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Research Article

Welland's type inequalities for fractional operators of convolution with kernels satisfying a Hörmander type condition and its commutators

Gladis Pradolini[a] and Jorgelina Recchi[b,c]

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[a]Facultad de Ingeniería Química, CONICET-UNL, Santa Fe, Argentina; [b]Instituto de Matemática, CONICET-UNS, Buenos Aires, Argentina; [c]Departamento de Matemática, UNS, Bahía Blanca, Buenos Aires, Argentina

ABSTRACT

Let μ be a non-negative Ahlfors n-dimensional measure on $\mathbb{R}^d$. In this context we shall consider convolution type operators $T_\alpha f = K_\alpha * f$, $0 < \alpha < n$, where the kernels K_α are supposed to satisfy certain size and regularity conditions. We prove Welland's type inequalities for the operator T_α and its commutator $[b, T_\alpha]$, with $b \in BMO$, that include the case $T_\alpha = I_\alpha$. As far as we know both estimates are new even in the case of the Lebesgue measure. We shall also give sufficient conditions on a pair of weights that guarantee the boundedness of $[b, T_\alpha]$ between two different weighted Lebesgue spaces when the underlying measure is Ahlfors n-dimensional.

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1. Introduction and statements of the main results

In many applications in Harmonic Analysis, it is well known that certain inequalities relating different operators are important tools in order to derive some continuity properties of them. Moreover it results interesting to find extensions of these inequalities to other frameworks, which leads to a deep knowledge of the behaviour of the operator considered. A very useful example is the Welland inequality that involves two important operators such that the fractional maximal operator and the fractional integral operator, defined, for $0 < \alpha < d$, by

$$M_\alpha f(x) = \sup_{B \ni x} |B|^{\alpha/d-1} \int_B |f(x)| \, dx \quad \text{and} \quad I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(x)}{|x-y|^{d-\alpha}} \, dy,$$

respectively. Concretely, this inequality establishes that, if $0 < \epsilon < \min\{\alpha, d-\alpha\}$, then

$$|I_\alpha f(x)| \le (M_{\alpha-\epsilon} f(x) M_{\alpha+\epsilon} f(x))^{1/2}.$$

Thus, by the boundedness properties of M_α we can derive boundedness results for I_α.

On the other hand it is well known the influence of the study of the continuity properties of the commutators of singular and fractional operators in partial differential equations, which allow us to obtain integrability properties of the derivatives of the solutions related

CONTACT Jorgelina Recchi drecchi@uns.edu.ar

[1–5]. This fact leads to the study of the boundedness of the commutators of integral operators of fractional type in different contexts and sometimes it involves the study of the maximal operators that govern the behaviour of them, which are fractional maximal operators associated to a Young function of $L \log L$ type. This control appears not only by means of the norm in the space where these operators act (see, e.g. [6–9]), but also by means of pointwise inequalities between them [10–13]. Is it this last point where we are interested in, that is relate the commutator of fractional type operators with maximal operators via a point-wise inequality and, particularly, it would be interesting to find a Welland type inequality relating both operators. As far as we know there is no such an estimate so, in this paper, we shall try to obtain one.

There is a wide class of operator of fractional type T_α which are the convolution with a kernel satisfying certain size and regularity conditions (see below). These operators were introduced in [14] and appear in connection with the ergodic theory. They generalize the fractional integral associated to a multipliers [15] and fractional integrals whose kernels are associated to a homogeneous function (for more details see [16–18]). The kernels are less regular than the kernel of the fractional integral operator and the regularity condition involves Young functions which determine the maximal operators related with them. In this paper we shall also give a Welland type inequality for the operator T_α which allow us to give two weighted norm estimates for this operator between Lebesgue spaces with different integrability. Moreover, and as a consequence, we can derive certain Welland type inequality for the composition of the sharp maximal operator with the commutator of T_α. This is a surprising result that can be employed to derive two weighted norm estimates for the commutator $[b, T_\alpha]$, $b \in BMO$. The techniques used to obtain it are related with the classical estimates of the sharp maximal function of a commutator in order to reduce the order of this last operator and then use an induction argument to derive continuity properties. But at this point we are not interested in reducing the order but we want to obtain maximal operators which govern the behaviour of $[b, T_\alpha]$. Then, the Welland type inequality for T_α plays an important role.

Throughout this paper we shall also be considering the Euclidean context $\mathbb{R}^d$ provided with a non-negative Ahlfors n-dimensional measure μ, that is, a Borel measure satisfying

$$c_1 l(Q)^n \le \mu(Q) \le c_2 l(Q)^n \tag{1.1}$$

for some positive constants c_1 and c_2 and for any cube $Q \subset \mathbb{R}^d$ with sides parallel to the coordinate axes, where $l(Q)$ stands for the side length of Q and n is a fixed real number such that $0 < n \le d$. Besides, for $r > 0$, rQ will mean the cube with the same centre as Q and with $l(rQ) = rl(Q)$.

Given $0 < \alpha < n$, the fractional maximal function is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)^{1-\alpha/n}} \int_Q |f(y)| \, d\mu(y).$$

When $\alpha = 0$, we write $\mathcal{M}_0 = \mathcal{M}$ to denote the Hardy–Littlewood maximal function with respect to measure μ.

Given a Young function A, we define $L^A_\mu(\mathbb{R}^d)$ as the set of all measurable functions f for which there exists a positive number λ such that

$$\int_{\mathbb{R}^d} A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) < \infty.$$

The fractional type maximal operator associated to a Young function A and measure μ is defined by

$$\mathcal{M}_{\alpha,A}(f)(x) = \sup_{Q\ni x} \mu(Q)^{\alpha/n}\|f\|_{A,Q}, \quad 0 \le \alpha < n,$$

where, for a cube Q,

$$\|f\|_{A,Q} = \inf\{\lambda > 0 : \frac{1}{\mu(Q)} \int_Q A\left(\frac{|f(x)|}{\lambda}\right) \mathrm{d}\mu(x) \le 1\}$$

is the Luxemburg type average associated to μ. When $A(t) = t^q$, with $1 \le q < \infty$, then $\|f\|_{A,Q} = ((1/\mu(Q)) \int_Q |f|^q \,\mathrm{d}\mu)^{1/q}$. When $\alpha = 0$, we write $\mathcal{M}_{0,A} = \mathcal{M}_A$.

In this paper, we shall consider convolution type operators $T_\alpha f = K_\alpha * f$, $0 < \alpha < n$, where the kernels K_α are supposed to satisfy conditions that ensure certain control on their size and their smoothness. From now on, we adopt the following convention: $|x| \sim s$ will stand for the set $\{s < |x| \le 2s\}$ and, for a Young function Φ, $\|f\|_{\Phi,|x|\sim s}$ will stand for $\|f\chi_{|x|\sim s}\|_{\Phi,B(0,2s)}$.

Definition 1.1: Let B be a Young function and let $0 < \alpha < n$. The kernel K_α is said to satisfy the $S_{\alpha,B}$ condition, and we denote $K_\alpha \in S_{\alpha,B}$, if there exists a positive constant C such that

$$\|K_\alpha\|_{B,|x|\sim s} \le Cs^{\alpha-n}. \tag{1.2}$$

When $\alpha = 0$ we simply write $S_{0,B} = S_B$ and when $B(t) = t$ we write $S_{\alpha,B} = S_\alpha$.

