The Representation Dimension of a Selfinjective Algebra of Euclidean Type

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Abstract

We prove that the representation dimension of a selfinjective algebra of euclidean type is equal to three, and give an explicit construction of the Auslander generator of its module category.

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1. Introduction

The homological dimensions are useful algebraic invariants, measuring how much an algebra or a module deviates from a situation considered to be ideal. The representation dimension of an Artin algebra was introduced by Auslander in the early seventies [10]. It measures the least global dimension of the endomorphism rings of modules which are at the same time generators and cogenerators of the module category, thus expressing the complexity of the morphisms in this category. Part of the interest in this invariant comes from its relation with the finitistic dimension conjecture: it was proved by Igusa and Todorov that, if the representation dimension is at most three, then its finitistic dimension is finite [24]. Iyama has shown that the representation dimension of any algebra is finite [25] and Rouquier has shown that, for any positive integer n, there exists an algebra having n as representation dimension [33]. There were several attempts to understand this invariant and to compute it for classes of algebras, see for instance [6, 4, 17, 20, 29, 32]. In particular, it was shown in [16] that the representation dimension of the trivial extension of an (iterated) tilted algebra equals three, and in [21] that the representation dimension of a special biserial algebra is at most three.

In the present paper, we are interested in tame algebras. It was shown by Auslander that an algebra is representation-finite if and only if its representation dimension equals two [10]. Because Auslander's expectation was that this invariant would measure how far an algebra is from being representation-finite, it is natural to conjecture that the representation dimension of a tame algebra is at most three.

Among the best known and most studied classes of tame algebras are the tame selfinjective algebras, see [38]. In fact, it is shown in [12] that the selfinjective algebras socle equivalent to weakly symmetric algebras of euclidean type have representation dimension three. Our objective here is to determine the representation dimension of selfinjective algebras of euclidean type (over an algebraically closed field). We recall that, if $\vec{\Delta}$ is an euclidean quiver, then an algebra A is called selfinjective of type $\vec{\Delta}$ whenever there exists a tilted algebra B of type $\vec{\Delta}$ and an infinite cyclic admissible group G of automorphisms of the repetitive category $\hat{B}$ of B such that $A \cong \hat{B}/G$. Because of the main result of [36], this class of algebras coincides with the class of representation infinite domestic selfinjective algebras which admit simply connected Galois coverings, in the sense of [8]. We prove the following theorem.

Theorem. *Let A be a selfinjective algebra of euclidean type. Then the representation dimension of A is equal to three.*

Our strategy is the following. We start by considering a class of algebras which we call domestic quasitube algebras, and prove that their representation dimension equals three. These algebras are considered as building blocks for the repetitive category of a tilted algebra of euclidean type, which we obtain from them (and from the tilted algebra) by successive gluings. Therefore, we show how to glue domestic quasitube algebras between them or to tilted algebras of euclidean type and prove that this does not change the representation dimension. Finally, using Galois coverings, we prove our main theorem. At each step, our proof is constructive: we give explicitly the generator-cogenerator of the module category for which the representation dimension is attained.

The paper is organised as follows. After an introductory section devoted to fixing the notation and recalling useful facts about the representation dimension, section 3 is devoted to domestic quasitube algebras and section 4 to the gluings of such algebras. We next recall in section 5 the necessary facts about the selfinjective algebras of euclidean type, then prove our main theorem in section 6.

2. The Representation Dimension

2.1. Notation

Throughout this paper, $\mathbf{k}$ denotes an algebraically closed field. By algebra A is meant a basic, connected, associative finite dimensional $\mathbf{k}$-algebra with an identity. Thus, there exists a connected bound quiver (Q_A, I) and an isomorphism $A \cong \mathbf{k}Q_A/I$. Equivalently, A may be considered as a $\mathbf{k}$-category with object class A_0 the set of points in Q_A, and with set of morphisms $A(x, y)$ from x to y the quotient of the $\mathbf{k}$-vector space $\mathbf{k}Q_A(x, y)$ of linear combinations of paths in Q_A from x to y by $I(x, y) = I \cap \mathbf{k}Q_A(x, y)$, see [15]. A full subcategory C of A is *convex* if, for each $x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_n$ in A with $x_0, x_n \in C_0$, we have $x_i \in C_0$ for each i. The algebra A is *triangular* if Q_A is acyclic.

Here, A-modules will mean finitely generated right A-modules. We denote by $\operatorname{mod} A$ the category of A-modules and by $\operatorname{ind} A$ a full subcategory consisting of a complete set of representatives of the isomorphism classes (isoclasses) of indecomposable A-modules. For a point x in Q_A, we denote by $P(x)$ (or $I(x)$, or $S(x)$) the indecomposable projective (or injective, or simple, respectively) A-module corresponding to x. The projective (or injective) dimension of a module M will be denoted by $\operatorname{pd} M$ (or $\operatorname{id} M$, respectively) and the global dimension of A by $\operatorname{gl.dim} A$. For a module M, the notation $\operatorname{add} M$ stands for the additive full subcategory of $\operatorname{mod} A$ with objects the direct sums of direct summands of M. For two full subcategories $\mathscr{C}, \mathscr{D}$ of $\operatorname{ind} A$, the notation $\operatorname{Hom}_A(\mathscr{C}, \mathscr{D}) = 0$ means $\operatorname{Hom}_A(M, N) = 0$ for all M in $\mathscr{C}$, N in $\mathscr{D}$. We then denote by $\mathscr{D} \vee \mathscr{C}$ the full subcategory of $\operatorname{ind} A$ having as objects those of $\mathscr{C}_0 \cup \mathscr{D}_0$. Finally, we denote by $D = \operatorname{Hom}_\mathbf{k}(-, \mathbf{k})$ the usual duality between $\operatorname{mod} A$ and $\operatorname{mod} A^{\operatorname{op}}$.

A *path* in $\operatorname{ind} A$ from M to N is a sequence of non-zero morphisms

$$M = M_0 \longrightarrow M_1 \longrightarrow \cdots \longrightarrow M_t = N \tag{*}$$

with all M_i indecomposable. We then say that M is a *predecessor* of N, or that N is a *successor* of M.

We use freely properties of the Auslander-Reiten translations $\tau_A = D \operatorname{Tr}$ and $\tau_A^{-1} = \operatorname{Tr} D$ and the Auslander-Reiten quiver $\Gamma(\operatorname{mod} A)$ of A for which we refer to [11, 7]. We identify points in $\Gamma(\operatorname{mod} A)$ with the corresponding A-modules and (parts of) components in $\Gamma(\operatorname{mod} A)$ with the corresponding full subcategories of $\operatorname{ind} A$. For tubes, tubular extensions and coextensions, we refer the reader to [30], and for tame algebras we refer to [37, 34, 35].

2.2. Representation dimension

The notion of representation dimension was introduced in [10] to which we refer for the original definition. Hence, we use as definition the following characterisation, also from [10].

Definition. Let A be a non-semisimple algebra. Its *representation dimension* $\operatorname{rep.dim} A$ is the infimum of the global dimensions of the algebras $\operatorname{End} M$, where the module M is at the same time a generator and a cogenerator of $\operatorname{mod} A$.

Note that, if M is a generator and a cogenerator of $\operatorname{mod} A$, then it can be written as $M = A \oplus D A \oplus M'$, where M' is an A-module. If M is such a module and moreover $\operatorname{rep.dim} A = \operatorname{gl.dim} \operatorname{End} M$, then M is called an *Auslander generator* of $\operatorname{mod} A$.

For studying the representation dimension, it is convenient to use a functorial language. A contravariant functor $F : (\mathrm{add}\, M)^{\mathrm{op}} \longrightarrow Ab$ is called *finitely presented*, or *coherent*, if there exists a morphism $f : M_1 \to M_0$, with M_1, M_0 in $\mathrm{add}\, M$, which induces an exact sequence of functors

$$\mathrm{Hom}_A(-, M_1) \xrightarrow{\ \mathrm{Hom}_A(-,f)\ } \mathrm{Hom}_A(-, M_0) \longrightarrow F \longrightarrow 0.$$

It is shown in [10] that the category $\mathscr{F}_M$ of finitely presented functors from $(\mathrm{add}\, M)^{\mathrm{op}}$ to Ab is equivalent to $\mathrm{mod}\,\mathrm{End}\, M$ and, in particular, is abelian.

In this paper, we are particularly interested in algebras of representation dimension 3. We recall that an algebra A is representation-finite if and only if $\mathrm{rep.dim.}\, A = 2$, see [10]. Therefore, if A is representation-infinite, then $\mathrm{rep.dim.}\, A \geqslant 3$. We have the following well-known characterisation of algebras with representation dimension 3, see [10, 16, 21, 39].

Lemma. *Let M be an A-module which is a generator and a cogenerator of $\mathrm{mod}\, A$. Then $\mathrm{gl.dim.}(\mathrm{End}\, M) \leqslant 3$ if and only if, for each A-module X, there exists a short exact sequence*

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with M_0, $M_1 \in \mathrm{add}\, M$, such that the induced sequence of functors

$$0 \longrightarrow \mathrm{Hom}_A(-, M_1) \longrightarrow \mathrm{Hom}_A(-, M_0) \longrightarrow \mathrm{Hom}_A(-, X) \longrightarrow 0$$

is exact in $\mathscr{F}_M$. In this case, $\mathrm{rep.dim.}\, A \leqslant 3$. $\qquad\qquad\square$

In the situation of the lemma, not only does M generate X, but also $\mathrm{Hom}_A(-, M)$ generates $\mathrm{Hom}_A(-, X)$. This leads to consider the case where the morphism $\mathrm{Hom}_A(-, M_0) \to \mathrm{Hom}_A(-, X)$ is a projective cover. It was proved in [6](1.4) that if X is generated by M, then there exists an epimorphism $f_0 : M_0 \to X$, where $M_0 \in \mathrm{add}\, M$, such that $\mathrm{Hom}_A(-, f_0) : \mathrm{Hom}_A(-, M_0) \to \mathrm{Hom}(-, X)$ is a projective cover.

Accordingly, a short exact sequence $0 \to M_1 \to M_0 \xrightarrow{f_0} X \to 0$ such that $\mathrm{Hom}_A(-, f_0) : \mathrm{Hom}_A(, M_0) \to \mathrm{Hom}(-, X)$ is a projective cover, is called an $\mathscr{F}_M$-*resolution* of X.

In this terminology, the previous lemma says that, if M is a generator-cogenerator of $\mathrm{mod}\, A$, then $\mathrm{gl.dim.}(\mathrm{End}\, M) \leqslant 3$ if and only if each A-module admits an $\mathscr{F}_M$-resolution.

2.3. Approximations

An equivalent language is useful. Let M be any A-module. Given an A-module X, a morphism $f_0 : M_0 \longrightarrow X$ with $M_0 \in \mathrm{add}\, M$ is an $\mathrm{add}\, M$-*approximation* if, for any morphism $f_1 : M_1 \to X$ there exists $g : M_1 \to M_0$ such that $f_1 = f_0 g$:

Equivalently, $f_0 : M_0 \longrightarrow X$ is an $\mathrm{add}\, M$-approximation if and only if $\mathrm{Hom}_A(-, f_0) : \mathrm{Hom}_A(-, M_0) \longrightarrow \mathrm{Hom}(-, X)$ is surjective in $\mathrm{add}\, M$.

Note that, if X is generated by M, then any $\mathrm{add}\, M$-approximation $f_0 : M_0 \to X$ of X is surjective. Indeed, let $f_1 : M_1 \to X$ with $M_1 \in \mathrm{add}\, M$ be surjective. Then there exists $g : M_1 \to M_0$ such that $f_1 = f_0 g$. The surjectivity of f_1 implies that of f_0.

