Shorting selfadjoint operators in Hilbert spaces

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Abstract

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and a (bounded) selfadjoint operator B acting on $\mathcal{H}$, a min-max representation of the shorted operator (or Schur complement) of B to $\mathcal{S}$ is obtained under compatibility hypotheses. Also, an extension of Pekarev's formula is given.

Key words: shorted operator, Schur complement, selfadjoint operator
1991 MSC: 47A64, 47A07, 47B65

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1 Introduction

Let $\mathcal{H}$ be a (complex) Hilbert space and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Given a positive (semidefinite) operator $A \in L(\mathcal{H})$, the shorted operator of A to a closed subspace $\mathcal{S}$ of $\mathcal{H}$ is defined as

$$A_{/\mathcal{S}} = \max\{X \in L(\mathcal{H}) : \ 0 \leq X \leq A, \ \ R(X) \subseteq \mathcal{S}^{\perp}\},$$

where the maximum is taken in the natural order of positive operators.

Although the above definition is due to W. N. Anderson Jr. and G. E. Trapp [3], M. Krein proved the existence of this maximum previously, in his work

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about the theory of extensions of Hermitian operators [14]. Also in [3] the shorted operator $A_{/\mathcal{S}}$ was characterized as the greatest lower bound of a set:

$$A_{/\mathcal{S}} = \inf_{N(Q)=\mathcal{S}} Q^*AQ,$$

where $Q \in L(\mathcal{H})$ is a projection. Later, E. L. Pekarev [17] obtained an explicit formula for $A_{/\mathcal{S}}$ in terms of the square root of A and the orthogonal projection onto $\mathcal{M}^{\perp}$:

$$A_{/\mathcal{S}} = A^{1/2} P_{\mathcal{M}^{\perp}} A^{1/2},$$

where $\mathcal{M}$ is the closure of the subspace $A^{1/2}(\mathcal{S})$.

There are definitions of shorted operators (or Schur complements) for broader classes of bounded operators, see for example [4,5]. In particular, given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, T. Ando [4] defined the Schur complement of $\mathcal{S}$-complementable selfadjoint operators. Then, G. Corach et al. [9] shown that a selfadjoint operator $B \in L(\mathcal{H})$ is $\mathcal{S}$-complementable if and only if the pair $(B, \mathcal{S})$ is compatible, i.e. there exists a projection $P \in L(\mathcal{H})$ with range $\mathcal{S}$ which is selfadjoint respect to the sesquilinear form induced by B:

$$\langle x, y \rangle_B = \langle Bx, y \rangle, \qquad x, y \in \mathcal{H}.$$

Under this hypothesis, the Schur complement of B to $\mathcal{S}$ is given by $B_{/\mathcal{S}} = B(I - P)$. Furthermore, they shown that, if $A \in L(\mathcal{H})$ is positive, the pair $(A, \mathcal{S})$ is compatible if and only if

$$A_{/\mathcal{S}} = \min_{N(Q)=\mathcal{S}} Q^*AQ,$$

where $Q \in L(\mathcal{H})$ is a projection.

Following these ideas, P. Massey and D. Stojanoff [16] showed that, given a selfadjoint operator $B \in L(\mathcal{H})$ and a B-nonnegative closed subspace $\mathcal{S}$ of $\mathcal{H}$ (i.e. $\langle Bx, x \rangle \geq 0$ for every $x \in \mathcal{S}$), if $(B, \mathcal{S})$ is compatible then

$$B_{/\mathcal{S}} = \min_{N(Q)=\mathcal{S}} Q^*BQ,$$

where $Q \in L(\mathcal{H})$ is a projection.

The purpose of this work is to show that, given a selfadjoint operator $B \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible,

$$B_{/\mathcal{S}} = \min_{N(Q_+)=\mathcal{S}_+} \left(\max_{N(Q_-)=\mathcal{S}_-} Q_+^*(Q_-^*BQ_-)Q_+ \right),$$

where $Q_{\pm}$ are projections in $L(\mathcal{H})$, $\mathcal{S} = \mathcal{S}_+ \dot{+} \mathcal{S}_-$ is a suitable decomposition of $\mathcal{S}$ with B-definite subspaces $\mathcal{S}_{\pm}$ and the natural order induced by the cone $L(\mathcal{H})^+$ is considered.

We also obtain an extension of Pekarev's formula for selfadjoint operators: if $(B, \mathcal{S})$ is compatible then

$$B_{/\mathcal{S}} = JA^{1/2} P_{J(\mathcal{M}^{\perp})//\mathcal{M}} A^{1/2},$$

where $B = JA$ is the polar decomposition of B (with A positive and $J = J^* = J^{-1}$) and $P_{J(\mathcal{M}^{\perp})//\mathcal{M}}$ is the (possibly unbounded) projection with range $J(\mathcal{M}^{\perp})$ and nullspace $\mathcal{M}$.

Section 2 contains the preliminaries; mostly results concerning compatibility conditions and the definition of the shorted operator of a selfadjoint operator given by T. Ando. For the proof of these results see [8,9,10,15].

Section 3 is devoted to prove the above mentioned min-max representation of the shorted of a selfadjoint operator B to a B-indefinite subspace $\mathcal{S}$. It is divided in four subsection. The first one deals with extremal characterizations of the shorted operator to B-definite subspaces. In subsection 3.2 conditions for shorting a shorted operator are given and the shorted operator $B_{/\mathcal{S}}$ to a B-indefinite subspace $\mathcal{S}$ is shown to be equal to $(B_{/\mathcal{S}_+})_{/\mathcal{S}_-}$, where $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is a suitable decomposition of the subspace $\mathcal{S}$. Finally, the min-max representation of the shorted operator is presented.

Given a selfadjoint operator $B \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ which are compatible, a generalization of Pekarev's formula is developed in Section 4. Also, this formula allows to extend Ando's definition of shorted operators to a broader class of selfadjoint operators, namely, those for which the operator given by the formula is well defined and bounded. Furthermore, it is shown that the sharper descriptions of the range and nullspace of the shorted operator obtained in [10] hold only under compatibility hypothesis.

2 Preliminaries

Along this work $\mathcal{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle \ , \ \rangle$. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $L(\mathcal{H}, \mathcal{K})$ is the algebra of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. If $T \in L(\mathcal{H})$ then T^* denotes the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its nullspace.

Given a Hilbert space $\mathcal{H}$, let $L(\mathcal{H})^+$ be the cone of (semidefinite) positive operators in $L(\mathcal{H})$, $L(\mathcal{H})^s$ the (real) vector space of selfadjoint operators in $L(\mathcal{H})$ and denote by $\mathcal{Q}$ the set of projections in $L(\mathcal{H})$, i.e., $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$. If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{H}$, denote by $\mathcal{S} \dotplus \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}$. If $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$, the oblique projection onto $\mathcal{S}$ along $\mathcal{T}$,

$P_{\mathcal{S}//\mathcal{T}}$, is the projection with $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^\perp}$ is the orthogonal projection onto $\mathcal{S}$.

