Automatica 93 (2018) 224–230

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Optimizing Kalman optimal observer for state affine systems by input selection ☆

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ARTICLE INFO

Article history:
Received 9 May 2017
Received in revised form 11 January 2018
Accepted 11 February 2018

Keywords:
Non-uniformly observable systems
State affine systems
Persistent input
Kalman-like observers
Input optimization

ABSTRACT

In this paper, a new algorithm to build an optimal input for state reconstruction in the class of state-affine systems is proposed, in the sense that it enhances the performances of a Kalman-like observer, as well as it guarantees the system observability. The approach relies on the fact that for a state-affine system, as soon as the input is defined as a function of time, Kalman filtering theory can be applied. In fact, it is first highlighted how an appropriate choice of the system input can improve the Kalman filtering performance in this case. It is then emphasized how this input selection amounts to a control problem, which can be solved by an appropriate optimization algorithm. Finally, the algorithm is applied to a case of fault detection in a pipeline as an illustrative example, with some simulation results showing the observer performance improvement with the proposed input.

1. Introduction

The well-known Kalman Filter has been widely used since it was first proposed in Kalman (1960) for linear time-varying systems, including extensions to *nonlinear systems* via approximate linearization (*Extended Kalman Filter*). For the particular case of nonlinear systems which are *linear in the state*, or *state-affine systems*, it was shown that Kalman results can even provide *exact observers* (Hammouri & de Leon Morales, 1990). However in that case, the required observability property in general depends on the applied input, corresponding to a specific problem of *non uniformly observable* systems (see e.g. Besançon, 2007).

Unlike the case of *observability for any input* (Gauthier & Bornard, 1981), which has been very largely studied since early observer results of Gauthier, Hammouri, and Othman (1992) and Tornambe (1992), less attention has been paid to systems for which this property is not satisfied. In fact, for this latter situation, appropriate inputs (*persistent inputs*) have been well characterized for some classes of systems (starting with state affine ones Besançon, Bornard, & Hammouri, 1996; Bornard, Couenne, & Celle, 1989), but this characterization is not of easy direct use. Usually in practice, inputs are designed heuristically (e.g. as in Ţiclea and Besançon (2006b) for a motor, or in Torres, Besançon, and Georges (2009) for a pipeline), and the property is checked afterwords. This leaves the selection of appropriate inputs with regards to observability in such a case, as a challenge. Notice that our former studies of Rubio Scola, Besançon, and Georges (2013a, b) proposed preliminary methods in that direction, based on *Gramian characterization* of observability (see also Rubio Scola, 2015). In Qian, Dufour, and Nadri (2013), Qian, Nadri, and Dufour (2017) and Qian, Nadri, Morosan, and Dufour (2014), the authors developed a *closed-loop optimal experiment design for on-line parameter identification* in nonlinear systems. Their main contribution consists in combining Lyapunov stability theory with an existing closed-loop identification approach, in order to maximize the information content in the experiment, while asymptotically stabilizing the closed-loop system.

In the present paper instead, we focus on the maximization of the observability of the whole nonlinear system, as well as the enhancement of Kalman optimal state estimation. The primary interest is thus a priori for observer applications which are not control ones — such as fault detection, or parameter estimation, but the combination with control purposes can be part of further extensions. Our approach is here to optimize the input by directly considering the observer equations (including Riccati equation), following our idea of Rubio Scola, Besançon, and Georges (2016). Notice that a first approach in order to build inputs based on Riccati matrix can be found in Winstead and Kolmanovsky (2005a, b), but the results there, being based on the Extended Kalman Filter, are only local ones, without any convergence guarantee. In the present

☆ The material in this paper was partially presented at the 15th annual European Control Conference, June 29–July 1, 2016, Aalborg, Denmark. This paper was recommended for publication in revised form by Associate Editor Gianluigi Pillonetto under the direction of Editor Torsten Söderström.

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https://doi.org/10.1016/j.automatica.2018.03.060
0005-1098/

paper instead, we propose a study that guarantees simultaneously global observability and global convergence. In addition, taking advantage of Kalman filtering theory, it provides a solution that improves filtering performances. In practice, the input construction amounts to some extent to a control problem, which can be solved by optimization. Going here beyond the presentation of Rubio Scola et al. (2016), we propose a numerical algorithm to achieve the optimal solution for this observer-oriented optimal control problem. An application to a fault detection issue in a pipeline is presented as an illustrative example, for which simulation results are provided.