An $\mathrm{add}\, M$-approximation is *(right) minimal* if each morphism $g : M_0 \to M_0$ such that $f_0 g = f_0$ is an isomorphism. Because of [11](I.2.1), if there exists an $\mathrm{add}\, M$-approximation, then there exists an $\mathrm{add}\, M$-approximation which is minimal and is then called a *minimal* $\mathrm{add}\, M$-*approximation*.

A short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \xrightarrow{f_0} X \longrightarrow 0$$

with $M_1, M_0 \in \text{add}\, M$ is an add M-*approximating sequence* if $f_0 : M_0 \to X$ is an add M-approximation of X. It is a *minimal* add M-*approximating sequence* if moreover f_0 is minimal. We need [4](1.7) which we reprove here because it is central to our considerations.

Lemma. *Let M, X be A-modules. If there exists an* add M-*approximating sequence of X, then there exists a minimal* add M-*approximating sequence which is moreover a direct summand of any* add M-*approximating sequence.*

Proof. Let M, X be A-modules and $0 \longrightarrow M_1 \longrightarrow M_0 \overset{f_0}{\longrightarrow} X \longrightarrow 0$ be an add M-approximating sequence. Because of [11](I.2.1), there exists a minimal add M-approximation $f_0' : M_0' \to X$. Also, f_0' is surjective as observed above. Let $M_1' = \text{Ker}\, f_0'$. Because each of f_0, f_0' is an add M-approximation of M, we get a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M_1' & \longrightarrow & M_0' & \overset{f_0'}{\longrightarrow} & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_0 & \overset{f_0}{\longrightarrow} & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle v'} & & \| & & \\
0 & \longrightarrow & M_1' & \longrightarrow & M_0' & \overset{f_0'}{\longrightarrow} & X & \longrightarrow & 0 \,.
\end{array}
$$

Because f_0' is minimal, $v'v$ is an isomorphism. Hence, so is $u'u$. Therefore, u and v are sections. $\qquad\square$

2.4. Approximating sequences and $\mathscr{F}_M$-resolutions

We now prove that these two terminologies are equivalent.

Lemma. *Let M, X be A-modules and $0 \longrightarrow M_1 \longrightarrow M_0 \overset{f_0}{\longrightarrow} X \longrightarrow 0$ be exact with $M_1, M_0 \in \text{add}\, M$. Then this sequence is an $\mathscr{F}_M$-resolution if and only if it is a minimal* add M-*approximating sequence.*

Proof. It suffices to prove that an add M-approximation $f_0 : M_0 \to X$ is minimal if and only if $\text{Hom}_A(-, f_0) : \text{Hom}_A(-, M_0) \to \text{Hom}_A(-, X)$ is a projective cover in $\mathscr{F}_M$.

Indeed, $\text{Hom}_A(-, f_0)$ is a projective cover if and only if, for any epimorphism $\text{Hom}_A(-, f') : \text{Hom}_A(-, M') \to \text{Hom}_A(-, X)$ with $M' \in \text{add}\, M$, there exists a retraction $\text{Hom}_A(-, g) : \text{Hom}_A(-, M') \to \text{Hom}_A(-, M_0)$ such that $\text{Hom}_A(-, f_0)\, \text{Hom}_A(-, g) = \text{Hom}_A(-, f')$. This is equivalent to saying that there exists a retraction $g : M' \to M_0$ such that $f_0 g = f'$, or to saying that among all add M-approximations of X, f_0 is the one whose domain M_0 has least length. Because of [11](I.2.2), this is the same as requiring that f_0 is minimal. $\qquad\square$

2.5. Tilted Algebras

We recall the definition of a tilted algebra. Let A be an algebra. A module T_A is said to be a *tilting module* if $\text{pd}\, T \le 1$, $\text{Ext}_A^1(T, T) = 0$ and there exists a short exact sequence $0 \longrightarrow A_A \longrightarrow T_A' \longrightarrow T_A'' \longrightarrow 0$ with T_A', T_A'' in add T. Let H be a hereditary algebra. An algebra A is *tilted of type H* if there exists a tilting H-module T such that $A = \text{End}\, T_H$. If H is the path algebra of a Dynkin (or euclidean, or wild) quiver Q, then we say that A is tilted of Dynkin (or euclidean, or wild, respectively) type Q. For tilting theory we refer the reader to [7].

Tilted algebras are characterised by the existence of slices in their module categories. We recall from [31](Appendix) that a class Σ in ind A is called a *complete slice* if:

(1) Σ is sincere, that is, if P is any projective A-module, then there exists $U \in \Sigma$ such that $\text{Hom}_A(P, U) \ne 0$;

(2) Σ is convex, that is, if $U_0 \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_t$ is a path in ind A with $U_0, U_t \in \Sigma$ then $U_i \in \Sigma$ for all i;

(3) if $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$ is an almost split sequence, then at most one of U, W lies in Σ. Moreover, if an indecomposable summand of V lies in Σ, then U or W lies in Σ.

It is shown (see, for instance, [30]) that an algebra is tilted if and only if it contains a complete slice Σ. In this case, the module $T = \bigoplus_{U \in \Sigma} U$ is a tilting module called the *slice module* of Σ. We need the following fact from [6].

Lemma. *Let A be a tilted algebra, and T the slice module of a complete slice. Then*

(a) *for any A-module X generated by T, there exists a minimal $\mathrm{add}(T \oplus \mathrm{D}A)$-approximating sequence for X of the form*

$$0 \longrightarrow T_1 \longrightarrow T_0 \oplus I_0 \longrightarrow X \longrightarrow 0$$

with $T_1, T_0 \in \mathrm{add}\, T$ and I_0 injective;

(b) *the module $M = A \oplus \mathrm{D}A \oplus T$ is an Auslander generator for $\mathrm{mod}\, A$ and $\mathrm{rep.\,dim.}\, A \leqslant 3$.*

Proof. (a) Existence of a minimal $\mathrm{add}(T \oplus \mathrm{D}A)$-approximation follows from [6](1.4) and [11](I.2.2). That the kernel of this approximation lies in $\mathrm{add}\, T$ follows from [6](2.2)(f).

(b) This is [6](2.3). □

3. Domestic Quasitube Algebras

3.1. The definition

In this section, we define domestic quasitube algebras and prove that their representation dimension is 3. First, we recall that a family of pairwise orthogonal generalised standard components $\mathscr{C} = (\mathscr{C}_i)_{i \in I}$ in the Auslander-Reiten quiver of an algebra A is called a *separating family* of components if the indecomposables not in $\mathscr{C}$ split into two classes $\mathscr{P}$ and $\mathscr{Q}$ such that:

(a) $\mathrm{Hom}_A(\mathscr{Q}, \mathscr{P}) = 0$, $\mathrm{Hom}_A(\mathscr{Q}, \mathscr{C}) = 0$ and $\mathrm{Hom}_A(\mathscr{C}, \mathscr{P}) = 0$, and

(b) any morphism from $\mathscr{P}$ to $\mathscr{Q}$ factors through $\mathrm{add}\, C$.

We thus have $\mathrm{ind}\, A = \mathscr{P} \vee \mathscr{C} \vee \mathscr{Q}$.

For the admissible operations, we refer to [9] or [27].

Given a tame concealed algebra C, an algebra A is a *quasitube enlargement* of C if A is obtained from C by an iteration of the admissible operations ad 1), ad 1*), ad 2), ad 2*) either on a stable tube of $\Gamma(\mathrm{mod}\, C)$ or on a quasitube obtained from a stable tube by means of the operations done so far. A quasitube enlargement A of C is a *domestic quasitube enlargement* provided A is a domestic algebra.

Definition. An algebra A is a *domestic quasitube algebra* provided A is a domestic quasitube enlargement of a tame concealed algebra such that all projectives, and all injectives in its quasitubes are projective-injective.

Thus, a quasitube in a domestic quasitube algebra becomes a stable tube after deletion of the projective-injectives and all arrows incident to them.

From now on, and until the end of the paper, we use the term quasitube algebra in this particular restricted sense.

Specialising theorems 3.5, 4.1 and Corollary 4.2 of [9] to our situation, we get the following structure theorem for domestic quasitube algebras and their module categories.

Theorem. *Let A be a domestic quasitube algebra obtained as a quasitube enlargement of a tame concealed algebra C. Then:*

(a) *A has a sincere separating family $\mathscr{C}$ of quasitubes obtained from the stable tubes of C by the corresponding sequence of admissible operations;*

(b) *there is a unique maximal branch coextension A^- of A which is a full convex subcategory of A, and which is a tilted algebra of euclidean type;*

(c) *there is a unique maximal branch extension A^+ of A which is a full convex subcategory of A, and which is a tilted algebra of euclidean type;*

(d) ind $A = \mathscr{P} \vee \mathscr{C} \vee \mathscr{Q}$, *where $\mathscr{P}$ is the postprojective component of $\Gamma(\mathrm{mod}\,A^-)$ while $\mathscr{Q}$ is the preinjective component of $\Gamma(\mathrm{mod}\,A^+)$.*

We may observe that an algebra A is a domestic quasitube algebra if and only if it is a domestic algebra with a separating family of quasitubes: this indeed follows easily from Theorems A and F of [28].

It follows easily from the description of the theorem that A^- is the left support algebra of A, while A^+ is its right support algebra, in the sense of [3]. We recall that support algebras of subcategories have been used in [6, 4] in order to calculate the representation dimension. It is then natural to use them in our context.

Example. Let A be given by the quiver
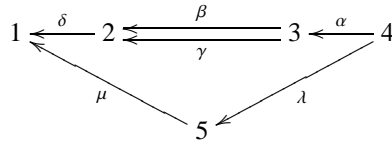
bound by $\alpha\beta = 0, \beta\delta = 0, \alpha\gamma\delta = \lambda\mu$. In this example A^- and A^+ are the full subcategories of A with object classes $A_0^- = A_0 \setminus \{4\}$ and $A_0^+ = A_0 \setminus \{1\}$, respectively. Both are tilted of type $\tilde{\mathbb{A}}$. The Auslander-Reiten quiver $\Gamma(\mathrm{mod}\,A)$ has the form
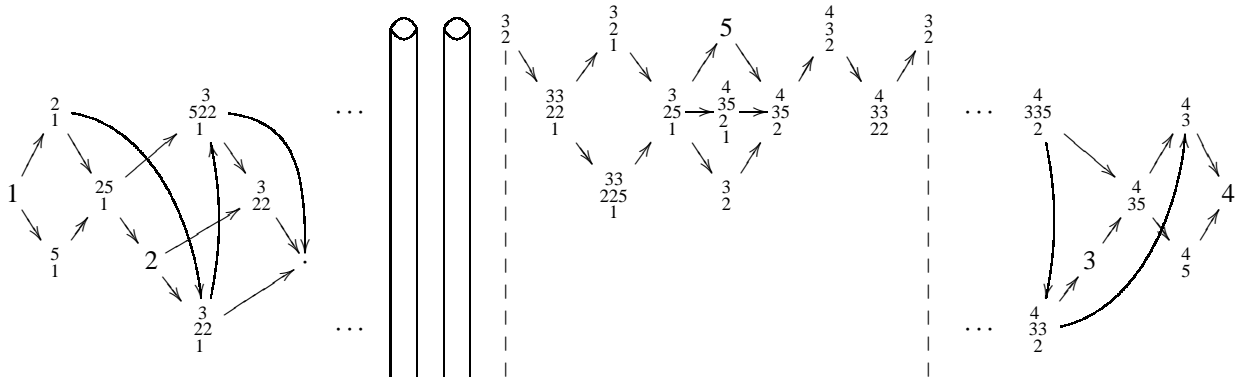
where the indecomposables are represented by their Loewy series. The shown quasitube is obtained by identifying along the vertical dotted lines.

