n}, \succ_{w_1}, \ldots, \succ_{w_m}) = ((\succ_f)_{f \in F}, (\succ_w)_{w \in W})$. We denote by $a \in F \cup W$ a generic agent of either set. Given a preference relation of an agent $\succ_a$, the subsets of partners preferred to the empty set by a are called *acceptable*.

To express preference relations in a concise manner, and since only acceptable sets of partners will matter, we will represent preference relations as lists of acceptable partners. For instance, $\succ_{f_i} = \{w_1, w_3\}, \{w_2\}, \{w_1\}, \{w_3\}$ and $\succ_{w_j} = \{f_1, f_3\}, \{f_1\}, \{f_3\}$ indicate that $\{w_1, w_3\} \succ_{f_i} \{w_2\} \succ_{f_i} \{w_1\} \succ_{f_i} \{w_3\} \succ_{f_i} \emptyset$ and $\{f_1, f_3\} \succ_{w_j} \{f_1\} \succ_{w_j} \{f_3\} \succ_{w_j} \emptyset$.

The *assignment problem* consists of matching workers with firms keeping the bilateral nature of their relationship and allowing for the possibility that both, firms and workers, may remain unmatched. Formally,

Definition 2.1. A **matching** μ is a mapping from the set $F \cup W$ into the set of all subsets of $F \cup W$ such that, for all $w \in W$ and $f \in F$:

1. $\mu(f) \in 2^W$.
2. $\mu(w) \in 2^F$ and $|\mu(w)| \leq s_w$.
3. $w \in \mu(f)$ if and only if $f \in \mu(w)$.[8]

We say that an agent a is *single* in a matching μ if $\mu(a) = \emptyset$. Otherwise, the agent is *matched*. A matching is said to be *one-to-one* (known as the *marriage problem*) if firms can hire at most one worker, and workers can work for at most one firm. A matching is said to be *many-to-one* if workers can work for at most one firm but firms may hire many workers.

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[7]This limitation may arise from, for example, technological, legal, or budgetary reasons. The *college admissions problem* (Roth [12]) incorporates the quota restriction of each college by imposing a limit on the number of students that a college may admit. However, from the point of view of stability, this is equivalent to supposing that all sets of students with cardinality larger than the quota are unacceptable for the college.

[8]We will often abuse notation by omitting the brackets to denote a set with a unique element.

Suppose each worker w gives its ranking of individual firms and orders subsets of firms in a *responsive* manner; namely, to add "good" firms to a set leads to a better set, whereas to add "bad" firms to a set leads to a worst set. In addition, for any two subsets that differ in only one firm, a worker prefers the subset containing the most preferred firm. Given an ordered list of quotas $\mathbf{s} = (s_1, \ldots, s_m)$ of the workers, we state the definition formally, as follow,

Definition 2.2. The preference relation $\succ_w$ over 2^F is **responsive** if satisfies the following conditions:

1. For all $T \subseteq F$ such that $|T| > s_w$, we have that $\emptyset \succ_w T$.
2. For all $T \subseteq F$ such that $|T| < s_w$ and $f \notin T$, we have that $T \cup \{f\} \succ_w T$ if and only if $\{f\} \succ_w \emptyset$.
3. For all $T \subseteq F$ such that $|T| < s_w$ and $f, f' \notin T$, we have that $T \cup \{f\} \succ_w T \cup \{f'\}$ if and only if $\{f\} \succ_w \{f'\}$.

A preference profile $(\succ_w)_{w \in W}$ is *responsive* if each $\succ_w$ satisfies responsiveness.

Given a set of firms $S \subseteq F$, each worker $w \in W$ can determine which subset of S would most prefer to hire. We will call this the w's *choice set* from S, and denote it by $Ch(S, \succ_w)$. Formally,

$$Ch(S, \succ_w) = \max_{\succ_w} \{T : T \subseteq S\}.$$

Symmetrically, given a set of workers $S \subseteq W$, let $Ch(S, \succ_f)$ denote firm f's most preferred subset of S according to its preference relation $\succ_f$. Formally,

$$Ch(S, \succ_f) = \max_{\succ_f} \{T : T \subseteq S\}.$$

We assume that firms' preferences for groups of workers are such that the firms regard individual workers more as *substitutes* for each other than as complements. See Chapter 6 in Roth and Sotomayor [15]'s book for the complete bibliography. Formally,

Definition 2.3. The preference relation $\succ_f$ over 2^W is **substitutable** if for any set $S \subseteq W$ containing workers w and $\bar{w}$ $(w \neq \bar{w})$, $w \in Ch(S, \succ_f)$ implies $w \in Ch(S \setminus \{\bar{w}\}, \succ_f)$.

That is, if f has substitutable preferences, then if its preferred set of employees from S includes w, so will its preferred set of employees from any subset of S that still includes w.[9] A preference profile $(\succ_f)_{f \in F}$ is *substitutable* if each $\succ_f$ satisfies substitutability. Note that substitutability is a weaker condition than responsiveness.

A matching μ is *blocked by agent* a if $\mu(a) \neq Ch(\mu(a), \succ_a)$. A matching μ is *individually rational* if it is not blocked by any individual agent. A matching μ is *blocked by a worker-firm pair* (w, f) if $w \notin \mu(f)$, $w \in Ch(\mu(f) \cup \{w\}, \succ_f)$, and $f \in Ch(\mu(w) \cup \{f\}, \succ_w)$. A matching μ is *stable* if it is not blocked by any individual agent or any worker-firm pair.

Let $\mathbf{M} = (F, W, \mathbf{s}, \succ)$ be a specific many-to-many matching problem such that firms have substitutable preferences and workers have responsive preferences.[10] Let

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[9]Remember that the quota of worker w imposes only a restriction on the maximal number of firms to which w can be assigned. But, it is important to emphasize that the quota of worker w does not have any effect about how a firm ranks subsets of workers.

[10]An example of this many-to-many problem is a labor market where workers can worker for a maximum number of firms, and firms hire workers but without any capacity restriction. We

$\mathcal{M}$ be the set of all matching in $\mathbf{M}$. Let $\mathcal{S}(\mathbf{M})$ be the set of all stable matchings in $\mathbf{M}$.

Remark 1. When $\mu \in \mathcal{M}$ is *blocked by a worker-firm pair* (w, f), and since we assume that $(\succ_w)_{w \in W}$ is responsive, an equivalent formulation of the condition $f \in Ch(\mu(w) \cup \{f\}, \succ_w)$ is the following:
a. If $|\mu(w)| < s_w$, then $f \succ_w \emptyset$.
b. If $|\mu(w)| = s_w$ then, there exists $f' \in \mu(w)$, such that $f \succ_w f'$.

3. **The connection between the models.** Gale and Sotomayor [5] give a formal proof of the equivalence between the *college admissions problem* and its *related marriage problem.* In the present section, we study if it is possible to extend in a natural way the methodology developed by Gale and Sotomayor [5] to study the equivalence between the many-to-many matching model and its *related* many-to-one matching model.

3.1. **The related many-to-one problem.** Given the many-to-many matching problem $\mathbf{M}$ we can consider its *related* many-to-one problem, in which each worker w_j with quota s_j is broken into s_j "pieces" of itself. In the *related* market each worker has a quota of one. In other words, we replace each worker w_j by s_j positions (copies) of w_j, denoted by $w_j^1, \ldots, w_j^{s_j}$, where the superscript of w_j^t indicates the t-th copy of w_j.[11] Each one of them, in that related many-to-one problem, has preferences over $F \cup \emptyset$ that are identical with those of w_j with a quota of one. We denote by $W^{\mathbf{s}}$ the set of copies of W; that is,

$$W^{\mathbf{s}} = \left\{ w_1^1, \ldots, w_1^{s_1}, w_2^1, \ldots, w_2^{s_2}, \ldots, w_m^1, \ldots, w_m^{s_m} \right\}.$$

Observe that, $|W^{\mathbf{s}}| = \sum_{j=1}^{m} s_j$. In addition, in the *related* market, each firm has preferences over $W^{\mathbf{s}} \cup \emptyset$ as follow: the preference relation of each firm $f \in F$ is modified replacing w_j, where this appears, by the list $w_j^1, \ldots, w_j^{s_j}$ in that order. That is, if a firm f in the original many-to-many problem prefers w to w', then the firm f in the *related* many-to one problem, will prefer all of the positions of w over all of the positions of w'. Since every firm $f \in F$ has preferences over the set of positions (copies) of workers, we will suppose that, if $w_j \succ_f \emptyset$, then $w_j^t \succ_f w_j^{t'}$ if and only if $t < t'$. We denote by $I_j = \{1, \ldots, s_j\}$ the set of indexes of the copies of every worker w_j.