The remainder of the paper is organized as follows: Section 2 first recalls the role of inputs in observability of state affine systems, while Section 3 then formulates the main idea of input selection as a control optimization problem. Section 4 subsequently proposes a numerical algorithm to solve the optimization, illustrated in Section 5 for a case of fault detection in pipelines, with corresponding simulation results. Section 6 finally concludes the paper and gives some perspectives.

2. State affine systems & observability condition for input selection

In this work, we consider the class of discrete-time systems which can basically be described by a state space representation of the following form (so-called *state-affine systems*):

$$x_{k+1} = A(u_k)x_k + B(u_k),\ y_k = C(u_k)x_k \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector,[1] $u \in \mathbb{R}^m$ is the vector of known inputs, that can be used for observation purposes, and $y \in \mathbb{R}^p$ is the measurements vector.

Notice first that such a system may a priori admit inputs for which observability is lost (that is the system is *not uniformly observable* Besançon, 2007).

In fact observability can be characterized from the linear time-varying system obtained as soon as an input sequence is chosen, which can be written in the following form (for example as in Ţiclea & Besançon, 2013):

$$x_{k+1} = A_k x_k + B_k,\ y_k = C_k x_k \tag{2}$$

where A_k, B_k, C_k result from $A(u_k),\ B(u_k),\ C(u_k)$.

Let us assume that those matrices are uniformly bounded (this is always true if u_k is bounded and matrices A, B and C are continuous w.r.t. their arguments).

For system (2), observability can classically be explicitly characterized from the output energy obtained when the system freely evolves from some state x_k at time k up to some time $k+\sigma$:

$$L_y = \sum_{l=k}^{k+\sigma} \|y_l\|^2 = \sum_{l=k}^{k+\sigma} y_l^T y_l. \tag{3}$$

Considering the so-called *transition matrix* defined by:

$$\Phi(k, k_0) = A_{k-1}A_{k-2}\dots A_{k_0},\ \Phi(k_0, k_0) = I_d \tag{4}$$

one easily obtains:

$$y_l = C_l\Phi(l, k)x_k \quad \text{and then} \quad L_y = x_k^T\Gamma(k, \sigma)x_k, \tag{5}$$

$$\text{with}:\ \Gamma(k, \sigma) = \sum_{l=k}^{k+\sigma} \Phi(l, k)^T C_l^T C_l \Phi(l, k). \tag{6}$$

[1] or an extended state vector including possible uncertain parameters, as in Ţiclea and Besançon (2006a, 2009).

This quantity Γ corresponds to the so-called *observability Gramian* of the system, and by definition, it is symmetric and semi-positive definite. From this, its eigenvalues are positive or zero, and if one of them vanishes, some state information is lost in L_y (see (5)). Consequently, observability can be characterized by all eigenvalues of Γ remaining away from zero, and *regularly persistent* inputs are in turn defined for (1) as follows (e.g. as in Ţiclea & Besançon, 2009, or originally Bornard et al., 1989 for continuous-time systems):

Definition 1. An input sequence $(u_k)_{k\geq 0}$ is *regularly persistent* for system (1) if for the corresponding time-varying system (2), there exists a fixed natural number σ such that for any time k:

$$0 < \alpha I_d \leq \Gamma(k, \sigma) \leq \beta I_d, \tag{7}$$

for some real α and β, and I_d the identity matrix.

Notice that for bounded matrices A_k, C_k, the upper bound condition in (7) is satisfied, and in order to obtain observability for system (1), one just needs an input u that guarantees a lower bound ($\alpha > 0$) on the eigenvalues of $\Gamma(k, \sigma)$ (remember that u modifies Γ through $A(u_k)$). In our previous works of Rubio Scola et al. (2013a, b) we provided some algorithms to find an input satisfying such a condition — and with the lowest energy, by relying on definition (6), and referring to available Kalman-like observers for which this condition guarantees convergence (Ţiclea & Besançon, 2013).

In the present paper, the idea is to go further in two ways:

- Rephrase the observability target as an optimal control problem;
- Combine it with observer equations and Kalman theory to further enhance filtering performances.

