Hopf-Rinow Theorem in the Sato Grassmannian*

Esteban Andruchow and Gabriel Larotonda

Abstract

Let $U_2(\mathcal{H})$ be the Banach-Lie group of unitary operators in the Hilbert space $\mathcal{H}$ which are Hilbert-Schmidt perturbations of the identity 1. In this paper we study the geometry of the unitary orbit

$$\{upu^* : u \in U_2(\mathcal{H})\},$$

of an infinite projection p in $\mathcal{H}$. This orbit coincides with the connected component of p in the Hilbert-Schmidt restricted Grassmannian $Gr_{res}(p)$ (also known in the literature as the Sato Grassmannian) corresponding to the polarization $\mathcal{H} = p(\mathcal{H}) \oplus p(\mathcal{H})^{\perp}$. It is known that the components of $Gr_{res}(p)$ are differentiable manifolds. Here we give a simple proof of the fact that $Gr^0_{res}(p)$ is a smooth submanifold of the affine Hilbert space $p + \mathcal{B}_2(\mathcal{H})$, where $\mathcal{B}_2(\mathcal{H})$ denotes the space of Hilbert-Schmidt operators of $\mathcal{H}$. Also we show that $Gr^0_{res}(p)$ is a homogeneous reductive space. We introduce a natural metric, which consists in endowing every tangent space with the trace inner product, and consider its Levi-Civita connection. This connection has been considered before, for instance its sectional curvature has been computed. We show that the Levi-Civita connection coincides with a linear connection induced by the reductive structure, a fact which allows for the easy computation of the geodesic curves. We prove that the geodesics of the connection, which are of the form $\gamma(t) = e^{tz}pe^{-tz}$, for z a p-codiagonal anti-hermitic element of $\mathcal{B}_2(\mathcal{H})$, have minimal length provided that $\|z\| \leq \pi/2$. Note that the condition is given in terms of the usual operator norm, a fact which implies that there exist minimal geodesics of arbitrary length. Also we show that any two points $p_1, p_2 \in Gr^0_{res}(p)$ are joined by a minimal geodesic. If moreover $\|p_1 - p_2\| < 1$, the minimal geodesic is unique. Finally, we replace the 2-norm by the k-Schatten norm ($k > 2$), and prove that the geodesics are also minimal for these norms, up to a critical value of t, which is estimated also in terms of the usual operator norm. In the process, minimality results in the k-norms are also obtained for the group $U_2(\mathcal{H})$. [1]

1 Introduction

Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\mathcal{B}(\mathcal{H})$ the space of bounded linear operators acting in $\mathcal{H}$. Denote by $Gl(\mathcal{H})$ and $U(\mathcal{H})$ the groups of invertible and unitary operators in $\mathcal{H}$, and by $\mathcal{B}_2(\mathcal{H})$ the space of Hilbert-Scmidt operators, that is

$$\mathcal{B}_2(\mathcal{H}) = \{a \in \mathcal{B}(\mathcal{H}) : Tr(a^*a) < \infty\},$$

*2000 MSC. Primary 22E65; Secondary 58E50, 58B20.

[1]**Keywords and phrases:** Sato Grassmannian, Infinite projections, Hilbert-Schmidt operators

where Tr is the usual trace in $\mathcal{B}(\mathcal{H})$. This space is a two sided ideal of $\mathcal{B}(\mathcal{H})$, and a Hilbert space with the inner product

$$< a, b >= Tr(b^*a).$$

The norm induced by this inner product is called the 2-norm, and denoted by

$$\|a\|_2 = Tr(a^*a)^{1/2}.$$

Throughout this paper, $\| \ \|$ denotes the usual operator norm. Consider the following groups of operators:

$$Gl_2(\mathcal{H}) = \{g \in Gl(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\},$$

and

$$U_2(\mathcal{H}) = \{u \in U(\mathcal{H}) : u - 1 \in \mathcal{B}_2(\mathcal{H})\},$$

here $1 \in \mathcal{B}(\mathcal{H})$ denotes the identity operator. These groups are examples of what in the literature [16] is called a *classical* Banach-Lie group. They have differentiable structure when endowed with the metric $\|g_1 - g_2\|_2$ (note that $g_1 - g_2 \in \mathcal{B}_2(\mathcal{H})$).
Fix a selfadjoint infinite projection $p \in \mathcal{B}(\mathcal{H})$. The aim of this paper is the geometric study of the set

$$Gr^0_{res}(p) = \{upu^* : u \in U_2(\mathcal{H})\},$$

the connected component of p in the (Hilbert-Schmidt) restricted Grassmannian corresponding to the polarization $\mathcal{H} = R(p) \oplus R(p)^\perp$ [29]. Since both $R(p)$ and $R(p)^\perp$ are infinite dimensional, the $U_2(\mathcal{H})$-orbit of p lies inside $Gr^0_{res}(p)$, [25]. The fact that the group $U_2(\mathcal{H})$ acts transitively on $Gr^0_{res}(p)$ was proved by Stratila and Voiculescu in [31] (see also [9]).
The Hopf-Rinow Theorem is not valid in infinite dimensional complete manifolds: two points in a Hilbert-Riemann complete manifold may not be joined by a minimal geodesic [12], [21], or even a geodesic [3]. The main results in this paper establish the validity of the Hopf Rinow Theorem for $Gr^0_{res}(p)$ (4.11, 4.12). In the process we prove minimality results for $U_2(\mathcal{H})$, which are perhaps well known but for which we could find no references. We also prove minimality results, both in $U_2(\mathcal{H})$ and $Gr^0_{res}(p)$, for the Finsler metrics given by the Schatten k-norms ($k \geq 2$).
If one chooses unitaries ω_n, $n \in \mathbb{Z}$, in the different components of $\mathcal{U}_{res}(\mathcal{H})$, then the connected components of $Gr_{res}(p)$ are

$$Gr^n_{res}(p) = \omega_n Gr^0_{res}(p)\omega_n^* = \{\omega_n upu^*\omega_n^* : u \in U_2(\mathcal{H})\},$$

so that the results described above are valid also in the other components of $Gr_{res}(p)$.
The restricted Grasamannian is related to several areas of mathematics and physics: loop groops [29], [25], integrable systems [27], [29], [22], [32], group representation theory [31], [9], [28], string theory [1], [5].
Unitary orbits of operators, and in particular projections, have been studied before in ([10], [24], [8], [7], [18], [2]). In this particular framework, restricting the action to classical groups, certain results can be found in [9], [8] and [7]. In the latter paper, the author considered the orbit of a finite rank projection.

If p has infinite rank and corank, then p and $\mathcal{B}_2(\mathcal{H})$ are linearly independent. We shall denote by

$$p + \mathcal{B}_2(\mathcal{H}) = \{p + a : a \in \mathcal{B}_2(\mathcal{H})\}.$$

Note that every element x in $p + \mathcal{B}_2(\mathcal{H})$ has a unique decomposition $x = p + a$, $a \in \mathcal{B}_2(\mathcal{H})$. We shall endow this affine space with the metric induced by the 2-norm: if $x = p + a$ and $y = p + b$, $\|x - y\|_2 := \|a - b\|_2$. Apparently, $p + \mathcal{B}_2(\mathcal{H})$ is a Hilbert space.
The orbit $Gr^0_{res}(p)$ sits inside $p + \mathcal{B}_2(\mathcal{H})$:

$$\begin{aligned} q &= upu^* = (1 + (u-1))p(1 - (u-1)^*) \\ &= p + (u-1)p + p(u-1)^* + (u-1)p(u-1)^* \in p + \mathcal{B}_2(\mathcal{H}). \end{aligned}$$