The following example shows that the natural correspondence used by Gale and Sotomayor [5] to prove the equivalence between the *college admissions problem* and its *related marriage problem* can not be extended in a natural way to many-to-many matching problems and their *related* many-to-one matching problems.

Example 1. Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$ and $\mathbf{s} = (2, 2)$. The firms have the following substitutable preferences,

$$\{w_1, w_2\} \succ_{f_1} \{w_1\} \succ_{f_1} \{w_2\} \succ_{f_1} \emptyset$$
$$\{w_1, w_2\} \succ_{f_2} \{w_1\} \succ_{f_2} \{w_2\} \succ_{f_2} \emptyset$$

The workers have the following responsive preferences,

$$\{f_1, f_2\} \succ_{w_1} \{f_1\} \succ_{w_1} \{f_2\} \succ_{w_1} \emptyset$$

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observe that our many-to-many problem is more general that the many-to-many problem with capacity restrictions in both sides of the market.

[11]If some worker w_j has quota equal to one, then simply will denote its unique copie as w_j^1.

$$\{f_1, f_2\} \succ_{w_2} \{f_1\} \succ_{w_2} \{f_2\} \succ_{w_2} \emptyset$$

Consider the *related* many-to-one problem in which each worker w_j is replaced by two positions (copies) of himself, then $W^{\mathbf{s}} = \{w_1^1, w_1^2, w_2^1, w_2^2\}$. Now, preferences of each w_j^t are over $F \cup \emptyset$, and preferences of each f_i are over $W^{\mathbf{s}} \cup \emptyset$. Each copy has the same preferences over firms that are identical with those of w_j with a quota of one, that is, $\{f_1\} \succ_{w_j^t} \{f_2\} \succ_{w_j^t} \emptyset$ for all $j = 1, 2$ and $t = 1, 2$. To construct firms' preferences is only needed the firms preference on singleton subsets of workers. Thus, each firm f_i has the following preferences:

$$\{w_1^1\} \succ_{f_i} \{w_1^2\} \succ_{f_i} \{w_2^1\} \succ_{f_i} \{w_2^2\} \succ_{f_i} \emptyset.$$

Let ν be the following stable matching of the *related* many-to-one problem,

$$\nu = \begin{pmatrix} f_1 & f_2 \\ \{w_1^1, w_1^2\} & \{w_2^1, w_2^2\} \end{pmatrix}.$$

Following Gale and Sotomayor [5], note that since all copies w_1^1, w_1^2 corresponding to worker w_1 have the same preferences then, in order for the matching ν in the related many-to-one problem, corresponds to a matching μ in the many-to-many problem, it must be that the most preferred firm in $\mu(w_1)$ is matched to w_1^1 and the second most preferred to w_1^2. Thus, $\mu(w_1) = f_1$. Similarly, $\mu(w_2) = f_2$. To sum up,

$$\mu = \begin{pmatrix} f_1 & f_2 \\ w_1 & w_2 \end{pmatrix}.$$

However, matching μ is not stable in the many-to-many problem because is *blocked*, for example, by the worker-firm pair (w_2, f_1) since the quota of worker w_2 is two. $\qquad\square$

Example 1 shows that the straightforward extended of the procedure proposed by Gale and Sotomayor [5] for the college admission problem and the marriage problem does not preserve the *stability* of the matchings between the many-to-many matching problem and the corresponding *related* many-to-one problem. This negative result, led us to modify the agents' preferences in the *related* many-to-one model and investigate if such equivalence exists.

Definition 3.1. Given $S \subseteq W^{\mathbf{s}}$, we will say that S **has clones** if there exists w_j and $t, t' \in I_j$ $(t \neq t')$ such that $\left\{w_j^t, w_j^{t'}\right\} \subseteq S$. If $S \subseteq W^{\mathbf{s}}$ has no clones, we call it a **simple set**.

Definition 3.1 specifies that a subset of copies of workers, $S \subseteq W^{\mathbf{s}}$, has *clones* if there exists at least two copies of a same worker in $S \subseteq W^{\mathbf{s}}$. Given $S \subseteq W^{\mathbf{s}}$, let

$$\overline{S} = \{w_j \in W \mid \exists\, t \in I_j \text{ such that } w_j^t \in S\}.$$

We define for each firm a strict preference relation $\succ_f^*$ over 2^{W^s} derived from $\succ_f$ over 2^W as follows,

Definition 3.2. Let S, $S' \subseteq W^{\mathbf{s}}$ and $\succ_f$ over 2^W be given. Denote by $\succ_f^*$ any complete preference relation [12] over $2^{W^{\mathbf{s}}}$ that satisfies the following properties:

1. If S is not a simple set, then $\emptyset \succ_f^* S$.

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[12]We note that, given the complete preference $\succ_f$ over 2^W, the complete preference $\succ_f^*$ over $2^{W^{\mathbf{s}}}$ is not necessarily unique.

2. If S and S' are simple sets:
   (a) *If* $w_j^t \in S$ *and* $w_j^{t'} \notin S$ *are such that* $t < t'$, *and* $w_j \succ_f \emptyset$, *then*

   $$S \succ_f^* \left(S \backslash \left\{w_j^t\right\}\right) \cup \left\{w_j^{t'}\right\}.$$

   (b) *If* $\overline{S} \neq \overline{S}'$, *then* $S \succ_f^* S'$ *if and only if* $\overline{S} \succ_f \overline{S}'$.

The following lemmata characterize the *choice set* of firm $f \in F$ in the *related* many-to-one market.

Lemma 3.3. *Let S be a subset of $W^{\mathbf{s}}$. Then,*

$$Ch(S, \succ_f^*) = \left\{w_j^t \in S \mid w_j \in Ch(\overline{S}, \succ_f) \text{ and } t \leq t' \text{ for all } w_j^{t'} \in S\right\}.$$

Proof. Suppose that $S \subseteq W^{\mathbf{s}}$ is not a simple set. Let

$$A = \left\{w_j^t \in S \mid w_j \in Ch(\overline{S}, \succ_f) \text{ and } t \leq t' \text{ for all } w_j^{t'} \in S\right\}.$$

We observed that $Ch(S, \succ_f^*)$ is a simple set, otherwise, $\emptyset \succ_f^* Ch(S, \succ_f^*)$.
Claim. $Ch(\overline{S}, \succ_f) = \overline{Ch(S, \succ_f^*)}$.

Proof of Claim. Since $Ch(S, \succ_f^*)$ is a simple set, then for all $w_j^{t'} \in S$ such that $t < t'$ we obtain that $w_j^{t'} \notin Ch(S, \succ_f^*)$ because $w_j^t \succ_f^* w_j^{t'}$. So, by the definition of the choice set

$$Ch(S, \succ_f^*) \succ_f^* S' \text{ for all } S' \subseteq S \text{ such that } S' \neq Ch(S, \succ_f^*)$$

which implies that if S' is a simple set and if $\overline{Ch(S, \succ_f^*)} \neq \overline{S}'$ then, by definition of $\succ_f^*$, $\overline{Ch(S, \succ_f^*)} \succ_f \overline{S}'$ for all $\overline{S}' \subseteq \overline{S}$; that is, $Ch(\overline{S}, \succ_f) = \overline{Ch(S, \succ_f^*)}$. If S' is not a simple set, by definition of $\succ_f^*$, we have $\emptyset \succ_f^* S'$ and then by definition of the choice set $Ch(S, \succ_f^*) \succeq_f^* \emptyset$, thus by transitivity of $\succ_f^*$

$$Ch(S, \succ_f^*) \succ_f^* S' \text{ for all } S' \subseteq S \text{ such that } S' \neq Ch(S, \succ_f^*) \tag{1}$$

which implies, by definition of $\succ_f^*$, $\overline{Ch(S, \succ_f^*)} \succ_f \overline{S}'$ for all $\overline{S}' \subseteq \overline{S}$; that is,

$$Ch(\overline{S}, \succ_f) = \overline{Ch(S, \succ_f^*)}.$$

$\square$

Let $w_j^t \in Ch(S, \succ_f^*)$. By definition, $w_j \in \overline{Ch(S, \succ_f^*)}$, thus by the Claim, $w_j \in Ch(\overline{S}, \succ_f)$. Hence, $w_j^t \in A$.

