Characterization of unitary operators by elementary operators and unitarily invariant norms

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Abstract

In this work we characterize unitary operators via inequalities of elementary operators with unitarily invariant norms.

1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $(B(\mathcal{H}), \|\cdot\|)$ the $C^*$-algebra of all bounded linear operators on $\mathcal{H}$ with the usual norm. We denote by $Gl(\mathcal{H})$ the group of invertible elements of $B(\mathcal{H})$, $U(\mathcal{H})$ the unitary operators and $Gl_s(\mathcal{H})$ the set of all invertible and selfadjoint operators.

A linear operator $R : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ defined by $R(X) = \sum_{i=1}^{n} A_i X B_i$, where $A_i, B_i \in B(\mathcal{H})$, with $1 \leq i \leq n$, is called an elementary operator on $B(\mathcal{H})$; and we denote by $R = R_{AB}$, where $A = (A_1, \ldots, A_n)$ and $B = (B_1, \ldots, B_n)$. This class of operators includes many important operators of $B(\mathcal{H})$ such as the inner derivation $\delta_A(X) = AX -XA$, the multiplication operator $M_{A,B}(X) = AXB$, the symmetrized two-sided multiplication $U_{A,B}(X) = AXB + BXA$ and the operator $V_{A,B} = AXB - BXA$. We denote by $\Phi_S$ the operator $U_{S,S^{-1}}$.

In [8], Nakamoto proved that a bounded linear operator $A$ on $\mathcal{H}$ is normal if and only if $\|\delta_A(X)\|_2 = \|\delta_A^*(X)\|_2$ for all $X \in B_2(\mathcal{H})$ (Hilbert-Schmidt class). In [9], A. Seedik characterizes the operators $S$ for which the Corach-Porta-Recht inequality ([4], [1]) holds, more precisely he proved that an invertible operator $S$ is a non zero complex multiple of some selfadjoint operator if and only if $\|\Phi_S(X)\| \geq 2\|X\|$ for all $X \in B(\mathcal{H})$.

On the other hand, in [7], the authors ask whether the same characterization obtained on [9] is true for other unitarily invariant norms. They proved that $S$ is necessarily a normal operator if $2\|X\|_I \leq \|\Phi_S(X)\|_I$ for all $X \in B(\mathcal{H})$, with rank one (Corollary 2.2). Furthermore, in this work Magajna et al. obtained that if $I$ a norm ideal and we denote by $\overline{A} = (tS, \frac{1}{t}S^{-1})$ and $\overline{B} = (S^{-1}, S)$ for $t > 0$, then

$$\gamma S \in Gl_s(\mathcal{H}), \lambda \in \mathbb{C} - \{0\} \text{ if and only if } \inf_{t > 0} \|R_{\lambda,t\overline{A},\overline{B}}(X)\|_I \geq 2\|X\|_I$$

for all $X \in I$ of rank 1.

In a recent work [11], A. Seedik obtains some characterizations of some subclasses of normal operators in $B(\mathcal{H})$ by inequalities or equalities (associated with elementary operators).

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Motivated by these results, in [3] we characterized the normal invertible operators of $B(\mathcal{H})$ via unitarily invariant norms and elementary operators. The purpose of this work is to find the set for a given norm ideal $\mathcal{I}$

$$E_{\mathcal{I}} = \{ S \in \text{Gl}(\mathcal{H}) : \|M_{S,S-1}(X)\|_{\mathcal{I}} + \|M_{S-1,S}(X)\|_{\mathcal{I}} = 2\|X\|_{\mathcal{I}} \text{ for every } X \in \mathcal{I} \}.$$ 

By a result obtained in [3] this subset is contained in $N(\mathcal{H}) = \{ T \in B(\mathcal{H}) : TT^* = T^*T \}$.

## 2 Preliminaries

We recall that $\mathcal{I}$ is a norm ideal of $B(\mathcal{H})$ if $\mathcal{I}$ is a two-sided ideal of $B(\mathcal{H})$ and a Banach space with respect to the norm $\| \cdot \|_{\mathcal{I}}$ satisfying:

1. $\|XTY\|_{\mathcal{I}} \leq \|X\|\|T\|\|Y\|$ for $T \in \mathcal{I}$ and $X,Y \in B(\mathcal{H})$,
2. $\|X\|_{\mathcal{I}} = \|X\|$ if $X$ is of rank one.

