Isoperimetric surfaces and area-angular momentum inequality in a rotating black hole in new massive gravity

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We study the existence and stability of isoperimetric surfaces in a family of rotating black holes in new massive gravity. We show that the stability of such surfaces is determined by the sign of the hair parameter. We use the isoperimetric surfaces to find a geometric inequality between the area and the angular momentum of the black hole, conjecturing geometric inequalities for more general black holes.

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I. INTRODUCTION

Since its proposition in 2009 by Bergshoeff, Hohm, and Townsend [1], new massive gravity (NMG) has received a great deal of attention, particularly due to its properties in the context of the AdS/CFT correspondence conjecture and because a variety of exact solutions have been found (see for example [2–4]). The theory describes gravity in a vacuum $(2+1)$ spacetime with a massive graviton. The action in this fourth-order derivative theory is given by

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left[R - 2\lambda - \frac{1}{m^2}K\right], \qquad (1)$$

where $K = R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2$, while m is a mass parameter. NMG admits solutions of constant curvature and possesses a unique maximally symmetric solution of constant curvature $\Lambda = 2\lambda$ when $\lambda = m^2$. Static and stationary solutions have been found for this last case [4]. In the case of a negative cosmological constant, the static solution found describes an asymptotically AdS black hole with a gravitational hair parameter.

In this paper, we focus on the rotating solution also found in [4] that is also asymptotically AdS. It has a hair parameter and the rotational parameter satisfies $|a| < l$, where the parameter l is related with the cosmological constant as $\Lambda = -1/l^2$. The extreme rotating case of this NMG black hole can be included after making a change in the hair parameter as suggested in [5]. The extreme case is obtained when $|a| = l$. We are interested in the search of geometric inequalities such as the one presented in [6] for the Kerr black hole. We do this by finding the isoperimetric surfaces and analyzing their stability. This method has been applied to Reissner-Nordström in [7]. Geometric inequalities are an important method to obtain physically relevant properties of metric theories, as they relate quantities of physical interest and tell us what type of phenomena is allowed within the theory. For a recent review of geometric inequalities in the context of general relativity, please refer to [8].

The paper is organized as follows. In Sec. II, the family of rotating black holes in NMG is presented. Then the isoperimetric surfaces are found and the stability condition determined in Sec. III. The stable and unstable cases are determined in Sec. IV. In Sec. V, we explore the geometric inequalities of area, mass and angular momentum for the NMG rotating black hole and conjecture inequalities for the general case. Finally, the conclusions are presented in Sec. VI.

II. THE NMG ROTATING BLACK HOLE

As said, NMG is a theory that describes gravity in a vacuum $(2+1)$ spacetime with a massive graviton [1]. A family of asymptotically AdS rotating black hole solutions have been found in [4] and it contains a gravitational hair parameter b'. The rotational parameter a is bounded by $-l < a < l$, where the parameter l is related with the cosmological constant in the usual way, $\Lambda = -1/l^2$.

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The solutions in the form presented in [4] do not include the extremely rotating case $|a| = l$, which we call, from now on, the extreme case. In order to include it, it is necessary to redefine the hair parameter as suggested in [5]. The new parameter is $b \coloneqq b'\xi^{-1}$, where ξ is defined below. The family of rotating black hole solutions that includes the extreme case is given by the metric

$$ds^2 = -NFdt^2 + \frac{dr^2}{F} + r^2(d\phi + N^\phi dt)^2, \qquad (2)$$

with

$$N = \left[1 + \frac{bl^2}{4\sigma}(1-\xi)\right]^2, \qquad (3)$$

$$N^\phi = -\frac{a}{2r^2}(\mu - b\sigma), \qquad (4)$$

$$F = \frac{\sigma^2}{r^2}\left[\frac{\sigma^2}{l^2} + \frac{b}{2}(1+\xi)\sigma + \frac{b^2 l^2}{16}(1-\xi)^2 - \mu\xi\right], \qquad (5)$$