Let $w_j^t \in A \subseteq S$; that is, $w_j \in Ch(\overline{S}, \succ_f)$ and $t \leq t'$ for all $w_j^{t'} \in S$. It follows from the Claim that $w_j \in \overline{Ch(S, \succ_f^*)}$, and there exists $t \in I_j$ such that $w_j^t \in Ch(S, \succ_f^*)$ and $t < t'$ for all $w_j^{t'} \in S$. Thus, $w_j^t \in Ch(S, \succ_f^*)$.

Particularly, if $S \subseteq W^{\mathbf{s}}$ is a simple set, we obtain that

$$Ch(S, \succ_f^*) = \left\{w_j^t \in S \mid w_j \in Ch(\overline{S}, \succ_f)\right\}.$$

$\square$

Therefore, denote the many-to-one problem *related* to $\mathbf{M}$ by:

$$\mathbf{M}^{\mathbf{s}} = (F, W^{\mathbf{s}}, \mathbf{s}, (\succ_f^*)_{f \in F}, (\succ_{w^t})_{w^t \in W^{\mathbf{s}}})$$

where, for each $f \in F$, $\succ_f^*$ is a preference relation over $2^{W^{\mathbf{s}}}$ and, for each $w^t \in W$, $\succ_{w^t}$ is the preference relation over 2^F generated by $\succ_w$ over 2^F. Let $\mathcal{M}^{\mathbf{s}}$ be the set of all matchings in $\mathbf{M}^{\mathbf{s}}$.

For simplicity of notation we will use the letter μ to denote a matching in $\mathbf{M}$ and will use the letter ν to denote a matching in $\mathbf{M}^{\mathbf{s}}$.

A matching $\nu \in \mathcal{M}^{\mathbf{s}}$ is *individually rational* if it is not blocked by any individual agent. In this case, a matching $\nu \in \mathcal{M}^{\mathbf{s}}$ is blocked by a position $w^t \in W^{\mathbf{s}}$, if $\emptyset \succ_{w^t} \nu(w^t)$; that is, if the agent w^t prefers remaining alone to be matched to $\nu(w^t)$.

Lemma 3.4. *Let $\nu \in \mathcal{M}^{\mathbf{s}}$ be an individually rational matching. Then $\nu(f)$ is a simple set for all $f \in F$.*

Proof. Suppose that $\nu(f)$ is not a simple set. From Lemma 3.3, $Ch(\nu(f), \succ_f^*)$ is a simple set. Since ν is individually rational, $\nu(f) = Ch(\nu(f), \succ_f^*)$. Thus, $\nu(f)$ is a simple set. □

Note that a matching ν in $\mathcal{M}^{\mathbf{s}}$ is *blocked by a worker-firm pair* (w^t, f) if $w^t \notin \nu(f)$, $w^t \in Ch(\nu(f) \cup \{w^t\}, \succ_f^*)$, and $f \succ_{w^t} \nu(w^t)$. A matching $\nu \in \mathcal{M}^{\mathbf{s}}$ is *stable* if it is not blocked by any individual agent or any worker-firm pair. Let $\mathcal{S}(\mathbf{M}^{\mathbf{s}})$ be the set of stable matchings in $\mathbf{M}^{\mathbf{s}}$.

We will define a correspondence between matchings in the original many-to-many model and matchings in the *related* many-to-one model as follows. We say that matching $\mu \in \mathcal{M}$ corresponds to matching $\nu \in \mathcal{M}^{\mathbf{s}}$ if f is the w's most preferred firm in $\mu(w)$, then $\nu(w^1) = f$, where w^1 is the first copy of worker w. Analogously, if $\overline{f}$ is the second w's most preferred firm in $\mu(w)$, then $\nu(w^2) = \overline{f}$, where w^2 is the second copy of worker w, and so forth. Also, if w does not fill his quota at $\mu(w)$, i.e., $|\mu(w)| = r < s_w$, $\nu(w^t) = \emptyset$ for those copies w^t of w, with $t > r$. Formally:

Let $\mu \in \mathcal{M}$ and $w_j \in W$ be such that $\mu(w_j) = \{f_{i_1}, f_{i_2}, \ldots, f_{i_r}\}$, $r \leq s_j$ and

$$\{f_{i_1}\} \succ_{w_j} \{f_{i_2}\} \succ_{w_j} \ldots \succ_{w_j} \{f_{i_r}\}.$$

Define,

$$\phi : \mathcal{M} \to \mathcal{M}^{\mathbf{s}},$$

by denoting $\phi(\mu) \equiv \phi_\mu$, as follows;

$$\phi_\mu(w_j^t) = \begin{cases} f_{i_t} \text{ if } f_{i_t} \in \mu(w_j), \\ \emptyset \text{ if } r < t \leq s_j. \end{cases} \tag{2}$$

$$\phi_\mu(f_{i_t}) = \{w_j^t \mid \phi_\mu(w_j^t) = f_{i_t}\}.$$

By definition, ϕ_μ is a matching.

Since each worker $w_j \in W$ has *responsive* preferences over 2^F, we have that $\phi_\mu(w_j^t) \succ_{w_j} \phi_\mu(w_j^{t+1})$ for all $t \in I_j$. The next theorem states that the sets of stable matchings of the many-to-many problem $\mathbf{M}$, and its *related* many-to-one problem $\mathbf{M}^{\mathbf{s}}$, are equivalent. Thus, we obtain a generalization of the result of Gale and Sotomayor [5].[13]

Theorem 3.5. *Let $\phi : \mathcal{M} \to \mathcal{M}^{\mathbf{s}}$ be defined as in (2). Let $\mathbf{M} = (F, W, \mathbf{s}, \succ)$ be a many-to-many problem such that workers' preferences are responsive and let $\mathbf{M}^{\mathbf{s}} = (F, W^{\mathbf{s}}, \mathbf{s}, (\succ_f^*)_{f \in F}, (\succ_{w^t})_{w^t \in W^{\mathbf{s}}})$ be its related many-to-one problem. Then, $\mu \in \mathcal{M}$ is stable if and only if $\phi(\mu) \in \mathcal{M}^{\mathbf{s}}$ is stable.*

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[13]Also see Roth and Sotomayor [15], Lemma 5.6.

It is important to highlight that Theorem 3.5 does not require the condition of substitutability for the firms' preferences.

The proof of Theorem 3.5 will use the following lemmata:

Lemma 3.6. *The function ϕ is injective.*

Proof. Let μ and $\mu' \in \mathcal{M}$ be such that $\mu \neq \mu'$. Consequently, there exists $f \in F$ such that $\mu(f) \neq \mu'(f)$; that is, there exists $w \in W$ such that, $w \in \mu(f)$ and $w \notin \mu'(f)$. Thus, by definition, there exist a copy of w, $w^t \in W^{\mathbf{s}}$ such that $w^t \in \phi_\mu(f)$ but, $w^t \notin \phi_{\mu'}(f)$. We thus get,

$$\phi_\mu(f) \neq \phi_{\mu'}(f). \tag{3}$$

By the bilateral nature of μ, $\mu' \in \mathcal{M}$ and definition of ϕ, $\phi_\mu(w^t) = f$ and $\phi_{\mu'}(w^t) \neq f$. Thus, $\phi_\mu(w^t) \neq \phi_{\mu'}(w^t)$. Finally, the last inequality and (3) yield that $\phi_\mu \neq \phi_{\mu'}$. $\qquad\square$

Remark 2. Note that when firms can employ only one worker, the natural injective correspondence defined by Gale and Sotomayor [5] is a particular case of our injective application $\phi : \mathcal{M} \to \mathcal{M}^{\mathbf{s}}$.