Therefore we shall consider $Gr^0_{res}(p)$ with the topology induced by the 2-metric of $p + \mathcal{B}_2(\mathcal{H})_2$. Throughout this paper, L_2 denotes the length functional for piecewise smooth curves, either in $U_2(\mathcal{H})$ or $Gr^0_{res}(p)$, measured with the 2-norm:

$$L_2(\alpha) = \int_{t_0}^{t_1} \|\dot{\alpha}(t)\|_2 \, dt.$$

We use the subscript h (resp. ah) to denote the sets of hermitic (resp. anti-hermitic.) operators, e.g. $\mathcal{B}_2(\mathcal{H})_{ah} = \{x \in \mathcal{B}_2(\mathcal{H}) : x^* = -x\}$.
Let us describe the contents and main results of the paper.
In section 2 we prove (Theorem 2.4) that $Gr^0_{res}(p)$ is a smooth submanifold of the affine Hilbert space $p + \mathcal{B}_2(\mathcal{H})$, and that the map $\pi_p : U_2(\mathcal{H}) \to Gr^0_{res}(p)$, $\pi_p(u) = upu^*$ is a submersion.
In section 3 we introduce a linear connection, which is the Levi-Civita connection of the trace inner product in $Gr^0_{res}(p)$. This connection was considered in [23], where its sectional curvature was computed, and shown to be non negative. It is presented here as the connection obtained from the reductive structure for the action of $U_2(\mathcal{H})$:

$$\mathcal{B}_2(\mathcal{H})_{ah} = \{y \in \mathcal{B}_2(\mathcal{H})_{ah} : py = yp\} \oplus \{z \in \mathcal{B}_2(\mathcal{H})_{ah} : pzp = (1-p)z(1-p) = 0\},$$

regarded as the decomposition of the Lie algebra of $U_2(\mathcal{H})$ (equal to $\mathcal{B}_2(\mathcal{H})_{ah}$), the first subspace is the Lie algebra of the isotropy group, and the second subspace is its orthogonal complement with respect to the trace. Therefore the geodesics can be explicitely computed.
In section 4 we prove the main results (Theorems 4.8, 4.11 and 4.12) on minimality of geodesics with given initial (respectively boundary) data. These results show that any pair of points in $Gr^0_{res}(p)$ can be joined by a minimal geodesic, and that there are mimimal geodesics which have arbitrary length.
In section 5 we consider the minimality problem, when the length of a curve is measured with (the Finsler metric given by) the Schatten k-norms, for $2 < k < \infty$. Here we obtain (Theorem 5.5) that for these metrics, the geodesics of the connection have minimal length up to a critical value of t (which depends on the usual operator norm of the initial data). In both settings, $k = 2$ and $k > 2$, the minimality results are proved first in $U_2(\mathcal{H})$, and then derived for $Gr^0_{res}(p)$ via a natural inmersion of projections as unitaries (more specifically, symmetries).

2 Differentiable structure of $Gr^0_{res}(p)$

As said above, it is known that $Gr^0_{res}(p)$, being a connected component of $Gr_{res}(p)$ (the component of virtual dimension 0 [29]), is a differentiable manifold. Here we show that $Gr^0_{res}(p) \subset p+\mathcal{B}_2(\mathcal{H})_h$ is a differentiable (real analytic) submanifold. The action of $U_2(\mathcal{H})$ induces the map

$$\pi_p : U_2(\mathcal{H}) \to Gr^0_{res}(p) \ , \quad \pi_p(u) = upu^*.$$

This map, regarded as a map on $p + \mathcal{B}_2(\mathcal{H})$, is real analytic. Its differential at the identity is the linear map

$$\delta_p : \mathcal{B}_2(\mathcal{H})_{ah} \to \mathcal{B}_2(\mathcal{H})_h, \ \delta_p(x) = xp - px.$$

Here we have identified the Banach-Lie algebra of $U_2(\mathcal{H})$ with the space $\mathcal{B}_2(\mathcal{H})_{ah}$ of anti-hermitic elements in $\mathcal{B}_2(\mathcal{H})$, and used the fact that π_p takes hermitic values, i.e. in the set $\mathcal{B}_2(\mathcal{H})_h$ of hermitic elements of $\mathcal{B}_2(\mathcal{H})$.

Lemma 2.1. *The map δ_p^2 defines an idempotent operator acting on $\mathcal{B}_2(\mathcal{H})_h$. Moreover, it is symmetric for the trace inner product in $\mathcal{B}_2(\mathcal{H})_h$.*

Proof. Straightforward computations show that if one regards δ_p as a linear map from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$, then it verifies $\delta_p^3 = \delta_p$. Therefore δ_p^2 is an idempotent, whose range and kernel coincide with the range and kernel of δ_p. Note that

$$\delta_p^2(x) = xp - 2pxp + px.$$

Clearly δ_p^2 maps $\mathcal{B}_2(\mathcal{H})_h$ into $\mathcal{B}_2(\mathcal{H})_h$ and $\mathcal{B}_2(\mathcal{H})_{ah}$ into $\mathcal{B}_2(\mathcal{H})_{ah}$, so that in particular it defines an idempotent (real) linear operator acting in $\mathcal{B}_2(\mathcal{H})_h$. Finally, if $x, y \in \mathcal{B}_2(\mathcal{H})_h$,

$$\begin{aligned} < \delta_p^2(x), y > &= Tr(y(xp - 2pxp + px) = Tr(pyx) - 2Tr(pypx) + Tr(ypx) \\ &= Tr((py - 2pyp + yp)x) =< x, \delta_p^2(y) > . \end{aligned}$$

□

Next let us show that the map π_p is a fibration:

Proposition 2.2. *The map*

$$\pi_p : U_2(\mathcal{H}) \to Gr^0_{res}(p) \subset p + \mathcal{B}_2(\mathcal{H})_h$$

has continuous local cross sections. In particular, it is a locally trivial fibre bundle.

Proof. It is well known that if p, q are projections such that $\|p - q\| < 1$, then the element $s = qp + (1 - q)(1 - p)$ is invertible. If $q \in Gr^0_{res}(p)$, then $s \in Gl_2(\mathcal{H})$. Indeed, there exists $u \in U_2(\mathcal{H})$ (i.e. a unitary such that $u_0 = u - 1 \in \mathcal{B}_2(\mathcal{H})$) such that $q = upu^*$. Then

$$\begin{aligned} s \ &= \ p + u_0p + pu_0^*p + u_0pu_0^*p + 1 - p + u_0(1 - p) + (1 - p)u_0^*(1 - p) \\ &\quad +u_0(1 - p)u_0^*(1 - p) \in 1 + \mathcal{B}_2(\mathcal{H}). \end{aligned}$$

Morever, s verifies $sp = qp = qs$. Let $s = w|s|$ be the polar decomposition of s. Note that $sp = qs$ implies that $ps^* = s^*q$, and then s^*s commutes with p. Therefore

$$wpw^* = s|s|^{-1}p|s|^{-1}s^* = s(s^*s)^{-1/2}p(s^*s)^{-1/2}s^* = sp(s^*s)^{-1}s^* = sps^{-1} = q.$$

We claim that $w \in U_2(\mathcal{H})$. Indeed, $\mathbb{C}1 + \mathcal{B}_2(\mathcal{H})$ is a *-Banach algebra (it is the unitization of $\mathcal{B}_2(\mathcal{H})$) with the 2-norm: $\|\lambda 1 + a\|_2^2 = |\lambda|^2 + \|a\|_2^2$. Since s lies in $Gl_2(\mathcal{H})$, in particular it is an invertible element of this Banach algebra, and therefore by the Riesz functional calculus, $w = s|s|^{-1} \in \mathbb{C} + \mathcal{B}_2(\mathcal{H})$, so that $w = \mu 1 + w_0$ with $w_0 \in \mathcal{B}_2(\mathcal{H})$. On the other hand, note that s^*s, it is a positive operator which lies in the C*-algebra $\mathbb{C}1 + \mathcal{K}(\mathcal{H})$, the unitization of the ideal $\mathcal{K}(\mathcal{H})$ of compact operators. Therefore its square root is of the form $r1 + b$, with $r \geq 0$ and b compact. Then $s^*s = (r1 + b)^2 = r^2 1 + b'$, with $b' \in \mathcal{K}(\mathcal{H})$. One has that $s^*s \in Gl_2(\mathcal{H})$, so that it is of the form 1 plus a compact operator, and since $\mathbb{C}1$ and $\mathcal{K}(\mathcal{H})$ are linearly independent, it follows that $r = 1$. Then $w = s|s|^{-1}$ is of the form 1 plus compact. By the same argument as above, this implies that $\mu = 1$.
The map sending an arbitrary invertible element $g \in Gl_2(\mathcal{H})$ to its unitary part $u \in U_2(\mathcal{H})$ is a continuous map between these groups. In fact, it is real analytic, by the regularity properties of the Riesz functional calculus.
Summarizing, consider the open set $\{q \in Gr^0_{res}(p) : \|q - p\|_2 < 1\}$ in $Gr^0_{res}(p)$. If q lies in this open set, then in particular $\|q - p\| \leq \|q - p\|_2 < 1$, so that s defined above lies in $Gl_2(\mathcal{H})$, and therefore its unitary part $u \in U_2(\mathcal{H})$ verifies $upu^* = q$. Denote by $u = \sigma_p(q)$. Clearly σ_p is continuous, being the composition of continuous maps. Thus it is a continuous local cross section for π_p on a neighbourhood of p. One obtains cross sections defined on neighbourhoods of other points of $Gr^0_{res}(p)$ by translating this one with the action of $U_2(\mathcal{H})$ in a standard fashion. □

Let us transcribe the following result contained in the appendix of the paper [26] by I. Raeburn, which is a consequence of the implicit function theorem in Banach spaces.

