Characterization of unitary operators by elementary operators and unitarily invariant norms *

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Abstract

In this work we characterize unitary operators via inequalities of elementary operators with unitarily invariant norms.[1]

1 Introduction

Let $\mathscr{H}$ be a complex Hilbert space, and let $(B(\mathscr{H}), \|.\|)$ the C^*-algebra of all bounded linear operators on $\mathscr{H}$ with the usual norm. We denote by $Gl(\mathscr{H})$ the group of invertible elements of $B(\mathscr{H})$, $U(\mathscr{H})$ the unitary operators and $Gl_s(\mathscr{H})$ the set of all invertible and selfadjoint operators.

A linear operator $R : B(\mathscr{H}) \to B(\mathscr{H})$ defined by $R(X) = \sum_{i=1}^{n} A_i X B_i$, where $A_i, B_i \in B(\mathscr{H})$, with $1 \leq i \leq n$, is called an elementary operator on $B(\mathscr{H})$; and we denote by $R = R_{\overline{A},\overline{B}}$, where $\overline{A} = (A_1, ..., A_n)$ and $\overline{B} = (B_1, ..., B_n)$. This class of operators includes many important operators of $B(\mathscr{H})$ such as the inner derivation $\delta_A(X) = AX - XA$, the multiplication operator $M_{A,B}(X) = AXB$, the symmetrized two-sided multiplication $U_{A,B}(X) = AXB + BXA$ and the operator $V_{A,B} = AXB - BXA$. We denote by Φ_S the operator $U_{S,S^{-1}}$.

In [8], Nakamoto proved that a bounded linear operator A on $\mathscr{H}$ is normal if and only if $\|\delta_A(X)\|_2 = \|\delta_{A^*}(X)\|_2$ for all $X \in B_2(\mathscr{H})$ (Hilbert-Schmidt class). In [9], A. Seedik characterizes the operators S for which the Corach-Porta-Recht inequality ([4], [1]) holds, more precisely he proved that an invertible operator S is a non zero complex multiple of some selfadjoint operator if and only if $\|\Phi_S(X)\| \geq 2\|X\|$ for all $X \in B(\mathscr{H})$.

On the other hand, in [7], the authors ask whether the same characterization obtained on [9] is true for other unitarily invariant norms. They proved that S is necessarily a normal operator if $2\|X\|_{\mathcal{I}} \leq \|\Phi_S(X)\|_{\mathcal{I}}$ for all $X \in B(\mathscr{H})$, with rank one (Corollary 2.2). Furthermore, in this work Magajna et al. obtained that if $\mathcal{I}$ a norm ideal and we denote by $\overline{A} = (tS, \frac{1}{t}S^{-1})$ and $\overline{B} = (S^{-1}, S)$ for $t > 0$, then

$$\gamma S \in Gl_s(\mathscr{H}), \lambda \in \mathbb{C} - \{0\} \text{ if and only if } \inf_{t>0} \|R_{\overline{A},\overline{B}}(X)\|_{\mathcal{I}} \geq 2\|X\|_{\mathcal{I}}$$

for all $X \in \mathcal{I}$ of rank 1.

In a recent work [11], A. Seedik obtains some characterizations of some subclasses of normal operators in $B(\mathscr{H})$ by inequalities or equalities (associated with elementary operators).

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Motivated by these results, in [3] we characterized the normal invertible operators of $B(\mathscr{H})$ via unitarily invariant norms and elementary operators. The purpose of this work is to find the set for a given norm ideal $\mathcal{I}$

$$E_{\mathcal{I}} = \{S \in Gl(\mathscr{H}) : \|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}} \text{ for every } X \in \mathcal{I}\}.$$

By a result obtained in [3] this subset is contained in $N(\mathscr{H}) = \{T \in B(\mathscr{H}) : TT^* = T^*T\}$.

2 Preliminaries

We recall that $\mathcal{I}$ is a norm ideal of $B(\mathscr{H})$ if $\mathcal{I}$ is a two-sided ideal of $B(\mathscr{H})$ and a Banach space with respect to the norm $\|.\|_{\mathcal{I}}$ satisfying:

1. $\|XTY\|_{\mathcal{I}} \leq \|X\|\|T\|_{\mathcal{I}}\|Y\|$ for $T \in \mathcal{I}$ and $X, Y \in B(\mathscr{H})$,
2. $\|X\|_{\mathcal{I}} = \|X\|$ if X is of rank one.