Lemma 3.7. *Let $\mu \in \mathcal{M}$ be a stable matching. Then, $\phi(\mu) \in \mathcal{M}^{\mathbf{s}}$ is a stable matching.*

Proof. Suppose that $\phi(\mu) \notin \mathcal{S}(\mathbf{M}^{\mathbf{s}})$ and denote $\nu \equiv \phi(\mu)$. We will consider the following two cases:

Case 1. The matching ν is not individually rational in $\mathbf{M}^{\mathbf{s}}$.

(a) If $w^t \in W^{\mathbf{s}}$ blocks ν, $\emptyset \succ_{w^t} \nu(w^t)$. Let $f_t = \nu(w^t)$; hence by definition, $f_t \in \mu(w)$. We obtain $\emptyset \succ_{w^t} f_t$, and so $\emptyset \succ_w f_t$.

Since $|\mu(w)| \leq s_w$, we have $|\mu(w) \backslash \{f_t\}| < s_w$. Since $\succ_w$ is responsive, $\mu(w) \backslash \{f_t\} \succ_w \mu(w)$, so $f_t \notin Ch(\mu(w), \succ_w)$. Finally, $\mu(w) \neq Ch(\mu(w), \succ_w)$; that is, worker $w \in W$ blocks μ, which is impossible. Therefore, w^t does not block ν.

(b) If $f \in F$ blocks ν, $\nu(f) \neq Ch(\nu(f), \succ_f^*)$. Note that

$$\overline{\nu(f)} = \{w \in W \mid w^t \in \nu(f)\} = \mu(f),$$

by the definition of ϕ. If $w^t \in W^{\mathbf{s}}$ is such that $w^t \in \nu(f)$ but $w^t \notin Ch(\nu(f), \succ_f^*)$, it follows that $w \in \mu(f)$ and from Lemma 3.3, $w \notin Ch(\mu(f), \succ_f)$. Thus, $\mu(f) \neq Ch(\mu(f), \succ_f)$. That is, firm $f \in F$ blocks μ, which contradicts our assumption. Therefore, f does not block ν.

Case 2. The matching ν is individually rational but is blocked by a worker-firm pair.

By assumption, there exists a pair (w_j^t, f_i) which blocks ν; that is, $w_j^t \notin \nu(f_i)$, and
(1) $w_j^t \in Ch(\nu(f_i) \cup \{w_j^t\}, \succ_{f_i}^*)$,
(2) $f_i \succ_{w_j^t} \nu(w_j^t)$.

From (1) and Lemma 3.3, $w_j \in Ch(\overline{\nu(f_i) \cup \{w_j^t\}}, \succ_{f_i})$ and $t < t'$ for all $w_j^{t'} \in \nu(f_i) \cup \{w_j^t\}$.

Claim. $\overline{\nu(f_i) \cup \{w_j^t\}} = \mu(f_i) \cup \{w_j\}$.

Proof of Claim. Remember that

$$\overline{\nu(f_i) \cup \{w_j^t\}} = \left\{ w_{j'} \mid \exists t \in I_{j'} \text{ such that } w_{j'}^{t'} \in \nu(f_i) \cup \{w_j^t\} \right\}.$$

($\subseteq$) Let $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$. If $j' \neq j$, there exists $t' \in I_{j'}$ such that $w_{j'}^{t'} \in \nu(f_i) \cup \{w_j^t\}$, hence $w_{j'}^{t'} \in \nu(f_i)$, thus $w_{j'} \in \mu(f_i)$. Accordingly, $w_{j'} \in \mu(f_i) \cup \{w_j\}$.

If $j' = j$, there exists $t' \in I_j$ such that $w_j^{t'} \in \nu(f_i) \cup \{w_j^t\}$. If $t' = t$, we have $w_j \in \mu(f_i) \cup \{w_j\}$. Otherwise, if $t' \neq t$, we have $w_j^{t'} \in \nu(f_i) \cup \{w_j^t\}$, and from (1) and Lemma 3.3, $t < t'$, which gives $\nu(w_j^{t'}) = f_i$. Then by (2), $\nu(w_j^{t'}) \succ_{w_j} \nu(w_j^t)$, thus $t' < t$, contrary to $t < t'$. This proves that there does not exist $t' \neq t$ such that $w_j^{t'} \in \nu(f_i) \cup \{w_j^t\}$.

Since $w_j^t \notin \nu(f_i)$, and there does not exist $t' \neq t$ such that $w_j^{t'} \in \nu(f_i) \cup \{w_j^t\}$, so $w_j \notin \mu(f_i)$. Therefore, $w_j \in \mu(f_i) \cup \{w_j\}$.

($\supseteq$) Let $w_{j'} \in \mu(f_i) \cup \{w_j\}$. If $j' = j$, then $w_{j'}^t \in \nu(f_i) \cup \{w_j^t\}$ holds trivially; hence, $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$. Otherwise, if $j' \neq j$, then $w_{j'} \in \mu(f_i)$; that is, $f_i \in \mu(w_{j'})$. Accordingly, there exists t' such that $w_{j'}^{t'} \in \nu(f_i)$, which implies $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$. □

This establishes that,

$$w_j \in Ch(\mu(f_i) \cup \{w_j\}, \succ_{f_i}). \tag{4}$$

Since $w_j \notin \mu(f_i)$, then $f_i \notin \mu(w_j)$. Suppose that $\nu(w_j^t) = f_{i'}$ for some $i' \neq i$; thus, $f_{i'} \in \mu(w_j)$, then from condition (2) we have $f_i \succ_{w_j^t} f_{i'}$; that is, $f_i \succ_{w_j} f_{i'}$; since $\succ_{w_j}$ is responsive. By Remark 1, we obtain:

$$f_i \in Ch(\mu(w_j) \cup \{f_i\}, \succ_{w_j}). \tag{5}$$

Finally, from (4) and (5), μ is blocked by a worker-firm pair (f_i, w_j), which is a contradiction. Therefore, $\nu \in \mathcal{S}(\mathbf{M^s})$. □

By Lemma 3.6 the function ϕ is injective. However, the function ϕ *is not surjective*. For example, let $F = \{f_1, f_2\}$, $W = \{w\}$ and $s_w = 2$, be such that w is acceptable by f_1 and f_2, and w has the following responsive preference,

$$\{f_1, f_2\} \succ_w \{f_1\} \succ_w \{f_2\} \succ_w \emptyset.$$

Consider the related many-to-one problem in which the worker w is replaced by two copies of himself; then $W^{\mathbf{s}} = \{w^1, w^2\}$. Now, preferences of each w^t are over $F \cup \emptyset$, and preferences of each f_i are over $W^{\mathbf{s}} \cup \emptyset$, for all $t = 1, 2$ and for all $i = 1, 2$. Each copy has the same preferences over firms that are identical with those of w with a quota of one, that is, $\{f_1\} \succ_{w^t} \{f_2\} \succ_{w^t} \emptyset$ for all $t = 1, 2$. In addition, each firm f_i has the following preferences:

$$\{w^1\} \succ_{f_i} \{w^2\} \succ_{f_i} \emptyset.$$

Let ν be a matching of the related many-to-one problem $\mathbf{M^s}$

$$\nu = \begin{pmatrix} f_1 & f_2 \\ w^2 & w^1 \end{pmatrix}.$$

Let μ be a matching of the many-to-many problem $\mathbf{M}$

$$\mu = \begin{pmatrix} \{f_1, f_2\} \\ w \end{pmatrix}.$$

Then,

$$\phi_\mu = \left(\begin{array}{cc} f_1 & f_2 \\ w^1 & w^2 \end{array} \right).$$

Thus, $\nu \neq \phi_\mu$. Since μ is the unique matching in $\mathbf{M}$, we conclude that the function ϕ is not surjective.

Next, we will consider the restriction of ϕ over the set of stable matchings, denoted by ϕ_S. The following lemma states that ϕ_S is a surjective function.

Lemma 3.8. *The function $\phi_S : \mathcal{S}(\mathbf{M}) \to \mathcal{S}(\mathbf{M^s})$ is surjective.*

Proof. Fix $\nu \in \mathcal{S}(\mathbf{M^s})$ and define for each $f_i \in F$ and $w_j \in W$

$$\begin{aligned}
\mu(f_i) &= \left\{ w_j \mid \exists t \in I_j \text{ such that } \nu(w_j^t) = f_i \right\} \text{ and} \\
\mu(w_j) &= \left\{ \nu(w_j^t) \mid t \in I_j \right\}.
\end{aligned}$$

By definition, $\phi_S(\mu) \equiv \nu$. We show that $\mu \in \mathcal{S}(\mathbf{M})$. Suppose that $\mu \notin \mathcal{S}(\mathbf{M})$. We will distinguish between two cases:

Case 1. The matching μ is not individually rational.