It is easy to see that, if $K_\alpha \in S_{\alpha,B}$ then the operator T_α is well defined for example for L^∞_c functions. On the other hand, if K_α satisfies condition $S_{\alpha,B}$ then

$$\|K_\alpha\|_{B,B_s} \le C\,s^{\alpha-n},$$

where B_s denotes a ball of radius s.

Definition 1.2: Let B be a Young function. We say that the kernel K_α satisfies the $L^{\alpha,B}$-Hörmander condition, and we write $K \in H_{\alpha,B}$, if there exist $c \ge 1$ and $C > 0$ (depending on B and k) such that for all $y \in \mathbb{R}^n$ and $R > c|y|$,

$$\sum_{m=1}^{\infty} (2^m R)^{n-\alpha} m\|K_\alpha(. - y) - K_\alpha(.)\|_{B,|x|\sim 2^m R} \le C. \tag{1.3}$$

The operators above are controlled, in some sense, for maximal type operators associated to the Young function B. For more information see [14].

The following condition is related to the classical Lipschitz condition.

Definition 1.3: The kernel K_α is said to satisfy the $H^*_{\alpha,\infty}$ condition if there exist $c \geq 1$ and $C > 0$ such that

$$|K_\alpha(x-y) - K_\alpha(x)| \leq C\frac{|y|}{|x|^{n+1-\alpha}}, \quad |x| > c|y|.$$

It is easy to see that $H^*_{\alpha,\infty} \subset H_{\alpha,B}$ for every Young function B.

Given $b \in L^1_{loc}(\mathbb{R}^n)$, the commutator of T_α is defined by

$$[b, T_\alpha]f(x) = \int_{\mathbb{R}^d} \big(b(x) - b(y)\big) K_\alpha(x-y)f(y)\,\mathrm{d}y.$$

We shall be concerned with commutators with symbols belonging to BMO. A locally integrable functions b is said to belong to BMO if

$$\|b\|_{BMO} = \sup_Q \frac{1}{\mu(Q)} \int_Q |b(x) - b_Q|\,\mathrm{d}\mu(x) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ and b_Q denotes the average of b over Q.

The following two theorems give Welland's type inequalities for the operator T_α, that includes the case $T_\alpha = I_\alpha$, and for the commutators of T_α, respectively. As far as we know this last estimate is new even in the case of the Lebesgue measure and $T_\alpha = I_\alpha$.

We will denote $T \preceq H$ when there exists a constant c such that $T \leq c\,H$.

Theorem 1.4 (Welland type inequality): *Let $0 < \alpha < n$ and T_α be a convolution operator with kernel K_α such that $K_\alpha \in S_{\alpha,B}$. Let A, B and C Young functions such that $A^{-1}B^{-1} \preceq C^{-1}$, then*

$$|T_\alpha f(x)| \leq C(n,\alpha,\epsilon)(\mathcal{M}_{\alpha+\epsilon,A}f(x)\mathcal{M}_{\alpha-\epsilon,A}f(x))^{1/2},$$

for almost every x, where $0 < \epsilon < \min\{\alpha, n-\alpha\}$.

Remark 1.1: It is important to note that, when $T_\alpha = I_\alpha$, the hypothesis $K_\alpha \in S_{\alpha,B}$ in the theorem above is superfluous any Young function C is needed. Thus the theorem holds with $A(t) = t$, as it was proved in [11].

Remark 1.2: When $C(t) = t$ and $K_\alpha \in S_{\alpha,B}$ then we have that

$$|T_\alpha f(x)| \leq C(n,\alpha,\epsilon)\left(\mathcal{M}_{\alpha+\epsilon,\tilde{B}}f(x)\mathcal{M}_{\alpha-\epsilon,\tilde{B}}f(x)\right)^{1/2},$$

where $\tilde{B}$ is the conjugate function of B. In [14] it was shown that the maximal operators in the inequality above are precisely those that control T_α.

The sharp maximal function of f is defined by

$$\mathcal{M}^\sharp f(x) = \sup_{x\in Q} \inf_{a\in\mathbb{R}} \frac{1}{\mu(Q)} \int_Q |f(y) - a|\,\mathrm{d}\mu(y).$$

We will use $\mathcal{M}^\sharp_\delta(f)$ to denote $\mathcal{M}^\sharp(|f|^\delta)^{1/\delta}$. We use the sharp maximal operator defined above to obtain the Welland type estimate involving the commutator of T_α.

Theorem 1.5: *Let T_α be a convolution operator with kernel K_α, $0 < \alpha < n$ and $b \in BMO$. Let A and B be Young functions such that $A^{-1}B^{-1} \preceq t/\log(e+t)$ and such that $A(t)/t^{n/\alpha}$ is quasi-decreasing and $A(t)/t^{n/\alpha} \to 0$ as $t \to \infty$. Assume also that there exists $1 \leq r < n/\alpha$ such that $A(t)/t^r$ is quasi-decreasing. Then, if $K_\alpha \in S_{\alpha,B} \cap H_{\alpha,B}$, there exists a positive constant C such that*

$$\mathcal{M}_\delta^\sharp([b,T_\alpha]f)(x) \leq C\|b\|_{BMO}(\mathcal{M}_{\alpha+\epsilon,A}f(x)\mathcal{M}_{\alpha-\epsilon,A}f(x))^{1/2},$$

for almost every x, where $0 < \epsilon < \min\{\alpha, n-\alpha\}$.

Remark 1.3: The function $A(t) = (t\log(e+t))^\beta$ with $\beta < n/\alpha$ satisfies the hypothesis of the previous theorem.

It is easy to see that if $K_\alpha \in H^*_{\alpha,\infty}$ then Theorem 1.5 holds. Particularly, if $T_\alpha = I_\alpha$, we obtain the following corollary.

Corollary 1.6: *Let $A(t) = t\log(e+t)$ and let μ be an Ahlfors n-dimensional measure. Given $0 < \alpha < n$, $b \in BMO$ and a non-negative function f, there exists a constant C such that*

$$\mathcal{M}_\delta^\sharp([b,I_\alpha]f(x) \leq C\|b\|_{BMO}(\mathcal{M}_{\alpha+\epsilon,L\log L}f(x)\mathcal{M}_{\alpha-\epsilon,L\log L}f(x))^{1/2}, \quad (1.4)$$

for almost every x, where $0 < \epsilon < \min\{\alpha, n-\alpha\}$.

In the classical Lebesgue context it is well known that the commutator above is controlled, in some sense, for fractional type maximal operators associated to the Young function $A(t) = t\log(e+t)$. Thus, this corollary is another way of control for commutators with this type of maximal operator.

Before introducing the next result we give some previous definitions and examples.

A doubling Young function B satisfies the B_p condition, $1 < p < \infty$, if there is a positive constant c such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{dt}{t} < \infty.$$

For more information [see, 19].