Lemma 2.3. *Let G be a Banach-Lie group acting smoothly on a Banach space X. For a fixed $x_0 \in X$, denote by $\pi_{x_0} : G \to X$ the smooth map $\pi_{x_0}(g) = g \cdot x_0$. Suppose that*

1. *π_{x_0} is an open mapping, when regarded as a map from G onto the orbit $\{g \cdot x_0 : g \in G\}$ of x_0 (with the relative topology of X).*
2. *The differential $d(\pi_{x_0})_1 : (TG)_1 \to X$ splits: its kernel and range are closed complemented subspaces.*

Then the orbit $\{g \cdot x_0 : g \in G\}$ is a smooth submanifold of X, and the map $\pi_{x_0} : G \to \{g \cdot x_0 : g \in G\}$ is a smooth submersion.

This lemma applies to our situation:

Theorem 2.4. *The orbit $Gr^0_{res}(p)$ is a real analytic submanifold of $p + \mathcal{B}_2(\mathcal{H})$ and the map*

$$\pi_p : U_2(\mathcal{H}) \to Gr^0_{res}(p) \ , \quad \pi_p(u) = upu^*$$

is a real analytic submersion.

Proof. In our case, $G = U_2(\mathcal{H})$, $X = p + \mathcal{B}_2(\mathcal{H})$, $x_0 = p$. The above proposition implies that π_p is open. The differential $d(\pi_p)_1$ is δ_p, its kernel is complemented because it is a closed subspace of the real Hilbert space $\mathcal{B}_2(\mathcal{H})_{ah} = (TU_2(\mathcal{H}))_1$. The range of δ_p equals $\delta_p^2(\mathcal{B}_2(\mathcal{H})_h)$, and therefore it is closed and complemented in $\mathcal{B}_2(\mathcal{H})$, by Lemma 2.1. In our context, smooth means real analytic (the group and the action are real analytic). $\square$

3 Linear connections in $Gr^0_{res}(p)$ and $U_2(\mathcal{H})$

The tangent space of $Gr^0_{res}(p)$ at q is

$$(TGr^0_{res}(p))_q = \{xq - qx : x \in \mathcal{B}_2(\mathcal{H})_{ah}\},$$

or equivalently, the range of δ_q, the differential of π_p at q. As noted above, it is a closed linear subspace of $\mathcal{B}_2(\mathcal{H})_h$, and the operator δ_q^2 is the orthogonal projection onto $(TGr^0_{res}(p))_q$. It is natural to consider the Hilbert-Riemann metric in $Gr^0_{res}(p)$ which consists of endowing each tangent space with the trace inner product. Therefore the Levi-Civita connection of this metric is given by differentiating in the ambient space $\mathcal{B}_2(\mathcal{H})_h$ and projecting onto $TGr^0_{res}(p)$. That is, if X is a tangent vector field along a curve γ in $Gr^0_{res}(p)$, then

$$\frac{DX}{dt} = \delta_\gamma^2(\dot{X}).$$

This same connection can be obtained by other means, it is the connection induced by the action of $U_2(\mathcal{H})$ on $Gr^0_{res}(p)$ and the decomposition of the Banach-Lie algebra $\mathcal{B}_2(\mathcal{H})_{ah}$ of $U_2(\mathcal{H})$:

$$\mathcal{B}_2(\mathcal{H})_{ah} = \{y \in \mathcal{B}_2(\mathcal{H})_{ah} : yp = py\} \oplus \{z \in \mathcal{B}_2(\mathcal{H})_{ah} : pzp = (1-p)z(1-p) = 0\},$$

or, if one regards operators as 2×2 matrices in terms of p, the decomposition of $\mathcal{B}_2(\mathcal{H})_{ah}$ in diagonal plus codiagonal matrices. This type of decomposition, where the first subspace is the Lie algebra of the isotropy group of the action (at p), and the second subspace is invariant under the inner action of the isotropy group, is what in differential geometry is called a reductive structure of the homogeneous space [30]. We do not perform this construction here, it can be read in [10], where it is done in a different context, but with computations that are formally identical. This alternative description of the Levi-Civita connection of $Gr^0_{res}(p)$ allows for the easy computation of the geodesics curves of the connection. The unique geodesic δ of $Gr^0_{res}(p)$ satisfying

$$\delta(0) = q \quad \text{and} \quad \dot{\delta}(0) = xq - qx$$

is given by

$$\delta(t) = e^{tz} q e^{-tz}$$

where z is the unique codiagonal element in $\mathcal{B}_2(\mathcal{H})_{ah}$ ($qzq = (1-q)z(1-q) = 0$) such that

$$zq - qz = xq - qx.$$

Equivalently, $z = \delta_q(xp - px)$ is the projection of x in the decomposition

$$x = y + z \in \{y \in \mathcal{B}_2(\mathcal{H})_{ah} : yp = py\} \oplus \{z \in \mathcal{B}_2(\mathcal{H})_{ah} : pzp = (1-p)z(1-p) = 0\}.$$

Although our main interest in this paper are projections, it will be useful to take a brief look at the natural Riemannian geometry of the group $U_2(\mathcal{H})$. Namely, the metric given by considering real part of the trace inner product, and therefore, the 2-norm at each tangent space. The tangent spaces of $U_2(\mathcal{H})$ identify with

$$(TU_2(\mathcal{H}))_u = u\mathcal{B}_2(\mathcal{H})_{ah} = \mathcal{B}_2(\mathcal{H})_{ah}u.$$

As with $Gr^0_{res}(p)$, the covariant derivative consists of differentiating in the ambient space, and projecting onto $TU_2(\mathcal{H})$. Geodesics of the Levi-Civita connection are curves of the form

$$\mu(t) = ue^{tx},$$

for $u \in U_2(\mathcal{H})$ and $x \in \mathcal{B}_2(\mathcal{H})_{ah}$. The exponential mapping of this connection is the map

$$exp : \mathcal{B}_2(\mathcal{H})_{ah} \to U_2(\mathcal{H}), \quad exp(x) = e^x.$$

Remark 3.1.

1. The exponential map

$$exp : \mathcal{B}_2(\mathcal{H})_{ah} \to U_2(\mathcal{H})$$

is surjective. This fact is certainly well known. Here is a simple proof. If $u \in U_2(\mathcal{H})$, then it has a spectral decomposition $u = p_0 + \sum_{k\geq 1}(1+\alpha_k)p_k$, where α_k are the non zero eigenvalues of $u - 1 \in \mathcal{B}_2(\mathcal{H})_{ah}$. There exist $t_k \in \mathbb{R}$ with $|t_k| \leq \pi$ such that $e^{it_k} = 1 + \alpha_k$. One has the elementary estimate

$$|t_k|^2(1 - \frac{|t_k|^2}{12}) \leq |e^{it_k} - 1|^2 = |\alpha_k|^2,$$

which implies that the sequence (t_k) is square summable. Let $z = \sum_{k\geq 1} it_kp_k$ (note that p_k are finite rank pairwise orthogonal projections). Thus $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ and clearly $e^z = u$.

2. The exponential map is a bijection between the sets

$$\mathcal{B}_2(\mathcal{H})_{ah} \supset \{z \in \mathcal{B}_2(\mathcal{H})_{ah} : \|z\| < \pi\} \to \{u \in U_2(\mathcal{H}) : \|1 - u\| < 2\}.$$

Clearly if $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ with $\|z\| < \pi$, then $e^z \in U_2(\mathcal{H})$ and $\|e^z - 1\| < 2$. Suppose that $u \in U_2(\mathcal{H})$ with $\|u-1\| < 2$. Then there exist $x \in \mathcal{B}(\mathcal{H})$, $x^* = -x$ and $\|x\| < \pi$, and $z \in \mathcal{B}_2(\mathcal{H})_{ah}$, such that $e^x = e^z = u$. Since $\|x\| < \pi$, x equals a power series in $u = e^z$, which implies that z commutes with x. Then $e^{z-x} = 1$ and thus (note that $z - x$ is anti-hermitic) $z - x = \sum_{k\geq 1} 2k\pi i p_k$ for certain projections p_k. Let $z = \sum_{j\geq 1} \lambda_j q_j$ be the spectral decomposition of z. Note that $e^z = \sum_{j\geq 1} e^{\lambda_j} q_j$, and since x commutes with e^z, this implies that x commutes with q_j, and also with z. Also it is clear that q_j and p_k also commute, and that q_j have finite ranks. Then

$$x = \sum_{j\geq 1} \lambda_j q_j + \sum_{|k|\geq 1} 2k\pi i p_k.$$

The fact that $\|x\| < \pi$ implies that the terms $2k\pi i p_k$ are cancelled by some of the $\lambda_j q_j$, in order that none of the remaining λ_j verify $|\lambda_j| \geq \pi$. It follows that the p_k have finite ranks, and that there are finitely many. Thus we can define z' adding the remaining $\lambda_j q_j$. Clearly z' verifies $\|z'\| < \pi$ and $e^{z'} = e^z = u$.