$$\sigma = \left[r^2 - \frac{\mu}{2}l^2(1-\xi) - \frac{b^2 l^4}{16}(1-\xi)^2\right]^{1/2}, \qquad (6)$$

$$\xi^2 = 1 - \frac{a^2}{l^2}, \qquad (7)$$

where $\mu = 4GM$, the angular momentum is given by $J = Ma$, M is the mass measured with respect to the zero mass black hole and b is the hair parameter. The rotational parameter a satisfies $-l \le a \le l$, and the extreme case is obtained when $|a| = l$.

Before continuing, from (6) we see that σ can be taken with either sign. If we allow it to be negative, then we notice that making the change $b \to -b$ and $\sigma \to -\sigma$ takes the metric functions to themselves, that is $N \to N$, $N^\phi \to N^\phi$, $F \to F$. Therefore, this change does not present new metrics, and we take only σ as positive in this paper, which is also the right choice for the BTZ case (i.e., $b = 0$).

These solutions possess one or more event horizons. If $b \le 0$, the coordinate of the outermost horizon, r_+, is given by

$$r_+ = \frac{l}{\sqrt{8}}(1+\xi)^{\frac{1}{2}}[(b^2 l^2 + 4\mu)^{\frac{1}{2}} - bl\xi^{\frac{1}{2}}], \qquad (8)$$

and the parameters need to satisfy

$$\mu \ge \mu_0 \coloneqq -\frac{b^2 l^2}{4}. \qquad (9)$$

The condition (9) is presented in [5]. On the other hand, if $b > 0$, the expression of the outermost horizon depends on the value of the mass,

$$r_+ = \frac{l}{\sqrt{8}}(1+\xi)^{\frac{1}{2}}[(b^2 l^2 + 4\mu)^{\frac{1}{2}} - bl\xi^{\frac{1}{2}}] \quad \text{if } \mu \ge \mu_+, \qquad (10)$$

and

$$r_+ = \frac{l}{4}(1-\xi)^{\frac{1}{2}}[8\mu + b^2 l^2(1-\xi)]^{\frac{1}{2}} \quad \text{if } \mu_+ \ge \mu \ge \mu_-, \qquad (11)$$

where

$$\mu_+ \coloneqq \frac{b^2 l^2}{16}\frac{(1-\xi)^2}{\xi} \quad \text{and} \quad \mu_- \coloneqq -\frac{b^2 l^2}{8}(1-\xi). \qquad (12)$$

In general, there is a curvature singularity, always hidden by the event horizon. Also, for $b \le 0$, the extreme limit $|a| = l$ corresponds to a cylindrical end, produced by the overlapping of the inner Cauchy horizon and the event horizon, similar to what happens for the extreme Kerr. For details of this analysis, see Ref. [9].

Our focus is extremality of the black holes with respect to the angular momentum parameter. Another criteria for black hole extremality is the vanishing of its surface gravity, which means vanishing temperature and corresponds with the coincidence of the event horizon with an inner Cauchy horizon, and therefore with the existence of a cylindrical end. For $b \le 0$, both criteria coincide; i.e., the surface gravity of the black hole is

$$\kappa = \frac{\xi}{l}\sqrt{\frac{b^2 l^2 + 4\mu}{2(1+\xi)}}. \qquad (13)$$

Also, we see that, for these black holes, the same kind of extremality can be achieved taking $\mu = \mu_0$, which also corresponds to a cylindrical end [9] and in [5] is considered a stronger extremality than that due to rotation, as it is the only way to have vanishing entropy. None of this happens for $b > 0$, as in that case the surface gravity of the black hole does not vanish for any value of the parameters, agreeing with the absence of solutions with a cylindrical end, and consequently there are no solutions of vanishing black hole temperature. Therefore, for $b > 0$, the only criteria for extremality is with respect to the rotation parameter.