Definition 1.7: A Young function A belongs to the class $\mathcal{L}_\alpha^{p_0,p}$ if it satisfies the following properties:

(i) $A^{q_0/p_0} \in B_{q_0}$ for some $1 < p_0 \leq n/\alpha$ and $1/q_0 = 1/p_0 - \alpha/n$,
(ii) There exist two Young functions ϕ and φ such that $\varphi^{-1}(t)t^{\alpha/n} \preceq A^{-1}(t) \preceq \phi^{-1}(t)t^{\alpha/n}$,
(iii) There exist two Young functions H and J such that $H^{-1}J^{-1} \preceq A^{-1}$ with $J \in B_p$.

Definition 1.8: A Young function A belongs to the class $\mathcal{L}_{\alpha,L\log L}^{p_0,p}$ if $A \in \mathcal{L}_\alpha^{p_0,p}$ and there exists a Young function B such that $A^{-1}B^{-1} \preceq t/\log(e+t)$.

We now give some examples of functions A belonging to the class $\mathcal{L}_{\alpha,L\log L}^{p_0,p}$.

Example 1.9: Let $1 < p < \infty$, $0 < \alpha < n$ and let $A(t) = t^r$, $1 < r < \min\{p, p_o, n/\alpha\}$. Then $\phi(t) = \varphi(t) = t^{rn/(n-\alpha r)}$ and $H(t) = t^{rs/s-r}$ with $J(t) = t^s \in B_p$, $r < s < p$ satisfies (ii) and (iii), respectively. By the hypothesis on r, i) is also verified. On the other hand, if we take $B(t) = (t \log(e+t))^{r'}$ then $A = t^r$ belongs to $\mathcal{L}^{p_o,p}_{\alpha, L\log L}$.

Example 1.10: The function $A(t) = (t\log(e+t))^r$, $1 < r < \min\{p, p_o, n/\alpha\}$ satisfies (i), as it can be easily proved. Moreover, property (ii) is true by taking $\phi(t) = \varphi(t) = (t\log(e+t))^{rn/(n-\alpha r)}$ and (iii) holds by considering $H(t) = (t\log(e+t))^{rs/s-r}$ and $J(t) = t^s \in B_p$ with $r < s < p$. Finally, the function $B(t) = t^{r'}$ allows us to say that $A \in \mathcal{L}^{p_o,p}_{\alpha, L\log L}$.

Example 1.11: Let $A(t) = t^r(\log(e+t))^\gamma$, $r \neq \gamma$, $1 < r < \min\{p, p_o, n/\alpha\}$ and $0 < \gamma < n$. It easy to see that (i) is true. Taking $\phi(t) = \varphi(t) = t^{rn/(n-\alpha r)}\log(e+t)^{\gamma n/n-\alpha r}$, A satisfies (ii). Moreover, if $H(t) = t^{rs/s-r}\log(e+t)^{\gamma s/s-r}$ and $J(t) = t^s \in B_p$, $r < s < p$, then (iii) is satisfied. On the other hand, the Young function

$$B(t) = \left(\frac{t}{\log(e+t)^{\gamma/n}}\right)^{r'}$$

allows us to say that $A \in \mathcal{L}^{p_o,p}_{\alpha, L\log L}$.

Theorem 1.5 is an important tool in order to obtain the next result, which gives sufficient conditions on a pair of weights that guarantee the boundedness of $[b, T_\alpha]$ between two different weighted Lebesgue spaces when the measure involved is Ahlfors n-dimensional.

Theorem 1.12: *Let $1 < p < q < \infty$, $0 < \alpha < n$, $1 < p_0 \leq n/\alpha$ and let μ be an Ahlfors n-dimensional measure. Let A be a submultiplicative Young function such that $A \in \mathcal{L}^{p_o,p}_{\alpha, L\log L}$. Let T_α a convolution operator with kernel $K_\alpha \in S_{\alpha,B} \cap H_{\alpha,B}$. If (u, v) is a pair of weights for which there exists $r > 1$ such that for every cube Q,*

$$\mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u(x)^r \mathrm{d}\mu(x)\right)^{1/rq}\|v^{-1/p}\|_{H,Q} \leq C$$

and $u \in A_\infty$, then for every $f \in L^p_\mu(v)$ and $b \in BMO$, there exists a positive constant C such that

$$\|[b, T_\alpha]f\|_{L^q_\mu(u)} \quad \leq \quad C\|b\|_{BMO}\|f\|_{L^p_\mu(v)}.$$

The functions B and H are given in the definition of the class $\mathcal{L}^{p_o,p}_{\alpha, L\log L}$.

Corollary 1.13: *Let T_α a convolution operator with kernel $K_\alpha \in S_{\alpha,B} \cap H^*_{\alpha,\infty}$ and the same hypotheses as in the previous theorem. Then for every $f \in L^p_\mu(v)$ and $b \in BMO$, there exists a positive constant C such that*

$$\|[b, T_\alpha]f\|_{L^q_\mu(u)} \leq C\|b\|_{BMO}\|f\|_{L^p_\mu(v)}.$$

Remark 1.4: It is easy to check that the fractional integral operator I_α satisfies the hypothesis of the previous corollary.

2. Preliminaries and auxiliary theorems

2.1. Orlicz spaces

A function $B : [0, \infty) \to [0, \infty)$ is a Young function if it is convex and increasing, if $B(0) = 0$ and $B(t) \to \infty$ as $t \to \infty$. We also deal with submultiplicative Young functions, which means that $B(st) \leq B(s)B(t)$ for every $s, t > 0$. If B is a submultiplicative Young function, it follows that $B'(t) \simeq B(t)/t$ for every $t > 0$.

Given a Young function B and a cube Q, we define the Luxemburg average of f on Q associated to μ by

$$\|f\|_{B,Q} = \inf\left\{\lambda > 0 : \frac{1}{\mu(Q)}\int_Q B\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}. \tag{2.1}$$

The Luxemburg average has two rescaling properties which we will use repeatedly. Given any Young function A and $r > 0$,

$$\|f^r\|_{A,Q} = \|f\|^r_{B,Q},$$

where $B(t) = A(t^r)$. By convexity, if $\tau > 1$, $\|f\|_{A,Q} \leq \tau^n \|f\|_{A,\tau Q}$. The complementary Young function $\tilde{B}$ of a given Young function B, is defined by

$$\tilde{B}(t) = \sup_{s>0}\{st - B(s)\}, \quad t > 0.$$

It is well known that B and $\tilde{B}$ satisfy the inequality $t \leq B^{-1}(t)\tilde{B}^{-1} \leq 2t$. It is also easy to check that the following version on the Hölder inequality

$$\frac{1}{\mu(Q)}\int_Q |f(x)g(x)|\, d\mu(x) \leq 2\|f\|_{B,Q}\|g\|_{\tilde{B},Q}$$

holds. Moreover, there is a further generalization of the inequality above. If A, B and C are Young functions such that for every $t \geq t_0 > 0$,

$$B^{-1}(t)C^{-1}(t) \preceq A^{-1}(t),$$

then, the following inequality holds

$$\|fg\|_{A,Q} \leq K\|f\|_{B,Q}\|g\|_{C,Q}. \tag{2.2}$$

The following theorem also gives a sufficient condition on the function B that guarantees the continuity of the fractional type maximal operator $\mathcal{M}_{\alpha,B}$ between Lebesgue spaces with Ahlfors n-dimensional measure.