3. The argument above in fact shows that when one considers the exponential

$$exp : \{z \in \mathcal{B}_2(\mathcal{H})_{ah} : \|z\| \leq \pi\} \to U_2(\mathcal{H}),$$

then it is surjective.

If $x \in \mathcal{B}(\mathcal{H})$ with $\|x\| < \pi$ then it is well known that

$$\|e^x - 1\| = 2\sin(\frac{\|x\|}{2}). \tag{1}$$

There is a natural way to imbedd projections in the unitary group by means of the map $q \mapsto \epsilon_q = 2q - 1$. The unitary $\epsilon_q = 2q - 1$ is a symmetry, i.e. a selfadjoint unitary: $\epsilon^* = \epsilon$, $\epsilon^2 = 1$. See for instance [24], where this simple trick was used to characterize the minimality of geodesics in the Grassmann manifold of a C*-algebra. However, if $q \in Gr^0_{res}(p)$, ϵ_q does not belong to $U_2(\mathcal{H})$ (recall that q has infinite rank and corank). One can slightly modify this imbedding in order that it takes values in $U_2(\mathcal{H})$. Consider:

$$S : Gr^0_{res}(p) \to U_2(\mathcal{H}), \quad S(q) = (2q-1)(2p-1) = \epsilon_q\epsilon_p.$$

Clearly it takes unitary values. Let us show that these unitaries belong to $U_2(\mathcal{H})$. Note that if $q = upu^*$ with $u \in U_2(\mathcal{H})$, then

$$\epsilon_q = u\epsilon_p u^* = \epsilon_p + (u-1)\epsilon_p + \epsilon_p(u^*-1) + (u-1)\epsilon_p(u^*-1) \in \epsilon_p + \mathcal{B}_2(\mathcal{H}),$$

so that

$$\epsilon_q\epsilon_p \in (\epsilon_p + \mathcal{B}_2(\mathcal{H}))\epsilon_p = 1 + \mathcal{B}_2(\mathcal{H}).$$

Proposition 3.2. *The map S preserves geodesics, and its differential is 2 times an isometry.*

Proof. Let δ be a geodesic in $Gr^0_{res}(p)$, $\delta(t) = e^{tz}qe^{-tz}$ with $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ and z codiagonal in terms of q. This latter condition is equivalent to z anticommuting with ϵ_q: $z\epsilon_q = -\epsilon_q z$. Which implies, as remarked in [10], that $\epsilon_q e^{-tz} = e^{tz}\epsilon_q$, and thus $e^{tz}\epsilon_q e^{-tz} = e^{2tz}\epsilon_q$. Therefore

$$S(\delta(t)) = e^{2tz}\epsilon_q\epsilon_p,$$

which is a geodesic in $U_2(\mathcal{H})$.
The differential of S at q is given by

$$dS_q(v) = 2v\epsilon_p, \quad v \in (TGr^0_{res}(p))_q.$$

Right multiplication by a fixed unitary operator is isometric in $\mathcal{B}_2(\mathcal{H})$, therefore this map is 2 times an isometry. □

4 Minimality of geodesics

In this section we prove that the geodesics of the linear connection have minimal length up to a certain critical value of t. This could be derived from the general theory of Hilbert-Riemann manifolds. We shall prove it here, and in the process obtain a uniform lower bound for the geodesic radius, i.e. the radius of normal neighbourhoods. First we need minimality results in the group $U_2(\mathcal{H})$. These results are perhaps well known. We include proofs here for we could not find references for them, and they are central to our argument on $Gr^0_{res}(p)$.

Lemma 4.1. *Suppose that $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ has finite spectrum and $\|x\| \le \pi$, and let $u \in U_2(\mathcal{H})$. Then the (geodesic) curve $\mu(t) = ue^{tx}$, $t \in [0,1]$, has minimal length among all piecewise smooth curves in $U_2(\mathcal{H})$ joining the same endpoints.*

Proof. Since the action of left multiplication by u is an isometric isomorphism of $U_2(\mathcal{H})$, it suffices to consider the case $u = 1$. Let $\sigma(x) = \{\lambda_0 = 0, \lambda_1, \dots, \lambda_n\}$ be the spectrum of x. Then $x = \sum_{i=1}^n \lambda_i p_i$ for p_i finite rank projections, and denote by p_0 the projection onto the kernel of x. Note that $e^{tx} = p_0 + \sum_{i=1}^n e^{t\lambda_i} p_i$. Let $r_i^2 = Tr(p_i)$, $i = 1, \dots, n$, and denote by $\mathcal{S}_i$ the sphere in $\mathcal{B}_2(\mathcal{H})$ of radius r_i,

$$\mathcal{S}_i = \{a \in \mathcal{B}_2(\mathcal{H}) : Tr(a^*a) = r_i^2\},$$

with its natural Hilbert-Riemann metric induced by the (trace) inner product in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. Consider the following smooth map

$$\Phi : U_2(\mathcal{H}) \to \mathcal{S}_1 \times \ldots \times \mathcal{S}_n, \quad \Phi(u) = (p_1 u, \dots, p_n u).$$

Here the product of spheres is considered with the product metric. Apparently Φ is well defined and smooth. Note that the curve $\Phi(\mu(t))$ is a minimal geodesic of the manifold $\mathcal{S}_1 \times \ldots \times \mathcal{S}_n$. Indeed,

$$\Phi(\mu(t)) = (e^{t\lambda_1} p_1, \dots, e^{t\lambda_n} p_2),$$

where each coordinate $e^{t\lambda_i} p_i$ is a geodesic of the corresponding sphere $\mathcal{S}_i$, with length equal to $|\lambda_i| r_i \le \|x\| r_i \le \pi r_i$, and therefore it is minimal. Then $\Phi(\mu(t))$ is minimal, being the cartesian product of n minimal geodesics in the factors. Next, note that the length of $\Phi(\mu)$ equals the length of μ:

$$L_2(\Phi(\mu)) = \int_0^1 \|(\lambda_1 e^{t\lambda_1} p_1, \dots, \lambda_n e^{t\lambda_n} p_n)\| dt = \{\sum_{i=1}^n |\lambda_i|^2 r_i^2\}^{1/2} = \|x\|_2 = L_2(\mu).$$

If $\nu(t)$, $t \in [0,1]$ is any other smooth curve in $U_2(\mathcal{H})$, we claim that $L_2(\Phi(\nu)) \le L_2(\nu)$. Clearly this would prove the lemma. Indeed, since $\Phi(\mu)$ is minimal in $\mathcal{S}_1 \times \ldots \times \mathcal{S}_n$, one has $L_2(\Phi(\mu)) \le L_2(\Phi(\nu))$, and therefore

$$L_2(\nu) \ge L_2(\Phi(\nu)) \ge L_2(\Phi(\mu)) = L_2(\mu).$$

Note that

$$L_2(\Phi(\nu)) = \int_0^1 \{\sum_{i=1}^n \|\dot{\nu} p_i\|_2^2\}^{1/2} dt.$$

Since $\sum_{i=1}^{n} p_i = 1 - p_0$ and $\dot\nu^*(1-p_0)\dot\nu \leq \dot\nu^*\dot\nu$, one has that

$$\sum_{i=1}^{n} \|\dot\nu p_i\|_2^2 = \sum_{i=1}^{n} Tr(\dot\nu^* p_i \dot\nu) = Tr(\dot\nu^*(1-p_0)\dot\nu) \leq Tr(\dot\nu^*\dot\nu) = \|\dot\nu\|_2^2.$$

Therefore

$$L_2(\Phi(\nu)) \leq \int_0^1 \|\dot\nu\|_2 dt = L_2(\nu).$$

□

Theorem 4.2. *Let $u \in U_2(\mathcal{H})$ and $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ with $\|x\| \leq \pi$. Then the curve $\mu(t) = ue^{tx}$, $t \in [0,1]$ is shorter than any other pieceise smooth curve in $U_2(\mathcal{H})$ joining the same endpoints. Moreover, if $\|x\| < \pi$, then μ is unique with this property.*