Given $B \in L(\mathcal{H})^s$ consider the sesquilinear form in $\mathcal{H} \times \mathcal{H}$ defined by

$$\langle x, y \rangle_B = \langle Bx, y \rangle, \quad \text{for } x, y \in \mathcal{H}.$$

If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $B \in L(\mathcal{H})^s$, the B-orthogonal subspace to $\mathcal{S}$ is given by

$$\mathcal{S}^{\perp_B} := \{x \in \mathcal{H} \ : \ \langle x, s \rangle_B = 0 \ \text{ for every } s \in \mathcal{S}\}.$$

It holds that $\mathcal{S}^{\perp_B} = B^{-1}(\mathcal{S}^\perp) = B(\mathcal{S})^\perp$.

A vector $x \in \mathcal{H}$ is B-*positive* if $\langle x, x \rangle_B > 0$. A subspace $\mathcal{S}$ of $\mathcal{H}$ is B-positive if every $x \in \mathcal{S}$, $x \neq 0$ is a B-positive vector. B-nonnegative, B-neutral, B-negative and B-nonpositive vectors (and subspaces) are defined analogously.

An operator $T \in L(\mathcal{H})$ is B-*selfadjoint* if $\langle Tx, y \rangle_B = \langle x, Ty \rangle_B$ for every $x, y \in \mathcal{H}$. It is easy to see that T satisfies this condition if and only if $BT = T^*B$. The operator $T \in L(\mathcal{H})$ is B-*positive* if $\langle Tx, x \rangle_B \geq 0$ for every $x \in \mathcal{H}$, i.e. BT is a (semidefinite) positive operator. B-neutral and B-negative operators are defined in a similar way.

Definition 2.1 *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. The pair $(B, \mathcal{S})$ is* compatible *if there exists a B-selfadjoint projection with range $\mathcal{S}$, i.e. if the set*

$$\mathcal{P}(B, \mathcal{S}) = \{Q \in \mathcal{Q} \ : \ R(Q) = \mathcal{S}, \ BQ = Q^*B\}$$

is not empty.

A projection Q is B-selfadjoint if and only if its nullspace satisfies the inclusion $N(Q) \subseteq R(Q)^{\perp_B}$. Then, $(B, \mathcal{S})$ is compatible if and only if

$$\mathcal{H} = \mathcal{S} + B^{-1}(\mathcal{S}^\perp).$$

In this case it holds that $\mathcal{S} \cap B(\mathcal{S})^\perp = \mathcal{S} \cap N(B)$. Given a compatible pair $(B, \mathcal{S})$, define $\mathcal{N} = \mathcal{S} \cap N(B)$. Since $\mathcal{H} = \mathcal{S} \dotplus (B(\mathcal{S})^\perp \ominus \mathcal{N})$, consider the oblique projection

$$P_{B,\mathcal{S}} := P_{\mathcal{S}//B(\mathcal{S})^\perp \ominus \mathcal{N}}. \tag{2.1}$$

Observe that $P_{B,\mathcal{S}} \in \mathcal{P}(B, \mathcal{S})$ because $R(P_{B,\mathcal{S}}) = \mathcal{S}$ and $N(P_{B,\mathcal{S}}) \subseteq B(\mathcal{S})^\perp$. Moreover, if $\mathcal{N} = \{0\}$ then $\mathcal{P}(B, \mathcal{S}) = \{P_{B,\mathcal{S}}\}$.

Lemma 2.2 *Let $B \in L(\mathcal{H})^s$, $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $\mathcal{N} = \mathcal{S} \cap N(B)$. Then, $(B, \mathcal{S} \ominus \mathcal{N})$ is compatible if and only if $(B, \mathcal{S})$ is compatible.*

See [15] for the proofs of these facts.

An operator $T \in L(\mathcal{H})$ is a *B-contraction* if $\langle Tx, Tx \rangle_B \leq \langle x, x \rangle_B$. It is easy to see that T is a B-contraction if and only if $T^*BT \leq B$. An operator $T \in L(\mathcal{H})$ is a *B-expansion* if $\langle Tx, Tx \rangle_B \geq \langle x, x \rangle_B$ (i.e. $T^*BT \geq B$) and T is a *B-isometry* if $\langle Tx, Tx \rangle_B = \langle x, x \rangle_B$ (i.e. $T^*BT = B$).

S. Hassi and K. Nordström characterized those projections which are B-contractive (see [12, §3, Proposition 5]). A similar result holds for B-expansive projections.

Proposition 2.3 *If $Q \in \mathcal{Q}$ then the following conditions are equivalent:*

(1) Q is B-contractive;
(2) Q is B-selfadjoint and $N(Q)$ is B-nonnegative;
(3) $I - Q$ is B-positive.

The following paragraphs introduce the notion of shorted operator for selfadjoint operators acting on a Hilbert space $\mathcal{H}$.

Definition 2.4 (T. Ando) *Given an operator $T \in L(\mathcal{H})$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, T is $\mathcal{S}$-*complementable *if there exist operators $M_l, M_r \in L(\mathcal{H})$ such that:*

(1) $M_lP = M_l$ and $M_lTP = TP$;
(2) $PM_r = M_r$ and $PTM_r = PT$;

where $P = P_{\mathcal{S}}$ is the orthogonal projection onto $\mathcal{S}$.

In this case, TM_r only depends on T and $\mathcal{S}$. Therefore, the *compression of T to $\mathcal{S}$* is defined as $T_{\mathcal{S}} = TM_r$ and the *generalized Schur complement of T to $\mathcal{S}$* as $T_{/\mathcal{S}} = T(I - M_r)$, see [4].

In [9] it was shown that, if $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, B is $\mathcal{S}$-complementable if and only if the pair $(B, \mathcal{S})$ is compatible. Moreover, it was proved that, if $(B, \mathcal{S})$ is compatible and $Q \in \mathcal{P}(B, \mathcal{S})$, then $M_r = Q$ and $M_l = Q^*$ satisfy the $\mathcal{S}$-complementability definition. Therefore,

$$B_{\mathcal{S}} = BQ \quad \text{and} \quad B_{/\mathcal{S}} = B(I - Q). \tag{2.2}$$

Observe that these operators are selfadjoint because Q is a B-selfadjoint projection.

3 A min-max representation for shorted operators

If A is a positive operator, the shorted operator of A to $\mathcal{S}$ satisfies that

$$A_{/\mathcal{S}} = \inf\{Q^*AQ \ : \ Q \in \mathcal{Q}, \ N(Q) = \mathcal{S}\},$$

(see [3]). Furthermore, given $B \in L(\mathcal{H})^s$ and a B-nonnegative closed subspace $\mathcal{S}$ of $\mathcal{H}$, it was shown in [16] that, if $(B, \mathcal{S})$ is compatible then

$$B_{/\mathcal{S}} = \min\{Q^*BQ \ : \ Q \in \mathcal{Q}, \ N(Q) = \mathcal{S}\}.$$

The aim of this section is to generalize this formula for any closed subspace $\mathcal{S}$ of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible.