The following theorem gives sufficient conditions for strong type inequalities for $\mathcal{M}_{\alpha,B}$.

Theorem 2.1 ([20]): *Let $1 < p < q < \infty, 0 \leq \alpha < n$ and let μ be an Ahlfors n-dimensional measure in $\mathbb{R}^d$. Let A be a submultiplicative Young function such that $A \in \mathcal{L}_\alpha^{p_o,p}$ and (u, v) is a pair of weights such that for every cube Q,*

$$\mu(Q)^{\alpha/n-1/p}u(Q)^{1/q}\|v^{-1/p}\|_{H,Q} \leq K$$

then, there exists a positive constant C such that for every $f \in L^p_\mu(v)$.

$$\|\mathcal{M}_{\alpha,A}(f)\|_{L^q_\mu(u)} \leq C\,\|f\|_{L^p_\mu(v)}.$$

The functions B and H are given in the definition of the class $\mathcal{L}^{p_o,p}_{\alpha,L\log L}$.

The theorem above was proved in [20] in the more general context of upper Ahlfors n-dimensional measures, generalizing Theorem 3.1 in [11] which is a special case of this theorem by considering $H(t) = t^{rp'}$, $J(t) = t^{(rp')'}$ and $A(t) = t$. Examples 1.9, 1.10, 1.11 given previously satisfies the hypothesis of the theorems above.

Proof of Theorem 1.4: We decompose the operator T_α as follows,

$$\begin{aligned} |T_\alpha f(x)| &\le \int_{\mathbb{R}^n} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &\le \int_{B(x,s)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) + \int_{\mathbb{R}^n\setminus B(x,s)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &= I + II. \end{aligned}$$

Let $S_k(x) := B(x, 2^k s) \setminus B(x, 2^{k-1}s)$. Since $A^{-1}B^{-1}\tilde{C}^{-1} \preceq t$ then by the generalized Hölder inequality (2.2) and the condition $S_{\alpha,B}$ we obtain that

$$\begin{aligned} I &= \sum_{k=0}^{\infty} \int_{S_{-k}(x)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &= \sum_{k=0}^{\infty} \frac{\mu(B(x,2^{-k}s))}{\mu(B(x,2^{-k}s))} \int_{S_{-k}(x)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &\le C\sum_{k=0}^{\infty} \mu(B(x,2^{-k}s))\|\chi_{S_{-k}}\|_{\tilde{C},B(x,2^{-k}s)}\|K_\alpha\|_{B,|x|\sim 2^{-k-1}s}\|f\|_{A,B(x,2^{-k}s)} \\ &\le C\sum_{k=0}^{\infty} \mu(B(x,2^{-k}s))\left(2^{-k-1}s\right)^{\alpha-n}\|f\|_{A,B(x,2^{-k}s)} \\ &\le Cs^{\epsilon}\mathcal{M}_{\alpha-\epsilon,A}f(x)\sum_{k=0}^{\infty} 2^{-k\epsilon} \le Cs^{\epsilon}\mathcal{M}_{\alpha-\epsilon,A}f(x). \end{aligned}$$

In order to estimate II we proceed as follows.

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \int_{S_{k+1}(x)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &= \sum_{k=0}^{\infty} \frac{\mu(B(x,2^{k+1}s))}{\mu(B(x,2^{k+1}s))} \int_{S_{k+1}(x)} |K_\alpha(x-y)||f(y)|\,\mathrm{d}\mu(y) \\ &\le C\sum_{k=0}^{\infty} \mu(B(x,2^{k+1}s))\|\chi_{S_{k+1}}\|_{\tilde{C},B(x,2^{k+1}s)}\|K_\alpha\|_{B,|x|\sim 2^{k}s}\|f\|_{A,B(x,2^{k+1}s)} \\ &\le C\sum_{k=0}^{\infty} \mu(B(x,2^{k+1}s))\left(2^{k}s\right)^{\alpha-n}\mu(B(x,2^{k+1}s))^{\epsilon/n-\epsilon/n}\|f\|_{A,B(x,2^{k+1}s)} \\ &\le Cs^{-\epsilon}\mathcal{M}_{\alpha+\epsilon,A}f(x)\sum_{k=0}^{\infty} 2^{-k\epsilon} \le Cs^{-\epsilon}\mathcal{M}_{\alpha+\epsilon,A}f(x). \end{aligned}$$

Then, for $s > 0$

$$|T_\alpha f(x)| \leq C\left(s^{\epsilon}\mathcal{M}_{\alpha-\epsilon,A}f(x) + s^{-\epsilon}\mathcal{M}_{\alpha+\epsilon,A}f(x)\right)$$

and the result can be obtained by minimizing this expression in the variable s. ■

The proof the following lemma is similar to the case of the Lebesgue measure.

Lemma 2.2: *The following are true*:

(1) *If* $f \in BMO(\mu)$ *then*

$$\sup_Q \frac{1}{\mu(Q)} \int_Q \exp\left(\frac{|f(x) - f_Q|}{C\|f\|_{BMO}}\right) d\mu(x) < \infty.$$

(2) *Let* $0 < p < \infty$, *there exists a constant* C_p *such that*

$$\sup_Q \left(\frac{1}{\mu(Q)} \int_Q |f(x) - f_Q|^p \, d\mu(x)\right)^{1/p} \leq C_p\|f\|_{BMO}.$$

Remark 2.1: Note that the inequality from *1.* in the previous lemma implies that

$$\|f - f_Q\|_{\exp^L, Q} \leq C\,\|b\|_{BMO}.$$

The following theorem establishes the relation between M and $M^\sharp$ and the proof can be found in [9].

Theorem 2.3: *Let* $0 < p, \delta < \infty$ *and suppose that* $u \in A_\infty(\mathbb{R}^n, \mu)$. *Then there exists a constant* C *such that the inequality*

$$\|\mathcal{M}_\delta(f)\|_{L^p(u)} \leq C\|\mathcal{M}^\sharp_\delta(f)\|_{L^p(u)}$$

holds for every function f *for which the left-hand side is finite.*

Lemma 2.4: *Given* α, $0 < \alpha < n$, *let* A *be a Young function such that* $A(t)/t^{n/\alpha}$ *is quasi-decreasing and* $A(t)/t^{n/\alpha} \to 0$ *as* $t \to \infty$. *If there exists* $1 \leq r < n/\alpha$ *such that* $A(t)/t^r$ *is quasi-decreasing, then there exists a positive constant* C *such that*

$$\mathcal{M}\left((\mathcal{M}_{\alpha,A}f)^s\right) \leq C(\mathcal{M}_{\alpha,A}f)^s$$

for every $0 < s < rn/(n - \alpha)$.

When μ is the Lebesgue measure, this theorem was proved in [21].

The proof of Lemma 2.4 requires several lemmas. Throughout this part we will assume without loss generality that all functions f are non-negative.