In particular, condition 1. implies that the norm is unitarily invariant, $\|UXV^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$ for $X \in \mathcal{I}$ and any $U,V \in U(\mathcal{H})$. The most known examples of norm ideals of $B(\mathcal{H})$ are the so called $p$-Schatten class with $p \geq 1$ defined by

$$B_p(\mathcal{H}) = \{ X \in B_0(\mathcal{H}) : \{ s_j(X) \} \in l^p \},$$

where $\{ s_j(X) \}$ denotes the sequence of singular values of $X$, rearranged such that $s_1(X) \geq s_2(X) \geq \cdots$ with multiplicities counted, with norm given by $\|X\|_p = \left( \sum s_j(X)^p \right)^{1/p}$ and $B_0(\mathcal{H})$ is the ideal of compact operators. When $p = \infty$, the norm $\| \cdot \|_{\infty}$ coincides with the usual norm $\|X\| = s_1(X)$. For a complete account of the theory of unitarily invariant norms the reader is referred to [5].

For sake of completeness, we recall three statements that we will use in the following section. Given a norm ideal $\mathcal{I}$ and a linear operator $P : \mathcal{I} \to \mathcal{I}$ we denote by

$$\|P\|_{B(\mathcal{I})} = \sup \{ \|P(X)\|_{\mathcal{I}} : \|X\|_{\mathcal{I}} = 1 \}.$$ 

**Theorem 2.1.** ([10], Theorem 2.1.)

Let $S \in B(\mathcal{H})$ be an invertible and selfadjoint operator and $\mathcal{I}$ a norm ideal. Then we have the following inequality:

$$\|\Phi_S\|_{B(\mathcal{I})} \geq \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|}. \quad (1)$$

**Lemma 2.2.** ([11], Theorem 3.1.)

Let $S \in \text{Gl}(\mathcal{H})$. Then $\|S\|\|S^{-1}\| = 1$ if and only if $S = \|S\|V$, for some unitary operator $V$.

**Theorem 2.3.** ([3], Theorem 2.1.) Let $S \in \text{Gl}(\mathcal{H})$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent:

1. $S$ is normal,
2. $\|M_{S,S-1}(X)\|_{\mathcal{I}} + \|M_{S-1,S}(X)\|_{\mathcal{I}} = \|M_{S^*S-1}(X)\|_{\mathcal{I}} + \|M_{S^*-1,S^*}(X)\|_{\mathcal{I}}$ for every $X \in \mathcal{I},$
3. $\|M_{S,S-1}(X)\|_{\mathcal{I}} + \|M_{S-1,S}(X)\|_{\mathcal{I}} \geq 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I},$
4. $\|M_{S,S-1}(X)\|_{\mathcal{I}} + \|M_{S-1,S}(X)\|_{\mathcal{I}} \geq 2\|X\|_{\mathcal{I}}$ for every $X \in \mathcal{I}$, with rank 1.
Remark 2.4. Other characterization of normal invertible operators is given in the following statement.

Proposition 2.5. Let $S \in \text{Gl}({\mathcal{H}})$ and $I$ a norm ideal. Then the following conditions are equivalent:

1. $S$ is normal,
2. $\|M_{S,S^{-1}}(X)\|_I + \|M_{S^{-1},S}(X)\|_I \leq \|M_{S^*,S^{-1}}(X)\|_I + \|M_{S^{-1},S^*}(X)\|_I$ for every $X \in I$, and
3. $\|M_{S,S^{-1}}(X)\|_I + \|M_{S^{-1},S}(X)\|_I \leq \|M_{S^*,S^{-1}}(X)\|_I + \|M_{S^{-1},S^*}(X)\|_I$ for every $X \in I$, of rank 1.