Remark 2. Notice that a continuous-time version of such ideas can be derived in a very similar fashion.

3. Observability/filtering-based optimal control for input selection

3.1. Observability-oriented optimal control statement

Coming back to definition (6) of observability Gramian Γ, one can check that it is the symmetric solution at time $l = k$ of the following equation (compare with Georges, 1995, 2013 for a continuous-time version):

$$W_l = A_l^T W_{l+1} A_l + C_l^T C_l, \quad W_{k+\sigma+1} = 0. \tag{8}$$

Hence, observability condition (7) is equivalent to:

$$0 < \alpha I_d \leq W_k \leq \beta I_d \tag{9}$$

for solution W_k of Eq. (8).

If this equation is stabilizable via control u, then there exists a nonempty set $\mathcal{U}$ of inputs that guarantee condition $W_k \geq \alpha I_d$ or equivalently $\det(M_i(W_k - \alpha I)) \geq 0,\ \forall i \in [1, n]$ (where det and M_i respectively refer to the determinant and the ith principal minor). In this way, the observability problem turns into a *control* problem, in the sense that u is to be designed so that W_k remains bounded.

3.2. Filtering-enhancement-oriented optimal control statement

Let us now consider that the system is corrupted with additive state and output noises v and w, of covariance matrices $Q_k^{n\times n}$ and $R_k^{p\times p}$ respectively, assumed to be both positive definite. It can then be described as:

$$\begin{aligned} x_{k+1} &= A(u_k)x_k + B(u_k) + w_k \\ y_k &= C(u_k)x_k + v_k \end{aligned} \tag{10}$$

and again reduces to a linear time-varying model as soon as an input sequence is fixed, as:

$$x_{k+1} = A_k x_k + B_k + w_k,\ y_k = C_k x_k + v_k. \tag{11}$$

For this system, the optimal observer (predictor) in the sense of minimizing the mean least square estimation error $E\{(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)\}$ is given by (Kalman observer — from Kalman, 1960):

$$\begin{aligned} \hat{x}_{k+1} &= A_k\hat{x}_k + B_k + A_k K_k(y_k - C\hat{x}_k) \\ K_k &= P_k C_k^T (C_k P_k C_k^T + R_k)^{-1} \\ P_{k+1} &= A_k(P_k - K_k C_k P_k)A_k^T + Q_k \end{aligned} \tag{12}$$

and it is known that under condition (7) on the one hand, and appropriate controllability of the pair $(A_k, Q_k^{1/2})$ on the other hand, this observer is asymptotically stable (Jazwinski, 1970) (it provides an asymptotic observer for the deterministic case of system (2), and the mean estimation error asymptotically decays to zero for system (11)). For Q_k positive definite, stability condition reduces to (7).

It is also known (from Kalman, 1960) that this observer minimizes $E\{(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)\}$, which results to be the trace of the estimation error covariance P_k:

$$\min_{\hat{x}} E\{(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)\} = \min_{\hat{x}_k} \mathrm{Tr}(P_k). \tag{13}$$

In our case of system (10), the optimal solution (12) for this problem is a function of input u. This motivates us to find an optimal persistent input $\bar{u} \in \bar{\mathcal{U}} \subset \mathcal{U}$ that additionally minimizes (13). This leads to:

$$\min_{\bar{u}} \min_{\hat{x}_k} E\{(x_k - \hat{x}_k)^T(x_k - \hat{x}_k)\} = \min_{\bar{u}} \min_{\hat{x}_k} \mathrm{Tr}(P_k). \tag{14}$$

The solution of this problem gives an optimal input for the optimal filter w.r.t. noises w and v. The optimization (control) problem combining this with observability requirement is summarized in next subsection.