Lemma 2.5: *Given α, $0 < \alpha < n$, let A be a Young function such that $A(t)/t^{n/\alpha}$ is quasi-decreasing. Then for every Q and $x \in Q$,*

$$\mathcal{M}_{\alpha,A}(f\chi_Q)(x) \approx \sup_{x\in P\subset Q} \mu(P)^{\alpha/n}\|f\|_{A,P}, \tag{2.3}$$

where P is any cube such that $P \subset Q$.

Proof: Fix a function f. Clearly the supremum on the right-hand side is less than or equal to the left-hand side, so it will suffice to prove the opposite inequality.

Fix $x \in Q$ and a cube $P \not\subset Q$. There are two cases. If $l(P) < l(Q)$ then by translating P we can find another cube P' containing x such that $P' \subset Q$; $Q \cap P \subset Q \cap P'$, and $\mu(P') = c\mu(P)$. But then,

$$\mu(P)^{\alpha/n}\|f\chi_Q\|_{A,P} \le C\,\mu(P')^{\alpha/n}\|f\|_{A,P'} \le C \sup_{x\in P\subset Q} \mu(P)^{\alpha/n}\|f\|_{A,P}.$$

Now suppose $l(P) \ge l(Q)$. Let $s = (l(P)/l(Q))^n \ge 1$. Since $A(t)/t^{n/\alpha}$ is quasi-decreasing, for all λ positive,

$$\frac{1}{\mu(P)}\int_P A\left(\frac{f(x)\chi_Q}{\lambda}\right)dx \le C\,\frac{1}{\mu(Q)}\int_Q s^{-1}A\left(\frac{f(x)}{\lambda}\right)dx \le C\,\frac{1}{\mu(Q)}\int_Q A\left(\frac{C'f(x)}{s^{\alpha/n}\lambda}\right)dx.$$

Therefore,

$$\begin{aligned}\mu(P)^{\alpha/n}\|f\chi_Q\|_{A,P} &\le C\,s^{-\alpha/n}\mu(P)^{\alpha/n}\|f\|_{A,Q} = C\,\mu(Q)^{\alpha/n}\|f\|_{A,Q}\\ &\le C \sup_{x\in P\subset Q} \mu(P)^{\alpha/n}\|f\chi_Q\|_{A,P}.\end{aligned}$$

If we take the supremum over all P we get result. ■

Lemma 2.6: *Given α, $0 < \alpha < n$, and a Young function A, then for every Q and for every $x \in Q$,*

$$\mathcal{M}_{\alpha,A}(f\chi_{\mathbb{R}^n\setminus 3Q})(x) \approx \sup_{P:\,Q\subset P} \mu(P)^{\alpha/n}\|f\chi_{\mathbb{R}^n\setminus 3Q}\|_{A,P}.$$

Proof: The supremum on the right-hand side is less than or equal to the left-hand side, so it will suffice to prove the opposite inequality holds up to a constant. Fix $x \in Q$ and P_0 containing x such that $(\mathbb{R}^n \setminus 3Q) \cap P_o \neq \emptyset$. Then $l(Q) \le l(P_0)$ and so $Q \subset 3P_0$. Therefore,

$$\mu(P_0)^{\alpha/n}\|f\chi_{\mathbb{R}^n\setminus 3Q}\|_{A,P_0} \le C\,\mu(3P_0)^{\alpha/n}\|f\chi_{\mathbb{R}^n\setminus 3Q}\|_{A,3P_0} \le C \sup_{P:\,Q\subset P} \mu(P)^{\alpha/n}\|f\chi_{\mathbb{R}^n\setminus 3Q}\|_{A,P}.$$

Taking the supremum over all such cubes P_0 yields the desired estimate. ■

Lemma 2.7: *Given α, $0 < \alpha < n$, suppose the Young function A is such that $A(t)/t^{n/\alpha}$ is quasi-decreasing and $A(t)/t^{n/\alpha} \to 0$ as $t \to \infty$. If $\operatorname{supp}(f) \subset Q_0$ for some cube Q_0, then $\mu(Q)^{\alpha/n}\|f\|_{A,Q} \to 0$ as $\mu(Q) \to \infty$.*

Proof: Since $\|f\|_{A,Q} \le C\|f + \chi_{Q_0}\|_{A,Q}$, we may assume without loss of generality that $f(x) \ge 1$. Let $\|f\|_{A,Q_0} = M$; then

$$\int_Q A\left(\frac{f(x)}{M}\right)\,\mathrm{d}x \le \mu(Q_0) < \infty.$$

Since $\operatorname{supp}(f) \subset Q_0$, it follows that the integrand is in L^1.

Fix $\epsilon > 0$, we need to show that there exists $N > 0$ such that if $\mu(Q) > N$, then $\mu(Q)^{\alpha/n}\|f\|_{A,Q} \le \epsilon$; to obtain this it suffices to see that

$$\frac{1}{\mu(Q)}\int_Q A\left(\frac{f(x)\mu(Q)^{\alpha/n}}{\epsilon}\right)\mathrm{d}x \le 1. \tag{2.4}$$

Since $f(x) \ge 1$ and since $A(t)/t^{n/\alpha}$ is quasi-decreasing, for almost every $x \in Q_0$ and for $\mu(Q)$ sufficiently large,

$$\frac{1}{\mu(Q)}A\left(\frac{f(x)\mu(Q)^{\alpha/n}}{\epsilon}\right) \le \frac{C}{\epsilon^{n/\alpha}}A\left(\frac{f(x)}{M}\right) \in L^1.$$

Since $A(t)/t^{n/\alpha} \to 0$,

$$\frac{1}{\mu(Q)}A\left(\frac{f(x)\mu(Q)^{\alpha/n}}{\epsilon}\right) = \left(\epsilon^{-1}f(x)\right)^{n/\alpha}\frac{A\left(\epsilon^{-1}f(x)\mu(Q)^{\alpha/n}\right)}{\left(\epsilon^{-1}f(x)\mu(Q)^{\alpha/n}\right)^{n/\alpha}} \to 0$$

as $\mu(Q) \to \infty$. Therefore, by the dominated convergence theorem,

$$\frac{1}{\mu(Q)}\int_Q A\left(\frac{f(x)\mu(Q)^{\alpha/n}}{\epsilon}\right)\mathrm{d}x \to 0,$$

so there exists $N > 0$ such that if $\mu(Q) > N$, (2.4) holds. ■

Lemma 2.8: *Given α, $0 < \alpha < n$, let A be a Young function such that $A(t)/t^{n/\alpha}$ is quasi-decreasing and $A(t)/t^{n/\alpha} \to 0$ as $t \to \infty$. Then there exist constant C, c such that for all cubes Q with $\mu(Q) = 1$, and every $\lambda > 0$,*

$$\mu(\{x \in Q : \mathcal{M}_{\alpha,A}(f\chi_Q)(x) > \lambda\})^{(n-\alpha)/n} \le C\int_{\{x\in Q: f(x)\ge\lambda/c\}} A\left(\frac{f(x)}{\lambda}\right)\mathrm{d}\mu. \tag{2.5}$$

Similar results were proved in [7,22].