Proof. Again, by the same argument as in the previous lemma, we may suppose $u = 1$. Assume that μ is not minimal. Let $\gamma(t)$, $t \in [0,1]$ be a piecewise smooth curve in $U_2(\mathcal{H})$ with $L_2(\gamma) + \delta = \|x\|_2 = L_2(\mu)$, for some $\delta > 0$. Let $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ be a finite rank operator close enough to x in the 2-norm in order that

$$y = log(e^{-x}e^{z}) \text{ verifies } \|y\|_2 < \delta/4,$$

$$|\|x\|_2 - \|z\|_2| < \delta/4,$$

and

$$\|z\| < \pi.$$

Let $\rho(t) = e^x e^{ty}$, and consider $\gamma\#\rho$ the curve γ followed by ρ, which joins 1 to e^z. Then

$$L_2(\gamma\#\rho) = L_2(\gamma) + L_2(\rho) = L_2(\gamma) + \|y\|_2 < L_2(\gamma) + \delta/4 = \|x\|_2 - 3\delta/4 < \|z\|_2 - \delta/2,$$

which contradicts the minimality of the curve e^{tz} proved in the previous lemma, because $\|z\| \leq \pi$.
Suppose now that $\|x\| < \pi$. By the general theory of Hilbert-Riemann manifolds [17], any minimal curve starting at u is a geodesic of the linear connection, i.e. a curve of the form ue^{tw}. If it joins the same endpoints as μ, then it must be $e^w = e^x$. Since $\|x\| < \pi$, x is a power series in terms of e^x, and therefore w commutes with x. Then $e^{w-x} = 1$. Suppose that $w \neq x$, then

$$w - x = \sum_{k=1}^{m} 2k\pi i p_i,$$

for certain pairwise orthogonal (non nil) projectors $p_i \in \mathcal{B}_2(\mathcal{H})$. Then,

$$\|w - x\|_2^2 = \sum_{k=1}^{m} 4\pi^2 Tr(p_i)^2 \geq 4\pi^2.$$

Since $\|x\| < \pi$, this inequality clearly impies that $\|w\| \geq \pi$, therefore leading to a contradiction. □

Remark 4.3. The proof in Theorem 4.2 shows that if $x \in \mathcal{B}_2(\mathcal{H})_{ah}$, the curve e^{tx} remains minimal as long as $t\|x\| \leq \pi$. One has coincidence $\|x\| = \|x\|_2$ only for rank one operators. In general, the number $C_x = \|x\|_2/\|x\|$ can be arbitrarily large. Therefore, for a specific $x \in \mathcal{B}_2(\mathcal{H})_{ah}$, in terms of the 2-norm, e^{tx} will remain minimal as long as

$$t\|x\|_2 \leq C_x \pi.$$

Corollary 4.4. *There are in $U_2(\mathcal{H})$ minimal geodesics of arbitrary length. Thus the Riemannian diameter of $U_2(\mathcal{H})$ is infinite.*

Theorem 4.5. *Let $u_0, u_1 \in U_2(\mathcal{H})$. Then there exists a minimal geodesic curve joining them. If $\|u_0 - u_1\| < 2$, then this geodesic is unique.*

Proof. Again, using the isometric property of the left action of $U_2(\mathcal{H})$ on itself, we may suppose $u_0 = 1$. The first assertion follows from the surjectivity of the exponential map $exp : \{x \in \mathcal{B}_2(\mathcal{H})_{ah} : \|x\| \leq \pi\} \to Gr^0_{res}(p)$ in Remark 3.1, and Theorem 4.2. The uniqueness assertion also follows from Remark 3.1. $\square$

Denote by d_2 the geodesic distance, i.e. the metric induced by the 2-norm on the tangent spaces, both in $U_2(\mathcal{H})$ and $Gr^0_{res}(p)$.

Proposition 4.6. *If $u, v \in U_2(\mathcal{H})$ then*

$$\sqrt{1 - \frac{\pi^2}{12}}\, d_2(u, v) \leq \|u - v\|_2 \leq d_2(u, v).$$

In particular the metric space $(U_2(\mathcal{H}), d_2)$ is complete.

Proof. Since left multiplication by v^* is an isometry for both metrics, we may assume that $v = 1$. As in Remark 3.1, we may assume that $u = p_0 + \sum_{k\geq 1} e^{it_k} p_k$, with p_i mutually orthogonal projections and $| t_k | \leq \pi$. Then

$$\|u - 1\|_2^2 = \| \sum_{k\geq 1} (e^{it_k} - 1) p_k \|_2^2 = \sum_{k\geq 1} | e^{it_k} - 1 |^2 \, r_k^2 = \sum_{k\geq 1} 2(1 - \cos(t_k)) r_k^2,$$

where $r_k = Tr(p_k)$. Now

$$| t |^2 \geq 2(1 - \cos(t)) \geq | t |^2 \, (1 - \frac{| t |^2}{12}) \geq | t |^2 \, (1 - \frac{\pi^2}{12})$$

for any $t \in [-\pi, \pi]$. Let $z = \sum_{k\geq 1} i t_k p_k$; clearly $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ by the inequality above and $e^z = u$. If $\gamma(t) = e^{tz}$, then γ is a minimal geodesic in $U_2(\mathcal{H})$ joining 1 to u because $\|z\| \leq \pi$. Then $d_2(u, 1) = L_2(\gamma) = \|z\|_2$, and from the two inequalities above we obtain $\sqrt{1 - \frac{\pi^2}{12}}\, \|z\|_2 \leq \|u - v\|_2 \leq \|z\|_2$, which proves the assertion of the proposition.

Therefore, $U_2(\mathcal{H})$ is complete with the geodesic distance, because $(U_2(\mathcal{H}), \| \ \|_2)$ is complete. This fact is certainly well known. We include a short proof. Suppose that u_n is a Cauchy sequence in $U_2(\mathcal{H})$ for the 2-norm. Since the 2-norm bounds the operator norm, it follows that there exists a unitary operator u such that $\|u_n - u\| \to 0$. On the other hand $u_n - 1$ is a Cauchy sequence in $\mathcal{B}_2(\mathcal{H})$, and therefore it converges to some operator in $\mathcal{B}_2(\mathcal{H})$. Thus $u \in U_2(\mathcal{H})$. $\square$

Remark 4.7. If x, y are anti-hermitic operators with $\|x\|, \|y\| < \pi$, then $e^x = e^y$ implies $x = y$. If $\|x\| = \|y\| = \pi$, from Theorem 4.2, it follows that $e^x = e^y$ implies $\|x\|_2 = \|y\|_2$, because the curves e^{tx}, e^{ty} are both minimal geodesics joining the same endpoints, hence they have the same length.

Now our main results on minimal geodesics of $Gr^0_{res}(p)$ follow:

Theorem 4.8. *Let $z \in \mathcal{B}_2(\mathcal{H})_{ah}$ which is codiagonal with respect to $q \in Gr^0_{res}(p)$, and such that $\|z\| \le \pi/2$. Then the geodesic $\alpha(t) = e^{tz}qe^{-tz}$, $t \in [0,1]$ has minimal length among all piecewise smooth curves in $Gr^0_{res}(p)$ joining the same endpoints. Moreover, if $\|z\| < \pi/2$, then α is unique having this property.*

Proof. Let β be any other piecewise smooth curve in $Gr^0_{res}(p)$ having the same endpoints as α. Consider $S(\alpha)$ and $S(\beta)$ in $U_2(\mathcal{H})$. Note that $S(\alpha)(t) = e^{2tz}\epsilon_q\epsilon_p$, with $2\|z\| \le \pi$. Therefore

$$L_2(\alpha) = \frac{1}{2}L_2(S(\alpha)) \le \frac{1}{2}(S(\beta)) = L_2(\beta).$$

The uniqueness part is an easy consequence. □

Remark 4.9. Again, as remarked after 4.2, for specific z (of rank greater than one), the geodesic $e^{tz}qe^{tz}$ will remain minimal as long as

$$t\|z\|_2 \le \frac{\pi}{2}C_z,$$

where again $C_z = \frac{\|z\|_2}{\|z\|}$ can be arbitrarily large.

Analogously as for $U_2(\mathcal{H})$, one has

Corollary 4.10. *There are in $Gr^0_{res}(p)$ minimal geodesics of arbitrary length, thus $Gr^0_{res}(p)$ has infinite Riemannian diameter.*

Theorem 4.11. *Let $q_0, q_1 \in Gr^0_{res}(p)$ such that $\|q_0 - q_1\| < 1$. Then there exists a unique geodesic joining them, which has minimal length.*

Proof. The action of $U_2(\mathcal{H})$ on $Gr^0_{res}(p)$ is isometric, therefore we may suppose without loss of generality that $q_0 = p$. Then [24], [10] there exists $z \in \mathcal{B}(\mathcal{H})$, $z^* = -z$, $\|z\| < \pi/2$, z p-codiagonal such that $e^z pe^{-z} = q_1$. Therefore

$$\epsilon_{q_1} = e^z\epsilon_p e^{-z} = e^{2z}\epsilon_p,$$

and thus $z = \frac{1}{2}log(\epsilon_{q_1}\epsilon_p)$, where log is well defined because $\|1 - \epsilon_{q_1}\epsilon_p\| = \|\epsilon_{q_1} - \epsilon_p\| = 2\|p - q_1\| < 2$. On the other hand $\epsilon_{q_1}\epsilon_p \in U_2(\mathcal{H})$, therefore by Remark 3.1, $z \in \mathcal{B}_2(\mathcal{H})_{ah}$. Moreover,

$$2\|z\| \le \pi,$$

and therefore the curve $\mu(t) = e^{2tz}\epsilon_p$ is a minimal geodesic in $U_2(\mathcal{H})$. Again, as in the previous theorem, this implies that the geodesic curve $\delta(t) = e^{tz}pe^{-tz}$, which joins p and q_1, is minimal. □

Next let us consider the case when $\|q_1 - q_2\| = 1$. The problem of existence of minimal curves in this case, in the context of abstract C^*-algebras, and measuring with the operator norm, has been studied by Brown in [6].