3.1 Extremality of the shorted operator for definite subspaces

Along this section $\mathcal{S}$ is a B-definite subspace. Propositions 3.1 and 3.5 were partially stated in [16]. Although the results in this section are announced for every B-definite subspace, we only present the proof of the B-positive case.

Proposition 3.1 *Let* $(B, \mathcal{S})$ *be compatible. Then,*

(1) $B_{/\mathcal{S}} = \min_{\leq}\{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$ *if and only if* $\mathcal{S}$ *is* B*-nonnegative.*
(2) $B_{/\mathcal{S}} = \max_{\leq}\{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$ *if and only if* $\mathcal{S}$ *is* B*-nonpositive.*

Proof. Suppose that $(B, \mathcal{S})$ is compatible and $\mathcal{S}$ is B-nonnegative. By Eqs. (2.1) and (2.2), $B_{/\mathcal{S}} = B(I - P_{B,\mathcal{S}}) = (I - P_{B,\mathcal{S}})^*B(I - P_{B,\mathcal{S}})$, then it is clear that $B_{/\mathcal{S}} \in \{Q^*BQ \ : \ Q \in \mathcal{Q}, \ N(Q) = \mathcal{S}\}$.

Given $Q \in \mathcal{Q}$ with $N(Q) = \mathcal{S}$, consider $E = I - Q$. Then, if $x \in \mathcal{H}$,

$$\begin{aligned}\langle BQx, Qx \rangle &= \langle x - Ex, x - Ex \rangle_B = \\ &= \langle P_{B,\mathcal{S}}x + (I - P_{B,\mathcal{S}})x - Ex, P_{B,\mathcal{S}}x + (I - P_{B,\mathcal{S}})x - Ex \rangle_B = \\ &= \langle (I - P_{B,\mathcal{S}})x, (I - P_{B,\mathcal{S}})x \rangle_B + \langle P_{B,\mathcal{S}}x - Ex, P_{B,\mathcal{S}}x - Ex \rangle_B \geq \\ &\geq \left\langle B_{/\mathcal{S}}x, x \right\rangle.\end{aligned}$$

Therefore, $Q^*BQ \geq B_{/\mathcal{S}}$.

Conversely, suppose that $B_{/\mathcal{S}} = \min_{\leq}\{Q^*BQ \ : \ Q \in \mathcal{Q}, \ N(Q) = \mathcal{S}\}$. Then, given $x \in \mathcal{H} \setminus \mathcal{S}$ and $s \in \mathcal{S}$, there exists $E \in \mathcal{Q}$ with $R(E) = \mathcal{S}$ such that

$Ex = s$ so that

$$\begin{aligned}\left\langle B_{/\mathcal{S}}x, x \right\rangle &\leq \langle B(I-E)x, (I-E)x \rangle = \langle B(x-s), x-s \rangle = \\ &= \langle B((x - P_{B,\mathcal{S}}x) + (P_{B,\mathcal{S}}x - s)), (x - P_{B,\mathcal{S}}x) + (P_{B,\mathcal{S}}x - s) \rangle = \\ &= \langle B(x - P_{B,\mathcal{S}}x), x - P_{B,\mathcal{S}}x \rangle + \langle B(P_{B,\mathcal{S}}x - s), P_{B,\mathcal{S}}x - s \rangle = \\ &= \left\langle B_{/\mathcal{S}}x, x \right\rangle + \langle B(P_{B,\mathcal{S}}x - s), P_{B,\mathcal{S}}x - s \rangle, \qquad \text{for every } s \in \mathcal{S},\end{aligned}$$

i.e. $\langle B(P_{B,\mathcal{S}}x - s), P_{B,\mathcal{S}}x - s \rangle \geq 0$ for every $s \in \mathcal{S}$, or equivalently, $\mathcal{S}$ is B-nonnegative. □

Proposition 3.2 *Let $B \in L(\mathcal{H})^s$ and suppose that the pair $(B, \mathcal{S})$ is compatible. Then, the following conditions are equivalent:*

(1) $B_{/\mathcal{S}} \in L(\mathcal{H})^+$;
(2) $N(Q)$ is B-nonnegative for some $Q \in \mathcal{P}(B, \mathcal{S})$.
(3) $B^{-1}(\mathcal{S}^\perp)$ is a B-nonnegative subspace of $\mathcal{H}$;

Proof. (1) $\Leftrightarrow$ (2) : Let $Q \in \mathcal{P}(B, \mathcal{S})$, by Eq. (2.2) we have that $B_{/\mathcal{S}} = B(I-Q)$. If $x \in \mathcal{H}$,

$$\left\langle B_{/\mathcal{S}}x, x \right\rangle = \langle B(I-Q)x, x \rangle = \langle (I-Q)x, (I-Q)x \rangle_B . \tag{3.1}$$

Therefore, $B_{/\mathcal{S}} \in L(\mathcal{H})^+$ if and only if $R(I-Q) = N(Q)$ is B-nonnegative.

(2) $\Leftrightarrow$ (3) : Observe that, for any projection $Q \in \mathcal{P}(B, \mathcal{S})$, $N(Q) \perp_B \mathcal{N}$, $B^{-1}(\mathcal{S}^\perp) = N(Q) \dot{+} \mathcal{N}$ and $\mathcal{N}$ is a B-neutral subspace of $\mathcal{H}$. Therefore, $N(Q)$ is B-nonnegative if and only if $B^{-1}(\mathcal{S}^\perp)$ is B-nonnegative. □

The following is a consequence of Propositions 3.1 and 3.2.

Corollary 3.3 *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $B \in L(\mathcal{H})^s$ such that the pair $(B, \mathcal{S})$ is compatible. Then, $B \in L(\mathcal{H})^+$ if and only if $B_{/\mathcal{S}} \in L(\mathcal{H})^+$ and $\mathcal{S}$ is B-nonnegative.*

Remark 3.4 Given a Hermitian block matrix $M = \begin{pmatrix} A & C \\ C^* & D \end{pmatrix}$ (with $A \in \mathbb{C}^{k\times k}$, $C \in \mathbb{C}^{k\times m}$ and $D \in \mathbb{C}^{m\times m}$) the Schur complement of M in A is defined as $M_{/A} = D - C^* A^\dagger C$, where $A^\dagger$ stands for the Moore-Penrose inverse of A. In [1], A. Albert showed that M is semidefinite positive (which we denote by $M \geq 0$) if and only if $A \geq 0$, $M_{/A} \geq 0$ and $N(A) \subseteq N(C^*)$ (or equivalently, $R(C) \subseteq R(A)$). See also [7, Section 2.1] and [11, Theorem 1.10].