For the presented family of rotating solutions, we are interested in the search of geometric inequalities as the ones presented in [6] for the Kerr black hole.

III. ISOPERIMETRIC SURFACES AND STABILITY CONDITION

A hypersurface in a manifold is isoperimetric if its area is an extreme with respect to nearby hypersurfaces that enclose the same volume. This implies that a hypersurface is isoperimetric if and only if its mean extrinsic curvature is constant [10]. We are looking for isoperimetric surfaces in the slices of constant t of the spacetime. We denote these slices as Σ_{t_0} where t_0 is a constant. As the metric in Σ_{t_0} inherits the axial symmetry, then the hypersurfaces of constant r are necessarily isoperimetric. We denote these circles by Σ_{t_0,r_0}, with r_0 the coordinate radius of the isoperimetric surface in Σ_{t_0}.[1] The extrinsic curvature of a hypersurface is given by

$$\chi_{ab} = h_a{}^c \nabla_c n_b, \tag{14}$$

where $h_a{}^b$ is the projector tensor into the hypersurface, ∇_a is the covariant derivative operator on the manifold and n^a is the unit normal vector to the hypersurface (for details please refer to [14]). The mean curvature is the trace of the extrinsic curvature, $\chi = \chi_a{}^a$, and for Σ_{t_0,r_0} it is

$$\chi = \frac{\sqrt{F(r_0)}}{r_0}, \tag{15}$$

which does not depend on the coordinate ϕ on Σ_{t_0,r_0} and therefore confirms that Σ_{t_0,r_0} is an isoperimetric surface in Σ_{t_0}. It also shows that the horizon is a minimal surface.

An isoperimetric surface is called stable if its area is a minimum with respect to nearby surfaces that enclose the same volume. For further discussion on isoperimetric surfaces in the context of general relativity we refer to [8,15,16] and for the concept of stability to [10]. It can be shown that the non-negativity of the second variation of the area for an isoperimetric surface Σ, required for it to be stable, is equivalent to the following condition [10],

$$G(\alpha) \geq 0, \tag{16}$$

where

$$G(\alpha) = \int_\Sigma [-\alpha\Delta_\Sigma\alpha - \alpha^2(\chi_{AB}\chi^{AB} + R_{ab}n^a n^b)]dA_\Sigma, \tag{17}$$

and α is any function on Σ such that

$$\int_\Sigma \alpha dA_\Sigma = 0. \tag{18}$$

In (17) R_{ab} is the Ricci tensor in the Riemannian manifold, n^a is the normal vector to the surface Σ, χ_{AB} is the extrinsic curvature of the surface Σ, Δ_Σ is the Laplace operator on Σ and dA_Σ is the volume element in Σ.

Evaluating $G(\alpha)$ for our case,

$$\begin{aligned} G(\alpha) &= \int_{\Sigma_{t_0,r_0}} \left[-\alpha\Delta_{\Sigma_{t_0,r_0}}\alpha - \alpha^2\left(\frac{F(r_0)}{r_0^2} - \frac{1}{2r}\frac{\partial F}{\partial r}(r_0)\right)\right] dA_{\Sigma_{t_0,r_0}} \\ &= \frac{1}{r_0}\int_0^{2\pi}\left[-\alpha\partial_\phi^2\alpha - \alpha^2\left(F(r_0) - \frac{r_0}{2}\frac{\partial F}{\partial r}(r_0)\right)\right] d\phi. \end{aligned} \tag{19}$$

We recall now that if Δ_0 is the Laplace operator on the unit sphere, the eigenvalues λ_k of the operator $-\Delta_0$ are given by $\lambda_k = k(k+n-1)$ where $k = 0, 1, \ldots$ and n is the dimension of the n-spheres [17]. Then in this case the first nonvanishing eigenvalue of the operator $-\partial_\phi^2$ is $\lambda_1 = 1$, which implies that

$$\int_0^{2\pi} -\alpha\partial_\phi^2\alpha d\phi \geq \int_0^{2\pi} \alpha^2 d\phi \tag{20}$$

and therefore, from (19),

$$G(\alpha) \geq \frac{1}{r_0}\left(1 - F(r_0) + \frac{r_0}{2}\frac{\partial F}{\partial r}(r_0)\right)\int_0^{2\pi} \alpha^2 d\phi. \tag{21}$$