Theorem 4.12. *Let $q_0, q_1 \in Gr^0_{res}(p)$ with $\|q_0 - q_1\| = 1$. Then there exists a minimal geodesic joining them.*

Proof. Again, without loss of generality, we may suppose $q_0 = p$. Consider the following subspaces:

$$H_{00} = \ker p \cap \ker q_1\ ,\ \ H_{01} = \ker p \cap R(q_1)\ ,\ \ H_{10} = R(p) \cap \ker q_1\ ,\ \ H_{11} = R(p) \cap R(q_1)\ ,$$

and

$$H_0 = (H_{00} \oplus H_{01} \oplus H_{10} \oplus H_{11})^\perp.$$

These are the usual subspaces to regard when considering the unitary equivalence of two projections [11]. The space H_0 is usually called the generic part of H. It is invariant both for p and q_1. Also it is clear that H_{00} and H_{11} are invariant for p and q_1, and that p and q_1 coincide here. Thus in order to find a unitary operator e^z conjugating p and q_1, with $z \in \mathcal{B}_2(\mathcal{H})_{ah}$, which is codiagonal with respect to p, and such that $\|z\| \leq \pi/2$, one needs to focus on the subspaces H_0 and $H_{01} \oplus H_{10}$. Let us treat first H_0, denote by p' and q_1' the projections p and q_1 reduced to H_0. These projections are in what in the literature is called generic position. In [13] Halmos showed that two projections in generic position are unitarily equivalent, more specifically, he showed that there exists a unitary operator $w : H_0 \to K \times K$ such that

$$wp'w* = p'' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad wq_1'w* = q_1'' = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix},$$

where c, s are positive commuting contractions acting in K and satisfying $c^2 + s^2 = 1$. We claim that there exists an anti-hermitic operator y acting on $K \times K$, which is a co-diagonal matrix, and such that $e^y p'' e^{-y} = q_1''$. In that case, the element $z_0 = w^* y w$ is an anti-hermitic operator in H_0, which verifies $e^{z_0} p' e^{-z_0} = q_1'$, and is co-diagonal with respect to p'. Moreover. we claim that y is a Hilbert-Schmidt operator in $K \times K$ with $\|y\| \leq \pi/2$, so that z_0 is also a Hilbert-Schmidt operator in H_0 with $\|z_0\| \leq \pi/2$. Let us prove these claims. By a functional calculus argument, there exists a positive element x in the C* algebra generated by c, with $\|x\| \leq \pi/2$, such that $c = cos(x)$ and $s = sin(x)$. Since q_1'' lies in the Hilbert-Schmidt Grassmannian of p'', in particular one has that $q_1''|_{R(p'')}$ is a Hilbert-Schmidt operator. That is, the operator $cos(x)\ sin(x) + sin(x)^2$ is Hilbert-Schmidt in K. By a strightforward functional calculus argument, it follows that x is a Hilbert-Schmidt operator. Consider the operator

$$y = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$$

Clearly $y^* = -y$, $\|y\| \leq \pi/2$. A straightforward computation shows that

$$e^y p'' e^{-y} = p_1'',$$

and our claims follow.
Let us consider now the space $H_{01} \oplus H_{10}$. Recall [29] that an alternative definition of $Gr^0_{res}(p)$ states that if $q_1 \in Gr^0_{res}(p)$ then

$$pq_1|_{R(q_1)} : R(q_1) \to R(p)$$

is a Fredholm operator of index 0. Note that $H_{01} = \ker(pq_1|_{R(q_1)})$. Thus in particular $\dim H_{01} < \infty$. On the other hand, it is also apparent that $H_{10} \subset R(pq_1)^{\perp} \cap R(p)$, and therefore also $\dim H_{10} < \infty$. Therefore, the fact that $pq_1|_{R(q_1)}$ has zero index implies that

$$\dim H_{01} \leq \dim H_{10}.$$

The fact that q_1 lies in the connected component of p in the Sato Grassmannian corresponding to the polarization given by p, implies that, reciprocally, p lies in the component of q_1, in the Grassmannian corresponding to the polarization given by q_1. Thus, by symmetry,

$$\dim H_{01} = \dim H_{10}.$$

Let $v : H_{10} \to H_{01}$ be a surjective isometry, and consider

$$w : H_{01} \oplus H_{10} \to H_{01} \oplus H_{10} \ , w(\xi' + \xi'') = v^*\xi' + v\xi''.$$

In matrix form (in terms of the decomposition $H_{01} \oplus H_{10}$),

$$w = \begin{pmatrix} 0 & v \\ v^* & 0 \end{pmatrix}.$$

Apparently, $wp|_{H_{01} \oplus H_{10}} w^* = q_1|_{H_{01} \oplus H_{10}}$. Let

$$z_2 = \pi/2 \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix}.$$

Note that z_2 is an anti-hermitic operator in $H_{01} \oplus H_{10}$, with norm equal to $\pi/2$. A straightforward matrix computation shows that $e^{z_2} = w$. Consider now

$$z = z_0 + z_1 + z_2,$$

where $z_1 = 0$ in $H_{00} \oplus H_{11}$, and z_0 is the anti-hermitic operator in the generic part H_0 of H found above. Then it is clear that z is anti-hermitic, Hilbert-Schmidt ($\dim(H_{01} \oplus H_{10}) < \infty$), p-codiagonal, $\|z\| = \pi/2$, and $e^z p e^{-z} = q_1$. □

Also completeness of the geodesic metric follows:

Corollary 4.13. *The metric space* $(Gr^0_{res}(p), d_2)$ *is complete.*

Proof. Let q_n be a Cauchy sequence in $Gr^0_{res}(p)$. Since the map $S : Gr^0_{res}(p) \to U_2(\mathcal{H})$ of Proposition 3.2 is 2 times an isometry, then $S(q_n)$ is a Cauchy sequence in $U_2(\mathcal{H})$, and therefore converges to an element u of $U_2(\mathcal{H})$ in the metric d_2. Moreover, $S(Gr^0_{res}(p))$ is closed in $U_2(\mathcal{H})$ and then exists $q \in Gr^0_{res}(p)$ such that $S(q) = u$. Clearly $d_2(q_n, q) \to 0$. □

5 k-norms

In this section we study the minimality problem of geodesics in $Gr^0_{res}(p)$ measured in the k-norms, for $k \in \mathbb{R}$, $k > 2$. To do this, as with the case $k = 2$, we study first short curves in $U_2(\mathcal{H})$ with these norms. Minimality of geodesics in $Gr^0_{res}(p)$ will follow with arguments similar as in the previous section. We shall endow now the tangent spaces of $U_2(\mathcal{H})$ and $Gr^0_{res}(p)$ with the Schatten k-norm:

$$\|x\|_k = Tr(|x|^k)^{1/k} = Tr((x^*x)^{k/2})^{1/k}.$$

Note that since the tangent spaces live inside $\mathcal{B}_2(\mathcal{H})$, and $k > 2$, the k-norm of x is finite. We shall denote by L_k the functional which measures the length of a curve (either in $U_2(\mathcal{H})$ or $Gr^0_{res}(p)$) in the k-norm:

$$L_k(\alpha) = \int_{t_o}^{t_1} \|\dot{\alpha}(t)\|_k dt.$$

We are now then in the realm of (infinite dimensional) Finsler geometry.
To prove our results, two inequalities proved by Hansen and Pedersen in [14] will play a fundamental role. Let us transcribe these inequalities, called Jensen's inequalities.

1. The first is the version for C*-algebras ([14], Th. 2.7): if $f(t)$ is a convex continuous real function, defined on an interval I and and A is a C*-algebra with finite unital trace tr, then the inequality

$$tr(f(\sum_{i=1}^n b_i^* a_i b_i)) \leq tr(\sum_{i=1}^n b_i^* f(a_i) b_i) \tag{2}$$

is valid for every n-tuple $(a_1, \ldots, a_n)$ of selfadjoint elements in A with spectra contained in I and every n-tuple $(b_1, \ldots, b_n)$ in A with $\sum_{i=1}^n b_i^* b_i = 1$. We shall use it in a simpler form: if a is a selfadjoint element in a C^*-algebra with trace tr, then

$$tr(f(a)) \leq f(tr(a)) \tag{3}$$

for every convex continuous real function defined in the spectrum of a.