Some additional remarks are in order. Let A be a domestic quasitube algebra. In the notation of the theorem, the postprojective component $\mathscr{P}$ of $\Gamma(\mathrm{mod}\,A)$ coincides with the postprojective component of $\Gamma(\mathrm{mod}\,A^-)$. Now, we know that A^- is a branch coextension of a tame concealed algebra, and is also a tilted algebra of euclidean type. Therefore, $\mathscr{P}$ contains a complete slice Σ^- of $\mathrm{mod}\,A^-$. However, Σ^- is clearly not a complete slice in $\mathrm{mod}\,A$, because the quasitubes of the family $\mathscr{C}$ generally contain projective-injectives. It is a right section in the sense of [2]. We recall the definition. Let Γ be a translation quiver. A full subquiver Σ of Γ is called a *right section* if

(1) Σ is acyclic;

(2) for any $x \in \Gamma_0$ such that there exist $y \in \Sigma_0$ and a path from y to x in Γ, there exists a unique $n \geqslant 0$ such that $\tau^n x \in \Sigma_0$;

(3) Σ is convex in Γ.

Dually, one defines *left sections*. A subquiver which is at the same time a right and a left section is called a *section*, see [7].

It follows easily from its definition that Σ^- is a right section in the postprojective component $\mathscr{P}$ of $\Gamma(\mathrm{mod}\,A)$ and that $A^- = A/\mathrm{Ann}\,\Sigma^-$, where $\mathrm{Ann}\,\Sigma^- = \bigcap_{U \in \Sigma^-} \mathrm{Ann}\,U$.

Dually, the preinjective component $\mathscr{Q}$ of $\Gamma(\mathrm{mod}\,A)$ contains a complete slice Σ^+ in $\mathrm{mod}\,A^+$, which is not a slice in $\mathrm{mod}\,A$, but rather a left section. Moreover, $A^+ = A/\mathrm{Ann}\,\Sigma^+$.

3.2. Restriction of injectives

When dealing with a module over a domestic quasitube algebra, we need to consider its restriction to A^-, that is, the largest A^--submodule of this module. We recall from theorem 3.1 that each of A^- and A^+ is a full convex subcategory of A, with A^- closed under successors and A^+ closed under predecessors. The following lemma is useful.

Lemma. *Let A be an algebra, B a quotient of A, I an indecomposable injective A-module having socle in $\operatorname{mod} B$. Then the largest B-submodule I' of I is an indecomposable injective B-module.*

Proof. Let $f : X \to Y$ be a monomorphism and $g : X \to I'$ be a morphism in $\operatorname{mod} B$. Let also $j : I' \to I$ denote the canonical inclusion. Then we have a diagram as shown in $\operatorname{mod} A$. Because I is injective in $\operatorname{mod} A$, there exists a morphism $h : Y \to I$ such that $hf = jg$. Because Y is a B-module, $h(Y) \subseteq I'$. That is, there exists $h' : Y \to I'$ such that $h'f = g$.

$$
\begin{array}{ccccc}
0 & \longrightarrow & X & \xrightarrow{\ f\ } & Y \\
 & & \downarrow{\scriptstyle g} & & \\
 & & I' & & \\
 & & \downarrow{\scriptstyle j} & & \\
 & & I & &
\end{array}
$$

$\square$

3.3. Constructing the approximating sequence.

Let A be a domestic quasitube algebra. As seen before, there exist a right section Σ^- in the postprojective component of $\Gamma(\operatorname{mod} A)$ and a left section Σ^+ in the preinjective component. Moreover, Σ^- and Σ^+ are respectively complete slices in $\operatorname{mod} A^-$ and $\operatorname{mod} A^+$. Let T^- and T^+ denote the respective slice modules of Σ^- and Σ^+. We may choose the slice Σ^+ in such a way that every preinjective indecomposable A-module with support lying completely in the extensions branches of A^+ is a successor of Σ^+. This is possible because there are only finitely many such indecomposables. As an easy consequence, the restriction to A^- of any preinjective predecessor of Σ^+ is nonzero. We then set

$$ M = A \oplus \mathrm{D} A \oplus T^- \oplus T^+ \oplus \mathrm{D} A^- $$

We prove, in theorem 3.4 below, that M is an Auslander generator for $\operatorname{mod} A$.

Also useful is the module $N = A^- \oplus T^- \oplus \mathrm{D} A^-$. Indeed, we recall that, because of lemma 2.5, every indecomposable A^--module admits a minimal $\operatorname{add} N$-approximating sequence.

Proposition. *Let A be a domestic quasitube algebra and X be an indecomposable A-module whose restriction Y to A^- is nonzero. Let also*

$$ 0 \longrightarrow L \xrightarrow{\ r\ } N_0 \xrightarrow{\ q\ } Y \longrightarrow 0 $$

$$ 0 \longrightarrow L' \xrightarrow{\ i'\ } P \xrightarrow{\ p'\ } X/Y \longrightarrow 0 $$

be respectively a minimal $\operatorname{add} N$-approximating sequence and a projective cover of X/Y in $\operatorname{mod} A$. Then there exists an A-module K such that we have exact sequences

$$ 0 \longrightarrow K \xrightarrow{\ s\ } N_0 \xrightarrow{\ t\ } X \longrightarrow 0 $$

$$ 0 \longrightarrow L \xrightarrow{\ \ell\ } K \xrightarrow{\ \ell'\ } L' \longrightarrow 0. $$

Moreover, $K \cong L \oplus L'$. In particular, $K \in \operatorname{add} N$.

Proof. Let $0 \longrightarrow Y \overset{i}{\longrightarrow} X \overset{p}{\longrightarrow} X/Y \longrightarrow 0$ be exact. Because P is projective, there exists a morphism $g : P \to X$ such that $pg = p'$. We now claim that $t = (iq, g) : N_0 \oplus P \to X$ is an epimorphism. Let $x \in X$. Because p' is surjective, there exists $z \in P$ such that $p(x) = p'(z) = pg(z)$. Therefore, $x - g(z) \in \operatorname{Ker} p = \operatorname{Im} i = \operatorname{Im} iq$ because q is surjective. This establishes our claim.

Let (K, s) denote the kernel of t. The snake lemma yields a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
& & 0 & & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \overset{r}{\longrightarrow} & N_0 & \overset{q}{\longrightarrow} & Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle k=\binom{1}{0}} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & K & \overset{s}{\longrightarrow} & N_0 \oplus P & \overset{t=(iq,g)}{\longrightarrow} & X & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \ell'} & & \downarrow{\scriptstyle k'=(0\ 1)} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & L' & \overset{i'}{\longrightarrow} & P & \overset{p'}{\longrightarrow} & X/Y & \longrightarrow & 0. \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}
$$

Now observe that K is an A^--module. Indeed, because of lemma 2.5, $L \in \operatorname{add} N$ and, in particular, is an A^--module. On the other hand, L' is the largest A^--submodule of P. Because of lemma 3.2, L' is actually an injective A^--module. This establishes our claim.

Because A^- is a projective A-module, applying the exact functor $\operatorname{Hom}_A(A^-, -)$ to the middle row of the above diagram yields an exact sequence

$$
0 \longrightarrow K \overset{s}{\longrightarrow} N_0 \oplus L' \overset{t'}{\longrightarrow} Y \longrightarrow 0
$$

in $\operatorname{mod} A^-$, where we have used that K is an A^--module, hence $s(K) \subseteq N_0 \oplus L'$. We deduce a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \overset{r}{\longrightarrow} & N_0 & \overset{q}{\longrightarrow} & Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle k} & & \| & & \\
0 & \longrightarrow & K & \overset{s}{\longrightarrow} & N_0 \oplus L' & \overset{t'}{\longrightarrow} & Y & \longrightarrow & 0
\end{array}
$$

where we have used that $k(N_0) \subseteq N_0 \oplus L'$. Because $t' : N_0 \oplus L' \to Y$ is a morphism from a module in $\operatorname{add} N$ to Y, while q is a minimal $\operatorname{add} N$-approximation, there exists $k'' : N_0 \oplus L' \to N_0$ such that we have a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \overset{r}{\longrightarrow} & N_0 & \overset{q}{\longrightarrow} & Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle k} & & \| & & \\
0 & \longrightarrow & K & \overset{s}{\longrightarrow} & N_0 \oplus L' & \overset{t'}{\longrightarrow} & Y & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \ell''} & & \downarrow{\scriptstyle k''} & & \| & & \\
0 & \longrightarrow & L & \overset{r}{\longrightarrow} & N_0 & \overset{q}{\longrightarrow} & Y & \longrightarrow & 0
\end{array}
$$

where ℓ'' is deduced by passing to the kernels. Minimality of q yields that $k''k$ is an isomorphism. Therefore, so is $\ell''\ell$. In particular, ℓ is a section and the short exact sequence

$$
0 \longrightarrow L \overset{\ell}{\longrightarrow} K \overset{\ell'}{\longrightarrow} L' \longrightarrow 0
$$

splits, that is, $K \cong L \oplus L'$. Finally, $K \in \operatorname{add} N$ because $L \in \operatorname{add} N$, while $L' \in \operatorname{add} D A^- \subseteq \operatorname{add} N$. $\qquad\square$

3.4. Representation dimension

Theorem. *Let A be a domestic quasitube algebra. Then* $\mathrm{rep.\,dim.}\,A = 3$.

Proof. Because A is representation-infinite, it suffices to show that $\mathrm{rep.\,dim.}\,A \leqslant 3$. Let M be as in 3.3 above.

Let X be a postprojective A-module (thus postprojective A^--module). Because of lemma 2.5, there exists a minimal $\mathrm{add}(A^- \oplus T^- \oplus \mathrm{D}\,A^-)$-approximating sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0.$$

Because projective A^--modules are also projective A-modules, we have $M_1, M_0 \in \mathrm{add}\,M$. Let $f : M' \to X$ be a nonzero morphism, where we may assume, without loss of generality, that M' is an indecomposable summand of M. If X is a predecessor of Σ^-, then M' is projective and trivially f lifts to a morphism $M' \to M_0$. If X is a successor of Σ^-, then $M' \in \mathrm{add}(A^- \oplus T^- \oplus \mathrm{D}\,A^-)$, hence f factors through T^-. This shows that we have an $\mathrm{add}\,M$-approximating sequence.

Let now X be a nonpostprojective A-module whose restriction Y to A^- is nonzero. In particular, this is the case for all indecomposables which belong either to the separating family of quasitubes or to the predecessors of Σ^+ in the preinjective component (this is due to our special choice of the slice Σ^+ defining T^+). Because of proposition 3.3, there exists an exact sequence

$$0 \longrightarrow K \xrightarrow{\;s\;} T_0 \oplus I_0 \oplus P \xrightarrow{\;t\;} X \longrightarrow 0$$

with $K, T_0 \oplus I_0 \oplus P \in \mathrm{add}(A^- \oplus T^- \oplus \mathrm{D}\,A^-)$. We show that it is an $\mathrm{add}\,M$-approximating sequence. Let $f : M' \to X$ be a nonzero morphism with $M' \in \mathrm{add}\,M$ indecomposable. Because $f \neq 0$, we have $M' \notin \mathrm{add}(T^+ \oplus \mathrm{D}\,A^+)$. Therefore, $M' \in \mathrm{add}(A \oplus T^- \oplus \mathrm{D}\,A^-)$. If M' is projective then f trivially lifts to a morphism $M' \to T_0 \oplus I_0 \oplus P$. If $M' \in \mathrm{add}(T^- \oplus \mathrm{D}\,A^-)$, then $f(M')$ lies in the restriction Y of X to A^-. Therefore, f lifts to a morphism $M' \to T_0 \oplus I_0$ and consequently to a morphism $M' \to T_0 \oplus I_0 \oplus P$.