In particular, condition 1. implies that the norm is unitarily invariant, $\|UXV^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$ for $X \in \mathcal{I}$ and any $U, V \in U(\mathscr{H})$. The most known examples of norm ideals of $B(\mathscr{H})$ are the so called p-Schatten class with $p \geq 1$ defined by

$$B_p(\mathscr{H}) = \{X \in B_0(\mathscr{H}) : \{s_j(X)\} \in l^p\},$$

where $\{s_j(X)\}$ denotes the sequence of singular values of X, rearranged such that $s_1(X) \geq s_2(X) \geq \cdots$ with multiplicies counted, with norm given by $\|X\|_p = (\sum s_j(X)^p)^{1/p}$ and $B_0(\mathscr{H})$ is the ideal of compact operators. When $p = \infty$, the norm $\|.\|_\infty$ coincides with the usual norm $\|X\| = s_1(X)$. For a complete account of the theory of unitarily invariant norms the reader is referred to [5].

For sake of completness, we recall three statements that we will use in the following section. Given a norm ideal $\mathcal{I}$ and a linear operator $P : \mathcal{I} \to \mathcal{I}$ we denote by

$$\|P\|_{B(\mathcal{I})} = \sup\{\|P(X)\|_{\mathcal{I}} : \|X\|_{\mathcal{I}} = 1\}.$$

Theorem 2.1. *([10], Theorem 2.1.)*
Let $S \in B(\mathscr{H})$ be an invertible and selfadjoint operator and $\mathcal{I}$ a norm ideal. Then we have the following inequality:

$$\|\Phi_S\|_{B(\mathcal{I})} \geq \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}. \tag{1}$$

Lemma 2.2. *([11], Theorem 3.1.)*
Let $S \in Gl(\mathscr{H})$. Then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$, for some unitary operator V.

Theorem 2.3. *([3], Theorem 2.1.) Let $S \in Gl(\mathscr{H})$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent:*

1. *S is normal,*
2. *$\|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} = \|M_{S^*,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
3. *$\|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} \geq 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
4. *$\|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} \geq 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$, with rank 1.*

Remark 2.4. Other characterization of normal invertible operators is given in the following statement.

Proposition 2.5. *Let $S \in Gl(\mathscr{H})$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent:*

1. *S is normal,*
2. *$\|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} \leq \|M_{S^*,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$,*
3. *$\|M_{S,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S}(X)\|_{\mathcal{I}} \leq \|M_{S^*,S^{-1}}(X)\|_{\mathcal{I}} + \|M_{S^{-1},S^*}(X)\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$, of rank 1.*

Proof. The implication *1* $\Rightarrow$ *2* follows inmediately from Theorem 2.3 and *2* $\Rightarrow$ *3* is trivial. *3* $\Rightarrow$ *1* We consider rank one operators $X = x \otimes y \in \mathcal{I}$ with $x, y \in \mathscr{H}$, then it follows that the following inequality holds

$$\|M_{S,S^{-1}}(x \otimes y)\|_{\mathcal{I}} + \|M_{S^{-1},S}(x \otimes y)\|_{\mathcal{I}} \leq \|M_{S^*,S^{-1}}(x \otimes y)\|_{\mathcal{I}} + \|M_{S^{-1},S^*}(x \otimes y)\|_{\mathcal{I}}$$

or equivalently,

$$\|S(x \otimes y)S^{-1}\|_{\mathcal{I}} + \|S^{-1}(x \otimes y)S\|_{\mathcal{I}} \leq \|S^*(x \otimes y)S^{-1}\|_{\mathcal{I}} + \|S^{-1}(x \otimes y)S^*\|_{\mathcal{I}}.$$

It is easy to see that $A(u \otimes v)B = Au \otimes B^*v$ and $\|u \otimes v\|_{\mathcal{I}} = \|u \otimes v\| = \|u\|\|v\|$, for all $u, v \in \mathscr{H}$ and $A, B \in B(\mathscr{H})$. Then

$$\|S(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S^*(y)\| \leq \|S^*(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S(y)\|. \tag{2}$$