2. The second inequality is valid for finite matrices ([14], Th. 2.4): let f be a convex continuous function defined on I and let m and n be natural numbers, then

$$Tr(f(\sum_{i=1}^n a_i^* x_i a_i)) \leq Tr(\sum_{i=1}^n a_i^* f(x_i) a_i) \tag{4}$$

for every n-tuple $(x_1, \ldots, x_n)$ of selfadjoint $m \times m$ matrices with spectra contained in I and every n-tuple $(a_1, \ldots, a_n)$ of $m \times m$ matrices with $\sum_{i=1}^n a_i^* a_i = 1$. We shall need a simpler version, namely if $r \in \mathbb{R}$, $r \geq 1$, then

$$Tr(a^r) = Tr((\sum_{j=0}^n p_j a^r p_j) \geq Tr((\sum_{j=0}^n p_j a p_j)^r) = Tr(\sum_{j=0}^n (p_j a p_j)^r), \tag{5}$$

for $p_0, p_1, \dots, p_n$ projections with $\sum_{j=0}^n p_j = 1$ and $p_1, \dots, p_n$ of finite rank, and a a positive trace class operator. A simple aproximation argument shows that one can obtain (5) from (4). Indeed, let $\{\xi_1^j, \dots, \xi_{k_j}^j\}$ be an orthonormal basis for the range of p_j, $j = 1, \dots, n$ and $\{\varphi_i, \varphi_2, \dots\}$ be an orthonormal basis for the range of p_0. For any integer $N \geq 1$, let e_N denote the orthogonal projection onto the subspace generated by $\{\xi_i^j, j = 1, \dots, n,\ i = 1, \dots k_j\} \cup \{\varphi_k, k = 1, \dots, N\}$. Clearly e_N is a finite rank projection such that $p_j \leq e_N$, for $j = 1, \dots, n$ and such that $e_N p_0 e_N = p_{0,N}$ is also a projection. Let $a_N = e_N a e_N$. Then the following facts are apparent:

(a) $p_j a_N p_j = p_j a p_j$ for $j = 1, \dots, n$.

(b) $a_N \to a$ and $p_{0,N} a_N p_{0,N} \to p_0 a p_0$ in $\| \ \|_1$, and therefore $p_j a_N^r p_j \to p_j a^r p_j$ for $j = 0, 1, \dots, n$ and $r \geq 1$ in $\| \ \|_1$.

It follows that one can reduce to prove (5) for the operator a_N and the projections $p_{0,N}, p_1, \dots, p_n$, all of which are operators in the range of e_N, which is finite dimensional.

Let us first state the following lemma which is a simple consequence of (3).

Lemma 5.1. *Let $a \in \mathcal{B}(\mathcal{H})$ be a positive operator and p a finite rank projection. Then, if $r \in \mathbb{R}$, $r \geq 1$*

$$Tr(pap)^r \leq Tr(p)^{r-1} Tr((pap)^r).$$

Proof. If $p = 0$ the result is trivial. Suppose $Tr(p) \neq 0$. Consider the finite C*-algebra $p\mathcal{B}(\mathcal{H})p$, with unit p and normalized finite trace $tr(pxp) = \frac{Tr(pxp)}{Tr(p)}$. Then by Jensen's trace inequality for the map $f(t) = t^r$,

$$\frac{Tr(pap)^r}{Tr(p)^r} \leq \frac{Tr((pap)^r)}{Tr(p)},$$

which is the desired inequality. □

Denote by $\mathcal{S}_R^k$ the unit sphere of $\mathcal{B}_k(\mathcal{H})$:

$$\mathcal{S}_R^k = \{x \in \mathcal{B}_k(\mathcal{H}) : \|x\|_k = R\}.$$

If $\mu(t)$ is a curve of unitaries in $U_2(\mathcal{H})$, and p is finite rank projection with $Tr(p) = R^k$, then $\mu(t)p$ is a curve in $\mathcal{S}_R^k$: $\|\mu p\|_k = Tr((p\mu^*\mu p)^{k/2})^{1/k} = R$.

Lemma 5.2. *Let p be a finite rank projection with $Tr(p) = R^k$ and $\mu(t)$ be a smooth curve in $U_2(\mathcal{H})$, such that $\mu(0)p = p$ and $\mu(1)p = e^{\alpha}p$ with $-\pi \leq \alpha \leq \pi$. Then the curve μp of $\mathcal{S}_R^k$, measured with the k-norm, is longer than the curve $\epsilon(t) = e^{it\alpha}p$.*

Proof. The length of μp is (in the k-norm) measured by

$$\int_0^1 \|\dot{\mu}(t)p\|_k dt = \int_0^1 Tr((p\dot{\mu}(t)^*\dot{\mu}(t)p)^{k/2})^{1/k} dt.$$

by the inequality in the above lemma,

$$L_k(\mu p) \geq Tr(p)^{\frac{1-k/2}{k}} \int_0^1 Tr(p\dot{\mu}(t)^*\dot{\mu}(t)p)^{1/2}dt.$$

This last integral measures the length of the curve μp in the 2-sphere $\mathcal{S}^2_{R^{k/2}}$ of radius $R^{k/2}$ in the Hilbert space $\mathcal{B}_2(\mathcal{H})$. The curves $\epsilon(t) = e^{t\alpha}p$ are minimizing geodesics of these spheres, provided that $|\alpha|R^{k/2} \leq \pi R^{k/2}$, which holds because $|\alpha| \leq \pi$. It follows that

$$\int_0^1 Tr(p\dot{\mu}(t)^*\dot{\mu}(t)p)^{1/2} \geq L_2(\epsilon) = |\alpha| Tr(p)^{1/2}.$$

Then

$$L_k(\mu p) \geq |\alpha| Tr(p)^{\frac{1-k/2}{k}} Tr(p)^{1/2} = |\alpha| R = L_k(\epsilon).$$

□

Lemma 5.3. *Let $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ with finite spectrum, $x = \sum_{i=1}^n \alpha_i p_i$ with $\sum_{i=1}^n p_i = 1 - p_0$ (p_0 the kernel projection of x) and $-\pi \leq \alpha_i \leq \pi$ (i.e. $\|x\| \leq \pi$). Then the curve $\delta(t) = e^{itx}$, $t \in [0,1]$ is the shortest curve in $U_2(\mathcal{H})$ joining its endpoints, when measured with the k-norm.*

Proof. Let $Tr(p_i) = R_i^k$, $i = 1, \dots, n$. Note that the kernel projection p_0 has infinite rank. The length $L_k(\mu)$ of μ is measured by $\int_0^1 \|\dot{\mu}(t)\|_k dt$. Then, by inequality (5), with $a = \dot{\mu}(t)^*\dot{\mu}(t) \geq 0$ in $\mathcal{B}_1(\mathcal{H})$, one has

$$\|\dot{\mu}(t)\|_k \geq \{\sum_{j=0}^n Tr((p_i\dot{\mu}(t)^*\dot{\mu}(t)p_i)^{k/2})\}^{1/k} = \{\sum_{i=0}^n \|\dot{\mu}(t)p_i\|_k^k\}^{1/k}. \tag{6}$$

On the other hand, note that

$$\|\dot{\delta}\|_k = \{\sum_{i=1}^n |\alpha_i|^k R_i^k)\}^{1/k}.$$

Trivially, $\{\sum_{j=0}^n Tr((p_i\dot{\mu}(t)^*\dot{\mu}(t)p_i)^{k/2})\}^{1/k} \geq \{\sum_{j=1}^n Tr((p_i\dot{\mu}(t)^*\dot{\mu}(t)p_i)^{k/2})\}^{1/k}$, (i.e. we omit the term corresponding to the projection p_0, which has infinite trace). We finish the proof by establishing that

$$\{\sum_{j=1}^n Tr((p_i\dot{\mu}(t)^*\dot{\mu}(t)p_i)^{k/2})\}^{1/k} \geq \{\sum_{i=1}^n |\alpha_i|^k R_i^k)\}^{1/k} = L_k(\delta).$$

There is a classic Minkowski type inequality (see inequality **201** of [15]) which states that if $f_1, \dots, f_n$ are non negative functions, then

$$\int_0^1 \{\sum_{i=1}^n f_i^k(t)\}^{1/k} dt \geq (\sum_{i=1}^n \{\int_0^1 f_i(t)\}^k)^{1/k}.$$

In our case $f_i(t) = \|\dot{\mu}(t)p_i\|_k$:

$$\int_0^1 \{\sum_{i=1}^n \|\dot{\mu}(t)p_i\|_k^k)\}^{1/k} dt \geq (\sum_{i=1}^n \{\int_0^1 \|\dot{\mu}(t)p_i\|_k dt\}^k)^{1/k} \geq \{\sum_{i=1}^n |\alpha_i|^k R_i^k\}^{1/k},$$

where in the last inequality we use the previous lemma: $\int_0^1 \|\dot\mu(t)\|_k dt \geq |\alpha_i| R_i$ for $i = 1, \ldots, n$. ∎