Let finally X be a successor of T^+. Because of lemma 2.5, there exists a minimal $\mathrm{add}(T^+ \oplus \mathrm{D}\,A^+)$-approximating sequence

$$0 \longrightarrow T_1' \longrightarrow T_1 \oplus I_1 \longrightarrow X \longrightarrow 0.$$

Because A^+-injectives are also A-injectives, we have $T_1', T_1 \oplus I_1 \in \mathrm{add}\,M$. Let $f : M' \to X$ be a nonzero morphism with $M' \in \mathrm{add}\,M$ indecomposable. If $M' \in \mathrm{add}(T^+ \oplus \mathrm{D}\,A^+)$, then clearly f lifts to a morphism $M' \to T_1 \oplus I_1$. If $M' \notin \mathrm{add}(T^+ \oplus \mathrm{D}\,A^+)$, then f must factor through T^+ and thus also lifts to a morphism $M' \to T_1 \oplus I_1$. This finishes the proof. $\qquad\square$

Example. In example 3.1, we may take $\Sigma^- = \left\{ {}^2_1, {}^{25}_{1}, 2, {}^{\;3}_{2}{}_2 \right\}$ and $\Sigma^+ = \left\{ {}^4_3, {}^{\;4}_{3}{}_5, 3, {}^{3}_{3}{}_2 \right\}$. Indeed, the only indecomposables with support in the extension branch which are preinjective, namely ${}^4_5$ and 4, are successors of Σ^+.

On the other hand, $\mathrm{D}\,A^- = {}^{\;3}_{2}{}_5 \oplus {}^{33}_{\;2} \oplus 3 \oplus 5$, so that the Auslander generator is

$$M = 1 \oplus {}^2_1 \oplus {}^5_1 \oplus {}^{\;3}_{2}{}_2 \oplus {}^{\;4}_{\;3}{}_5 \oplus {}^{4}_{3}{}_3 \oplus {}^4_3 \oplus {}^4_5 \oplus 4 \oplus 3 \oplus 5 \oplus 2 \oplus {}^{25}_{1} \oplus {}^{\;3}_{2}{}_5 \oplus {}^{33}_{\;2} \oplus {}^{\;4}_{3}{}_5.$$

4. Gluings of Algebras

4.1. Finite gluings

The purpose of this section is to show how to glue together algebras having representation dimension three in order to construct larger algebras having the same representation dimension. We need to introduce a notation. Let A be a representation-infinite algebra, having a right section Σ in a postprojective component, or a left section, also denoted by Σ, in a preinjective component of $\Gamma(\mathrm{mod}\,A)$. We denote by $\overrightarrow{\Sigma}$ the set of all indecomposable A-modules X which are predecessors of Σ, that is, such that there exist Y in Σ and a path in $\mathrm{mod}\,A$ from X to Y. Dually, we denote by $\overleftarrow{\Sigma}$ the set of all indecomposable A-modules which are successors of Σ.

Definition. We say that an algebra A is a *finite gluing* of two algebras B and C, in symbols $A = B * C$, if

(FG1) $\Gamma(\mathrm{mod}\,B)$ has a unique preinjective component $\mathcal{Q}_B$ containing a left section Σ_B^+ and $\Gamma(\mathrm{mod}\,C)$ has a unique postprojective component $\mathscr{P}_C$ containing a right section Σ_C^-;

(FG2) $\Gamma(\mathrm{mod}\,A)$ has a separating component $\mathscr{G}$ such that:

    (1) $\mathscr{G}$ contains a left section isomorphic to Σ_B^+ and the indecomposable A-modules in $\mathscr{G}$ which precede it are exactly those of $\overleftarrow{\Sigma}_B^+ \cap \mathcal{Q}_B$,

    (2) $\mathscr{G}$ contains a left section isomorphic to Σ_C^- and the indecomposable A-modules in $\mathscr{G}$ which succede it are exactly those of $\overrightarrow{\Sigma}_C^- \cap \mathscr{P}_C$,

    (3) $(\overleftarrow{\Sigma}_B^+ \cap \mathcal{Q}_B) \cup (\overrightarrow{\Sigma}_C^- \cap \mathscr{P}_C)$ is cofinite in $\mathscr{G}$;
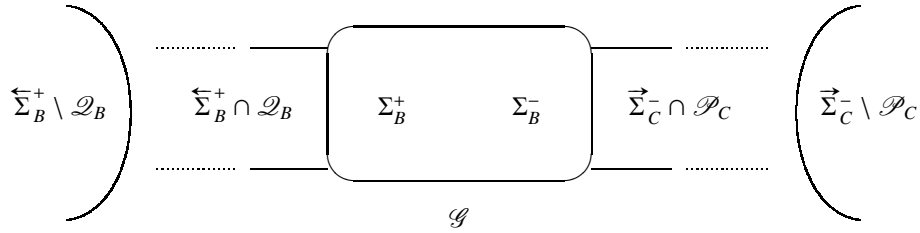
(FG3) the remaining indecomposable A-modules belong to one of two classes:

    (1) those which precede $\mathscr{G}$ are the indecomposable B-modules in $\overleftarrow{\Sigma}_B^+ \setminus \mathcal{Q}_B$,

    (2) those which succede $\mathscr{G}$ are the indecomposable C-modules in $\overrightarrow{\Sigma}_C^- \setminus \mathscr{P}_C$.
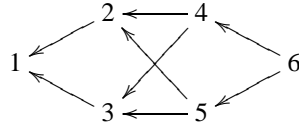
Thus we have

$$\mathrm{ind}\,A = (\overleftarrow{\Sigma}_B^+ \setminus \mathcal{Q}_B) \vee \mathscr{G} \vee (\overrightarrow{\Sigma}_C^- \setminus \mathscr{P}_C).$$
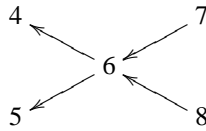
The component $\mathscr{G}$ is called the *glued component* of $\Gamma(\mathrm{mod}\,A)$. The latter may be visualised as
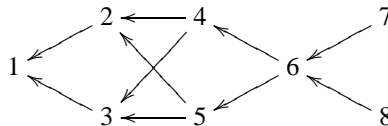

Example. Let B be the domestic quasitube algebra given by the fully commutative quiver

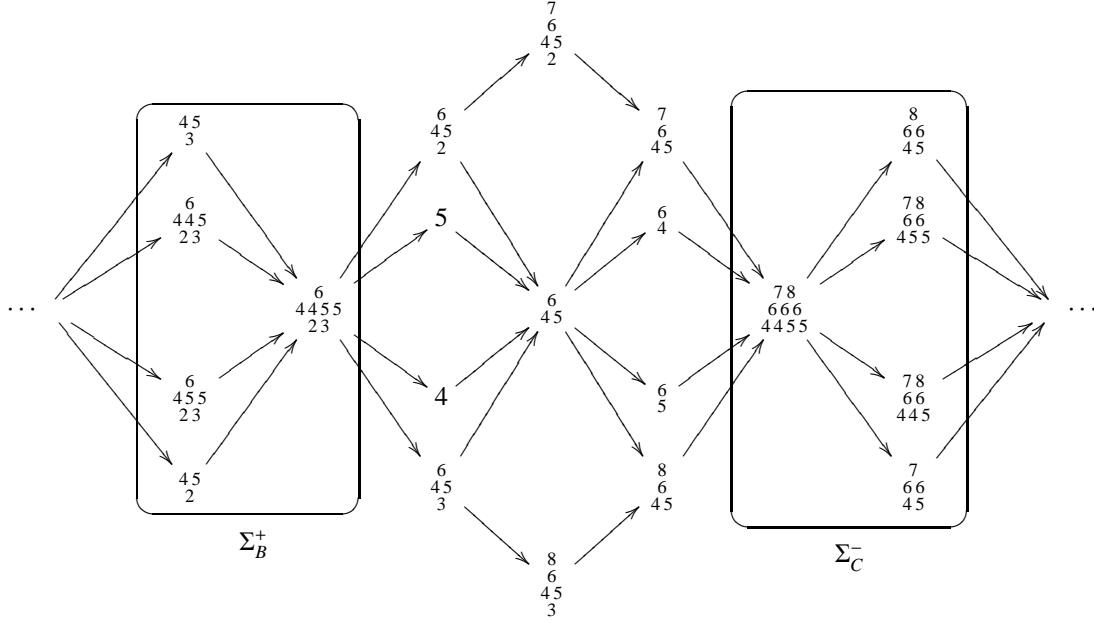and C be the hereditary algebra given by the quiver

Then the algebra A given by the quiver

bound by $\mathrm{rad}^4 A = 0$, two zero-relations of length three from 7 to 3 and from 8 to 2, respectively, and all possible commutativity relations is a finite gluing of B and C. We draw below the central part of the glued component of

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Γ(mod A)

The diagram shows an Auslander–Reiten quiver with the following module symbols:

Top:
$\begin{smallmatrix}7\\6\\4\,5\\2\end{smallmatrix}$

Inside box Σ_B^+ (left): $\begin{smallmatrix}4\,5\\3\end{smallmatrix}$, $\begin{smallmatrix}6\\4\,4\,5\\2\,3\end{smallmatrix}$, $\begin{smallmatrix}6\\4\,4\,5\,5\\2\,3\end{smallmatrix}$, $\begin{smallmatrix}6\\4\,5\,5\\2\,3\end{smallmatrix}$, $\begin{smallmatrix}4\,5\\2\end{smallmatrix}$

Middle: $\begin{smallmatrix}6\\4\,5\\2\end{smallmatrix}$, 5, $\begin{smallmatrix}7\\6\\4\,5\end{smallmatrix}$, $\begin{smallmatrix}6\\4\end{smallmatrix}$, $\begin{smallmatrix}6\\4\,5\end{smallmatrix}$, 4, $\begin{smallmatrix}6\\5\end{smallmatrix}$, $\begin{smallmatrix}6\\4\,5\\3\end{smallmatrix}$, $\begin{smallmatrix}8\\6\\4\,5\end{smallmatrix}$, $\begin{smallmatrix}8\\6\\4\,5\\3\end{smallmatrix}$

Inside box Σ_C^- (right): $\begin{smallmatrix}8\\6\,6\\4\,5\end{smallmatrix}$, $\begin{smallmatrix}7\,8\\6\,6\\4\,5\,5\end{smallmatrix}$, $\begin{smallmatrix}7\,8\\6\,6\,6\\4\,4\,5\,5\end{smallmatrix}$, $\begin{smallmatrix}7\,8\\6\,6\\4\,4\,5\end{smallmatrix}$, $\begin{smallmatrix}7\\6\,6\\4\,5\end{smallmatrix}$

Σ_B^+ Σ_C^-

4.2. Representation dimension of gluings

We now prove that a finite gluing of two algebras having representation dimension three also has representation dimension three under a reasonable hypothesis.

Proposition. *Let $A = B * C$. We assume that* rep. dim. $B = 3$, rep. dim. $C = 3$ *and the slice module Σ_C^- is a direct summand of an Auslander generator for mod C. Then* rep. dim. $A = 3$.