Since shorted operators are an infinite dimensional generalization of Schur complements, the above corollary can be understood as an extension of [1, Theorem 1] (recall that, in [8], it was proven that $(B, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$, where

$$B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

is the matrix representation of B induced by $\mathcal{S}$). See Corollary 2 (to Theorem 1.7) in [18] for a similar result in this direction.

If $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, the shorted operator of A to $\mathcal{S}$ was defined as

$$A_{/\mathcal{S}} = \max_{\leq}\{X \in L(\mathcal{H})^+ : \ X \leq A, \ R(X) \subseteq \mathcal{S}^\perp\},$$

(see [14,3]). Given $B \in L(\mathcal{H})^s$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$, consider the sets

$$\mathcal{M}^-(B, \mathcal{S}^\perp) = \{X \in L(\mathcal{H})^s : \ X \leq B, \ R(X) \subseteq \mathcal{S}^\perp\},$$
$$\mathcal{M}^+(B, \mathcal{S}^\perp) = \{X \in L(\mathcal{H})^s : \ B \leq X, \ R(X) \subseteq \mathcal{S}^\perp\}.$$

Proposition 3.5 *Let $(B, \mathcal{S})$ be a compatible pair. Then,*

(1) $\mathcal{S}$ is B-nonnegative if and only if $B_{/\mathcal{S}} = \max_{\leq} \mathcal{M}^-(B, \mathcal{S}^\perp)$.
(2) $\mathcal{S}$ is B-nonpositive if and only if $B_{/\mathcal{S}} = \min_{\leq} \mathcal{M}^+(B, \mathcal{S}^\perp)$.

Proof. Suppose that $\mathcal{S}$ is B-nonnegative. Then, if $Q = I - P_{B,\mathcal{S}}$ it follows, by Proposition 2.3, that Q is B-contractive. Therefore, by Eq. (2.2), $B_{/\mathcal{S}} = BQ = Q^*BQ \leq B$ and $R(B_{/\mathcal{S}}) \subseteq \mathcal{S}^\perp$, i.e. $B_{/\mathcal{S}} \in \mathcal{M}^-(B, \mathcal{S}^\perp)$. Furthermore, if $X \in \mathcal{M}^-(B, \mathcal{S}^\perp)$ then $X = Q^*X = XQ = Q^*XQ \leq Q^*BQ = BQ = B_{/\mathcal{S}}$. Thus, $B_{/\mathcal{S}} = \max_{\leq} \mathcal{M}^-(B, \mathcal{S}^\perp)$.

Conversely, suppose that $B_{/\mathcal{S}} = \max_{\leq} \mathcal{M}^-(B, \mathcal{S}^\perp)$. Then, since $R(B_{/\mathcal{S}}) \subseteq \mathcal{S}^\perp$, $B_{/\mathcal{S}} = Q^*B_{/\mathcal{S}}Q \leq Q^*BQ$, for every $Q \in \mathcal{Q}$ with $N(Q) = \mathcal{S}$. On the other hand, $B_{/\mathcal{S}} = (I - P_{B,\mathcal{S}})^*B(I - P_{B,\mathcal{S}})$. Therefore,

$$B_{/\mathcal{S}} = \min_{\leq}\{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}.$$

Applying Proposition 3.1 it follows that $\mathcal{S}$ is B-nonnegative. $\square$

Given a selfadjoint operator $B \in L(\mathcal{H})$ and two closed subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{H}$, we are interested in obtaining (whenever it is possible) the shorted operator to $\mathcal{S}_2$ of $B_{/\mathcal{S}_1}$ and compare it with the shorted operator to $\mathcal{S}_1$ of $B_{/\mathcal{S}_2}$.

First, we need to prove that $B_{/\mathcal{T}} = B_{/\mathcal{S}\ominus\mathcal{N}}$ for a family of subspaces of $\mathcal{H}$.

Lemma 3.6 *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace $\mathcal{S}$ of $\mathcal{H}$. If $(B,\mathcal{S})$ is compatible and $\mathcal{T}$ is a closed subspace of $\mathcal{H}$ such that*

$$\mathcal{S} \ominus \mathcal{N} \subseteq \mathcal{T} \subseteq \mathcal{S} + N(B)$$

then $(B,\mathcal{T})$ is compatible and $B_{/\mathcal{T}} = B_{/\mathcal{S}\ominus\mathcal{N}}$.

Proof. Suppose that $(B,\mathcal{S})$ is compatible. Then, by Lemma 2.2, $(B,\mathcal{S}\ominus\mathcal{N})$ is compatible. Observe that $\mathcal{H} = \mathcal{S}\ominus\mathcal{N} + B(\mathcal{S})^\perp \subseteq \mathcal{T} + B(\mathcal{T})^\perp$ because $B(\mathcal{T}) = B(\mathcal{S}\ominus\mathcal{N})$. Therefore, $(B,\mathcal{T})$ is compatible.

It only remains to prove that $B_{/\mathcal{T}} = B_{/\mathcal{S}\ominus\mathcal{N}}$. Notice that $R(I - P_{B,\mathcal{S}\ominus\mathcal{N}}) = N(P_{B,\mathcal{S}\ominus\mathcal{N}}) = B(\mathcal{S}\ominus\mathcal{N})^\perp = B(\mathcal{T})^\perp = N(P_{B,\mathcal{T}}) \dot{+} \mathcal{T}\cap N(B)$. Then, $BP_{B,\mathcal{T}}(I - P_{B,\mathcal{S}\ominus\mathcal{N}}) = 0$ and $B_{\mathcal{T}} = BP_{B,\mathcal{T}} = BP_{B,\mathcal{T}}P_{B,\mathcal{S}\ominus\mathcal{N}} = BP_{B,\mathcal{S}\ominus\mathcal{N}} = B_{\mathcal{S}\ominus\mathcal{N}}$. Therefore, $B_{/\mathcal{T}} = B_{/\mathcal{S}\ominus\mathcal{N}}$. $\square$

Proposition 3.7 *Let $B \in L(\mathcal{H})^s$ and consider two closed subspaces $\mathcal{S}_1$, $\mathcal{S}_2$ of $\mathcal{H}$ such that $\mathcal{S}_1 \perp_B \mathcal{S}_2$ and $(B,\mathcal{S}_i)$ is compatible for $i = 1,2$. Then, $(B_{/\mathcal{S}_i},\mathcal{S}_j)$ is compatible and*

$$\mathcal{P}(B,\mathcal{S}_j) = \mathcal{P}(B_{/\mathcal{S}_i},\mathcal{S}_j) \quad \text{(for } i \neq j\text{).}$$

Proof. Suppose that $(B,\mathcal{S}_i)$ is compatible and consider $Q_i \in \mathcal{P}(B,\mathcal{S}_i)$, for $i = 1,2$. Since $\mathcal{S}_1 \perp_B \mathcal{S}_2$ it follows that $BQ_iQ_j = Q_i^*BQ_j = 0$ if $i \neq j$. Therefore,

$$B_{/\mathcal{S}_i}Q_j = B(I - Q_i)Q_j = BQ_j(I - Q_i) = Q_j^*B_{/\mathcal{S}_i},$$

i.e. $Q_j \in \mathcal{P}(B_{/\mathcal{S}_i},\mathcal{S}_j)$. The other inclusion follows in a similar way. $\square$