Theorem 5.4. *Let $x \in \mathcal{B}_2(\mathcal{H})_{ah}$ with $\|x\| \leq \pi$, and $v \in U_2(\mathcal{H})$. Then the curve $\delta(t) = ve^{tx}$ has minimal length among piecewise smooth curves in $U_2(\mathcal{H})$ joining the same endpoints, measured with the k-norm.*

Proof. There is no loss of generality if we suppose $v = 1$. Indeed, for any curve μ of unitaries, $L_k(\mu) = L_k(v^*\mu)$. Suppose that there exists a piecewise C^1 curve of unitaries μ which is strictly shorter than δ, $L_k(\mu) < L_k(\delta) - \epsilon = \|x\|_k - \epsilon$. The element x can be approximated in the k-norm topology of $\mathcal{B}_k(\mathcal{H})$ by anti-hermitic elements $z \in \mathcal{B}_k(\mathcal{H})$, with finite spectrum and the following conditions:

1. $\|z\| \leq \|x\| \leq \pi$.
2. $\|x\|_k - \epsilon/2 < \|z\|_k \leq \|x\|_k$.
3. There exists a C^∞ curve of unitaries joining e^x and e^z of k-length L_k less than $\epsilon/2$.

The first two are clear. The third condition can be obtained as follows. By the third condition $e^{-x}e^z = e^y$, with $y \in \mathcal{B}_2(\mathcal{H})_{ah}$. Moreover z can be adjusted so as to obtain y of arbitrarily small k-norm. Then the curve of unitaries $\gamma(t) = e^x e^{ty}$ is C^∞, joins e^x and e^z, with k-length $\|y\|_k < \epsilon/2$.
Consider now the curve μ', which is the curve μ followed by the curve $e^x e^{ty}$ above. Then clearly

$$L_k(\mu') \leq L_k(\mu) + \|y\|_k < L_k(\mu) + \epsilon/2.$$

Therefore $L_k(\mu') < \|x\|_k - \epsilon/2$. On the other hand, since μ' joins 1 and e^z, by the lemma above, it must have length greater than or equal to $\|z\|_k$. It follows that

$$\|z\|_k \leq \|x\|_k - \epsilon/2,$$

a contradiction. ∎

One obtains minimality of geodesics in $Gr^0_{res}(p)$ for the k-norm analogously as in the previous section:

Theorem 5.5. *Let $z \in \mathcal{B}_2(\mathcal{H})_{ah}$, codiagonal with respect to $q \in Gr^0_{res}(p)$, with $\|z\| \leq \pi/2$. Then the geodesic $\alpha(t) = e^{tz}qe^{-tz}$, $t \in [0,1]$, has minimal length for the k-norm among all piecewise smooth curves in $Gr^0_{res}(p)$ having the same endpoints. If $\|z\| < \pi/2$, this curve α is unique with this property.*

Proof. The proof follows as in the analogous result for the 2 norm in the previous section, noting that the map S is also isometric for the k-norms. ∎

Theorem 5.6. *Let $q_1, q_2 \in Gr^0_{res}(p)$, then there exists a geodesic joining them, which has minimal length fot the k-norm.*

Proof. The proof follows as in the above result, the geodesic $\alpha(t) = e^{tz}q_1e^{-tz}$ with $\|z\| \leq \pi/2$ exists by virtue of (4.11) and (4.12). ∎

References

[1] L. Alvarez-Gaumé, C. Gomez, C. Reina, Loop groups, Grassmannians and string theory, Phys. Lett. B 190 (1987), no. 1-2, 55–62.

[2] E. Andruchow, D. Stojanoff, Geometry of unitary orbits, J. Operator Theory 26 (1991), no. 1, 25–41.

[3] C.J. Atkin, The Hopf-Rinow theorem is false in infinite dimensions, Bull. London Math. Soc. 7 (1975) 261–266

[4] R. Bhatia, Matrix analysis, Graduate Texts in Mathematics, **169**. Springer-Verlag, New York (1997).

[5] M. J. Bowick, S. G. Rajeev, String theory as the Kähler geometry of loop space, Phys. Rev. Lett. 58 (1987), no. 6, 535–538.

[6] L.G. Brown, The rectifiable metric on the set of closed subspaces of a Hilbert space, Trans. Amer. Math. Soc. 337 (1993), no. 1, 279–289.

[7] P. Bóna, Some considerations on topologies of infinite dimensional unitary coadjoint orbits. J. Geom. Phys. **51** (2004) n^{o}2, 256-268.

[8] D. Beltiţă, T.S. Ratiu, A.B. Tumpach, The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits , J. Functional Analysis 247 (2007), no. 1, 138-168.

[9] Carey, A. L., Some homogeneous spaces and representations of the Hilbert Lie group $U(H)_2$, Rev. Roumaine Math. Pures Appl. 30 (1985), no. 7, 505–520.

[10] G. Corach, H. Porta and L. Recht, The geometry of spaces of projections in C^*-algebras, Adv. in Math. **41** (1997) n^{o}1, 54-76.

[11] Dixmier, J., Position relative de deux variétés linéaires fermées dans un espace de Hilbert (French), Revue Sci. 86, (1948). 387–399.

[12] N. Grossman, Hilbert manifolds without epiconjugate points, Proc. Amer. Math. Soc. 16 (1965) 1365–1371.

[13] P. Halmos, Two subpaces, Trans. Amer. Math. Soc. 144, (1969), 381–389.

[14] F. Hansen, G.K. Pedersen, Jensen's operator inequality, Bull. London Math. Soc. 35 (2003), no. 4, 553–564

[15] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge University Press, London, 1934.

[16] P. de la Harpe, Classical Banach-Lie Algebras and Banach-Lie Groups of Operators in Hilbert Space, Lecture Notes in Mathematics **285**, Springer, Berlin (1972).

[17] S. Lang, Differential and Riemannian manifolds. Third edition. Graduate Texts in Mathematics, 160. Springer-Verlag, New York (1995).

[18] G. Larotonda, Unitary orbits in a full matrix algebra, Integ. Equat. Oper. Th. 54 (2006), no. 4, 511–523.

[19] J.M. Lee, Riemannian Geometry, Graduate Texts in Mathematics, Springer-Verlag **176** (1997).

[20] L. Mata-Lorenzo and L. Recht, Infinite dimensional homogeneous reductive spaces, Acta Cient. Venezolana 43 (1992), no. 2, 76–90.

[21] J. McAlpin, Infinite Dimensional Manifolds and Morse Theory, PhD Thesis, Columbia University, (1965).

[22] M. Mulase, Solvability of the super KP equation and a generalization of the Birkhoff decomposition, Invent. Math. 92 (1988), no. 1, 1–46.

[23] Ö. Pekonen, On the Kähler geometry of the Hilbert-Schmidt Grassmannian, Manuscripta Math. 63 (1989), no. 1, 21–27.

[24] H. Porta, and L. Recht, Minimality of geodesics in Grassmann manifolds. Proc. Amer. Math. Soc. **100** (1987) n^{o}3, 464-466.

[25] A. Pressley, G. Segal, Loop groups, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.

[26] I. Raeburn, The relationship between a commutative Banach algebra and its maximal ideal space, J. Functional Analysis 25 (1977), no. 4, 366–390.

[27] M. Sato, Y. Sato, Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold, Nonlinear partial differential equations in applied science (Tokyo, 1982), 259–271, North-Holland Math. Stud., 81, North-Holland, Amsterdam, 1983.

[28] G. Segal, Unitary representations of some infinite-dimensional groups, Comm. Math. Phys. 80 (1981), no. 3, 301–342.

[29] G. Segal, G. Wilson, Loop groups and equations of KdV type, Inst. Hautes Études Sci. Publ. Math. No. 61 (1985), 5–65.

[30] R. W. Sharpe, Differential geometry. Cartan's generalization of Klein's Erlangen program. With a foreword by S. S. Chern. Graduate Texts in Mathematics, 166. Springer-Verlag, New York, 1997.

[31] S. Stratila, D.V. Voiculescu, On a class of KMS states for the group $U(\infty)$, Math. Ann. 235 (1978), 87–110.

[32] M. I. Zelikin, The ζ-function of the cross-ratio on Sato's Grassmannian as an integral invariant of the Kadomtsev-Petviashvili hierarchy, (Russian) Dokl. Akad. Nauk 389 (2003), no. 2, 159–163.

Esteban Andruchow and Gabriel Larotonda
Instituto de Ciencias
Universidad Nacional de Gral. Sarmiento
J. M. Gutierrez 1150
(1613) Los Polvorines
Argentina
e-mails: eandruch@ungs.edu.ar, glaroton@ungs.edu.ar