Proof. We introduce some notation. Let M denote an Auslander generator for mod B, and N denote an Auslander generator for C having the slice module of Σ_C^- as a direct summand. Finally, let L denote the direct sum of the (finitely many) indecomposable modules in the glued component of $\Gamma(\text{mod}\,A)$ which are successors of Σ_B^+ and predecessors of Σ_C^-. We claim that

$$\overline{M}_A = A \oplus \mathrm{D}A \oplus L \oplus M \oplus N$$

is an Auslander generator for mod A.

Let indeed X be an indecomposable A-module. Assume first that $X \in \overleftarrow{\Sigma}_B^+$. In particular, X is an indecomposable B-module and has a minimal add M_B-approximating sequence

$$0 \longrightarrow M_1 \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

with $M_1, M_0 \in \text{add}\,M \subseteq \text{add}\,\overline{M}$. We claim that this sequence is also an add $\overline{M}$-approximating sequence in mod A. Let $f : M' \to X$ be a nonzero morphism with $M' \in \text{add}\,\overline{M}$ indecomposable. Because f is nonzero, $M' \in \overleftarrow{\Sigma}_B^+ \setminus \mathcal{D}_B$. Therefore $M' \in \text{add}\,M$. But then f lifts to a morphism $M' \to M_0$.

Let $X \in \text{add}\,L$, then there is nothing to show.

Finally, let $X \in \overrightarrow{\Sigma}_C^-$. Then X is a C-module and has a minimal add N_C-approximating sequence

$$0 \longrightarrow N_1 \longrightarrow N_0 \longrightarrow X \longrightarrow 0$$

with $N_1, N_0 \in \text{add}\,N \subseteq \text{add}\,\overline{M}$. We claim that this sequence is also an add $\overline{M}$-approximating sequence. Let $f : M' \to X$ be a nonzero morphism with M' indecomposable. If M' is not a successor of Σ_C^-, then the morphism f lifts to a morphism $M' \to N_0$. If M' is a successor of Σ_C^- in add $\overline{M}$, then M' is an indecomposable C-module. So f lifts to a morphism $M' \to N_0$ because the above sequence is an add N-approximating sequence. $\qquad\square$

4.3. Induction

Let $A = B * C$ be a finite gluing of B and C as above. If A has a unique preinjective component containing a left section Σ_A^+, then we can define in the same way a finite gluing of A with an algebra D having a unique postprojective component containing a right section. Inductively, assuming that $B_1, \cdots B_n$ are a finite sequence of algebras having a unique preinjective component containing a left section $\Sigma_{B_i}^+$ and a unique postprojective component containing a right section $\Sigma_{B_{i+1}}^-$, for $1 \leq i < n$, then we say that $A = B_1 * \cdots * B_n$ is a finite gluing of the B_i if $A = (B_1 * \cdots * B_{n-1}) * B_n$.

Corollary. *Let* $A = B_1 * \cdots * B_n$ *where we assume that* rep. dim. $B_i = 3$ *and the slice modules of* $\Sigma_{B_i}^+, \Sigma_{B_{i+1}}^-$ *for* $1 \leq i < n$ *are direct summands of an Auslander generator for* mod B_{i+1}. *Then* rep. dim. $A = 3$.

Proof. This is done by induction on $n \geqslant 2$, the case $n = 2$ being Proposition 4.2. $\qquad\square$

4.4. Duplicated and replicated algebras

We end this section with an application of gluing to a class of algebras of finite global dimension, which are closely related to selfinjective algebras. Let B be an algebra and consider the matrix algebra

$$\overline{B} = \begin{pmatrix} B & 0 \\ \mathrm{D}\,B & B \end{pmatrix}$$

with the ordinary matrix addition and the multiplication induced from the bimodule structure of $\mathrm{D}\,B$.

This algebra is called the *duplicated algebra* of B, see for instance [1].

Proposition. *Let B be a tilted algebra of euclidean type.*

- (a) *If B admits a complete slice in its postprojective component, then there exist domestic quasitube algebras C_1, C_2 such that $\overline{B} = B * C_1 * C_2$.*

- (b) *If B admits a complete slice in its preinjective component, then there exist domestic quasitube algebras C_1, C_2 such that $\overline{B} = C_1 * C_2 * B$.*

In particular, rep. dim. $\overline{B} = 3$.

Proof. We only prove (a), because the proof of (b) is similar. We use the description of $\Gamma(\mathrm{mod}\,\overline{B})$ given in [1]. Let Σ be a complete slice in $\Gamma(\mathrm{mod}\,B)$. Then Σ embeds as a section in the stable part of the Auslander-Reiten quiver of the trivial extension $T(B) = B \ltimes \mathrm{D}\,B$ of B by its minimal injective cogenerator bimodule. Then, an exact fundamental domain for $\Gamma(\mathrm{mod}\,T(B))$ inserts in between the predecessors and the successors of $\mathrm{D}\,B$ in $\Gamma(\mathrm{mod}\,B)$ to yield $\Gamma(\mathrm{mod}\,\overline{B})$. Thus $\Gamma(\mathrm{mod}\,\overline{B})$ has the following shape
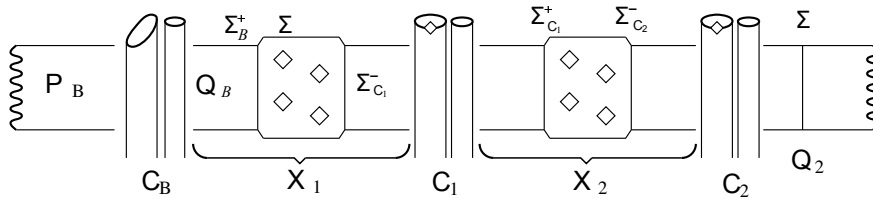
where the diamonds represent possible occurrence of projective-injectives. It thus consists of:

- (a) a postprojective component $\mathscr{P}_B$ which is the postprojective component of $\Gamma(\mathrm{mod}\,B)$;

- (b) a separating tubular family of (extended) tubes $\mathscr{C}_B$, which are the tubes in $\Gamma(\mathrm{mod}\,B)$;

- (c) a transjective component $\mathscr{X}_1$ generally containing projective-injectives and having Σ as left section. The predecessors of Σ in $\mathscr{X}_1$ are exactly the predecessors of Σ in the preinjective component of $\Gamma(\mathrm{mod}\,B)$;

- (d) a separating family of quasitubes $\mathscr{C}_1$;

- (e) a second transjective component $\mathscr{X}_2$ also generally containing projective-injectives;

(f) another separating family of quasitubes $\mathscr{C}_2$;

(g) a preinjective component $\mathscr{Q}_2$ having a left section isomorphic to Σ. The successors of this left section in $\mathscr{Q}_2$ are exactly the successors of Σ in the preinjective component of $\Gamma(\text{mod}\, B)$.

Thus we have $\text{ind}\, \overline{B} = \mathscr{P}_B \vee \mathscr{C}_B \vee \mathscr{X}_1 \vee \mathscr{C}_1 \vee \mathscr{X}_2 \vee \mathscr{C}_2 \vee \mathscr{Q}_2$.

Let C_1 be the support algebra of the family $\mathscr{C}_1$ of quasitubes inside $\overline{B}$. That is, C_1 is the full subcategory of $\overline{B}$ consisting of those $x \in \overline{B}_0$ such that there exists a module M in $\mathscr{C}_1$ satisfying $M(x) \neq 0$. Because every C_1-module is also a $\overline{B}$-module, $\mathscr{C}_1$ is a separating family of quasitubes in $\text{mod}\, C_1$ and moreover C_1 is domestic. Therefore, C_1 is a domestic quasitube algebra. Similarly, the support algebra C_2 of the family $\mathscr{C}_2$ of quasitubes is a domestic quasitube algebra.

We now show how to realise $\overline{B}$ as a finite gluing of B, C_1 and C_2. As observed, there exist a finite number of projective-injectives in the component $\mathscr{X}_1$. First, let Σ_B^+ denote the left section isomorphic to Σ in $\mathscr{X}_1$. It precedes all the projective-injectives. Let next $\Sigma_{C_1}^-$ be any right section in $\mathscr{X}_1$ which succedes these projective-injectives. Similarly, let $\Sigma_{C_1}^+$ and $\Sigma_{C_2}^-$ be respectively a left section and a right section in $\mathscr{X}_2$, preceding and following the projective-injectives in that component. Looking at the definition of finite gluing now shows that $\overline{B} = B * C_1 * C_2$.

As for the last assertion, it follows directly from the main result of [6], theorem 3.4 and corollary 4.3, that rep. dim. $\overline{B} = 3$. $\qquad\square$

The preceding proposition can easily be generalised. For $n \geqslant 2$

$$B^{(n)} = \begin{pmatrix} B_1 & 0 & & & 0 \\ E_1 & B_2 & & & \\ & E_2 & B_3 & & \\ & & \ddots & \ddots & \\ 0 & & & E_{n-1} & B_n \end{pmatrix}$$

be the lower triangular matrix algebra, where $B_i = B$ and $E_i = \text{D}\, B$ for each i. The addition is the usual addition of matrices and the multiplication induced from the bimodule structure of $\text{D}\, B$ and the morphisms $\text{D}\, B \otimes_B \text{D}\, B \longrightarrow 0$. This algebra is called the *n-replicated algebra* of B. It is easily shown, as in the proposition above, that, if B is tilted of euclidean type, then rep. dim. $B^{(n)} = 3$ for any n.

5. Orbit Algebras of Repetitive Categories

5.1. Repetitive categories

The selfinjective algebras of euclidean type are orbit algebras of repetitive categories. In this section, we recall the definitions and results on these algebras that are needed in the proof of our main theorem.

Let B be a basic and connected algebra and $\{e_1, \cdots, e_n\}$ a complete set of primitive orthogonal idempotents for B.

Following [23], the *repetitive category* $\hat{B}$ of B is the category having as objects $e_{m,i}$ with $(m, i) \in \mathbb{Z} \times \{1, \cdots, n\}$ and where the morphism spaces are defined by

$$\hat{B}(e_{r,i}, e_{s,j}) = \begin{cases} e_j B e_i & \text{if}\, s = r \\ \text{D}(e_i B e_j) & \text{if}\, s = r + 1 \\ 0 & \text{otherwise.} \end{cases}$$

The repetitive category is a connected locally bounded selfinjective category. A group G of automorphisms of $\hat{B}$ is called *admissible* if G acts freely on the objects of $\hat{B}$ and has finitely many orbits.

We define the Nakayama automorphism $\nu_{\hat{B}}$ of $\hat{B}$ to be the automorphism defined on the objects by

$$\nu_{\hat{B}}(e_{m,i}) = e_{m,i+1}$$

for every $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$ and in the obvious way on the morphisms. Then the infinite cyclic group $(\nu_{\hat{B}})$ generated by $\nu_{\hat{B}}$ is an admissible group of automorphisms of $\hat{B}$.

Let now φ be an automorphism of the category $\hat{B}$. Thus φ is said to be:

(a) *positive* if, for each $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,i}) = e_{p,j}$ for some $p \geqslant m$ and $j \in \{1, 2, \ldots, n\}$.

(b) *rigid* if, for each $(m, i) \in \mathbb{Z} \times \{1, \ldots, n\}$, we have $\varphi(e_{m,j}) = e_{m,j}$ for some $j \in \{1, 2, \ldots, n\}$.

(c) *strictly positive* if it is positive but not rigid.

For instance, $\nu_{\hat{B}}$ is a strictly positive automorphism of $\hat{B}$.

The following structure theorem for admissible groups of automorphisms is a consequence of the results of [13, 14, 36].