Proof: We will first show that $\mathcal{M}_{\alpha,A}(\chi_Q)(x) \le C$ for every $x \in Q$. By Lemma 2.5, it suffices to show that for all cubes $P \subset Q$, $\mu(P)^{\alpha/n}\|\chi_Q\|_{A,P} \le C$. Fix such a P; then $\mu(P) \le \mu(Q) = 1$ and by the definition of the norm,

$$\mu(P)^{\alpha/n}\|\chi_Q\|_{A,P} \le \|\chi_Q\|_{A,P} \le C.$$

Now write $f\chi_Q$ as $f_1 + f_2$ where $f_1 = f\chi_{\{x\in Q: f\le 1\}}$. By the above observation, if $x \in Q$, $\mathcal{M}_{\alpha,A}f_1(x) \le \mathcal{M}_{\alpha,A}(\chi_Q)(x) \le C$. By Lemma 2.5 it follows that $\mathcal{M}_{\alpha,A}f_2(x) \le$

$C\mathcal{M}^{Q}_{\alpha,A}f_2(x)$, where

$$\mathcal{M}^{Q}_{\alpha,A}f_2(x) = \sup_{x\in P\subset Q} \mu(P)^{\alpha/n}\|f_2\|_{A,P}.$$

Therefore, there exists a constant $C_0 > 0$ such that

$$\begin{aligned}\{x \in Q : \mathcal{M}_{A,\alpha}(f\chi_Q)(x) > 2C_0\} &\subset \{x \in Q : \mathcal{M}_{A,\alpha}f_2(x) > C_0\}\\ &\subset \{x \in Q : \mathcal{M}^{Q}_{A,\alpha}f_2(x) > 1\} = E.\end{aligned}$$

For each $x \in E$, there exists a cube $P_x \subset Q$ containing x such that $\mu(P_x)^{\alpha/n}\|f_2\|_{A,P_x} > 1$. By Lemma 2.7, $\mu(Q)^{\alpha/n}\|f_2\|_{A,Q} \to 0$ as $\mu(Q) \to \infty$. Therefore, we can adapt the proof of the fractional Calderón–Zygmund decomposition in Proposition A.7 in Appendix A in 21] to show that there exist a collection of disjoint dyadic cubes $\{P_j\}_j$ such that $l(P_j) \le 2$, $E \subset \cup_j 3P_j$, and $\mu(P_j)^{\alpha/n}\|f_2\|_{A,P_j} > \beta > 0$ for some $\beta < 1$.

Since A is convex, for each cube P_j,

$$1 \le \frac{1}{\mu(P_j)}\int_{P_j} A\left(\frac{\mu(P_j)^{\alpha/n}f_2(x)}{\beta}\right)\mathrm{d}x \le \frac{C}{\mu(P_j)^{1-\alpha/n}}\int_{P_j} A\left(\frac{C'f_2(x)}{\beta}\right)\mathrm{d}x.$$

Therefore, since $A(t)/t^{n/\alpha}$ is quasi-decreasing and the P_j's are disjoint,

$$\begin{aligned}\mu(\{x \in Q : \mathcal{M}_{\alpha,A}(f\chi_Q)(x) > 2C_0\})^{(n-\alpha)/n} &\le \mu(E)^{(n-\alpha)/n} \le \sum_j \mu(3P_j)^{(n-\alpha)/n}\\ &\le C\sum_j \int_{P_j} A\left(\frac{C'f_2(x)}{\beta}\right)\mathrm{d}x\\ &\le C\int_{\{x\in Q: f\ge 1\}} A\left(\frac{f(x)}{2C_0}\right)\mathrm{d}x.\end{aligned}$$

Inequality 2.8 follows by homogeneity, replacing f by $2C_0f/\lambda$. ■

Proof of Lemma 2.4: We will first show that if Q is a cube such that $\mu(Q) = 1$, then for any $x \in Q$,

$$\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{A,\alpha}(f\chi_Q)(x)^s\,\mathrm{d}x \le C\|f\|^{s}_{A,Q}. \tag{2.6}$$

By homogeneity we may assume $\|f\|_{A,Q} = 1$, and so, in particular, that

$$\int_Q A\left(f(x)\right)\mathrm{d}x = \frac{1}{\mu(Q)}\int_Q A\left(f(x)\right)\mathrm{d}x \le 1.$$

Therefore, by Lemma 2.8, the fact that $A(t)/t^{n/\alpha}$ is quasi-decreasing and $0 < s < rn/(n-\alpha)$,

$$\begin{aligned}\int_Q \mathcal{M}_{\alpha,A}\left(f\chi_Q\right)(x)^s\,\mathrm{d}x &= \int_0^\infty s\lambda^s \mu\left(\{x \in Q : \mathcal{M}_{\alpha,A}(f\chi_Q)(x) > \lambda\}\right)\frac{\mathrm{d}\lambda}{\lambda}\\ &\le c^s + C\int_c^\infty \lambda^s\left(\int_{Q\cap\{f\ge\lambda/c\}} A\left(\frac{f(x)}{\lambda}\right)\mathrm{d}x\right)^{n/(n-\alpha)}\frac{\mathrm{d}\lambda}{\lambda}\\ &\le C + C\int_c^\infty \lambda^s\left(\int_Q \frac{1}{\lambda^r}A\left(f(x)\right)\mathrm{d}x\right)^{n/(n-\alpha)}\frac{\mathrm{d}\lambda}{\lambda} \le C.\end{aligned}$$

This yields (2.6).

We will now prove via a homogeneity argument that (2.6) extends to arbitrary cubes: for all Q,

$$\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha,A}\left(f\chi_Q\right)(x)^s\mathrm{d}x \le C\,\mu(Q)^{s\alpha/n}\|f\|_{A,Q}^s. \tag{2.7}$$

Fix a cube Q; by translation invariance we may assume without loss of generality that Q is centred at the origin. Let $l = l(Q)$, and let $f_l(x) = f(lx)$. If P is any cube contained in Q with centre x_P, let P_l be the cube centred at x_P/l with side-length $l(P)/l$. Note that $l(Q) = 1$ and every cube contained in Q_l is of the form P_l for some $P \subset Q$. Therefore, if we make the change of variables $x = ly$, we get

$$\begin{aligned}\mu(P)^{\alpha/n}\|f\|_{A,P} &= \mu(P)^{\alpha/n}\inf\left\{\lambda > 0 : \frac{1}{\mu(P)}\int_P A\left(\frac{f(x)}{\lambda}\right)\mathrm{d}x \le 1\right\}\\ &= l^\alpha\mu(P)^{\alpha/n}\inf\left\{\lambda > 0 : \frac{1}{\mu(P_l)}\int_{P_l} A\left(\frac{f(x)}{\lambda}\right)\mathrm{d}x \le 1\right\}\\ &= l^\alpha\mu(P_l)^{\alpha/n}\|f_l\|_{A,P_l}.\end{aligned}$$

Since $x \in P$ if and only if $x/l \in P_l$, this identity combined with Lemma 2.5 shows that $\mathcal{M}_{\alpha,A}(f\chi_Q)(x) \le Cl^\alpha\mathcal{M}_{\alpha,A}(f_l\chi_{Q_l})(x/l)$. hence, if we make the change of variables $y = x/l$, it follows from (2.6)that