3.3. Overall control problem for input selection

In order to gather observability requirement and filtering enhancement, we propose to consider an optimal control problem over a sliding time window as follows (see for instance Trélat, 2000, 2005 for details on optimal control):

$$\min_{u_k..u_{k+N-1}} q\,\mathrm{Tr}(P_{k+N}) + \sum_{l=k}^{k+N-1} \{\mathrm{Tr}(P_l) + \frac{r}{2}\|u_l\|^2\} \tag{15}$$

under the following dynamics:

$$\begin{aligned} P_{l+1} &= A_l P_l A_l^T + Q_l \\ &\quad - A_l P_l C_l^T (C_l P_l C_l^T + R_l)^{-1} C_l P_l A_l^T \end{aligned} \tag{16}$$

$$W_l = A_l^T W_{l+1} A_l + C_l^T C_l \tag{17}$$

with initial, Gramian and inputs conditions as:

$$P_k = P_k^T \succ 0 \tag{18a}$$

$$W_{k+N} = 0 \tag{18b}$$

$$\begin{aligned} &-\det(M_i(W_k - \alpha I_d)) \le 0, \\ &\alpha > 0 \text{ and } \forall i \in [1, \dim(W)] \end{aligned} \tag{18c}$$

$$u_l - u_{max} \le 0,\ u_{min} - u_l \le 0 \qquad \forall l \tag{18d}$$

$$\left.\begin{aligned} u_l - u_{l-1} - \Delta u_{max} \le 0 \\ \Delta u_{min} - u_l + u_{l-1} \le 0 \end{aligned}\right\} \qquad \forall l \ne k \tag{18e}$$

for some weights $q, r > 0$. Here u_{max} and u_{min} are the maximum and minimum of the input, and Δu_{max} and Δu_{min} its maximum and minimum variations. Notice that the input boundedness constraint ensures the upper bound for the Gramian, but also some admissible operation of the system, at least in the case of a BIBS one.

The considered criterion clearly aims at minimizing the optimal estimation error already guaranteed by Kalman, through the term $Tr(P_l)$.

In addition here, the energy of the input signal is minimized in order to find a bounded optimal solution, leading to a persistent input (thanks to condition (7)) of minimum energy.

Notice that the size of optimization window N must be chosen large enough so as to guarantee enough controllability of Eqs. (16)–(17) (we need stabilizability of the eigenvalues of P_l).

This persistent condition, combined with controllability of $(A_k, Q_k^{1/2})$ (with $Q_k \succ 0$), ensures the convergence of the observer (Jazwinski, 1970) (even in case of sub-optimal solutions).

Remark 3. The optimization problem can also be extended to a general case of non negative matrix Q_k, by adding the related controllability constraint.

Notice that in practice, the optimization problem can be solved in a moving horizon fashion: solve it first over its N-step window using initial guesses u_0 and P_0. Then, retain only the first element of the optimal input sequence u_1, and use it in the Riccati equation to get a new matrix P_1, which becomes the next initial condition for the Riccati equation in the optimization, and so on.

Remark 4. One could think of extending the approach to state affine systems *with output injection* (i.e. with $A = A(u_k, y_k)$), but in that case the prediction of output evolution is needed, which makes the problem more complex. This is left for future studies.

4. Numerical implementation

Let us propose here some numerical scheme for an actual solving of the previous optimization problem: using indeed the technique presented for instance in Cohen and Zhu (1984), optimization of (15) can be addressed by including the inequality constraint in an *augmented Lagrangian* as:

$$\begin{aligned} L_c &= q\,\mathrm{Tr}(P_{k+N}) + \sum_{l=k}^{k+N-1} \{\mathrm{Tr}(P_l) + \frac{r}{2}\|u_l\|^2\} \\ &+ \frac{1}{2c}((\max(0, p^i + cI(W_k, \{u_l\}_{k\le l\le k+N-1})))^2 - (p^i)^2) \end{aligned} \tag{19}$$

where the p^i's are the Lagrangian multipliers associated to the constraint at dual iteration i, $c > 0$ is the coefficient of Lagrangian augmentation, $I(W_k, \{u_l\}_{k\le l\le k+N-1})$ means a vector with the left-hand side values of inequality (18c), (18d) and (18e) and max, respectively exponent 2, denotes the element-wise maximum, respectively element-wise square.