Theorem. *Let B be a tilted algebra of euclidean type and G be a torsion-free admissible group of automorphisms of $\hat{B}$. Then G is an infinite cyclic group generated by a strictly positive automorphism of one of the forms*

(a) $\sigma\nu_{\hat{B}}^k$ *for a rigid automorphism σ and some $k \geqslant 0$, or*

(b) $\mu\varphi^{2k+1}$ *for a rigid automorphism μ, a strictly positive automorphism φ such that $\varphi^2 = \nu_{\hat{B}}$ and some $k \geqslant 0$.*

Note that, if B is tilted of type $\tilde{\mathbb{E}}$, then $\hat{B}$ does not admit a strictly positive automorphism φ such that $\varphi^2 = \nu_{\hat{B}}$, see [26]. We refer to [13, 14] for a complete description of the repetitive categories $\hat{B}$ of the tilted algebras B of types $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{D}}$ with $\varphi^2 = \nu_{\hat{B}}$ for some strictly positive automorphism φ of $\hat{B}$.

5.2. Orbit categories and the Auslander-Reiten quiver of a repetitive category

We now define orbit categories [22]. Let B be a basic and connected algebra and G be an admissible group of automorphisms of $\hat{B}$. The *orbit category* $\hat{B}/G$ has as objects the G-orbits of objects of $\hat{B}$. Given $a, b \in \left(\hat{B}/G\right)_0$, the morphism space $\hat{B}/G(a, b)$ is defined as

$$\hat{B}/G(a, b) = \left\{ (f_{yx}) \in \prod_{(x,y) \in a \times b} \hat{B}/G(x, y) \mid g(f_{yx}) = f_{g(x),g(y)}, \text{ for all } g \in G, x \in a, y \in b \right\}.$$

In this situation, there exists a natural functor $F : \hat{B} \to \hat{B}/G$ called the associated *Galois covering functor*, which assigns to any object $x \in \hat{B}_0$ its G-orbit Gx, and maps a morphism $\xi \in \hat{B}(x, y)$ to the family $F\xi$ such that

$$(F\xi)_{h(y),g(x)} = \begin{cases} g(\xi) & \text{if } h = g \\ 0 & \text{if } h \neq g. \end{cases}$$

Moreover, the functor F induces **k**-linear isomorphisms

$$\bigoplus_{Fx=a} \hat{B}(x, y) \cong \hat{B}/G(a, Fy) \quad ; \quad \bigoplus_{Fy=b} \hat{B}(x, y) \cong \hat{B}/G(Fx, b)$$

for all $x, y \in \hat{B}_0$; $a, b \in \left(\hat{B}/G\right)_0$. Because G is admissible, $\hat{B}/G$ is a category with finitely many objects and we may (and shall) identify it to the finite dimensional algebra $\bigoplus \left(\hat{B}/G\right)$ which is the sum of all the morphism spaces in $\hat{B}/G$.

For instance, the orbit algebra of $\hat{B}$ by the (admissible) automorphism group $(\nu_{\hat{B}})$ generated by the Nakayama automorphism $\nu_{\hat{B}}$ is the trivial extension $T(B)$ of B by $\mathrm{D}\,B$.

Now we consider $\hat{B}$-modules. We denote by $\operatorname{mod} \hat{B}$ the category of all the contravariant functors from $\hat{B}$ to $\operatorname{mod} \mathbf{k}$, which we call finite dimensional $\hat{B}$-modules. An admissible group G of automorphisms of $\hat{B}$ also acts on $\operatorname{mod} \hat{B}$ by

$${}^g M = M g^{-1}$$

for all $\hat{B}$-modules M and all $g \in G$.

The Galois covering $F : \hat{B} \longrightarrow \hat{B}/G$ induces the so-called *pushdown functor* $F_\lambda : \operatorname{mod} \hat{B} \longrightarrow \operatorname{mod} \hat{B}/G$ such that

$$(F_\lambda M)(a) = \bigoplus_{x \in a} M(x)$$

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for all $\hat{B}$-modules M and all $a \in (\hat{B}/G)_0$, see [15, 22].

Assume that the group G is torsion-free. Because of [22], the functor F_λ preserves almost split sequences and induces an embedding from the set of G-orbits $(\text{ind } \hat{B})/G$ of isoclasses of indecomposable $\hat{B}$-modules into the set $\text{ind}(\hat{B}/G)$ of isoclasses of indecomposable $\hat{B}/G$-modules.

The density theorem proved in [18, 19] says that F_λ is dense whenever the category $\hat{B}$ is *locally support-finite*, that is, for each $a \in \hat{B}_0$, the full subcategory $\hat{B}_a$ of $\hat{B}$, given by the supports of all M in ind $\hat{B}$ such that $M(a) \neq 0$, is finite.

If F_λ is dense, then it induces an isomorphism between the orbit quiver $\Gamma(\text{mod } \hat{B})/G$ of $\Gamma(\text{mod } \hat{B})$ under the action of G and the Auslander-Reiten quiver $\Gamma(\text{mod}(\hat{B}/G))$ of $\hat{B}/G$.

Moreover, we are able to describe the Auslander-Reiten quiver of the repetitive category of a tilted algebra of euclidean type [5].

Theorem. *Let B be a tilted algebra of euclidean type $\vec{\Delta}$. Then the Auslander-Reiten quiver of $\hat{B}$ is of the form $\Gamma(\text{mod } \hat{B}) = \bigvee_{q \in \mathbb{Z}} (\mathscr{X}_q \vee \mathscr{C}_q)$ where for each $q \in \mathbb{Z}$,*

    (a) *$\mathscr{X}_q$ is an acyclic component whose stable part is of the form $\mathbb{Z}\vec{\Delta}$,*

    (b) *$\mathscr{C}_q$ is a family $(\mathscr{C}_{q,\lambda})_{\lambda \in \mathbb{P}_1(\mathbf{k})}$ of quasitubes,*

    (c) *$\nu_{\hat{B}}(\mathscr{X}_q) = \mathscr{X}_{q+2}$ and $\nu_{\hat{B}}(\mathscr{C}_q) = \mathscr{C}_{q+2}$,*

    (d) *$\mathscr{X}_q$ separates $\bigvee_{p<q}(\mathscr{X}_p \vee \mathscr{C}_p)$ from $\mathscr{C}_q \vee \left(\bigvee_{p>q}(\mathscr{X}_p \vee \mathscr{C}_p)\right)$,*

    (e) *$\mathscr{C}_q$ separates $\bigvee_{p<q}(\mathscr{X}_p \vee \mathscr{C}_p) \vee \mathscr{X}_q$ from $\bigvee_{p>q}(\mathscr{X}_p \vee \mathscr{C}_p)$.*

The description of the previous theorem is said to be the *canonical decomposition* of $\Gamma(\text{mod } \hat{B})$.

5.3. Structure of the repetitive category

We need one more concept from [23]. Let B be a triangular algebra, then $\hat{B}$ is also triangular. We identify B with the full subcategory of $\hat{B}$ with object set $\{e_{i,0} \mid i \in \{1, \ldots, n\}\}$. Let now i be a sink in Q_B. The *reflection* $S_i^+ B$ of B at i is the full subcategory of $\hat{B}$ given by the objects $e_{0,j}$ with $j \in \{1, \ldots, n\} \setminus \{i\}$ and $e_{1,i} = \nu_{\hat{B}}(e_{0,i})$. In this case, the quiver $\sigma_i^+ Q_B = Q_{S_i^+ B}$ of $S_i^+ B$ is called the *reflection* of Q_B at i. Observe that $\hat{B} = \widehat{S_i^+ B}$.

A *reflection sequence of sinks* is a sequence $i_1, \ldots, i_t$ of points in Q_B such that, for each $s \in \{1, \ldots, t\}$, the point i_s is a sink in the quiver $\sigma_{i_{s-1}}^+ \cdots \sigma_{i_1}^+ Q_B$.

Finally, for a sink i in Q_B, we denote by $T_i^+ B$ the full subcategory of $\hat{B}$ having as objects those of B and $e_{1,i} = \nu_{\hat{B}}(e_{0,i})$. We note that $T_i^+ B$ is isomorphic to the one-point extension $B[I(i)_B]$ of B by the indecomposable injective B-module $I(i)$ at the point i.

We are now able to state the following result [5].

Theorem. *Let B be a tilted algebra of euclidean type $\vec{\Delta}$ and let*

$$\Gamma(\text{mod } \hat{B}) = \bigvee_{q \in \mathbb{Z}} (\mathscr{X}_q \vee \mathscr{C}_q)$$

be the canonical decomposition of $\Gamma(\text{mod } \hat{B})$. For any $q \in \mathbb{Z}$, we have:

    (a) *The support algebra B_q of $\mathscr{C}_q$ is a domestic quasitube algebra, which is a quasitube enlargement of a tame concealed full convex subcategory C_q of $\hat{B}$.*

    (b) *$\Gamma(\text{mod } B_q) = \mathscr{P}^{B_q} \vee \mathscr{C}_q \vee \mathscr{Q}^{B_q}$, where $\mathscr{P}^{B_q}$ and $\mathscr{Q}^{B_q}$ are respectively a postprojective and a preinjective component, both of euclidean type, and $\mathscr{C}_q$ separates $\mathscr{P}^{B_q}$ from $\mathscr{Q}^{B_q}$.*

    (c) *The support algebra B_q^- of $\mathscr{P}^{B_q}$ is a domestic tubular coextension of C_q and the support algebra B_q^+ of $\mathscr{Q}^{B_q}$ is a domestic tubular extension of C_q.*

(d) *There is a reflection sequence of sinks $i_1, \ldots, i_r$ of $Q_{B_q^-}$ (possibly empty) such that $B_q^+ = S_{i_r}^+ \cdots S_{i_1}^+ B_q^-$ and $B_q = T_{i_r}^+ \cdots T_{i_1}^+ B_q^-$.*

(e) *There is a reflection sequence of sinks $j_1, \ldots, j_s$ of $Q_{B_q^+}$ (nonempty) such that $B_{q-1}^- = S_{j_s}^+ \cdots S_{j_1}^+ B_{q-1}^+$ and $D_q = T_{j_s}^+ \cdots T_{j_1}^+ B_{q-1}^+$ is the support algebra of $\mathscr{X}_q$. Hence, $\mathscr{X}_q$ contains at least one projective module.*

(f) *There is a cofinite full translation subquiver $\mathscr{X}_q^- = (-\mathbb{N})\vec{\Delta}$ of $\mathscr{P}^{B_q}$ which is a full translation subquiver of $\mathscr{X}_q$ closed under successors.*

(g) *There is a cofinite full translation subquiver $\mathscr{X}_q^+ = \mathbb{N}\vec{\Delta}$ of $\mathscr{Q}^{B_q}$ which is a full translation subquiver of $\mathscr{X}_{q+1}$ closed under predecessors.*

In particular, $\hat{B}$ is locally support-finite. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6. Selfinjective Algebras of Euclidean Type

Definition. A selfinjective algebra A is said to be of *euclidean type* if there exist a tilted algebra B of euclidean type and an admissible infinite cyclic group G of automorphisms of $\hat{B}$ such that $A = \hat{B}/G$.

Examples of selfinjective algebras of euclidean type are provided by trivial extensions of tilted algebras of euclidean type.

It has been proven in [36] that a selfinjective algebra is of euclidean type if and only if it is representation-infinite, domestic and admits a simply connected Galois covering (in the sense of [8]).

We are now able to prove the main result of the paper.

Theorem. *Let A be a selfinjective algebra of euclidean type. Then* rep. dim. $A = 3$.