Proposition 3.8 *Let $B \in L(\mathcal{H})^s$ and consider two closed subspaces $\mathcal{S}_1$, $\mathcal{S}_2$ of $\mathcal{H}$ such that $\mathcal{S}_1 \perp_B \mathcal{S}_2$, $\mathcal{S} = \mathcal{S}_1 \dot{+} \mathcal{S}_2$ is closed and $(B,\mathcal{S}_i)$ is compatible for $i = 1,2$. Then, $(B,\mathcal{S})$ is compatible and*

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_1})_{/\mathcal{S}_2} = (B_{/\mathcal{S}_2})_{/\mathcal{S}_1}.$$

Proof. Consider $\mathcal{N}_1 = \mathcal{S}_1 \cap N(B)$. Observe that $\mathcal{N}_1 \cap \mathcal{S}_2 = \{0\}$ and $\mathcal{N}_1 + \mathcal{S}_2$ is a closed subspace of $B(\mathcal{S}_1)^\perp$. If $\mathcal{T}_1 = B(\mathcal{S}_1)^\perp \ominus (\mathcal{N}_1 + \mathcal{S}_2)$, then $\mathcal{T}_1 + \mathcal{S}_2$ is closed and $\mathcal{T}_1 + \mathcal{N}_1 + \mathcal{S}_2 = B(\mathcal{S}_1)^\perp$. Let $Q_1 = P_{\mathcal{S}_1//\mathcal{T}_1+\mathcal{S}_2}$, notice that $Q_1 \in \mathcal{P}(B, \mathcal{S}_1)$ and $Q_1 P = 0$ for every $P \in \mathcal{Q}$ with $R(P) = \mathcal{S}_2$.

Analogously, consider $\mathcal{N}_2 = \mathcal{S}_2 \cap N(B)$, $\mathcal{T}_2 = B(\mathcal{S}_2)^\perp \ominus (\mathcal{N}_2 + \mathcal{S}_1)$ and $Q_2 = P_{\mathcal{S}_2//\mathcal{T}_2+\mathcal{S}_1} \in \mathcal{P}(B, \mathcal{S}_2)$. Therefore, $Q = Q_1 + Q_2 \in \mathcal{Q}$ satisfies $BQ = Q^*B$ and $R(Q) = \mathcal{S}$, i.e. $(B, \mathcal{S})$ is compatible. Noticing that $I - Q = (I - Q_1)(I - Q_2)$, it holds that

$$B_{/\mathcal{S}} = B(I-Q) = B(I-Q_1)(I-Q_2) = B_{/\mathcal{S}_1}(I-Q_2) = B_{/\mathcal{S}_1}(I-Q_2) = (B_{/\mathcal{S}_1})_{/\mathcal{S}_2}.$$

Analogously, $B_{/\mathcal{S}} = (B_{/\mathcal{S}_2})_{/\mathcal{S}_1}$. □

To obtain a representation of $B_{/\mathcal{S}}$ for a general closed subspace $\mathcal{S}$, we decompose $\mathcal{S}$ as $\mathcal{S} = \mathcal{S}_+ \dotplus \mathcal{S}_-$, where $\mathcal{S}_+$ is a B-nonegative subspace and $\mathcal{S}_-$ is a B-nonpositive subspace.

The following theorem is a rewriting of the decomposition of B-selfadjoint projections given in [15]. See [15, Theorem 5.1 and Proposition 5.2] for the proof.

Theorem 3.9 *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, $(B, \mathcal{S})$ is compatible if and only if there exists a (unique) decomposition of $\mathcal{S} \ominus \mathcal{N}$ as $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_+$ is a (closed) B-positive subspace, $\mathcal{S}_-$ is a (closed) B-negative subspace, $(B, \mathcal{S}_\pm)$ is compatible and $\mathcal{S}_+ \perp_B \mathcal{S}_-$.*

The following is a corollary of Proposition 3.8 (considering the decomposition of Theorem 3.9).

Proposition 3.10 *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and $B \in L(\mathcal{H})^s$. If $(B, \mathcal{S})$ is compatible then*

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = (B_{/\mathcal{S}_-})_{/\mathcal{S}_+},$$

where $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is the decomposition given in Theorem 3.9.

Proof. If $(B, \mathcal{S})$ is compatible then, by Lemma 3.6, $B_{/\mathcal{S}} = B_{/\mathcal{S}\ominus\mathcal{N}}$. If $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is the decomposition given in Theorem 3.9 then $(B, \mathcal{S}_\pm)$ is compatible and $\mathcal{S}_+ \perp_B \mathcal{S}_-$. Therefore, applying Proposition 3.8 we get that

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = (B_{/\mathcal{S}_-})_{/\mathcal{S}_+}. \qquad \square$$

Suppose that $(B, \mathcal{S})$ is compatible and consider the decomposition of Theorem

3.9. Then, Proposition 3.5 says that

$$B_{/\mathcal{S}_+} \leq B \leq B_{/\mathcal{S}_-}.$$

The following corollary shows that $B_{/\mathcal{S}}$ also belongs to this interval of selfadjoint operators.

Corollary 3.11 *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $(B, \mathcal{S})$ is compatible then*

$$B_{/\mathcal{S}_+} \leq B_{/\mathcal{S}} \leq B_{/\mathcal{S}_-}, \tag{3.2}$$

where $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is the decomposition given in Theorem 3.9.

Proof. If $(B, \mathcal{S})$ is compatible consider the decomposition given in Theorem 3.9. Then, applying Propositions 3.10 and 3.5, it follows that $B_{/\mathcal{S}} = (B_{/\mathcal{S}_-})_{/\mathcal{S}_+} = \max_{\leq} \mathcal{M}^-(B_{/\mathcal{S}_-}, \mathcal{S}_+^\perp) \leq B_{/\mathcal{S}_-}$. Analogously, $B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = \min_{\leq} \mathcal{M}^+(B_{/\mathcal{S}_+}, \mathcal{S}_-^\perp) \geq B_{/\mathcal{S}_+}$. $\square$

3.3 A min-max representation for B-indefinite subspaces

Theorem 3.12 *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $(B, \mathcal{S})$ is compatible then*

$$B_{/\mathcal{S}} = \min_{N(Q_+)=\mathcal{S}_+} \max_{N(Q_-)=\mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ = \max_{N(Q_-)=\mathcal{S}_-} \min_{N(Q_+)=\mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_-, \tag{3.3}$$

where $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is the decomposition given in Theorem 3.9.