Assume that S is not a normal operator. Consequently there exist vector $x \in \mathscr{H}$, $\|x\| = 1$ such that $\|Sx\| > \|S^*x\|$ (or $\|Sx\| < \|S^*x\|$). It follows, from (2), that for all $y \in \mathscr{H}$ with $\|y\| = 1$, $\|Sy\| > \|S^*y\|$ (or $\|Sy\| < \|S^*y\|$), so we have from (2) that

$$0 < (\|Sx\| - \|S^*x\|) \leq (\|Sy\| - \|S^*y\|)\|(S^{-1})(x)\|\|(S^*)^{-1}(y)\|^{-1} \leq (\|Sy\| - \|S^*y\|)\|S^{-1}\|\|S^*\|.$$

Hence, for all $y \in \mathscr{H}$ with $\|y\| = 1$

$$\|Sx\| + \|S^{-1}\|\|S\|\|S^*y\| \leq \|S^*x\| + \|S^{-1}\|\|S\|\|Sy\|$$

Thus $\|Sx\| + \|S^{-1}\|\|S\|\|S^*\| \leq \|S^*x\| + \|S^{-1}\|\|S\|\|S\|$. It follows that $\|Sx\| \leq \|S^*x\|$, which it is a contradiction. Therefore S is a normal operator. $\square$

In [6], Kittaneh obtained the generalization of the Corach-Porta-Recht inequality in any norm ideal $\mathcal{I}$. More precisely, for Hilbert-space operators T, R, X with T, R invertible operators and a unitarily invariant norm $\mathcal{I}$, the inequality

$$2\|X\|_{\mathcal{I}} \leq \|R^*XT^{-1} + R^{-1}XT^*\|_{\mathcal{I}}, \tag{3}$$

holds for all $X \in \mathcal{I}$.

In this work, we consider the polar decomposition of $S \in Gl(\mathscr{H})$ given by $S = U|S|$ with $|S| = (S^*S)^{1/2}$ positive and $U \in U(\mathscr{H})$.

3 Main results

Proposition 3.1. *If $S \in Gl_s(\mathscr{H})$ and $\|\Phi_S(X)\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$, then $S = \|S\|V$ with $V \in U(\mathscr{H})$.*

Proof. Since $\|\Phi_S(X)\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ for all $X \in \mathcal{I}$, it follows that $\|\Phi_S\|_{B(\mathcal{I})} \leq 2$.

By (1) we have that $2 \geq \|\Phi_S\|_{B(\mathcal{I})} \geq \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} \geq 2$.

From this we derive that $\|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} = 2$, so it follows immediately that $\|S\|\|S^{-1}\| = 1$. So from [11] , Lemma 2 it turns out that $S = \|S\|V$ with V an unitary operator. □

Now, we obtain a generalization of [11], Th. 8.

Corollary 3.2. *Let $\mathcal{I}$ a norm ideal, then*

$$U_s(\mathscr{H}) = \{S \in Gl_s(\mathscr{H}) : \|S\| = 1 \text{ and } \|\Phi_S(X)\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\}.$$

where $U_s(\mathscr{H})$ denotes the unitary selfadjoint operators in $B(\mathscr{H})$.

Proof. If $S \in U_s(\mathscr{H})$ then $S \in Gl_s(H), \|S\| = 1$ and for any $X \in \mathcal{I}$ we have

$$\|\Phi_S(X)\|_{\mathcal{I}} = \|SXS^{-1} + S^{-1}XS\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}},$$

by the unitary invariance of the norm. Thus, the result follows immediately from Proposition 3.1. □

In the previous statement, if we omit the hypothesis $\|S\| = 1$ we obtain a characterization of $\mathbb{R}^* U_s(\mathscr{H})$, with $\mathbb{R}^* = \mathbb{R} - \{0\}$.

If $S = U|S| \in U(\mathscr{H})$ then necessarily $|S| \in U_s(\mathscr{H})$ and in consequence $|S|$ is characterized by the previous corollary. In the following result we prove that a condition which holds for the modules of S is a sufficient condition for determinate if S is an unitary operator in $B(\mathscr{H})$.

Theorem 3.3. *Let $\mathcal{I}$ a norm ideal then*

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1, S = U|S| \text{ and } \|\Phi_{|S|}(X)\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\}.$$

Proof. By the hypothesis, $\|\Phi_{|S|}\|_{B(\mathcal{I})} \leq 2$. By (1) we have the following lower bound for the operator $\Phi_{|S|}$,

$$\|\Phi_{|S|}\|_{B(\mathcal{I})} \geq \| \, |S| \, \| \, \| \, |S|^{-1} \, \| + \frac{1}{\| \, |S| \, \| \, \| \, |S|^{-1} \, \|} \geq 2.$$

From this inequality and the condition obtained above, we get that

$$\|\Phi_{|S|}\|_{B(\mathcal{I})} = \| \, |S| \, \| \, \| \, |S|^{-1} \, \| + \frac{1}{\| \, |S| \, \| \, \| \, |S|^{-1} \, \|} = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} = 2.$$

In other words, $\|S\|\|S^{-1}\| = 1$. Then the result follows immediately from the Lemma 2.2.