Proof. Because A is representation-infinite, it suffices to prove that rep. dim. $A \leqslant 3$. Let B be a tilted algebra of euclidean type $\vec{\Delta}$ and G be an infinite cyclic admissible group of automorphisms of $\hat{B}$ such that $A = \hat{B}/G$. Because of theorem 5.1, G is generated by a strictly positive automorphism g of $\hat{B}$ and, because of theorem 5.2, $\Gamma(\mathrm{mod}\,\hat{B})$ admits a canonical decomposition

$$\Gamma(\mathrm{mod}\,\hat{B}) = \bigvee_{q \in \mathbb{Z}} (\mathscr{X}_q \vee \mathscr{C}_q).$$

Furthermore, for each $q \in \mathbb{Z}$, we have algebras B_q^-, B_q and B_q^+ which satisfy the conditions of theorem 5.3.

Because G also acts on the translation quiver $\Gamma(\mathrm{mod}\,\hat{B})$, there exists $m > 0$ such that $g(\mathscr{X}_q) = \mathscr{X}_{q+m}$ and $g(\mathscr{C}_q) = \mathscr{C}_{q+m}$ for each $q \in \mathbb{Z}$. Then it follows from the definitions of B_q^-, B_q, B_q^+ that we also have

$$g(B_q^-) = B_{q+m}^-, \quad g(B_q) = B_{q+m} \quad \text{and } g(B_q^+) = B_{q+m}^+$$

for each $q \in \mathbb{Z}$.

Because of theorem 5.3, we may chose in $\mathscr{P}^{B_q} = \mathscr{P}^{B_q^-}$ an euclidean right section Σ_q^- of type $\vec{\Delta}$ such that the full translation subquiver $\mathscr{X}_q^-$ of $\mathscr{P}^{B_q}$ given by all successors of Σ_q^- in $\mathscr{P}^{B_q}$ consists of modules having nonzero restrictions to the tame concealed full convex subcategory C_q, and is a full translation subquiver of $\mathscr{X}_q$ closed under successors.

Similarly, we may chose in $\mathscr{Q}^{B_q} = \mathscr{Q}^{B_q^+}$ an euclidean left section Σ_q^+ of type $\vec{\Delta}$ such that the full translation subquiver $\mathscr{X}_q^+$ of $\mathscr{Q}^{B_q}$ given by all predecessors of Σ_q^+ in $\mathscr{Q}^{B_q}$ consists of modules having nonzero restrictions to C_q, and is a full translation subquiver of $\mathscr{X}_{q+1}$ closed under predecessors.

We may assume that $g\left(\Sigma_q^-\right) = \Sigma_{q+m}^-$ and $g\left(\Sigma_q^+\right) = \Sigma_{q+m}^+$. Consequently, $g(\mathscr{X}_q^-) = \mathscr{X}_{q+m}^-$ and $g(\mathscr{X}_q^-) = \mathscr{X}_{q+m}^+$ for each $q \in \mathbb{Z}$.

For a given $q \in \mathbb{Z}$, denote by $\mathscr{Y}_q$ the finite translation subquiver of $\mathscr{X}_q$ consisting of all modules which are successors of Σ_{q-1}^+ and predecessors of Σ_q^-. Observe that every projective module of $\mathscr{X}_q$ lies in $\mathscr{Y}_q$. Moreover, we have $g(\mathscr{Y}_q) = \mathscr{Y}_{q+m}$ for any q.

Now, for each q, let M_q denote the direct sum of all modules in $\mathscr{Y}_q$, all injective B_q^--modules lying in $\mathscr{C}_q$ and all projective $\hat{B}$-modules lying in $\mathscr{C}_q$. Then, clearly ${}^g M_q = M_{q+m}$ for any $q \in \mathbb{Z}$.

Finally, we set $M = \bigoplus_{i=0}^{m-1} M_i$.

Let $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} A$ be the pushdown functor associated to the Galois covering $F : \hat{B} \to \hat{B}/G = A$. We shall prove that $F_\lambda(M)$ is an Auslander generator for A.

First, note that A is a direct summand of $F_\lambda(M)$. Indeed, any indecomposable projective A-module is of the form $F_\lambda(P)$, for some indecomposable projective $\hat{B}$-module P. By definition of M, there exists $r \in \mathbb{Z}$ such that P is a direct summand of ${}^{g^r} M$. But then $F_\lambda(P)$ is a direct summand of $F_\lambda({}^{g^r} M) = F_\lambda(M)$. We now prove that gl. dim. $\operatorname{End} M \leqslant 3$, which will complete the proof.

Let Z be an indecomposable A-module which is not a direct summand of $F_\lambda(M)$. Because the pushdown functor is dense, there exists i such that $0 \leqslant i < m$ and an indecomposable module $X \in (\mathscr{X}_i^- \setminus \Sigma_i^-) \vee \mathscr{C}_i \vee (\mathscr{X}_i^+ \setminus \Sigma_i^+)$ such that $Z = F_\lambda(X)$. Moreover, if $X \in \mathscr{C}_i$, then X is neither a projective B_i-module, nor an injective B_i^--module. Because of theorem 3.5, there exists an add M_i-minimal approximating sequence

$$0 \longrightarrow U \overset{u}{\longrightarrow} V \overset{v}{\longrightarrow} X \longrightarrow 0$$

in $\operatorname{mod} \hat{B}$. Applying the exact functor F_λ yields an exact sequence

$$0 \longrightarrow F_\lambda(U) \overset{F_\lambda(u)}{\longrightarrow} F_\lambda(V) \overset{F_\lambda(v)}{\longrightarrow} F_\lambda(X) \longrightarrow 0$$

with $F_\lambda(U), F_\lambda(V) \in \operatorname{add} F_\lambda(M)$. We recall also that $F_\lambda(X) = Z$. We claim that $F_\lambda(v) : F_\lambda(V) \to Z$ is an add $F_\lambda(M)$-approximation. Let $h : F_\lambda(M) \to F_\lambda(X) = Z$ be a nonzero morphism. The pushdown functor $F_\lambda : \operatorname{mod} \hat{B} \to \operatorname{mod} A$ is a Galois covering of module categories. In particular, it induces a vector space isomorphism

$$\operatorname{Hom}_A(F_\lambda(M), F_\lambda(X)) \cong \bigoplus_{r \in \mathbb{Z}} \operatorname{Hom}_{\hat{B}}({}^{g^r} M, X).$$

Therefore, for each $r \in \mathbb{Z}$, there exists a morphism $f_r : {}^{g^r} M \to X$, all but finitely many of the f_r being zero, such that $h = \sum_{r \in \mathbb{Z}} F_\lambda(f_r)$.

We claim that, for any $r \geqslant 1$, we have $\operatorname{Hom}_{\hat{B}}({}^{g^r} M, X) = 0$. Indeed, $X \in \mathscr{X}_i \vee \mathscr{C}_i \vee \mathscr{X}_{i+1}$ for some i with $0 \leqslant i < m$. On the other hand, for $r \geqslant 1$, the module ${}^{g^r} M$ is a direct sum of modules lying in $\bigvee_{j=0}^{m-1}(\mathscr{X}_{j+mr} \vee \mathscr{C}_{j+mr})$. This establishes our claim.

Let now $f_r : {}^{g^r} M \to X$ be a nonzero morphism in $\operatorname{mod} \hat{B}$ for some $r \geqslant 0$. Applying theorem 5.2, we conclude that f_r factors through a module in add M_i. Because v is an add M_i-approximation, there exists a morphism $w_t : {}^{g^r} M \to V$ in $\operatorname{mod} \hat{B}$ such that $f_r = v w_r$. But then $F_\lambda(f_r) = F_\lambda(v) F_\lambda(w_r)$ with $F_\lambda(w_r) : F_\lambda(M) \to F_\lambda(V)$ because $F_\lambda({}^{g^r} M) = F_\lambda(M)$. Summing up, there exists a morphism $w : F_\lambda(M) \to F_\lambda(V)$ such that $h = F_\lambda(v) w$. This concludes the proof. $\qquad \square$

6.1. Example

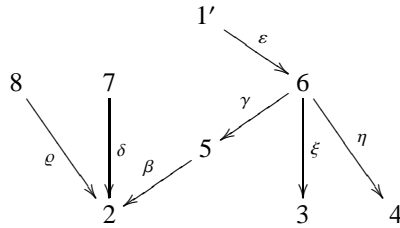Let B be the algebra given by the quiver Q
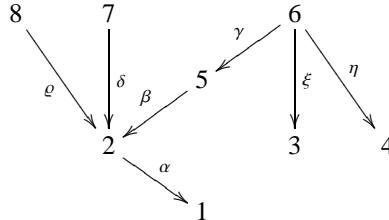
bound by the relations $\varrho \alpha = 0$ and $\delta \alpha = 0$. Then B is a tilted algebra of euclidean type $\widetilde{\mathbb{D}}_7$, being the one-point coextension of the hereditary algebra $H = K \overrightarrow{\Delta}$ of type $\mathbb{D}_6$, given by the full subquiver $\overrightarrow{\Delta}$ of Q formed by the points

$2, 3, 4, 5, 6, 7, 8$, by the (uniserial) simple regular module

$$R = \begin{matrix} 6 \\ 5 \\ 2 \end{matrix}$$

lying on the mouth of the unique stable tube of Γ_H of rank 4. Then $1, 2, 3, 4, 5, 6, 7, 8$ is a reflection sequence of sinks in $Q_B = Q$ such that

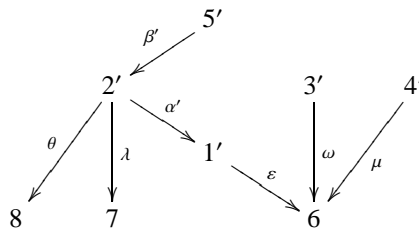- $S_1^+ B$ is the algebra given by the quiver $\sigma_1^+ Q$ of the form

  8   7 $1'$ ε 6 η
  ϱ δ β γ 5 ξ
  2   3   4

  bound by the relations $\varepsilon\xi = 0$ and $\varepsilon\eta = 0$, which is a tilted algebra of type $\widetilde{\mathbb{D}}_7$, being the one-point extension of H by the simple regular module R;

- $S_4^+ S_3^+ S_2^+ S_1^+ B$ is the algebra given by the quiver $\sigma_4^+ \sigma_3^+ \sigma_2^+ \sigma_1^+ Q$ of the form

  $2'$ α' $3'$ $4'$
  θ λ $1'$ ω μ
  8   7 ε 6
  γ 5
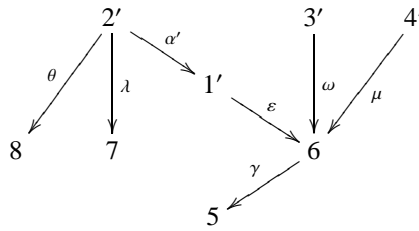
  bound by the relations $\omega\gamma = 0$ and $\mu\gamma = 0$, which is a tilted algebra of type $\widetilde{\mathbb{D}}_7$, and isomorphic to B;

- $S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B$ is the algebra given by the quiver $\sigma_5^+ \sigma_4^+ \sigma_3^+ \sigma_2^+ \sigma_1^+ Q$ of the form

  β' $5'$
  $2'$ α' $3'$ $4'$
  θ λ $1'$ ω μ
  8   7 ε 6

  bound by the relations $\beta'\theta = 0$ and $\beta'\lambda = 0$, which is a tilted algebra of type $\widetilde{\mathbb{D}}_7$, and isomorphic to $S_1^+ B$;

- $S_8^+ S_7^+ S_6^+ S_5^+ S_4^+ S_3^+ S_2^+ S_1^+ B$ is the algebra given by the quiver $\sigma_8^+ \sigma_7^+ \sigma_6^+ \sigma_5^+ \sigma_4^+ \sigma_3^+ \sigma_2^+ \sigma_1^+ Q$ of the form

  $8'$ $7'$ γ' $6'$
  ϱ' δ' β' $5'$ ξ' η'
  $2'$ $3'$ $4'$
  α' $1'$

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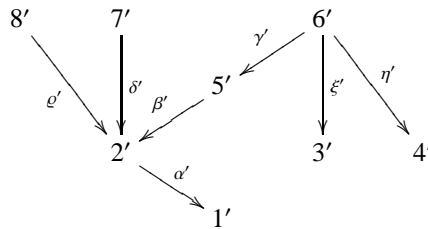
bound by the relations $\varrho'\alpha' = 0$ and $\delta'\alpha' = 0$, which is a tilted algebra of type $\widetilde{\mathbb{D}}_7$, and isomorphic to B.