(a) If $w \in W$ blocks μ, $\mu(w) \neq Ch(\mu(w), \succ_w)$. Let $f \in \mu(w)$ but $f \notin Ch_w(\mu(w), \succ_w)$, then $\mu(w) \setminus \{f\} \succ_w \mu(w)$. Since $|\mu(w)| \leq s_w$, we have $|\mu(w) \setminus \{f\}| < s_w$. Then, since $\succ_w$ is responsive, $\emptyset \succ_{w^t} f$ for all $t \in I_w$. As $f \in \mu(w)$, there exists $t \in I_w$ such that $f = \nu(w^t)$. Since $\emptyset \succ_{w^t} \nu(w^t)$, worker w^t blocks the matching ν, which is a contradiction. Therefore, $\mu \in \mathcal{S}(\mathbf{M})$.

(b) If $f \in F$ blocks μ, $\mu(f) \neq Ch(\mu(f), \succ_f)$. Let $w \in \mu(f)$ but $w \notin Ch(\mu(f), \succ_f)$. Then, there exists $t \in I_w$ such that $w^t \in \nu(f)$. We observe that $\mu(f) = \overline{\nu(f)}$. Accordingly, $w \notin Ch\left(\overline{\nu(f)}, \succ_f\right)$ and thus, by Lemma 3.3, $w^t \notin Ch(\nu(f), \succ_f^*)$. Finally, $\nu(f) \neq Ch(\nu(f), \succ_f^*)$; that is, firm f blocks ν, which is a contradiction. Therefore, $\mu \in \mathcal{S}(\mathbf{M})$.

Case 2. The matching μ is individually rational but it is blocked by a worker-firm pair (w_j, f_i); that is, $f_i \notin \mu(w_j)$, and
(1) $f_i \in Ch(\mu(w_j) \cup \{f_i\}, \succ_{w_j})$,
(2) $w_j \in Ch(\mu(f_i) \cup \{w_j\}, \succ_{f_i})$.

Since $w_j \notin \mu(f_i)$, we have that for all $t \in I_j$, $w_j^t \notin \nu(f_i)$. By assumption, $\nu \in \mathcal{S}(\mathbf{M^s})$, therefore ν is individually rational and, by Lemma 3.4, $\nu(f_i)$ is a simple set. Thus, $\nu(f_i) \cup \{w_j^t\}$ is a simple set, which implies that, $w_j^t \in \nu(f_i) \cup \{w_j^t\}$ and $w_j \in Ch(\mu(f_i) \cup \{w_j\}, \succ_{f_i})$.

Claim. $\mu(f_i) \cup \{w_j\} = \overline{\nu(f_i) \cup \{w_j^t\}}$.

Proof of Claim. Remember that $\overline{\nu(f_i) \cup \{w_j^t\}} = \{w_{j'} \mid \exists t' \in I_{j'} \text{ such that } w_{j'}^{t'} \in \nu(f_i) \cup \{w_j^t\}\}$.

($\subseteq$) Let $w_{j'} \in \mu(f_i) \cup \{w_j\}$. If $j' = j$, then $w_{j'}^t \in \nu(f_i) \cup \{w_j^t\}$, hence $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$. Assume that $j' \neq j$, then $w_{j'} \in \mu(f_i)$. Accordingly, there exists $t' \in I_{j'}$ such that $w_{j'}^{t'} \in \nu(f_i)$, which implies $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$.

($\supseteq$) Let $w_{j'} \in \overline{\nu(f_i) \cup \{w_j^t\}}$. If $j' = j$, $w_{j'} \in \mu(f_i) \cup \{w_j^t\}$. Assume that $j' \neq j$; then, there exists $t' \in I_{j'}$ such that $w_{j'}^{t'} \in \nu(f_i) \cup \{w_j^t\}$, and so $w_{j'}^{t'} \in \nu(f_i)$; thus, by definition, $w_{j'} \in \mu(f_i)$. Accordingly, $w_{j'} \in \mu(f_i) \cup \{w_j^t\}$.   □

We conclude from Lemma 3.3 that, for all $t \in I_j$,

$$w_j^t \in Ch(\nu(f_i) \cup \{w_j^t\}, \succ_f^*). \tag{6}$$

Since $f_i \in Ch(\mu(w_j) \cup \{f_i\}, \succ_{w_j})$, $f_i \notin \mu(w_j)$.

Assume first that $|\mu(w_j)| = s_j$. Then, since $\succ_{w_j}$ is responsive, from Remark 1, there exists $f_{i'} \in \mu(w_j)$ such that $f_i \succ_{w_j} f_{i'}$. Thus, $f_i \succ_{w_j^t} f_{i'}$ for all $t \in I_j$.

Since $w_j \in \mu(f_{i'})$, this implies there exists $\widehat{t} \in I_j$ such that $\nu(w_j^{\widehat{t}}) = f_{i'}$; hence, $w_j^{\widehat{t}} \in \nu(f_{i'})$. By Lemma 3.4, $\nu(f_i)$ is a simple set for all $f_i \in F$. Hence, there does not exist $t \neq \widehat{t}$ such that $w_j^t \in \nu(f_{i'})$.

Since, $f_i \succ_{w_j^t} f_{i'}$ for all $t \in I_j$, then

$$f_i \succ_{w_j^{\widehat{t}}} f_{i'}. \tag{7}$$

We conclude from (6) and (7) that the pair $(w_j^{\widehat{t}}, f_i)$ blocks ν, which is a contradiction. Finally, $\mu \in \mathcal{S}(\mathbf{M})$.

Assume now that $|\mu(w_j)| < s_j$. Then, since $\succ_{w_j}$ is responsive, from Remark 1, $f_i \succ_{w_j} \emptyset$. Also, there exists $t' \in I_j$ such that $\nu(w_j^{t'}) = \emptyset$ and $f_i \succ_{w_j} \emptyset$, so $f_i \succ_{w_j^t} \emptyset$ for all $t \in I_j$; particularly,

$$f_i \succ_{w_j^{t'}} \nu(w_j^{t'}). \tag{8}$$

We conclude from (6) and (8) that the pair $(w_j^{t'}, f_i)$ blocks ν, which is a contradiction. Thus, $\mu \in \mathcal{S}(\mathbf{M})$. $\qquad\square$

We are now ready to prove Theorem 3.5.

Proof of Theorem 3.5. By Lemma 3.6, $\phi : \mathcal{M} \to \mathcal{M}^{\mathbf{s}}$ is injective. By Lemma 3.7, the restriction $\phi_S : \mathcal{S}(\mathbf{M}) \to \mathcal{S}(\mathbf{M}^{\mathbf{s}})$ is well-defined, and by Lemma 3.8, ϕ_S is surjective. Therefore, ϕ_S is a bijective function and the result follows. $\qquad\square$

3.1.1. *Existence of stable matchings under substitutability.* We provide here an alternative proof of the existence of stable matchings. The result is already known in the literature.[14]

Theorem 3.9. *Every many-to-many problem such that firms have substitutable preferences and workers have responsive preferences, has a non-empty set of stable matchings.*

We will prove Theorem 3.9 using the application ϕ_S, defined before Lemma 3.8. Previously, we need the following lemma, which says that the modified firms' preferences, $\succ_f^*$, in the *related* many-to-one problem inherits *substitutability*.

Lemma 3.10. *Assume f's preferences $\succ_f$ are substitutable in $\mathbf{M}$. Then, f's preferences $\succ_f^*$ are substitutable in $\mathbf{M}^{\mathbf{s}}$.*

Proof. Fix $f \in F$ and let $S \subseteq W^{\mathbf{s}}$ be such that w_j^t, $w_{j'}^{t'} \in S$, with $w_j^t \neq w_{j'}^{t'}$, and assume $w_j^t \in Ch(S, \succ_f^*)$. We will distinguish between two cases:

Case 1. $j \neq j'$.