On the other hand, if $S = U|S| \in U(\mathscr{H})$ then for any $X \in \mathcal{I}$ we may write $X = U^*Y$ with $Y \in \mathcal{I}$ and since $\|S\| = 1$ we have that

$$\|\Phi_{|S|}(X)\|_{\mathcal{I}} = \||S|U^*Y|S|^{-1} + |S|^{-1}U^*Y|S|\|_{\mathcal{I}} = \|S^*YS^{-1} + S^{-1}YS^*\|_{\mathcal{I}} \leq 2\|Y\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}.$$

This concludes the proof. □

Remark 3.4. 1. The operator which characterize the unitary operators of $B(\mathscr{H})$ can be written as follows

$$\tau_{S^*,S} = SXS^{-1} + (S^*)^{-1}XS^* = SXS^{-1} + (S^{-1})^*XS^* = SXS^{-1} + (SX^*S^{-1})^*$$

in particular if X is a selfadjoint operator then

$$SXS^{-1} + (S^*)^{-1}XS^* = 2Re(SXS^{-1}),$$

where $Re(T) = \frac{1}{2}(T + T^*)$.

A natural question is if with the selfadjoint operators of $\mathcal{I}$ we can describe all $U(\mathscr{H})$, more precisely

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_{\mathcal{I}} \leq \|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}, X = X^*\}?$$

We give a particular example when $\mathcal{I}$ is the 2-Schatten ideal. Every $X \in B(\mathscr{H})$ can be written as $X = Re(X) + iIm(X)$, where $Re(X), Im(X)$ are selfadjoint operators and

$$Re(X) = \frac{1}{2}(X + X^*) \qquad \text{and} \qquad Im(X) = \frac{1}{2i}(X - X^*).$$

We call this the Cartesian decomposition of X.

In [2], Bhatia and Kittaneh prove sharp inequalities comparing the norm $\|X\|_p$ with $(\|Re(X)\|_p^2 + \|Im(X)\|_p^2)$. More precisely, for $p = 2$ we get

$$\|Re(X)\|_2^2 + \|Im(X)\|_2^2 = \|X\|_2^2.$$

Theorem 3.5.

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_2 \leq \|X\|_2 \text{ for all } X \in B_2(\mathscr{H}), X = X^*\}.$$

Proof. Let $Z = Re(Z) + iIm(Z) \in B_2(\mathscr{H})$ then

$$\begin{aligned}
\|SZS^{-1} + (S^*)^{-1}ZS^*\|_2^2 &= \|S(Re(Z) + iIm(Z))S^{-1} + (S^*)^{-1}(Re(Z) + iIm(Z))S^*\|_2^2 \\
&= \|2(Re(SRe(Z)S^{-1}) + iRe(SIm(Z)S^{-1}))\|_2^2 \\
&= 4\|Re(SRe(Z)S^{-1}) + iRe(SIm(Z)S^{-1})\|_2^2 \\
&= 4(\|Re(SRe(Z)S^{-1})\|_2^2 + \|Re(SIm(Z)S^{-1})\|_2^2) \\
&\leq 4(\|Re(Z)\|_2^2 + \|Im(Z)\|_2^2) = 4\|Z\|_2^2.
\end{aligned}$$

□

Since the norm of the operators $\Phi_{|S|}(X), SXS^{-1} + (S^*)^{-1}XS^*$ and $S^*YS^{-1} + S^{-1}YS^*$ are related via unitaries operators, more precisely for all $X \in \mathcal{I}$

$$\|\Phi_{|S|}(X)\|_{\mathcal{I}} = \||S|X|S|^{-1} + |S|^{-1}X|S|\|_{\mathcal{I}} = \|S^*(UX)S^{-1} + S^{-1}(UX)S^*\|_{\mathcal{I}}$$

and

$$\|\Phi_{|S|}(X)\|_{\mathcal{I}} = \|U^*(SXS^{-1} + (S^*)^{-1}XS^*)U\|_{\mathcal{I}} = \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}}$$

where $S = U|S|$, we obtain the following characterizations of $U(\mathscr{H})$.