The repetitive category $\widehat{B}$ of B is given by the quiver
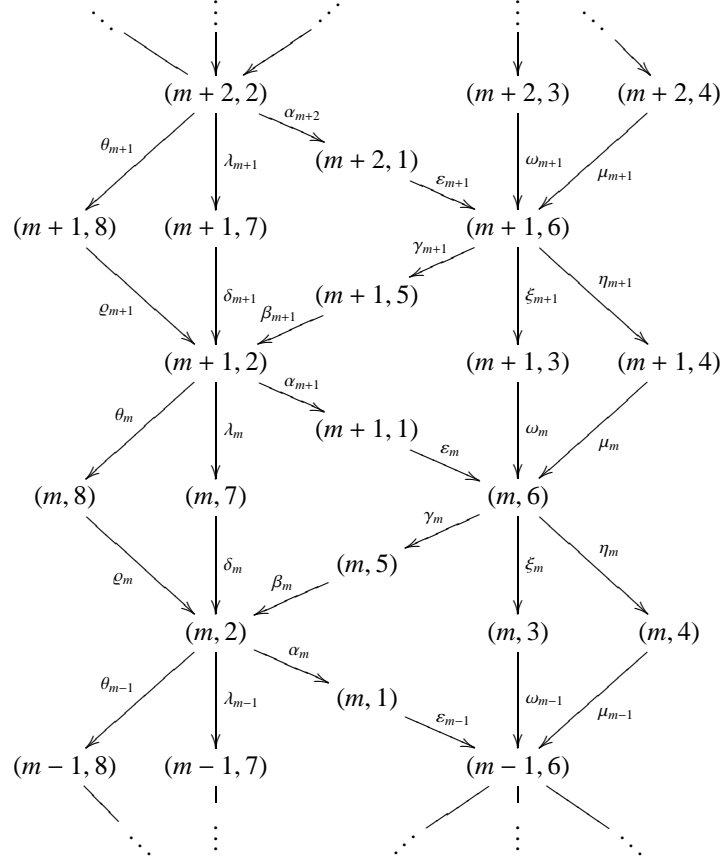
bound by the relations

$$\theta_m\varrho_m = \lambda_m\delta_m = \alpha_{m+1}\varepsilon_m\gamma_m\beta_m,\ \gamma_{m+1}\beta_{m+1}\alpha_{m+1}\varepsilon_m = \xi_{m+1}\omega_m = \eta_{m+1}\mu_m,\ \varrho_m\alpha_m = 0,\ \varrho_m\lambda_{m-1} = 0,\ \delta_m\alpha_m = 0,$$
$$\delta_m\theta_{m-1} = 0,\ \beta_m\theta_{m-1} = 0,\ \beta_m\lambda_{m-1} = 0,\ \varepsilon_m\xi_m = 0,\ \varepsilon_m\eta_m = 0,\ \omega_m\gamma_m = 0,\ \omega_m\eta_m = 0,\ \mu_m\gamma_m = 0,\ \mu_m\xi_m = 0,$$

for all $m \in \mathbb{Z}$. We identify the algebra B with the full subcategory of $\widehat{B}$ given by the objects $(0, i)$, $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Let $\varphi : \widehat{B} \to \widehat{B}$ be the automorphism of the category $\widehat{B}$ given by

$$\varphi((m, 1)) = (m, 5), \qquad \varphi((m, 2)) = (m, 6), \qquad \varphi((m, 3)) = (m, 7), \qquad \varphi((m, 4)) = (m, 8),$$
$$\varphi((m, 5)) = (m + 1, 1), \qquad \varphi((m, 6)) = (m + 1, 2), \qquad \varphi((m, 7)) = (m + 1, 3), \qquad \varphi((m, 8)) = (m + 1, 4),$$
$$\varphi(\alpha_m) = \gamma_m, \qquad\qquad \varphi(\beta_m) = \varepsilon_m, \qquad\qquad \varphi(\gamma_m) = \alpha_{m+1}, \qquad\qquad \varphi(\xi_m) = \lambda_m,$$
$$\varphi(\eta_m) = \theta_m, \qquad\qquad \varphi(\delta_m) = \omega_m, \qquad\qquad \varphi(\varrho_m) = \mu_m, \qquad\qquad \varphi(\lambda_m) = \xi_{m+1},$$
$$\varphi(\theta_m) = \eta_{m+1} \qquad\qquad \varphi(\varepsilon_m) = \beta_{m+1}, \qquad\qquad \varphi(\omega_m) = \delta_{m+1}, \qquad\qquad \varphi(\mu_m) = \varrho_{m+1},$$

for all $m \in \mathbb{Z}$. Observe that $\varphi^2 = \nu_{\widehat{B}}$.

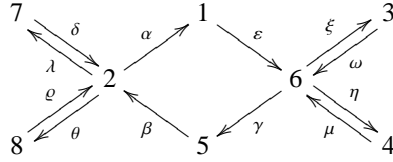The orbit algebra $A_1 = \widehat{B}/(\varphi)$ is the algebra given by the quiver
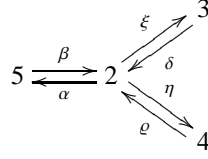
bound by the relations

$$\alpha\beta\alpha\beta = \xi\delta = \eta\varrho, \ \varrho\alpha = 0, \ \delta\alpha = 0, \ \beta\xi = 0, \ \beta\eta = 0, \ \delta\eta = 0, \ \varrho\xi = 0.$$

We note that A_1 is a symmetric one-parametric algebra.

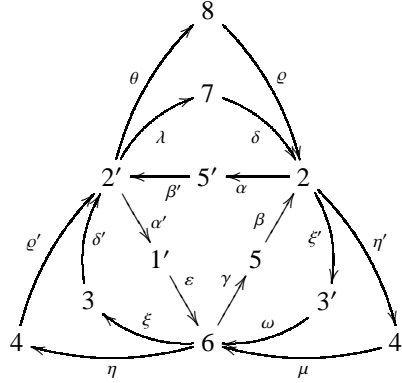The orbit algebra $A_2 = \widehat{B}/(\varphi^2) = \widehat{B}/(\nu_{\widehat{B}}) = \mathrm{T}(B)$ is the algebra given by the quiver

bound by the relations

$$\theta\varrho = \lambda\delta = \alpha\varepsilon\gamma\beta, \ \xi\omega = \eta\mu = \gamma\beta\alpha\varepsilon, \ \varrho\alpha = 0, \ \varrho\lambda = 0, \ \delta\alpha = 0, \ \delta\theta = 0,$$
$$\beta\theta = 0, \ \beta\lambda = 0, \ \varepsilon\xi = 0, \ \varepsilon\eta = 0, \ \omega\gamma = 0, \ \omega\eta = 0, \ \mu\gamma = 0, \ \mu\xi = 0.$$

We note that $A_2 = \mathrm{T}(B)$ is a 2-parametric symmetric algebra.

The orbit algebra $A_3 = \widehat{B}/(\varphi^3)$ is the algebra given by the quiver

bound by the relations

$$\theta\varrho = \lambda\delta = \alpha'\varepsilon\gamma\beta, \quad \alpha\beta'\alpha'\varepsilon = \xi'\omega = \eta'\mu, \quad \gamma\beta\alpha\beta' = \xi\delta' = \eta\delta',$$
$$\varrho\alpha = 0, \quad \delta\alpha = 0, \quad \varrho\xi' = 0, \quad \delta\eta' = 0, \quad \beta\xi' = 0, \quad \beta\eta' = 0,$$
$$\varepsilon\xi = 0, \quad \varepsilon\eta = 0, \quad \omega\gamma = 0, \quad \omega\eta = 0, \quad \mu\gamma = 0, \quad \mu\xi = 0, \quad \varrho'\alpha' = 0,$$
$$\delta'\alpha' = 0, \quad \varrho'\lambda = 0, \quad \delta'\theta = 0, \quad \beta'\theta = 0, \quad \beta'\lambda = 0.$$

We note that A_3 is a 3-parametric selfinjective algebra of euclidean type, which is not weakly symmetric.

More generally, for any positive integer $r \geq 3$, the orbit algebra $A_r = \widehat{B}/(\varphi^r)$ is an r-parametric selfinjective algebra of euclidean type, which is not weakly symmetric.

We mention that every domestic quasitube convex subcategory of $\widehat{B}$ is the algebra $\varphi^r(T_1^+ B)$, for some $r \in \mathbb{Z}$, and $T_1^+ B$ is given by the quiver
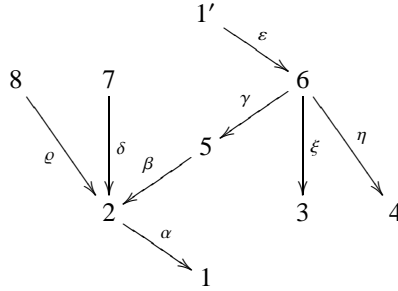
bound by the relations $\varrho\alpha = 0$, $\delta\alpha = 0$, $\varepsilon\xi = 0$ and $\varepsilon\eta = 0$. The algebras $\varphi^r(T_1^+ B)$, $r \in \mathbb{Z}$, are the support algebras of the families $\mathscr{C}_q = (\mathscr{C}_{q,\lambda})_{\lambda \in \mathbb{P}_1(K)}$, $q \in \mathbb{Z}$, of quasitubes of $\Gamma(\mathrm{mod}\,\widehat{B})$, described in theorem 5.2.

We also note that the support algebras of the acyclic components $\mathscr{X}_q$, $q \in \mathbb{Z}$, of $\Gamma(\mathrm{mod}\,\widehat{B})$, described in theorem 5.2, are of the forms $\varphi^r(T_4^+ T_3^+ T_2^+ S_1^+ B)$, $r \in \mathbb{Z}$, where $T_4^+ T_3^+ T_2^+ S_1^+ B$ is given by the quiver
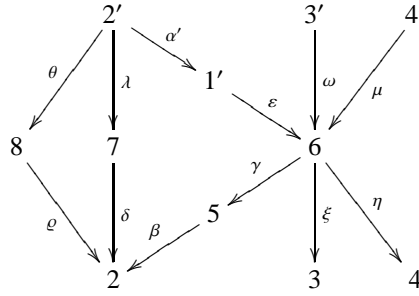
bound by the relations $\theta\varrho = \lambda\delta = \alpha'\varepsilon\gamma\beta$, $\varepsilon\xi = 0$, $\varepsilon\eta = 0$, $\omega\gamma = 0$, $\omega\eta = 0$, $\mu\gamma = 0$ and $\mu\xi = 0$. Moreover, $T_4^+ T_3^+ T_2^+ S_1^+ B$ is a gluing of the tilted algebras $S_1^+ B$ and $S_4^+ S_3^+ S_2^+ S_1^+ B$.

Acknowledgement

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