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[14]Blair [2] shows that in a many-to-many matching model such that agents have *susbstitutable* preferences, the set of stable matchings is non-empty.

If $w_j^t \in Ch(S, \succ_f^*)$ then, by Lemma 3.3, $t \leq \bar{t}$ for all $w_j^{\bar{t}} \in S$ and $w_j \in Ch(\overline{S}, \succ_f)$. Since $\succ_f$ is substitutable, and $w_j, w_{j'} \in \overline{S}$ are such that $w_j \neq w_{j'}$,

$$w_j \in Ch(\overline{S} \backslash \{w_{j'}\}, \succ_f). \tag{9}$$

Also,

$$w_j^t \in S \backslash \left\{ w_{j'}^{t'} \right\}. \tag{10}$$

Finally, from (9), (10) and Lemma 3.3, we have that $w_j^t \in Ch(S \backslash \left\{ w_{j'}^{t'} \right\}, \succ_f^*)$.

Case 2. $j = j'$.

By hypothesis $t \neq t'$ and $w_j^t \in Ch(S, \succ_f^*)$. By Lemma 3.3, $w_j^{t'} \notin Ch(S, \succ_f^*)$ for all $t \neq t'$. Since $w_j^t \in S \backslash \left\{ w_j^{t'} \right\}$, we conclude that $Ch(S, \succ_f^*) \subset S \backslash \left\{ w_j^{t'} \right\} \subset S$. Hence by a property of the choice set, established by Blair [2],[15]

$$Ch(S, \succ_f^*) = Ch(S \backslash \left\{ w_j^{t'} \right\}, \succ_f^*)$$

Finally, since $w_j^t \in Ch(S, \succ_f^*)$, we have $w_j^t \in Ch(S \backslash \left\{ w_j^{t'} \right\}, \succ_f^*)$. □

Proof of Theorem 3.9. Assume **M** is such that firms have substitutable preferences and workers have responsive preferences. Then, Lemma 3.10 implies that in its related many-to-one model, $\mathbf{M^s}$, firms' preferences are substitutable. But $\mathcal{S}(\mathbf{M^s})$ is non-empty,[16] and hence, by Theorem 3.5, we conclude that $\mathcal{S}(\mathbf{M})$ is non-empty, which completes the proof. □

4. **Lattice Structure.** We recall that a partially ordered set $(L, \succeq)$[17] define a *lattice* if every two elements of L have a *supremum* (also called the least upper bound or join) and an *infimum* (also called the greatest lower bound or meet). Given $a, b \in L$, they are denoted by $a \vee b$ and $a \wedge b$, respectively.

Let $(L, \succeq)$ and $(L', \succeq')$ be two lattices. A function $f : L \longrightarrow L'$ is a *lattice isomorphism* if, for all $a, b \in L$,

1. $f(a \vee b) = f(a) \vee' f(b)$.
2. $f(a \wedge b) = f(a) \wedge' f(b)$.
3. f is bijective.

Two lattices are *isomorphic* if there exists a lattice isomorphism $f : L \longrightarrow L'$. In this case, we say that $(L, \succeq)$ and $(L', \succeq')$ are *lattices equivalent*. A function $f : L \longrightarrow L'$ is an *isomorphism of ordered sets* if and only if f is bijective and f as well as f^{-1} preserve the order.

The following two results were established by Birkhoff [1].

Lemma 4.1. *A function $f : L \longrightarrow L'$ is a lattice isomorphism if and only if the function f is an isomorphism of ordered sets.*

Lemma 4.2. *Let $(L, \succeq)$ be a lattice and $f : L \longrightarrow L'$ be an isomorphism of ordered sets. Then, $(L', \succeq')$ is a lattice. Also, by Lemma 4.1, they are isomorphic lattices.*

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[15]Let A and B be subsets of agents and let i be an agent. If $Ch_i(A) \subset B \subset A$, then $Ch_i(B) = Ch_i(A)$.

[16]See Roth and Sotomayor [15] (Theorem 6.5). Without the restricction of *substitutability*, it is easy to construct examples of preference profiles with the property that the set of stable matchings is empty (see, for instance, Example 2.7 in Roth and Sotomayor [15]).

[17]We denote by $\succeq$ any partial order.

In the previous section we defined the function ϕ mapping matchings of the original problem $\mathbf{M}$ into matchings of the related problem $\mathbf{M^s}$ and we proved that both problems have sets of equivalent stable matchings. In this section, using the function ϕ_S, we will establish that the set of stable matchings of $\mathbf{M}$ has a lattice structure.[18]

Following Blair [2], let μ_1 and μ_2 two individually rational matchings for all firms in the many-to-many problem $\mathbf{M}$. Define the partial order $\succeq_F^B$, by setting

$$\mu_1 \succeq_F^B \mu_2 \iff Ch(\mu_1(f) \cup \mu_2(f), \succ_f) = \mu_1(f) \text{ for all } f \in F.$$

Similarly, we define the partial order $\succeq_W^B$ for all workers in the many-to-many model $\mathbf{M}$.

Let ν_1 and ν_2 two individually rational matchings for all firms in the related many-to-one model $\mathbf{M^s}$. Define the partial order $\succeq_F^{*B}$, by setting

$$\nu_1 \succeq_F^{*B} \nu_2 \iff Ch(\nu_1(f) \cup \nu_2(f), \succ_f^*) = \nu_1(f) \text{ for all } f \in F.$$

Finally, we define the partial order $\succeq_{W^s}$ in the related many-to-one model $\mathbf{M^s}$, by setting

$$\nu_1 \succeq_{W^s} \nu_2 \iff \nu_1(w) \succeq_w \nu_2(w) \text{ for all } w \in W^s.$$

Theorem 4.3. *Let* $\mathbf{M} = (F, W, \mathbf{s}, \succ)$ *be a many-to-many matching problem such that firms have substitutable preferences and workers have responsive preferences. Then, the partial orders* $\succeq_F^B$ *and* $\succeq_W^B$ *endow the set of stable matchings with two lattice structures.*

Corollary 1. *Let* $\mathbf{M} = (F, W, \mathbf{s}, \succ)$ *be a many-to-many matching problem such that firms have substitutable preferences and workers have responsive preferences. Then,*
i) $(\mathcal{S}(\mathbf{M}), \succeq_F^B)$ *and* $(\mathcal{S}(\mathbf{M^s}), \succeq_F^{*B})$ *are lattices equivalent.*
ii) $(\mathcal{S}(\mathbf{M}), \succeq_W^B)$ *and* $(\mathcal{S}(\mathbf{M^s}), \succeq_{W^s})$ *are lattices equivalent.*

We will prove Theorem 4.3 using the application ϕ_S. Since ϕ_S is a bijective function, the function $\phi_S^{-1} : \mathcal{S}(\mathbf{M^s}) \to \mathcal{S}(\mathbf{M})$ is bijective also and it is given by:

$$\begin{aligned}
\left[\phi_S^{-1}(\nu) \right](f_i) &= \left\{ w_j \mid \exists t \in I_j \text{ such that } \nu(w_j^t) = f_i \right\} \\
\left[\phi_S^{-1}(\nu) \right](w_j) &= \left\{ \nu(w_j^t) \mid t \in I_j \right\}.
\end{aligned}$$

Theorem 4.4. *The function* $\phi_S^{-1} : \mathcal{S}(\mathbf{M^s}) \to \mathcal{S}(\mathbf{M})$ *is an isomorphism of ordered sets.*

The proof of Theorem 4.4 will use lemmata below:

Lemma 4.5. *Let* $\nu, \nu' \in \mathcal{S}(\mathbf{M^s})$. *Then, for all* $f \in F$,

$$\nu(f) \succ_f^{*B} \nu'(f) \text{ if and only if } \phi_S^{-1}(\nu)(f) \succ_f^B \phi_S^{-1}(\nu')(f).$$

Proof. ($\Longrightarrow$) From Theorem 3.5, there exist $\mu, \mu' \in \mathcal{S}(\mathbf{M})$ such that, for all $f \in F$

$$\left[\phi_S^{-1}(\nu) \right](f) = \mu(f) \tag{11}$$

$$\left[\phi_S^{-1}(\nu') \right](f) = \mu'(f). \tag{12}$$

By definition of the Blair's partial ordering,

$$\nu(f) \succ_f^{*B} \nu'(f) \text{ if and only if } Ch(\nu(f) \cup \nu'(f), \succ_f^*) = \nu(f). \tag{13}$$

We denote $\overline{T} = \mu(f) \cup \mu'(f)$ and define $T = \left\{ w_j^t \mid w_j \in \mu(f) \cup \mu'(f) \right\}$.
Claim. $T = \nu(f) \cup \nu'(f)$.