Theorem 3.6. *Let $\mathcal{I}$ a norm ideal then*

$$\begin{aligned} U(\mathscr{H}) &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\} \\ &= \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}} \text{ for all } X \in \mathcal{I}\}. \end{aligned}$$

In particular, if $\mathcal{I} = B_2(\mathscr{H})$ we get

$$U(\mathscr{H}) = \{S \in Gl(\mathscr{H}) : \|S\| = 1 \text{ and } \|Re(SXS^{-1})\|_2 = \|X\|_2 \text{ for all } X \in B_2(\mathscr{H}), X = X^*\}.$$

We observe that if $S = \lambda V$ with $V \in U(\mathscr{H})$ and $\lambda \in \mathbb{R} - \{0\}$, then for every $X \in \mathcal{I}$

$$\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}. \tag{4}$$

Now motivated by the conclusion of Proposition 3.1 and the equality (4), we characterize the real multiples of some unitary operator.

Theorem 3.7. *Let $S \in Gl(\mathscr{H})$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent:*

1. $S = \lambda V$ *with* $\lambda \in \mathbb{R}^*$ *and* $V \in U(\mathscr{H})$*,*
2. $\|SXS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
3. $\|S^*XS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
4. $\|S^*XS^{-1}\|_{\mathcal{I}} + \|S^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
5. $\|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
6. $\|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
7. $\|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} = 2$ *for every* $X \in \mathcal{I}$*,* $\|X\|_{\mathcal{I}} = 1$.
8. $\|SXS^{-1}\|_{\mathcal{I}} + \|(S^*)^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
9. $\|SXS^{-1}\|_{\mathcal{I}} + \|(S^*)^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
10. $\|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}} \leq 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
11. $\|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}}$ *for every* $X \in \mathcal{I}$*,*
12. $\|SXS^{-1} + (S^*)^{-1}XS^*\|_{\mathcal{I}} = 2$ *for every* $X \in \mathcal{I}$*,* $\|X\|_{\mathcal{I}} = 1$.

Proof. The implications *1.* $\Rightarrow$ *2.*, *3.* $\Rightarrow$ *4.*, *4.* $\Rightarrow$ *5.* and *6.* $\Rightarrow$ *7.* are trivial.

2. $\Rightarrow$ *3*. This implication is a consequence of the unitary invariance of the norm and the fact that S is a normal operator (see Th. 2.3).

5. $\Rightarrow$ *6*. Let $X \in \mathcal{I}$ then

$$2\|X\|_{\mathcal{I}} \quad \geq \quad \|S^*XS^{-1} + S^{-1}XS^*\|_{\mathcal{I}} = \|\Phi_{|S|}(U^*X)\|_{\mathcal{I}} \geq 2\|U^*X\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}},$$

in the last inequality we use (3).

7. $\Rightarrow$ *1*. By the hypothesis and the inequality (3) we get that $\|\Phi_{|S|}\|_{B(\mathcal{I})} = 2$. In other words, $\|S\|\|S^{-1}\| = 1$. Then the result follows immediately.

We actually showed that the first seven conditions are equivalent. With a similar argument (using (3)) we obtain that the conditions *2*, *8*, *9*, *10*, *11* and *12* are also equivalent and this concludes the proof

□

Remark 3.8. 1. This theorem is a generalization of [11], Th. 6.

2. If $\mathcal{I} = \mathcal{I}_\phi$ is a norm ideal associated with a ϕ regular symmetric norming function (we refer to [5] for details on norm ideals generated by a symmetric norming function), that is

$$\lim_{n\to\infty} \phi(\xi_{n+1}, \xi_{n+2}, ..) = 0, \tag{5}$$

or the equivalent condition $\mathcal{I}_\phi^{(0)} = \mathcal{I}_\phi$ where $\mathcal{I}_\phi^{(0)}$ denotes the closure of th ideal of finite rank operators, $B_{0,0}(\mathscr{H})$, with respect to the norm $\|.\|_{\mathcal{I}}$, then in the previous results we can relax the hypothesis for all $X \in B_{0,0}(\mathscr{H})$. For example, the ideal $B_p(\mathscr{H})$ with $1 \leq p \leq \infty$ satisfies the condition (5).

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