Proof. If $(B, \mathcal{S})$ is compatible then, by Theorem 3.9, $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$, where $\mathcal{S}_+$ is a (closed) B-positive subspace, $\mathcal{S}_-$ is a (closed) B-negative subspace, $(B, \mathcal{S}_\pm)$ is compatible and $\mathcal{S}_+ \perp_B \mathcal{S}_-$. Therefore, applying Propositions 3.10 and 3.1, it follows that

$$\begin{aligned} B_{/\mathcal{S}} = (B_{/\mathcal{S}_-})_{/\mathcal{S}_+} &= \min_{N(Q_+)=\mathcal{S}_+} Q_+^* B_{/\mathcal{S}_-} Q_+ = \\ &= \min_{N(Q_+)=\mathcal{S}_+} Q_+^* \left(\max_{N(Q_-)=\mathcal{S}_-} Q_-^* B Q_- \right) Q_+ = \\ &= \min_{N(Q_+)=\mathcal{S}_+} \left(\max_{N(Q_-)=\mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right). \end{aligned}$$

Analogously, $B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = \max_{N(Q_-)=\mathcal{S}_-} (\min_{N(Q_+)=\mathcal{S}_+} Q_-^* (Q_+^* B Q_+) Q_-)$. $\square$

Corollary 3.13 *Let $B \in L(\mathcal{H})^s$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $(B, \mathcal{S})$ is compatible then, for every $x \in \mathcal{H}$,*

$$\left\langle B_{/\mathcal{S}}\, x, x \right\rangle = \min_{s \in \mathcal{S}_+} \max_{t \in \mathcal{S}_-} \langle\, B(x - (s+t)), x - (s+t) \,\rangle\,, \tag{3.4}$$

where $\mathcal{S} \ominus \mathcal{N} = \mathcal{S}_+ \oplus \mathcal{S}_-$ is a decomposition as in Theorem 3.9.

Proof. By Theorem 3.12 it holds that

$$B_{/\mathcal{S}} = \min_{N(Q_+)=\mathcal{S}_+} \max_{N(Q_-)=\mathcal{S}_-} Q_+^*(Q_-^* B Q_-)Q_+ = \min_{N(Q_+)=\mathcal{S}_+} Q_+^* B_{/\mathcal{S}_-} Q_+.$$

Then, given $x \in \mathcal{H}$,

$$\begin{aligned} \left\langle B_{/\mathcal{S}} x, x \right\rangle &= \left\langle (\min_{N(Q_+)=\mathcal{S}_+} Q_+^* B_{/\mathcal{S}_-} Q_+)x, x \right\rangle = \min_{N(Q_+)=\mathcal{S}_+} \left\langle Q_+^* B_{/\mathcal{S}_-} Q_+ x, x \right\rangle = \\ &= \min_{N(Q_+)=\mathcal{S}_+} \left\langle B_{/\mathcal{S}_-}(x - (I - Q_+)x), x - (I - Q_+)x \right\rangle = \\ &= \min_{s \in \mathcal{S}_+} \left\langle B_{/\mathcal{S}_-}(x - s), x - s \right\rangle. \end{aligned}$$

By Proposition 3.1, $B_{/\mathcal{S}_-} = \max\limits_{N(Q_-)=\mathcal{S}_-} Q_-^* B Q_-$ and a similar argument shows that

$$\begin{aligned} \left\langle B_{/\mathcal{S}} x, x \right\rangle &= \min_{s \in \mathcal{S}_+} \left\langle B_{/\mathcal{S}_-}(x - s), x - s \right\rangle = \\ &= \min_{s \in \mathcal{S}_+} \max_{t \in \mathcal{S}_-} \langle\, B(x - (s+t)), x - (s+t) \,\rangle \quad \text{for every } x \in \mathcal{H}. \end{aligned}$$ □

4 A formula for the shorted operator

In [17, Theorem 1.4], E. L. Pekarev proved that the shorted operator of an operator $A \in L(\mathcal{H})^+$ to a closed subspace $\mathcal{S}$ of $\mathcal{H}$ can be represented by the following formula:

$$A_{/\mathcal{S}} = A^{1/2} P_{\mathcal{M}^\perp} A^{1/2}, \tag{4.1}$$

where $P_{\mathcal{M}^\perp}$ is the orthogonal projection onto $\mathcal{M}^\perp$ and $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.

Along this section we are going to prove that, given $B \in L(\mathcal{H})^s$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ such that $(B, \mathcal{S})$ is compatible, there is a natural generalization of Pekarev's formula (4.1) for the shorted operator of B to $\mathcal{S}$. More generally, we will show that, if the given formula defines a bounded operator, then it can be used as the definition of $B_{/\mathcal{S}}$ and its basic properties are obtained.

In the following, if $B \in L(\mathcal{H})^s$ we use the polar decomposition of B given by $B = JA$, where $A = |B| \in L(\mathcal{H})^+$ and $J = J^* = J^{-1}$ satisfies that $Jx = x$ for every $x \in N(B)$.

Lemma 4.1 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$. If $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ and $(B, \mathcal{S})$ is compatible then $\mathcal{M} \cap J(\mathcal{M})^\perp = \{0\}$ and therefore $\mathcal{M} \dot{+} J(\mathcal{M})^\perp$ is dense in $\mathcal{H}$.*

Proof. See [15, Corollary 4.6].

Theorem 4.2 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Then, if $(B, \mathcal{S})$ is compatible, the $\mathcal{S}$-compression and the Schur complement of B to $\mathcal{S}$ can be written as*

$$B_{\mathcal{S}} = JA^{1/2} P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2} \quad \text{and} \quad B_{/\mathcal{S}} = JA^{1/2} P_{J(\mathcal{M})^\perp//\mathcal{M}} A^{1/2}, \tag{4.2}$$

where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.

Proof. If $(B, \mathcal{S})$ is compatible then, by Lemma 4.1, $P_{\mathcal{M}//J(\mathcal{M})^\perp}$ is densely defined. Furthermore, $P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2}$ is well defined, bounded and satisfies

$$P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2} = A^{1/2} Q,$$

where $Q \in \mathcal{P}(B, \mathcal{S})$. Indeed, if $x \in \mathcal{S} = R(Q)$ then $P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2} x = A^{1/2} x = A^{1/2} Q x$, and if $y \in N(Q)$ it follows that $A^{1/2} y \in J(\mathcal{M})^\perp$ and then $P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2} y = 0 = A^{1/2} Q y$. Therefore, if $Q \in \mathcal{P}(B, \mathcal{S})$ we have that $B_{\mathcal{S}} = BQ = JA^{1/2}(A^{1/2} Q) = JA^{1/2} P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2}$ and

$$B_{/\mathcal{S}} = B - B_{\mathcal{S}} = JA^{1/2} P_{J(\mathcal{M})^\perp//\mathcal{M}} A^{1/2}. \qquad \square$$

In particular, if $B \in L(\mathcal{H})^+$, the formula (4.2) for the Schur complement of B to $\mathcal{S}$ is just Pekarev's formula, see [17, Theorem 1.4].

The next example shows that Eq. (4.2) gives well defined and bounded operators in some cases where the definition of shorted operator given by T. Ando in [4] can not be applied.