Proof. The implication $1 \Rightarrow 2$ follows immediately from Theorem 2.3 and $2 \Rightarrow 3$ is trivial. $3 \Rightarrow 1$ We consider rank one operators $X = x \otimes y \in I$ with $x, y \in \mathcal{H}$, then it follows that the following inequality holds

$$\|M_{S,S^{-1}}(x \otimes y)\|_I + \|M_{S^{-1},S}(x \otimes y)\|_I \leq \|M_{S^*,S^{-1}}(x \otimes y)\|_I + \|M_{S^{-1},S^*}(x \otimes y)\|_I$$

or equivalently,

$$\|S(x \otimes y)S^{-1}\|_I + \|S^{-1}(x \otimes y)S\|_I \leq \|S^*(x \otimes y)S^{-1}\|_I + \|S^{-1}(x \otimes y)S^*\|_I.$$

It is easy to see that $A(u \otimes v)B = Au \otimes B^*v$ and $\|u \otimes v\|_I = \|u\|\|v\|$, for all $u, v \in \mathcal{H}$ and $A, B \in B(\mathcal{H})$. Then

$$\|S(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S^*(y)\| \leq \|S^*(x)\|\|(S^{-1})^*(y)\| + \|S^{-1}(x)\|\|S^*(y)\|.$$ (2)

Assume that $S$ is not a normal operator. Consequently there exist vector $x \in \mathcal{H}$, $\|x\| = 1$ such that $\|Sx\| > \|S^*x\|$ (or $\|Sx\| < \|S^*x\|$). It follows, from (2), that for all $y \in \mathcal{H}$ with $\|y\| = 1$, $\|Sy\| > \|S^*y\|$ (or $\|Sy\| < \|S^*y\|$), so we have from (2) that

$$0 < (\|Sx\| - \|S^*x\|) \leq (\|Sy\| - \|S^*y\|)(\|S^{-1}(x)\|\|(S^*)^{-1}(y)\|^{-1} \leq (\|Sy\| - \|S^*y\|)\|S^{-1}\|\|S^*\|.$$ Hence, for all $y \in \mathcal{H}$ with $\|y\| = 1$ 

$$\|Sx\| + \|S^{-1}\|\|S\|\|S^*y\| \leq \|S^*x\| + \|S^{-1}\|\|S\|\|Sy\|$$

Thus $\|Sx\| + \|S^{-1}\|\|S\|\|S^*\| \leq \|S^*x\| + \|S^{-1}\|\|S\|\|S\|$. It follows that $\|Sx\| \leq \|S^*x\|$, which it is a contradiction. Therefore $S$ is a normal operator. \hfill $\square$

In [6], Kittaneh obtained the generalization of the Corach-Porta-Recht inequality in any norm ideal $I$. More precisely, for Hilbert-space operators $T, R, X$ with $T, R$ invertible operators and a unitarily invariant norm $I$, the inequality

$$2\|X\|_I \leq \|R^*XT^{-1} + R^{-1}XT^*\|_I,$$ (3)

holds for all $X \in I$.

In this work, we consider the polar decomposition of $S \in \text{Gl}({\mathcal{H}})$ given by $S = U|S|$ with $|S| = (S^*S)^{1/2}$ positive and $U \in U(\mathcal{H})$. 

3
3 Main results

**Proposition 3.1.** If $S \in \text{Gl}_s(\mathcal{H})$ and $\|\Phi_S(X)\|_I \leq 2\|X\|_I$ for all $X \in I$, then $S = \|S\|V$ with $V \in U(\mathcal{H})$.

**Proof.** Since $\|\Phi_S(X)\|_I \leq 2\|X\|_I$ for all $X \in I$, it follows that $\|\Phi_S\|_{B(I)} \leq 2$.

By (1) we have that $2 \geq \|\Phi_S\|_{B(I)} \geq \|S\|\|\|S^{-1}\|\| + \frac{1}{\|S\|\|S^{-1}\|} \geq 2$.

From this we derive that $\|S\|\|\|S^{-1}\|\| + \frac{1}{\|S\|\|S^{-1}\|} = 2$, so it follows immediately that $\|S\|\|\|S^{-1}\|\| = 1$. So from [11], Lemma 2 it turns out that $S = \|S\|V$ with $V$ an unitary operator.