Proof of Claim. Let $w_j^t \in T$. Then, $w_j \in \mu(f) \cup \mu'(f)$. If $w_j \in \mu(f)$, by (11) there exist $t \in I_j$ such that $w_j^t \in \nu(f)$. Hence, $w_j^t \in \nu(f) \cup \nu'(f)$. If $w_j \in \mu'(f)$, we conclude from (12), in the same manner, that $w_j^t \in \nu(f) \cup \nu'(f)$. In the other hand, let $w_j^t \in \nu(f) \cup \nu'(f)$. If $w_j^t \in \nu(f)$, we conclude from (11) that $w_j \in \mu(f)$, thus $w_j \in \mu(f) \cup \mu'(f)$. Hence, $w_j^t \in T$. Suppose that $w_j^t \in \nu'(f)$, in the same manner, we can see that $w_j \in \mu(f) \cup \mu'(f)$. Then, $w_j^t \in T$.   □

We need to prove that $\mu(f) \succ_f^B \mu'(f)$, which is equivalent to show that

$$Ch(\mu(f) \cup \mu'(f), \succ_f) = \mu(f)$$

holds.

($\supseteq$) Let $w_j \in \mu(f)$ and so, there exists $t \in I_j$ such that $w_j^t \in \nu(f)$, but by (13) $Ch(\nu(f) \cup \nu'(f), \succ_f^*) = \nu(f)$, which implies that $w_j^t \in Ch(T, \succ_f^*)$. By definition of the choice set, $w_j \in Ch(\overline{T}, \succ_f)$; that is, $w_j \in Ch(\mu(f) \cup \mu'(f), \succ_f)$.

($\subseteq$) Suppose that $w_j \in Ch(\mu(f) \cup \mu'(f), \succ_f)$; that is, $w_j \in \mu(f) \cup \mu'(f)$. If $w_j \in \mu(f)$, then the inclusion follows. In the other case, if $w_j \in \mu'(f) \setminus \mu(f)$, by definition of ϕ_S, there exists $t \in I_j$ such that $w_j^t \in \nu'(f)$ and $w_j^t \notin \nu(f)$ for all $t \in I_j$; that is, from (13), $w_j^t \notin Ch(\nu(f) \cup \nu'(f), \succ_f^*)$ for all $t \in I_j$. Thus, Lemma 3.3 and Claim above would imply that $w_j \notin Ch(\mu(f) \cup \mu'(f), \succ_f)$, which is impossible by hypothesis. It might as well be the case that $w_j \in Ch(\mu(f) \cup \mu'(f), \succ_f)$ but then, would there exist $w_j^{t'} \in \nu(f) \cup \nu'(f)$ with $t' < t$ such that $w_j^{t'} \in Ch(\nu(f) \cup \nu'(f), \succ_f^*)$; that is, $w_j^{t'} \in \nu'(f)$ but also $w_j^t \in \nu'(f)$, which is again a contradiction, because $\nu'(f)$ is a simple set, since $\nu' \in \mathcal{S}(\mathbf{M^s})$. Finally, $w_j \in \mu(f)$. We conclude that $Ch(\overline{T}, \succ_f) = \mu(f)$.

($\Longleftarrow$) We will show that $\nu(f) \succ_f^{*B} \nu'(f)$; that is, $Ch(T, \succ_f^*) = \nu(f)$.

Let $w_j^t \in Ch(T, \succ_f^*)$. By Lemma 3.3, $w_j \in Ch_f(\overline{T}, \succ_f)$ and $t \leq t'$ for all $w_j^{t'} \in T$. Then, by hypothesis, $w_j \in \mu(f)$. Hence by definition $w_j^t \in \nu(f)$, hence,

$$Ch(T, \succ_f^*) \subseteq \nu(f). \tag{14}$$

To prove the other inclusion, suppose that for some $t \in I_j$, $w_j^t \in \nu(f)$. By hypothesis $\nu(f)$ is a simple set; that is, there does not exist $t' \neq t$ such that $w_j^{t'} \in \nu(f)$. Then, by definition of the function ϕ_S, $w_j \in \mu(f)$, so by hypothesis $w_j \in Ch(\overline{T}, \succ_f)$ and $w_j^t \in \nu(f) \cup \nu'(f)$. We suppose there exists $t' \neq t$ such that $w_j^{t'} \in \nu'(f) \setminus \nu(f)$; that is, $w_j \in \mu'(f)$ but $w_j \notin \mu(f)$, which contradicts the hypothesis. Thus, there does not exist $t' \neq t$ such that $t' < t$ and $w_j^{t'} \in \nu'(f)$. Finally by Lemma 3.3, $w_j^t \in Ch(T, \succ_f^*)$; that is,

$$\nu(f) \subseteq Ch(T, \succ_f^*). \tag{15}$$

We conclude from (14) and (15) that $Ch(T, \succ_f^*) = \nu(f)$.   □

Lemma 4.6. *Let ν, $\nu' \in \mathcal{S}(\mathbf{M^s})$. Then, for all $w \in W$ and $w^t \in W^{\mathbf{s}}$,*

$$\left[\phi_S^{-1}(\nu) \right](w) \succ_w^B \left[\phi_S^{-1}(\nu') \right](w) \text{ if and only if } \nu(w^t) \succ_{w^t} \nu'(w^t).$$

Proof. ($\Longrightarrow$) From Theorem 3.3, there exist μ, $\mu' \in \mathcal{S}(\mathbf{M})$ such that for all $w \in W$, $\left[\phi_S^{-1}(\nu) \right](w) = \mu(w)$ and $\left[\phi_S^{-1}(\nu') \right](w) = \mu'(w)$. By hypothesis, $\mu(w) \succ_w^B \mu'(w)$,

which is equivalent to $Ch(\mu(w) \cup \mu'(w), \succ_w) = \mu(w)$; that is, $\mu(w) \succ_w T$ for all $T \subseteq \mu(w) \cup \mu'(w)$.

Since $\succ_{w_j}$ is responsive, there do not exist $f' \in \mu'(w)\backslash\mu(w)$ and $f \in \mu(w)$ such that $f' \succ_w f$, otherwise we will have $|\mu(w)\backslash\{f\}| < s_w$ and $\mu(w)\backslash\{f\} \cup \{f'\} \succ_w \mu(w)$, which is impossible since $\mu(w)\backslash\{f\} \cup \{f'\} \subset \mu(f) \cup \mu'(f)$. Therefore, $f \succ_w f'$ for all $f' \in \mu'(w)\backslash\mu(w)$ and $f \in \mu(w)$. Then, $f \succ_{w^t} f'$ for all $t \in I_w$; that is, $\nu(w^t) \succ_{w^t} \nu'(w^t)$.