Considering the stability condition (16), we have that Σ_{t_0,r_0} is a stable isoperimetric surface if

$$H(r_0) \coloneqq 1 - F(r_0) + \frac{r_0}{2}\frac{\partial F}{\partial r}(r_0) \geq 0. \tag{22}$$

In the following, we drop the subscript 0 in r_0 and consider r as a parameter that indicates which isoperimetric surface we are considering. What is left is to determine the stability of the isoperimetric surfaces, and this is performed in the following section, but before we write (22) in two convenient forms. From (5) and (6), we have that H can be written as

$$H = 1 - \left(1 - \frac{A}{\sigma^2}\right)F + \frac{\sigma}{l^2}(\sigma + B), \tag{23}$$

where

$$\begin{aligned} A &= \frac{l^2}{16}(1-\xi)[b^2l^2(1-\xi) + 8\mu], \\ B &= \frac{bl^2}{4}(1+\xi), \\ C &= \frac{l^2}{4}\xi(b^2l^2 + 4\mu), \end{aligned} \tag{24}$$

$$\sigma = \sqrt{r^2 - A}, \qquad F = \frac{\sigma^2}{l^2 r^2}[(\sigma + B)^2 - C]. \tag{25}$$

[1]The main reason for choosing the slices of constant t to look for isoperimetric surfaces is that they respect the axial symmetry of the spacetime; that is, Σ_{t_0} inherits the axial symmetry. We could have chosen a different slicing respecting this symmetry, but we expect similar results, since the horizon is a minimal surface. If we drop the axial symmetry by choosing a different slicing, then it is not clear whether we would have been able to recover the isoperimetric profile, as results concerning isoperimetric surfaces without symmetry are sparse. Given that Σ_{t_0} is axially symmetric, the isoperimetric surfaces are known to be of three types, namely, circles of constant r, unduloids, and nodoids. Furthermore, checking conditions on the Gauß curvature for the case at hand, it can be seen that for $b < 0$ the only isoperimetric surfaces are circles of constant r. For the complete classification and results, please refer to [11–13]

H can also be written as

$$H = \frac{1+\xi}{4r^2} h, \tag{26}$$

with h a third degree polynomial in σ,

$$\begin{aligned} h = &-b\sigma^3 + \frac{4}{1+\xi}\left[1+\mu+\frac{b^2 l^2}{16}(1-\xi)^2\right]\sigma^2 \\ &+\frac{3}{16} b l^2 (1-\xi)[b^2 l^2 (1-\xi)+8\mu]\sigma \\ &+\frac{l^2}{4}\left(\frac{1-\xi}{1+\xi}\right)[b^2 l^2 (1-\xi)+8\mu] \\ &\times\left[1-\mu\xi+\frac{b^2 l^2}{16}(1-\xi)^2\right], \end{aligned} \tag{27}$$

and

$$H \geq 0 \Leftrightarrow h \geq 0. \tag{28}$$

IV. STABILITY OF THE ISOPERIMETRIC SURFACES

A. Stability for the BTZ black hole

If $b = 0$, then the BTZ black hole with mass μ presented in [18] is obtained. The stability condition (22) is $H_{b=0} \geq 0$ with

$$H_{b=0} = 1 + \mu - \frac{\mu^2 l^2}{2r^2}(1-\xi^2). \tag{29}$$

For this black hole, $\mu \geq 0$, so the stability condition is

$$r \geq r_c \coloneqq l\mu\sqrt{\frac{1-\xi^2}{2(1+\mu)}}, \tag{30}$$

where r_c is the critical radial position for the isoperimetric surfaces. On the other hand, from (8), the outer horizon is

$$r_+ = l\sqrt{\frac{\mu}{2}(1+\xi)}; \tag{31}$$

therefore, $r_+ > r_c$ and all axially symmetric isoperimetric surfaces in BTZ in the domain of outer communication are stable.