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha,A}\left(f\chi_Q\right)(x)^s\mathrm{d}x &\le C\,l^{s\alpha}\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha,A}\left(f_l\chi_{Q_l}\right)(x/l)^s\,\mathrm{d}x\\ &= C\,l^{s\alpha}\frac{1}{\mu(Q)}\int_{Q_l}\mathcal{M}_{\alpha,A}\left(f_l\chi_{Q_l}\right)(y)^s\,\mathrm{d}y\\ &\le Cl^{s\alpha}\|f_l\|_{A,Q_l}^s\\ &= C\mu(Q)^{s\alpha/n}\|f\|_{A,Q}^s.\end{aligned}$$

We can now finish the proof. Fix any cube Q. By (2.7) and Lemma 2.6, for every $y \in Q$,

$$\begin{aligned}
&\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha,\mathcal{A}}f(x)^s\,\mathrm{d}x \\
&\le C\,\frac{1}{\mu(3Q)}\int_{3Q}\mathcal{M}_{\alpha,A}\left(f\chi_{3Q}\right)(x)^s\,\mathrm{d}x+\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha,A}\left(f\chi_{\mathbb{R}^n\setminus 3Q}\right)(x)^s\,\mathrm{d}x \\
&\le C\mu(3Q)^{s\alpha/n}\|f\|^s_{A,3Q}+C\left(\sup_{P:Q\subset P}\mu(P)^{\alpha/n}\|f\chi_{\mathbb{R}^n\setminus 3Q}\|_{A,P}\right)^s \le C\,\mathcal{M}_{\alpha,\mathcal{A}}f(y)^s.
\end{aligned}$$

■

3. Proof of the main results

Proof of Theorem 1.5: Decompose f as f_1+f_2, where $f_1=f\chi_{Q^*}$, and Q^* is the cube centred in x which sides are $2\sqrt{d}$ times larger. Let $c_Q=(T_\alpha((b-b_{Q^*})f_2))_Q$. Then, since $[b,T_\alpha]f=[b-b_{Q^*},T_\alpha]f$,

$$\begin{aligned}
&\left(\frac{1}{\mu(Q)}\int_Q |[b,T_\alpha]f(y)-c_Q|^\delta\,\mathrm{d}\mu(y)\right)^{1/\delta} \\
&\le \left(\frac{1}{\mu(Q)}\int_Q |(b(y)-b_{Q^*})T_\alpha f(y)|^\delta\,\mathrm{d}\mu(y)\right)^{1/\delta} \\
&\quad+\left(\frac{1}{\mu(Q)}\int_Q |T_\alpha[(b(y)-b_{Q^*})f_1](y)|^\delta\,\mathrm{d}\mu(y)\right)^{1/\delta} \\
&\quad+\left(\frac{1}{\mu(Q)}\int_Q |T_\alpha[(b(y)-b_{Q^*})f_2](y)-c_Q|^\delta\,\mathrm{d}\mu(y)\right)^{1/\delta} \\
&=I_1+I_2+I_3.
\end{aligned}$$

For I_1, by the Hölder inequality with $\theta=1/\delta$ and $\theta'=1/1-\delta$

$$I_1\le\left(\frac{1}{\mu(Q)}\int_Q |b(y)-b_{Q^*}|^{\delta/(1-\delta)}\,\mathrm{d}\mu(y)\right)^{(1-\delta)/\delta}\left(\frac{1}{\mu(Q)}\int_Q T_\alpha f(y)\,\mathrm{d}\mu(y)\right).$$

Hence, by Lemma 2.2(2), and Theorem 1.4 with $C(t)=t\log(e+t)$,

$$\begin{aligned}
I_1&\le C\|b\|_{BMO}\frac{1}{\mu(Q)}\int_Q\left(\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f(y)\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f(y)\right)^{1/2}\,\mathrm{d}\mu(y) \\
&\le C\|b\|_{BMO}\left(\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f(y)\,\mathrm{d}\mu(y)\right)^{1/2}\left(\frac{1}{\mu(Q)}\int_Q \mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f(y)\,\mathrm{d}\mu(y)\right)^{1/2}.
\end{aligned}$$

By Lemma 2.4 with $s=1$, we get

$$I_1\le C\|b\|_{BMO}\left(\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f(x)\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f(x)\right)^{1/2}.$$

In order to estimate the second integral, we use that $K_\alpha \in S_{\alpha,B}$ and $A^{-1}B^{-1}\tilde{C}^{-1} < t$, to get

$$\begin{aligned} I_2 &= \left(\frac{1}{\mu(Q)} \int_Q |T_\alpha[(b(y) - b_{Q^*})f_1](y)|^\delta \, \mathrm{d}\mu(y) \right)^{1/\delta} \\ &\leq \frac{1}{\mu(Q)} \int_Q \left[\frac{\mu(Q^*)}{\mu(Q^*)} \int_{Q^*} |K_\alpha(y-z)(b(z) - b_{Q^*})f(z)| \, \mathrm{d}\mu(z) \right] \mathrm{d}\mu(y) \\ &\leq C \frac{\mu(Q^*)}{\mu(Q)} \int_Q \|K_\alpha(y - .)\|_{B,Q^*} \|b - b_{Q^*}\|_{\tilde{C},Q^*} \|f\|_{A,Q^*} \, \mathrm{d}\mu(y) \\ &\leq C \|b\|_{BMO} \mathcal{M}_{\alpha,A} f(x), \end{aligned}$$

where in the last inequality we have used (1) of Lemma 2.2.

We now proceed with the estimate of I_3. Let $Q_k = 2^{k+1}Q^*$, $S_{k+1} = 2^{k+1}Q^* \setminus 2^kQ^*$ and $b_k = b_{Q_k}$. By the Hölder inequality and condition $H_{\alpha,B}$, we obtain that

$$\begin{aligned} I_3 &\leq \frac{1}{\mu(Q)} \int_Q |T_\alpha[(b(y) - b_{Q^*})f_2](y) - \left(T_\alpha((b - b_{Q^*})f_2)\right)_Q | \, \mathrm{d}\mu(y) \\ &\leq \frac{1}{\mu(Q)^2} \int_Q \int_Q \sum_{k=0}^{\infty} \int_{S_{k+1}} |b(w) - b_k||f(w)||K_\alpha(y-w) \\ &\quad - K_\alpha(z-w)| \, \mathrm{d}\mu(w) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\quad + \frac{1}{\mu(Q)^2} \int_Q \int_Q \sum_{k=0}^{\infty} \int_{S_{k+1}} |b_k - b_{Q^*}||f(w)||K_\alpha(y-w) \\ &\quad - K_\alpha(z-w)| \, \mathrm{d}\mu(w) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\leq \frac{\|b\|_{BMO}}{\mu(Q)^2} \int_Q \int_Q \sum_{k=0}^{\infty} \mu(Q_k) \|f\|_{A,Q_k} \| \left(K_\alpha(y - .) - K_\alpha(z - .)\right) \|_{B,S_{k+1}} \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\quad + \frac{\|b\|_{BMO}}{\mu(Q)^2} \int_Q \int_Q \sum_{k=0}^{\infty} k\mu(Q_k) \|f\|_{A,Q_k} \| \left(K_\alpha(y - .) - K_\alpha(z - .)\right) \|_{B,S_{k+1}} \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\leq \frac{\|b\|_{BMO}}{\mu(Q)^2} \mathcal{M}_{\alpha,A} f(x) \int_Q \int_Q \sum_{k=0}^{\infty} k\mu(Q_k)^{1-\alpha/n} \|K_\alpha(. - y) - K_\alpha(.)\|_{B,S_{k+1}} \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(y) \\ &\leq C\|b\|_{BMO} \mathcal{M}_{\alpha,A} f(x). \end{aligned}$$