Now, we obtain a generalization of [11], Th. 8.

**Corollary 3.2.** Let $I$ a norm ideal, then

$$U_s(\mathcal{H}) = \{S \in \text{Gl}_s(\mathcal{H}) : \|S\| = 1 \text{ and } \|\Phi_S(X)\|_I \leq 2\|X\|_I \text{ for all } X \in I\},$$

where $U_s(\mathcal{H})$ denotes the unitary selfadjoint operators in $B(\mathcal{H})$.

**Proof.** If $S \in U_s(\mathcal{H})$ then $S \in \text{Gl}_s(H)$, $\|S\| = 1$ and for any $X \in I$ we have

$$\|\Phi_S(X)\|_I = \|SX\|S^{-1} + S^{-1}XS\|_I \leq 2\|X\|_I,$$

by the unitary invariance of the norm. Thus, the result follows immediately from Proposition 3.1.

In the previous statement, if we omit the hypothesis $\|S\| = 1$ we obtain a characterization of $\mathbb{R}^*U_s(\mathcal{H})$, with $\mathbb{R}^* = \mathbb{R} - \{0\}$.

If $S = U|S| \in U(\mathcal{H})$ then necessarily $|S| \in U_s(\mathcal{H})$ and in consequence $|S|$ is characterized by the previous corollary. In the following result we prove that a condition which holds for the modules of $S$ is a sufficient condition for determinate if $S$ is an unitary operator in $B(\mathcal{H})$.

**Theorem 3.3.** Let $I$ a norm ideal then

$$U(\mathcal{H}) = \{S \in \text{Gl}(\mathcal{H}) : \|S\| = 1, S = U|S| \text{ and } \|\Phi_{|S|}(X)\|_I \leq 2\|X\|_I \text{ for all } X \in I\}.$$ 

**Proof.** By the hypothesis, $\|\Phi_{|S|}\|_{B(I)} \leq 2$. By (1) we have the following lower bound for the operator $\Phi_{|S|}$,

$$\|\Phi_{|S|}\|_{B(I)} \geq \|\|S\|\|\|\|S^{-1}\|\| + \frac{1}{\|S\|\|S^{-1}\|} \geq 2.$$ 

From this inequality and the condition obtained above, we get that

$$\|\Phi_{|S|}\|_{B(I)} = \|\|S\|\|\|\|S^{-1}\|| + \frac{1}{\|S\|\|S^{-1}\|} = \|S\|\|S^{-1}\| + \frac{1}{\|S\|\|S^{-1}\|} = 2.$$ 

In other words, $\|S\|\|S^{-1}\| = 1$. Then the result follows immediately from the Lemma 2.

On the other hand, if $S = U|S| \in U(\mathcal{H})$ then for any $X \in I$ we may write $X = U^*Y$ with $Y \in I$ and since $\|S\| = 1$ we have that

$$\|\Phi_{|S|}(X)\|_I = \||S|U^*Y|S^{-1}| + |S^{-1}U^*Y|S||_I = \|S^*YS^{-1} + S^{-1}YS^*\|_I \leq 2\|Y\|_I = 2\|X\|_I.$$

This concludes the proof.
Remark 3.4. 1. The operator which characterize the unitary operators of $B(H)$ can be written as follows

$$\tau_{S, S} = SXS^{-1} + (S^*)^{-1}XS^* = SXS^{-1} + (S^{-1})^*XS^* = SXS^{-1} + (SX^*S^{-1})^*$$

in particular if $X$ is a selfadjoint operator then

$$SXS^{-1} + (S^*)^{-1}XS^* = 2\text{Re}(SS^{-1}),$$

where $\text{Re}(T) = \frac{1}{2}(T + T^*)$.

A natural question is if with the selfadjoint operators of $I$ we can describe all $U(H)$, more precisely

$$U(H) = \{ S \in \text{Gl}(H) : \|S\| = 1 \text{ and } \|\text{Re}(SS^{-1})\|_2 \leq \|X\|_2 \text{ for all } X \in I, X = X^* \}?$$

We give a particular example when $I$ is the 2-Schatten ideal. Every $X \in B(H)$ can be written as $X = \text{Re}(X) + i\text{Im}(X)$, where $\text{Re}(X), \text{Im}(X)$ are selfadjoint operators and

$$\text{Re}(X) = \frac{1}{2}(X + X^*) \quad \text{and} \quad \text{Im}(X) = \frac{1}{2i}(X - X^*).$$

We call this the Cartesian decomposition of $X$.