B. Stability in the asymptotic region and at the horizon

As an intermediate step, we want to see if the isoperimetric surfaces are stable in the asymptotic region, $r \to \infty$, and close to the horizon. For the asymptotic region, we see that σ behaves as r, and therefore the leading order of the function h is

$$h \underset{r\to\infty}{\to} -br^3, \tag{32}$$

which gives that, asymptotically,

$$h > 0 \quad \text{if } b < 0 \quad \text{and} \quad h < 0 \quad \text{if } b > 0. \tag{33}$$

So, in the asymptotic region, the isoperimetric surfaces are stable if $b < 0$ and unstable if $b > 0$.

At the horizon, we have

$$H_+ = 1 - \left(1 - \frac{A}{\sigma_+^2}\right)F_+ + \frac{\sigma_+}{l^2}(\sigma_+ + B), \tag{34}$$

where a subscript $+$ indicates that the function is evaluated at r_+. Given that $F_+ = 0$, if $\sigma_+ \neq 0$, we have

$$H_+ = 1 + \frac{\sigma_+}{l^2}(\sigma_+ + B). \tag{35}$$

There are two situations where $\sigma_+ > 0$, namely if $b < 0$, or if $b > 0$ and $\mu \geq \mu_+$ (for the explicit expressions for σ_+ corresponding to the different ranges of μ please refer to [9]). In both cases, also $\sigma_+ + B > 0$, which from (35) implies $H_+ > 0$. The remaining situation, $b > 0$ and $\mu_- < \mu < \mu_+$, has $\sigma_+ = 0$ and (35) cannot be used, as (34) is formally singular. Instead, we use (27) and (28), and evaluating h_+, we have

$$h_+ = \frac{l^2}{4}\left(\frac{1-\xi}{1+\xi}\right)[b^2 l^2(1-\xi)+8\mu]\left[1-\mu\xi+\frac{b^2 l^2}{16}(1-\xi)^2\right]. \tag{36}$$

Every factor in this expression is positive and then $h_+ > 0$. We conclude that in all cases the horizon is a stable isoperimetric surface. Also, given that all the involved functions are continuous, then there is always a neighborhood of r_+ where the isoperimetric surfaces are stable. It seems that the black hole stabilizes the isoperimetric surfaces in its neighborhood.

C. Stability for $b < 0$

Due to the fact that for $b > 0$ the isoperimetric surfaces in the asymptotic region are unstable, we focus on the case $b < 0$, which also is the one that possess the cylindrical limit. To prove that all isoperimetric surfaces for $b < 0$ are stable we perform the following steps. First, we consider the function h for the particular case of minimal mass, $\mu = \mu_0$. By taking its derivative with respect to σ and analyzing its roots, we show that it is an increasing function of σ, and this together with the stability near the horizon of the previous section shows that all isoperimetric surfaces for μ_0 are stable. Then we consider the general case $\mu \geq \mu_0$, but this time we show that h is an increasing function of μ, and knowing from the previous step that for μ_0 the function is positive, it yields that all isoperimetric surfaces are stable for all allowed values of the parameters.