($\Longleftarrow$) If $f_{i_t} = \nu(w^t) \succ_{w^t} \nu'(w^t) = f_{i'_t}$, we have $f_{i_t} \succ_w f_{i'_t}$. By hypothesis $\mu, \mu' \in \mathcal{S}(\mathbf{M})$, $|\mu(w)| \leq s_w$ and $|\mu'(w)| \leq s_w$, so $|\mu(w)\backslash\{f_{i_t}\}| < s_w$ and $\left|\mu'(w)\backslash\{f_{i'_t}\}\right| < s_w$. Also, we know that $\succ_{w_j}$ is responsive. Then, $\mu(w) \succ_{w_j} \mu(w)\backslash\{f_{i_t}\} \cup \{f_{i'_t}\}$. Since $\mu(w)\backslash\{f_{i_t}\} \cup \{f_{i'_t}\} \subset \mu(f) \cup \mu'(f)$, we have that

$$Ch(\mu(w) \cup \mu'(w), \succ_w) = \mu(w).$$

$\square$

Proof of Theorem 4.4. By Theorem 3.5, the mapping $\phi_S : \mathcal{S}(\mathbf{M}) \to \mathcal{S}(\mathbf{M^s})$ is bijective, therefore, ϕ_S^{-1} is bijective too. In addition, by Lemmata 4.5 and 4.6, we have that ϕ_S^{-1} preserves the Blair partial order of the agents. Then, by definition, $\phi_S^{-1} : \mathcal{S}(\mathbf{M^s}) \to \mathcal{S}(\mathbf{M})$ is an isomorphism of ordered sets. $\square$

We are ready now to prove Theorem 4.3

Proof of Theorem 4.3. By Theorem 4.4, we have proved that $\mathcal{S}(\mathbf{M})$ and $\mathcal{S}(\mathbf{M^s})$ are isomorphic as ordered sets. Also, by Theorem 10 of Echenique and Oviedo [3], we know that $(\mathcal{S}(\mathbf{M^s}), \succeq_F^{*B})$ and $(\mathcal{S}(\mathbf{M^s}), \succeq_W s)$ are non-empty lattices. We conclude from Lemma 4.2 that $(\mathcal{S}(\mathbf{M}), \succeq_F^B)$ and $(\mathcal{S}(\mathbf{M}), \succeq_W^B)$ have lattice structures as well. Finally, from Lemma 4.1 it follows that the sets of stable matchings of both problems are isomorphism lattices. $\square$

Proof of Corollary 1. By Theorem 4.4, $\phi_S^{-1} : \mathcal{S}(\mathbf{M^s}) \to \mathcal{S}(\mathbf{M})$ is an isomorphism of ordered sets. Moreover, by Theorem 4.3, $(\mathcal{S}(\mathbf{M}), \succeq_F^B)$ and $(\mathcal{S}(\mathbf{M}), \succeq_W^B)$ have lattice structures. Finally, from Lemma 4.1 it follows that i)$(\mathcal{S}(\mathbf{M}), \succeq_F^B)$ and $(\mathcal{S}(\mathbf{M^s}), \succeq_F^{*B})$ are lattices equivalent, and ii)$(\mathcal{S}(\mathbf{M}), \succeq_W^B)$ and $(\mathcal{S}(\mathbf{M^s}), \succeq_W s)$ are lattices equivalent. $\square$

Corollary 2. *Let μ, $\mu' \in \mathcal{S}(\mathbf{M})$. If for some w we have that $\mu(w) \succ_w^B \mu'(w)$, then $f \succ_w f'$ for all $f \in \mu(w)$ and $f' \in \mu'(w)\backslash\mu(w)$.*

Proof. If $\mu(w) \succ_w^B \mu'(w)$, then $Ch(\mu(w) \cup \mu'(w), \succ_w) = \mu(w)$, since $\succ_w$ is responsive, therefore, there is no $f' \in \mu'(w)\backslash\mu(w)$ and $f \in \mu(w)$ such that $f' \succ_w f$. Otherwise $\mu(w)\backslash\{f\} \cup \{f'\} \succ_w \mu(w)$, contradicting the hypothesis. Thus, $f \succ_w f'$ for all $f \in \mu(w)$ and $f' \in \mu'(w)\backslash\mu(w)$. $\square$

5. **Final Remarks.** This paper contributes to the literature by proposing a new tool (the bijection ϕ_S) to prove results in the many-to-many model that follow from results in the many-to-one model. In particular, we gave an alternative proof of the non-emptiness and lattice structure of the set of stable matchings in the many-to-many model with *substitutable* preferences for firms, and *responsive* preferences for workers. The proof follows from results that hold in the related many-to-one model. We finish the paper with two final remarks.

First, Lemma 3.10 shows that, if the preference relation $\succ_f$ is *substitutable* in the model $\mathbf{M}$, then the preference relation $\succ_f^*$ is *substitutable* in the model $\mathbf{M^s}$. Nevertheless, if we assume that the preference relation $\succ_f$ is *responsive* in the

model $\mathbf{M}$, then we can not assure that the preference relation $\succ_f^*$ is *responsive* in the model $\mathbf{M^s}$. The following example shows this possibility:

Example 1 (Continued). Let $F = \{f_1, f_2\}$, $W = \{w_1, w_2\}$ and assume that the quota of each agent is equal to two. The agents have the following responsive preferences over the other side of the market:

$$\{w_1, w_2\} \succ_{f_i} \{w_1\} \succ_{f_i} \{w_2\} \succ_{f_i} \emptyset \qquad \text{for } i = 1, 2.$$

$$\{f_1, f_2\} \succ_{w_j} \{f_1\} \succ_{w_j} \{f_2\} \succ_{w_j} \emptyset \qquad \text{for } j = 1, 2.$$

Consider the related many-to-one problem in which each worker w_j is replaced by two positions (copies) of himself, then $W^\mathbf{s} = \{w_1^1, w_1^2, w_2^1, w_2^2\}$. Now, preferences of each w_j^t are over $F \cup \emptyset$, and preferences of each f_i are over $W^\mathbf{s} \cup \emptyset$. Each copy has the same preferences over firms that are identical with those of w_j with a quota of one; that is, $\{f_1\} \succ_{w_j^t} \{f_2\} \succ_{w_j^t} \emptyset$ for all $j = 1, 2$ and $t = 1, 2$. For each firm f_i in the related many-to-one problem, we consider the modified preference relation $\succ_f^*$ over $2^{W^\mathbf{s}}$:

$$\{w_1^1, w_2^1\} \succ_{f_i}^* \{w_1^1, w_2^2\} \succ_{f_i}^* \{w_1^2, w_2^2\} \succ_{f_i}^* \{w_1^1\} \succ_{f_i}^* \{w_1^2\} \succ_{f_i}^* \{w_2^1\} \succ_{f_i}^* \{w_2^2\} \succ_{f_i}^* \emptyset.$$

Let $T = \{w_1^2\}$. Since $\{w_1^1\} \succ_{f_i}^* \{w_2^2\}$ holds, if the preference relation $\succ_{f_i}^*$ is responsive over $2^{W^\mathbf{s}}$ we have that $T \cup \{w_1^1\} \succ_{f_i}^* T \cup \{w_2^2\}$; that is,

$$\{w_1^1, w_1^2\} \succ_{f_i}^* \{w_1^2, w_2^2\},$$

But, this is not true since $\{w_1^1, w_1^2\}$ is not a simple set, therefore

$$\emptyset \succ_{f_i}^* \{w_1^1, w_1^2\}$$

which implies that the set $\{w_1^2, w_2^2\}$ is not acceptable for the firm f_i, which is a contradiction. $\qquad\qquad\square$

Gale and Sotomayor [5] proved that there exists a bijection between the set of stable matchings of the colleges admission problem (with responsive preferences) and the set of stable matchings of its related marriage problem. We prove the existence of a bijection, ϕ_S, between the set of stable matchings of a many-to-many matching problem (such that the workers preferences are responsive and firms preferences are substitutable), and the set of stable matchings of its related many-to-one model. This function allows us to give an alternative proof of the result which says that $\mathcal{S}(\mathbf{M})$ is non-empty and has a lattice structure. Nevertheless, if $\mathbf{M}$ is an instance such that all agents have responsive preferences, in Example 1 (Continued), we have showed that the partial order $\succ_f^*$ defined over $2^{W^\mathbf{s}}$ is not necessarily responsive in $\mathbf{M^s}$. Then, since the substitutability condition is weaker than the responsiveness condition, we can not assure the existence of a bijection between the sets of stable matchings of the many-to-many model and its related marriage problem.

Second, using the bijection ϕ_S, it would be interesting to ask whether one could define two binary operations in a many-to-many model as a consequence of the result that hold in the related many-to-one model. Thus, using these binary operations I would prove that the set of stable matchings has a lattice structure. In addition, Sotomayor [16] defined, in a many-to-many model, a new concept of stability called *setwise stability*. A matching μ is setwise stable if it is individually rational and cannot be blocked by a coalition that forms new links only among its members, but may preserve its links to agents outside of the coalition. Sotomayor [10] proposes

to find a sufficient condition for the existence of *setwise stable matchings*. Thus, it would be interesting to ask, using the bijection ϕ_S, whether one could prove that the *set of setwise stable matchings* is non-empty. But, we leave the answers of these questions for further research.

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