In [2], Bhatia and Kittaneh prove sharp inequalities comparing the norm $\|X\|_p$ with $(\|\text{Re}(X)\|_p^p + \|\text{Im}(X)\|_p^p)$. More precisely, for $p = 2$ we get

$$\|\text{Re}(X)\|_2^2 + \|\text{Im}(X)\|_2^2 = \|X\|_2^2.$$  

Theorem 3.5.

$$U(H) = \{ S \in \text{Gl}(H) : \|S\| = 1 \text{ and } \|\text{Re}(SS^{-1})\|_2 \leq \|X\|_2 \text{ for all } X \in B_2(H), X = X^* \}.$$  

Proof. Let $Z = \text{Re}(Z) + i\text{Im}(Z) \in B_2(H)$ then

$$\|SZS^{-1} + (S^*)^{-1}ZS^*\|_2^2 = \|S(\text{Re}(Z) + i\text{Im}(Z))S^{-1} + (S^*)^{-1}(\text{Re}(Z) + i\text{Im}(Z))S^*\|_2^2 = \|2(\text{Re}(S\text{Re}(Z)S^{-1}) + i\text{Re}(S\text{Im}(Z)S^{-1}))\|_2^2 = 4\|\text{Re}(S\text{Re}(Z)S^{-1}) + i\text{Re}(S\text{Im}(Z)S^{-1})\|_2^2 = 4(\|\text{Re}(S\text{Re}(Z)S^{-1})\|_2^2 + \|\text{Re}(S\text{Im}(Z)S^{-1})\|_2^2) \leq 4(\|\text{Re}(Z)\|_2^2 + \|\text{Im}(Z)\|_2^2) = 4\|Z\|_2^2.$$  

Since the norm of the operators $\Phi_{|S|}(X), SXS^{-1} + (S^*)^{-1}XS^*$ and $S^*YS^{-1} + S^{-1}YS^*$ are related via unitaries operators, more precisely for all $X \in I$

$$\|\Phi_{|S|}(X)\|_2 = \|S|X|S|^{-1} + |S|^{-1}X|S||_2 = \|S^*U\|S^{-1} + S^{-1}(U\|S)^S\|_2$$

and

$$\|\Phi_{|S|}(X)\|_2 = \|U\|SXS^{-1} + (S^*)^{-1}XS^*\|_2 = \|SS^{-1} + (S^*)^{-1}XS^*\|_2$$

where $S = U|S|$, we obtain the following characterizations of $U(H)$.  

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Theorem 3.6. Let $\mathcal{I}$ a norm ideal then

\[
U(\mathcal{H}) = \{ S \in Gl(\mathcal{H}) : \| S \| = 1 \text{ and } \| S^* X S^{-1} + S^{-1} X S^* \|_I \leq 2 \| X \|_I \text{ for all } X \in \mathcal{I} \}
\]

\[
= \{ S \in Gl(\mathcal{H}) : \| S \| = 1 \text{ and } \| S X S^{-1} + (S^*)^{-1} X S^* \|_I \leq 2 \| X \|_I \text{ for all } X \in \mathcal{I} \}
\]

\[
= \{ S \in Gl(\mathcal{H}) : \| S \| = 1 \text{ and } \| S X S^{-1} + S^{-1} X S^* \|_I = 2 \| X \|_I \text{ for all } X \in \mathcal{I} \}
\]

In particular, if $\mathcal{I} = B_2(\mathcal{H})$ we get

\[
U(\mathcal{H}) = \{ S \in Gl(\mathcal{H}) : \| S \| = 1 \text{ and } \| \text{Re}(S X S^{-1}) \|_I = \| X \|_I \text{ for all } X \in B_2(\mathcal{H}), X = X^* \}.
\]

We observe that if $S = \lambda V$ with $V \in U(\mathcal{H})$ and $\lambda \in \mathbb{R} - \{0\}$, then for every $X \in \mathcal{I}$

\[
\| S X S^{-1} \|_I + \| S^{-1} X S^* \|_I = 2 \| X \|_I. \tag{4}
\]

Now motivated by the conclusion of Proposition 3.1 and the equality (4), we characterize the real multiples of some unitary operator.