So we consider $b < 0$ and $\mu = \mu_0$. Then,

$$h_{\mu=\mu_0} = -b\sigma^3 + \frac{4}{1+\xi}\left[1 - \frac{b^2l^2}{16}(1+\xi)(3-\xi)\right]\sigma^2 \quad (37)$$

$$-\frac{3}{16}b^3l^4(1-\xi^2)\sigma - \frac{b^2l^4}{4}(1-\xi)\left[1+\frac{b^2l^2}{16}(1+\xi)^2\right]. \quad (38)$$

We know that for big enough σ, this is an increasing function. The derivative with respect to σ is

$$\partial_\sigma h_{\mu=\mu_0} = -3b\sigma^2 + \frac{8}{1+\xi}\left[1 - \frac{b^2l^2}{16}(1+\xi)(3-\xi)\right]\sigma - \frac{3}{16}b^3l^4(1-\xi^2), \quad (39)$$

and therefore the change from decreasing to increasing is at

$$\sigma_c = \frac{-16 + b^2l^2(1+\xi)(3-\xi) + 16\sqrt{\Delta_1}}{12(-b)(1+\xi)}, \quad (40)$$

where

$$\Delta_1 = 1 - \frac{b^2l^2}{8}(1+\xi)(3-\xi) - \frac{b^4l^4}{128}\xi(1+\xi)^2(3-5\xi). \quad (41)$$

It can be checked that

$$\sigma_+(\mu_0) = -\frac{bl^2}{4}(1+\xi) > \sigma_c, \quad (42)$$

and as $h_{\mu=\mu_0}(\sigma_+) > 0$, then $h_{\mu=\mu_0} > 0$ for $\sigma \geq \sigma_+$.

Now we consider $\mu \geq \mu_0$. The derivative of h with respect to μ is

$$\partial_\mu h = \frac{4}{1+\xi}\sigma^2 + \frac{3}{2}bl^2(1-\xi)\sigma + \frac{l^2}{8}\frac{1-\xi}{1+\xi}[16 - 32\mu\xi + b^2l^2(1-\xi)(1-3\xi)]. \quad (43)$$

The biggest root is

$$\sigma_c = \frac{l}{16}[-3bl(1-\xi^2) + \sqrt{\Delta_2}], \quad (44)$$

where

$$\Delta_2 = (1-\xi)[-128 + 256\mu\xi + b^2l^2(1-\xi)(1+42\xi+9\xi^2)]. \quad (45)$$

We can check directly that $\sigma_+ > \sigma_c$, and therefore h is an increasing function of μ for $\sigma \geq \sigma_+$. This completes the proof that, for $b < 0$, all isoperimetric surfaces outside the horizon are stable.

V. GEOMETRIC INEQUALITIES

We have shown that the surfaces Σ_{t_0,r_0} are stable isoperimetric surfaces for $b \leq 0$. We now use these surfaces in the search of geometric inequalities.

From the induced metric on Σ_{t_0,r_0}, we have that its area is simply

$$A = 2\pi r_0. \quad (46)$$

Then, the area of the horizon is $A_+ = 2\pi r_+$, and for $b \leq 0$, it takes the explicit form

$$A_+ = \frac{\pi l}{\sqrt{2}}(1+\xi)^{\frac{1}{2}}[(b^2l^2 + 4\mu)^{\frac{1}{2}} - bl\xi^{\frac{1}{2}}]. \quad (47)$$

Given that r_+ is the outermost horizon, then $r_0 \geq r_+$, and therefore $A \geq A_+$. Also A_+ is an increasing function of ξ, and we have

$$A_+ \geq A_e = \frac{\pi l}{\sqrt{2}}(b^2l^2 + 4\mu)^{\frac{1}{2}}, \quad (48)$$

with equality only in the extreme case.