Finally, observe that

$$\begin{aligned} \mu(Q)^{\alpha/n} \|f\|_{A,Q} &= \left(\mu(Q)^{\alpha/n} \|f\|_{A,Q}\right)^{1/2} \left(\mu(Q)^{\alpha/n} \|f\|_{A,Q}\right)^{1/2} \mu(Q)^{(\epsilon/n - \epsilon/n)1/2} \\ &= \left(\mu(Q)^{(\alpha+\epsilon)/n} \|f\|_{A,Q}\right)^{1/2} \left(\mu(Q)^{(\alpha-\epsilon)/n} \|f\|_{A,Q}\right)^{1/2} \\ &\leq \left(\mathcal{M}_{\alpha+\epsilon,A} f(x)\right)^{1/2} \left(\mathcal{M}_{\alpha-\epsilon,A} f(x)\right)^{1/2}. \end{aligned}$$

■

Proof of Theorem 1.12: Let $f \geq 0$ be a bounded function with compact support. Choose ϵ such that

$$0 < \epsilon < \min\left\{\alpha, n-\alpha, \frac{n}{q}, n\left(\frac{1}{p}-\frac{1}{q}\right), \frac{n}{qr'}\right\}.$$

Since $u \in A_\infty$, by Theorem 2.3 applying Theorem 1.5, we obtain that

$$\begin{aligned}
\|[b,T_\alpha]\|_{L^q(u)} &\leq \|M_\delta([b,T_\alpha])\|_{L^q(u)} \\
&\leq \|M_\delta^\sharp([b,T_\alpha])\|_{L^q(u)} \\
&\leq C\|b\|_{BMO}\left\{\int_{\mathbb{R}^d}\left(\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f(x)\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f(x)\right)^{q/2}u(x)\,\mathrm{d}\mu(x)\right\}^{1/q} \\
&\leq C\|b\|_{BMO}\left(\int_{\mathbb{R}^d}F(x)G(x)\,\mathrm{d}\mu(x)\right)^{1/q},
\end{aligned}$$

where $F(x) = (\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f(x)u(x)^{1/q})^{q/2}$ and $G(x) = (\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f(x)u(x)^{1/q})^{q/2}$. Let

$$\frac{1}{q_\epsilon^+} = \frac{1}{q}-\frac{\epsilon}{n}, \quad \frac{1}{q_\epsilon^-} = \frac{1}{q}+\frac{\epsilon}{n}, \quad q^+ = 2\frac{q_\epsilon^+}{q} \quad \text{and} \quad q^- = 2\frac{q_\epsilon^-}{q}.$$

From the way we choose ϵ, we have

$$1 < p < q_\epsilon^- < q < q_\epsilon^+ < \infty, \quad 1 < q^- < q^+ < \infty \quad \text{and} \quad \frac{1}{q^+}+\frac{1}{q^-} = 1.$$

Thus we use Hölder's inequality to get,

$$\begin{aligned}
\left(\int_{\mathbb{R}^d}F(x)G(x)\,\mathrm{d}\mu(x)\right)^{1/q} &\leq \|F\|_{L^{q^+}(\mu)}^{1/q}\|G\|_{L^{q^-}(\mu)}^{1/q} \\
&= \|\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f\|_{L^{q_\epsilon^+}(u^+)}^{1/2}\|\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f\|_{L^{q_\epsilon^-}(u^-)}^{1/2},
\end{aligned}$$

where $u^+ = u^{q_\epsilon^+/q}$ and $u^- = u^{q_\epsilon^-/q}$. Now, we will see that the pair of weights (u^+, v) satisfies the condition in Theorem 2.1 with $1 < q_\epsilon^+/q < r$ and α replaced by $\alpha+\epsilon$.

$$\begin{aligned}
&\mu(Q)^{(\alpha+\epsilon)/n-1/p}u^+(Q)^{1/q_\epsilon^+}\|v^{-1/p}\|_{Q,H} \\
&\quad = \mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u^+\,\mathrm{d}\mu\right)^{1/q_\epsilon^+}\|v^{-1/p}\|_{Q,H} \\
&\quad \leq \mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u^{q_\epsilon^+/q}\,\mathrm{d}\mu\right)^{1/q_\epsilon^+}\|v^{-1/p}\|_{Q,H} \\
&\quad \leq \mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u^r\,\mathrm{d}\mu\right)^{1/rq}\|v^{-1/p}\|_{Q,H} \leq C.
\end{aligned}$$

Note that in the last inequality we have used the hypothesis on the weights u and v. Then, Theorem 2.1 implies that

$$\|\mathcal{M}_{\alpha+\epsilon,\mathcal{A}}f\|_{L^{q_\epsilon^+}(u^+)} \leq \|f\|_{L^p(v)}.$$

Now, for the second term it is easy to prove the estimate for the weights, since $q_\epsilon^-/q < 1 < r$, and

$$\begin{aligned}
&\mu(Q)^{(\alpha-\epsilon)/n-1/p}u^-(Q)^{1/q_\epsilon^-}\|v^{-1/p}\|_{Q,H} \\
&= \mu(Q)^{1/q+\alpha/n-1/p}\left(\frac{1}{\mu(Q)}\int_Q u^-\,\mathrm{d}\mu\right)^{1/q_\epsilon^-}\|v^{-1/p}\|_{Q,H} \\
&= \mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u^{\frac{q_\epsilon^-}{q}}\,\mathrm{d}\mu\right)^{q/qq_\epsilon^-}\|v^{-1/p}\|_{Q,H} \\
&\leq \mu(Q)^{(1/q+\alpha/n-1/p)}\left(\frac{1}{\mu(Q)}\int_Q u^r\,\mathrm{d}\mu\right)^{1/rq}\|v^{-1/p}\|_{Q,H} \leq C.
\end{aligned}$$

In this way, the pair of weights (u^-, v) verifies the condition with $1 < p < q_\epsilon^- < \infty$ and $\alpha - \epsilon$. By Theorem 2.1,

$$\|\mathcal{M}_{\alpha-\epsilon,\mathcal{A}}f\|_{L^{q_\epsilon^-}(u^-)} \leq \|f\|_{L^p(v)}.$$

Then

$$\left(\int_{\mathbb{R}^d} F(x)G(x)\,\mathrm{d}\mu(x)\right)^{1/q} \leq C\|f\|_{L^p(v)}.$$

■

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