Example 4.3 Let $\mathcal{S}$ and $\mathcal{T}$ be closed subspaces of an infinite-dimensional Hilbert space $\mathcal{H}$ such that $\mathcal{S} + \mathcal{T}$ is not closed. Denote by $\mathcal{W}$ the orthogonal complement of $\mathcal{T}$ in $\mathcal{H}$. Consider the Hilbert space $\mathcal{H}_2 = \mathcal{H} \oplus \mathcal{H}$ and the operators $A, J \in L(\mathcal{H}_2)$ defined by

$$A = \begin{pmatrix} P_{\mathcal{W}} & 0 \\ 0 & I \end{pmatrix}, \quad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

in the matrix representation induced by $\mathcal{H} \oplus \{0\}$. Let $B = JA$. Observe that $A \in L(\mathcal{H}_2)^+$, $R(A) = \mathcal{W} \oplus \mathcal{H}$ is closed and $J = J^* = J^{-1}$. Since J and A commute, $B = JA$ is the polar decomposition of B and $B \in L(\mathcal{H}_2)^s$.

Given the closed subspace $\mathcal{S}_2 = \mathcal{S} \oplus \{0\}$ of $\mathcal{H}_2$ we show that the operators given in (4.2) are defined but $(B, \mathcal{S}_2)$ is not compatible. Observe that $\mathcal{M} = \overline{A^{1/2}(\mathcal{S}_2)} = \overline{P_{\mathcal{W}}(\mathcal{S})} \oplus \{0\} \subseteq \mathcal{H} \oplus \{0\}$ is invariant under J and $P_{J(\mathcal{M})^\perp//\mathcal{M}} = P_{\mathcal{M}^\perp} \in L(\mathcal{H}_2)$. Therefore, $B_{/\mathcal{S}_2}$ (as in Eq. (4.2)) is defined, in fact

$$B_{/\mathcal{S}_2} = \begin{pmatrix} P_{\mathcal{W}} P_{\mathcal{Z}} P_{\mathcal{W}} & 0 \\ 0 & -I \end{pmatrix},$$

where $\mathcal{Z}$ is the orthogonal complement of $P_{\mathcal{W}}(\mathcal{S})$ in $\mathcal{H}$. On the other hand, $A^{1/2}(\mathcal{S}_2) = P_{\mathcal{W}}(\mathcal{S}) \oplus \{0\}$ is not closed (because $\mathcal{S} + \mathcal{T}$ is not closed, see [6,13]) and by [15, Proposition 4.11] the pair $(B, \mathcal{S}_2)$ is not compatible.

Definition 4.4 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Consider $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$ and suppose that the operator $P_{J(\mathcal{M})^\perp//\mathcal{M}} A^{1/2}$ is well defined and bounded. Then the shorted operator of B to $\mathcal{S}$ is defined as*

$$B_{/\mathcal{S}} = J A^{1/2} P_{J(\mathcal{M})^\perp//\mathcal{M}} A^{1/2}$$

and the $\mathcal{S}$-compression as $B_{\mathcal{S}} = J A^{1/2} P_{\mathcal{M}//J(\mathcal{M})^\perp} A^{1/2}$.

Remark 4.5 *Observe that, if $P_{J(\mathcal{M})^\perp//\mathcal{M}} A^{1/2}$ is well defined and bounded, then $R(A^{1/2}) \subset \mathcal{M} + J(\mathcal{M})^\perp$ and $\mathcal{M} \cap J(\mathcal{M})^\perp = \{0\}$. Therefore, $P_{J(\mathcal{M})^\perp//\mathcal{M}}$ is a densely defined (possibly unbounded) operator and admits an adjoint.*

Proposition 4.6 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. If $B_{/\mathcal{S}}$ is given by Definition 4.4 then $B_{/\mathcal{S}}$ and $B_{\mathcal{S}}$ are selfadjoint operators.*

Proof. If $P = P_{J(\mathcal{M})^\perp//\mathcal{M}}$ it is easy to see that JP is a symmetric operator with domain $\mathcal{M} + J(\mathcal{M})^\perp$. Therefore, for every $x, y \in \mathcal{H}$,

$$\left\langle B_{/\mathcal{S}} x, y \right\rangle = \left\langle JPA^{1/2} x, A^{1/2} y \right\rangle = \left\langle A^{1/2} x, JPA^{1/2} y \right\rangle = \left\langle x, B_{/\mathcal{S}} y \right\rangle,$$

because $R(A^{1/2})$ is contained in the domain of JP. Then, $B_{/\mathcal{S}} \in L(\mathcal{H})^s$. □

Proposition 4.7 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$ and suppose that $B_{/\mathcal{S}}$ and $B_{\mathcal{S}}$ are given by Definition 4.4. Then,*

(1) $B(\mathcal{S}) \subseteq R(B_{\mathcal{S}}) \subseteq \overline{B(\mathcal{S})}$ and $N(B_{\mathcal{S}}) = B(\mathcal{S})^\perp$;
(2) $R(B) \cap \mathcal{S}^\perp \subseteq R(B_{/\mathcal{S}}) \subseteq R(A^{1/2}) \cap \mathcal{S}^\perp$ and $N(B_{/\mathcal{S}}) = A^{-1/2}(\mathcal{M})$.

Proof. *(1)* It is easy to see that $B(\mathcal{S}) = JA^{1/2}(A^{1/2}(\mathcal{S})) = B_{\mathcal{S}}(\mathcal{S}) \subseteq R(B_{\mathcal{S}}) \subseteq JA^{1/2}(\mathcal{M}) \subseteq \overline{B(\mathcal{S})}$. Since $\mathcal{M} \subseteq N(A)^\perp$, then $N(B_{\mathcal{S}}) = N(P_{\mathcal{M}//J(\mathcal{M})^\perp}A^{1/2}) = A^{-1/2}(J(\mathcal{M})^\perp) = B^{-1}(\mathcal{S}^\perp) = B(\mathcal{S})^\perp$.

(2) If $y \in R(B) \cap \mathcal{S}^\perp$ then there exists $x \in \mathcal{H}$ such that $y = Bx \in \mathcal{S}^\perp$. Notice that $A^{1/2}x \in J(\mathcal{M})^\perp$ and $B_{/\mathcal{S}}x = JA^{1/2}P_{J(\mathcal{M})^\perp//\mathcal{M}}(A^{1/2}x) = Bx = y$. Thus, $R(B) \cap \mathcal{S}^\perp \subseteq R(B_{/\mathcal{S}})$. On the other hand, $R(B_{/\mathcal{S}}) \subseteq JA^{1/2}(J(\mathcal{M})^\perp) = A^{1/2}(A^{-1/2}(\mathcal{S}^\perp)) = \mathcal{S}^\perp \cap R(A^{1/2})$. As in item *(1)*, notice that $N(B_{/\mathcal{S}}) = N(P_{J(\mathcal{M})^\perp//\mathcal{M}}A^{1/2}) = A^{-1/2}(\mathcal{M})$. □

In general, the inclusions in items *(1)* and *(2)* of the above proposition are strict. See the examples in [3] and [9].

Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The range and the nullspace of $A_{/\mathcal{S}}$ and $A_{\mathcal{S}}$ had been initially studied in [3], [14] and [17]. Under compatibility hypothesis, G. Corach et al. [10] obtain a sharper description, say, $R(A_{\mathcal{S}}) = A(\mathcal{S})$, $R(A_{/\mathcal{S}}) = R(A) \cap \mathcal{S}^\perp$ and $N(A_{/\mathcal{S}}) = N(A) + \mathcal{S}$. Moreover, these characterizations of the range and nullspace of $A_{/\mathcal{S}}$ and $A_{\mathcal{S}}$ are attained if and only if $(A, \mathcal{S})$ is compatible. The following theorem shows that the same result holds in the selfadjoint case.

Lemma 4.8 *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, and $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$. Suppose that $B_{/\mathcal{S}}$ is given by Definition 4.4 and consider $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$. Then, the following inclusions hold:*

$$J(\mathcal{M})^\perp \cap R(A^{1/2}) \subseteq R(P_{J(\mathcal{M})^\perp//\mathcal{M}}A^{1/2}) \subseteq J(\mathcal{M})^\perp \cap \overline{R(A^{1/2})}.$$

Proof. Let $P = P_{J(\mathcal{M})^\perp//\mathcal{M}}$. Suppose that $PA^{1/2}$ is everywhere defined and bounded. Hence, the first inclusion is trivial. If $y \in R(PA^{1/2})$, then $y \in J(\mathcal{M})^\perp$ and there exists $x \in \mathcal{H}$ such that $y = PA^{1/2}x$. It is easy to see that $\mathcal{M}^\perp + J(\mathcal{M})$ is dense in $\mathcal{H}$, it is contained in dom(P^*) and $P^*u = P_{\mathcal{M}^\perp//J(\mathcal{M})}u$ for every $u \in \mathcal{M}^\perp + J(\mathcal{M})$. Then, since $N(A^{1/2}) \subseteq \mathcal{M}^\perp$, $\langle y, z \rangle = \left\langle A^{1/2}x, P^*z \right\rangle = \left\langle A^{1/2}x, z \right\rangle = 0$ for every $z \in N(A^{1/2})$. Therefore, $y \in \overline{R(A^{1/2})} \cap J(\mathcal{M})^\perp$. □

Theorem 4.9 *Let $B \in L(\mathcal{H})^s$ with polar decomposition $B = JA$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. Suppose that $B_{/\mathcal{S}}$ is given by Definition 4.4. Then, the* *following conditions are equivalent:*

(1) $(B, \mathcal{S})$ *is compatible,*
(2) $R(B_{/\mathcal{S}}) = \mathcal{S}^{\perp} \cap R(B)$ *and* $N(B_{/\mathcal{S}}) = \mathcal{S} + N(B)$*;*
(3) $R(B_{\mathcal{S}}) = B(\mathcal{S})$*.*

Proof. *(1)* $\Rightarrow$ *(2)* admits the same proof given in Proposition 3.4 of [9].

(2) $\Rightarrow$ *(3)*: Let $P = P_{J(\mathcal{M})^{\perp}//\mathcal{M}}$. If $A^{-1/2}(\mathcal{M}) = \mathcal{S} + N(B) = N(B_{/\mathcal{S}})$ then $\mathcal{M} \cap R(A^{1/2}) = A^{1/2}(\mathcal{S} + N(B)) = A^{1/2}(\mathcal{S})$ and we have that $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$. If $R(B_{/\mathcal{S}}) = \mathcal{S}^{\perp} \cap R(B)$ then $A^{1/2}(R(JPA^{1/2})) = \mathcal{S}^{\perp} \cap R(A)$. Taking the counterimage of $A^{1/2}$ on the equation, we obtain $R(JPA^{1/2}) + N(A^{1/2}) = A^{-1/2}(\mathcal{S}^{\perp} \cap R(A)) = \mathcal{M}^{\perp} \cap (R(A^{1/2}) + N(A^{1/2}))$, and applying J we obtain $R(PA^{1/2}) + N(A^{1/2}) = J(\mathcal{M})^{\perp} \cap (R(A^{1/2}) + N(A^{1/2}))$.

Since $N(A^{1/2}) \subseteq J(\mathcal{M})^{\perp}$, it follows that

$$R(PA^{1/2}) + N(A^{1/2}) = J(\mathcal{M})^{\perp} \cap R(A^{1/2}) + N(A^{1/2})$$

and, by Lemma 4.8, we get that $J(\mathcal{M})^{\perp} \cap R(A^{1/2}) = R(PA^{1/2})$. Using this fact it is easy to show that $R((I-P)A^{1/2}) = \mathcal{M} \cap R(A^{1/2})$ and

$$R(A^{1/2}) = R(PA^{1/2}) + R((I-P)A^{1/2}) = J(\mathcal{M})^{\perp} \cap R(A^{1/2}) + A^{1/2}(\mathcal{S}).$$

Thus, $R(B_{\mathcal{S}}) = JA^{1/2}P_{\mathcal{M}//J(\mathcal{M})^{\perp}}(R(A^{1/2})) = JA^{1/2}(A^{1/2}(\mathcal{S})) = B(\mathcal{S})$.

(3) $\Rightarrow$ *(1)*: The identity $R(B_{\mathcal{S}}) = B(\mathcal{S})$ implies that

$$\mathcal{H} = B_{\mathcal{S}}^{-1}(R(B_{\mathcal{S}})) = B_{\mathcal{S}}^{-1}(B(\mathcal{S})) = B_{\mathcal{S}}^{-1}(B_{\mathcal{S}}(\mathcal{S})) = \mathcal{S} + N(B_{\mathcal{S}}) = \mathcal{S} + B^{-1}(\mathcal{S}^{\perp}),$$

because, by Proposition 4.7, $N(B_{\mathcal{S}}) = B^{-1}(\mathcal{S}^{\perp})$. Therefore, $(B, \mathcal{S})$ is compatible. $\square$

Remark 4.10 Some of the results of Section 3, stated for compatible pairs $(B, \mathcal{S})$, remain valid when the new definition of shorted operator is considered. For instance, it is not difficult to prove that, given $B \in L(\mathcal{H})^s$ and a closed subspace of $\mathcal{H}$ such that $B_{/\mathcal{S}}$ is well defined, the following conditions are equivalent:

(1) $(B, \mathcal{S})$ is compatible and $\mathcal{S}$ is B-nonnegative;
(2) $B_{/\mathcal{S}} = \min_{\leq}\{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$;
(3) $B_{/\mathcal{S}} = \max_{\leq}\{X \in L(\mathcal{H})^s : X \leq B,\ R(X) \subseteq \mathcal{S}^{\perp}\}$.

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