For the angular momentum, we have $J = \frac{\mu a}{4G}$ with $|a| \leq l$. Now it is convenient to separate the analysis according to the sign of μ. Let us start considering $\mu \geq 0$, then $\mu = \frac{4G}{l}|J_e|$, and

$$|J_e| \geq |J|, \quad (49)$$

where J is the angular momentum of a spacetime where the other parameters are the same as in the extreme case. Putting it all together, we have

$$A \geq A_+ \geq A_e = \frac{\pi l}{\sqrt{2}}\left(b^2l^2 + \frac{16G}{l}|J_e|\right)^{\frac{1}{2}} \geq \frac{\pi l}{\sqrt{2}}\left(b^2l^2 + \frac{16G}{l}|J|\right)^{\frac{1}{2}}; \quad (50)$$

therefore, for any isoperimetric surfaces in a spacetime with $b \leq 0$ and $\mu \geq 0$,

$$A \geq \frac{\pi l}{\sqrt{2}}\left(b^2l^2 + \frac{16G}{l}|J|\right)^{\frac{1}{2}}, \quad (51)$$

and the equality is only achieved at the horizon in the extreme case. So, a black hole with a given area cannot be rotating at any angular momentum because it has a maximal value depending on the hair parameter and the cosmological constant. For the BTZ black hole, from (51), we have that

$$A_{b=0} \geq \pi\sqrt{8Gl|J|}. \tag{52}$$

On the other hand, for $\mu_0 \leq \mu < 0$,

$$A \geq A_+ \geq A_e = \frac{\pi l}{\sqrt{2}}(b^2l^2 + 4\mu)^{\frac{1}{2}} = \frac{\pi l}{\sqrt{2}}(b^2l^2 - 4|\mu|)^{\frac{1}{2}}, \tag{53}$$

and given that $|\mu| = \frac{4G}{l}|J_e|$, so

$$A_e = \frac{\pi l}{\sqrt{2}}\left(b^2l^2 - \frac{16G}{l}|J_e|\right)^{\frac{1}{2}}. \tag{54}$$

Here again, with the other parameters unchanged, $|J_e| \geq |J|$, so

$$A_e \leq \frac{\pi l}{\sqrt{2}}\left(b^2l^2 - \frac{16G}{l}|J|\right)^{\frac{1}{2}}. \tag{55}$$

We see that, in this case, the horizon area for the extreme black hole is not bounded below by the angular momentum, and then the areas of the isoperimetric surfaces are not bounded by the angular momentum. In particular, for the extreme case with $\mu = \mu_0$ which has $A_e = 0$, it has a nonvanishing angular momentum given by $|J_e| = \frac{b^2l^3}{16G}$.

In [5,19], it is proposed that the mass and angular momentum should be measured with respect to the extreme case with $\mu = \mu_0$. Accordingly, we redefine the mass and the angular momentum as follows,

$$\mathcal{M} := M - M_0 = \frac{1}{16G}(b^2l^2 + 4\mu), \tag{56}$$

$$\mathcal{J} := J - J_0 = Ma - M_0a = \mathcal{M}a, \tag{57}$$

where $M_0 = \frac{\mu_0}{4G}$, and $J_0 = M_0a$. It can be noticed that $\mathcal{J} = J$ and $\mathcal{M} = M$ in the BTZ case. These new parameters satisfy

$$\mathcal{M} \geq 0, \qquad |\mathcal{J}| \leq \mathcal{M}l. \tag{58}$$

Then the area of the extreme black hole can be expressed as

$$A_e = \pi l\sqrt{8G\mathcal{M}}. \tag{59}$$

The angular momentum satisfies $|\mathcal{J}_e| \geq |\mathcal{J}|$ with $|\mathcal{J}_e| = \mathcal{M}l$, and fixing the other parameters, we finally have

$$A \geq \pi\sqrt{8Gl|\mathcal{J}|}. \tag{60}$$

It is indeed surprising that the definition of $\mathcal{M}$ and $\mathcal{J}$, which was motivated in [5,19] by black hole entropy considerations, is the right definition of mass and angular momentum regarding geometric inequalities.