Theorem 3.7. Let $S \in Gl(\mathcal{H})$ and $\mathcal{I}$ a norm ideal. Then the following conditions are equivalent:

1. $S = \lambda V$ with $\lambda \in \mathbb{R}^*$ and $V \in U(\mathcal{H})$,
2. $\| S X S^{-1} \|_I + \| S^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$,
3. $\| S X S^{-1} \|_I + \| S^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$,
4. $\| S X S^{-1} \|_I + \| S^{-1} X S^* \|_I \leq 2 \| X \|_I$ for every $X \in \mathcal{I}$,
5. $\| S X S^{-1} + S^{-1} X S^* \|_I \leq 2 \| X \|_I$ for every $X \in \mathcal{I}$,
6. $\| S^* X S^{-1} + S^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$,
7. $\| S^* X S^{-1} + S^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$
8. $\| S^* X S^{-1} \|_I + \| (S^*)^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$,
9. $\| S X S^{-1} \|_I + \| (S^*)^{-1} X S^* \|_I \leq 2 \| X \|_I$ for every $X \in \mathcal{I}$,
10. $\| S X S^{-1} + (S^*)^{-1} X S^* \|_I \leq 2 \| X \|_I$ for every $X \in \mathcal{I}$,
11. $\| S X S^{-1} + (S^*)^{-1} X S^* \|_I \leq 2 \| X \|_I$ for every $X \in \mathcal{I}$,
12. $\| S X S^{-1} + (S^*)^{-1} X S^* \|_I = 2 \| X \|_I$ for every $X \in \mathcal{I}$.

Proof. The implications 1. $\Rightarrow$ 2., 3. $\Rightarrow$ 4., 5. and 6. $\Rightarrow$ 7. are trivial.

2. $\Rightarrow$ 3. This implication is a consequence of the unitary invariance of the norm and the fact that $S$ is a normal operator (see Th. 2.3).

5. $\Rightarrow$ 6. Let $X \in \mathcal{I}$ then

\[
2 \| X \|_I \geq \| S^* X S^{-1} + S^{-1} X S^* \|_I = \| \Phi_{| \mathcal{S}}(U^* X) \|_I \geq 2 \| U^* X \|_I = 2 \| X \|_I,
\]

in the last inequality we use (3).

7. $\Rightarrow$ 1. By the hypothesis and the inequality (3) we get that $\| \Phi_{| \mathcal{S}} \|_{B(\mathcal{I})} = 2$. In other words, $\| S \| = \| S^{-1} \| = 1$. Then the result follows immediately.

We actually showed that the first seven conditions are equivalent. With a similar argument (using (3)) we obtain that the conditions 2, 8, 9, 10, 11 and 12 are also equivalent and this concludes the proof.
Remark 3.8. 1. This theorem is a generalization of [11], Th. 6.

2. If $\mathcal{I} = \mathcal{I}_\phi$ is a norm ideal associated with a $\phi$ regular symmetric norming function (we refer to [5] for details on norm ideals generated by a symmetric norming function), that is

$$\lim_{n \to \infty} \phi(\xi_{n+1}, \xi_{n+2}, \ldots) = 0,$$

or the equivalent condition $\mathcal{I}^{(0)}_\phi = \mathcal{I}_\phi$ where $\mathcal{I}^{(0)}_\phi$ denotes the closure of the ideal of finite rank operators, $B_{0,0}(\mathcal{H})$, with respect to the norm $\| \cdot \|_\mathcal{I}$, then in the previous results we can relax the hypothesis for all $X \in B_{0,0}(\mathcal{H})$. For example, the ideal $B_p(\mathcal{H})$ with $1 \leq p \leq \infty$ satisfies the condition (5).

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