To summarize, and as a conjecture for dynamical black hole solutions of NMG, we have the following inequalities:

$$\mathcal{M} \geq 0, \tag{61}$$

$$|\mathcal{J}| \leq \mathcal{M}l, \tag{62}$$

$$A \geq \pi\sqrt{8Gl|\mathcal{J}|}. \tag{63}$$

To complete the analysis, we consider the case $b > 0$. Here, for $\mu < \mu_+$, we have that the horizon area is a decreasing function of ξ, which means that the minimum of the area is obtained for the minimum value of the angular momentum. Also, for $\mu < 0$, the minimum angular momentum does not correspond to the static case (i.e., $\xi = 1$, $J = 0$) but to $\xi = 1 + \frac{8\mu}{b^2l^2}$, and the area of the horizon is zero (for details see [9]). This situation can be remedied by redefining again the mass and angular momentum as

$$\mathcal{M} := M - M_- = \frac{1}{4G}(\mu - \mu_-), \tag{64}$$

$$\mathcal{J} := J - J_- = Ma - M_-a = \mathcal{M}a, \tag{65}$$

and they satisfy

$$\mathcal{M} \geq 0, \qquad -\mathcal{M}l \leq \mathcal{J} \leq \mathcal{M}l. \tag{66}$$

To simplify the notation, we define the mass parameter

$$\nu := \frac{16G}{b^2l^2}\mathcal{M}. \tag{67}$$

The area of the horizon for the corresponding mass and angular momentum is

$$A_+ = \frac{\pi l^2 b}{\sqrt{2}}(1-\xi)^{\frac{1}{2}}\nu^{\frac{1}{2}}, \qquad 0 < \nu < \nu_+, \tag{68}$$

$$A_+ = \frac{\pi l^2 b}{\sqrt{2}}(1+\xi)^{\frac{1}{2}}\left[\left(\nu + \frac{1+\xi}{2}\right)^{\frac{1}{2}} - \xi^{\frac{1}{2}}\right], \qquad \nu_+ \leq \nu, \tag{69}$$

with

$$\nu_+ = \frac{1-\xi^2}{4\xi}. \tag{70}$$

We need to consider two ranges of the mass parameter. For $0 < \nu < \nu_c$, the minimum of the area of the horizon with respect to the angular momentum once the mass and other parameters are fixed is obtained for $\xi = 1$, that is, for the static case, where

$$\nu_c \approx 5.64. \tag{71}$$

For $\nu \geq \nu_c$, the minimum area is obtained for the angular momentum parameter satisfying

$$\nu = \frac{1 + 2\xi + 2\xi^2 + (1+2\xi)\sqrt{1 + 2\xi + 2\xi^2}}{2\xi}. \tag{72}$$

Therefore, the minimum of the area is never obtained in the extremely rotating case, and hence an inequality in the spirit of (60) can not be obtained. This can be expected from the solution not having a cylindrical end, which makes the extreme case a not typical extreme case, and from the isoperimetric surfaces in the constant t slices not being all stable.

VI. CONCLUSIONS

We have analyzed the existence and stability of isoperimetric surfaces in the $t =$ constant slices of the black hole solutions of NMG. We concluded that the determining factor deciding the stability of the isoperimetric surfaces is the sign of the hair parameter b being stable for $b \leq 0$ and unstable for $b > 0$. Also, for either sign of the hair parameter, the isoperimetric surfaces are stable close to the horizon, which seems to indicate that the horizon performs a stabilizing function.

It needs to be pointed out that, previous to the work [9], particular attention to the case $b > 0$ was not paid. The conclusions of the present work support what was seen in [9]—that the behavior of the solutions is radically different for $b > 0$.

We have found geometric inequalities among the physical parameters of the solution. It is important to state that the particular selection of the physical parameters is crucial at this step, and that it is surprising that a choice based on the analysis of thermodynamical properties ([5,19]) is the one suited to the geometric inequalities.

To continue the analysis, that is, to prove that the geometric inequalities conjectured hold for more general solutions of NMG, the constraint equations in the theory need to be analyzed. In that setting, it would be particularly interesting to see how the hair parameter appears and how it is related to other stability properties.

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