It can here be recalled how the use of an augmented Lagrangian approach is an efficient regularization technique introduced to overcome duality gaps occurring in nonconvex optimization problems, such as the one studied in this paper (see Bertsekas, 1982). In particular, for regularization coefficient c large enough, nonconvex constrained optimization problems exhibit local saddle points (and therefore avoidance of duality gaps), which allows the convergence of duality theory-based algorithms such as algorithm (22)–(23) presented below. The convergence of the here-proposed duality-based algorithm can be proved mainly under convexity assumptions on both the cost function and the constraints (see theorem 15 in Cohen & Zhu, 1984). Clearly the here-addressed problem is not convex, but it is well known that under local convexity assumptions (ensured by the existence of a local saddle point for c large enough), nonlinear nonconvex optimization problems can be successfully solved by using convex optimization methods.[2] However, only local minima of the problem (for which Kuhn–Tucker's first-order necessary conditions and second-order sufficient conditions hold locally) will be available. In our case, the obtention of a local minimum (thanks to the existence of a local saddle point) is not an issue since any feasible suboptimal solution will ensure convergence of the observer.

Formally, two fundamental results derived from Cohen and Zhu (1984) and Zhu (2003) are thus just needed:

Lemma 5. *If the pair* $(\{u_l\}_{k\le l\le k+N-1}, p^i)$*, (called a local saddle point of the problem), is solution of the following min–max problem:*

$$\min_{\{u_l\}_{k\le l\le k+N-1}} \max_{p^i} L_c = \max_{p^i} \min_{\{u_l\}_{k\le l\le k+N-1}} L_c \tag{20}$$

for L_c *given by (19), then* $\{u_l\}_{k\le l\le k+N-1}$ *is a local minimum of problem (15)–(18c).*

Lemma 6. *A local saddle point of augmented Lagrangian (19) always exists provided* c *is chosen large enough (Bertsekas, 1982).*

Therefore, we are brought to solve problem (20), that is:

$$\max_{p^i} \min_{\{u_l\}_{k\le l\le k+N-1}} L_c = \max_{p^i} w_c(p^i) \tag{21}$$

where $w_c(p^i) = \min_{\{u_l\}_{k\le l\le k+N-1}} L_c$ is called *dual function* of the problem. An algorithm to find w_c can be as follows (among other options):

Algorithm 1.

1. Start with $(\{u_l^0\}_{k\le l\le k+N-1}, p^0)$, for step $i = 0$.
2. At each iteration $i+1$, solve the problem:

$$\min_{u^{i+1}_{k\le l\le k+N-1}} \sum_{l=k}^{k+N-1} \{\mathrm{Tr}(P_l) + \frac{r}{2}\|u_l^{i+1}\|^2\} + \frac{1}{2c}[(\max(0, p^i + cI(W_k, \{u_l\}_{k\le l\le k+N-1})))^2 \cdots - (p^i)^2] + q\,\mathrm{Tr}(P_{k+N}) \tag{22}$$

subject to dynamics (16)–(17) and initial conditions (18a)–(18b).
If $\{u_l^{i+1}\}_{k\le l\le k+N-1}$ is a solution of (22), then we have $w_c(p^i)$ (and also W_k^{i+1} from Eq. (17)).
3. Update p with the gradient method:

$$p^{i+1} = (1 - \frac{\rho}{c})p^i + \frac{\rho}{c}\max(0, p^i + cI(W_k^{i+1}, \{u_l^{i+1}\}_{k\le l\le k+N-1})) \text{ with } 0 < \rho \le c. \tag{23}$$

4. If $\|u_l^{i+1} - u_l^i\| < \epsilon\ \forall l = k, ..k+N$, and $\|p^{i+1} - p^i\| < \epsilon$ then stop;
If not, then go to step 2 with now i incremented by 1.

The solution of step 2 (calculation of the dual function) is done using the adjoint state which is detailed below:

Consider first vectorized forms of matrices P_k and W_k. Knowing that both of them are symmetric, let us call z_l and w_l the vectors which contain the inferior (superior) triangular part of P_l and W_l respectively, and $F_P(z_l, u_l)$, $F_W(w_{l+1}, u_l)$ the vectors which contain the inferior (superior) triangular part of P_{l+1} from Eq. (16) and W_l from Eq. (17) respectively. We thus obtain vector dynamics as:

$$z_{l+1} = F_P(z_l, u_l), \quad w_l = F_W(w_{l+1}, u_l)$$

Re-writing $\mathrm{Tr}(P_l)$ as $T(z_l)$, the optimization problem is:

$$\min_{\{u_l\}_{k\le l\le k+N-1}} qT(z_{k+N}) + \sum_{l=k}^{k+N-1} \{T(z_l) + \frac{r}{2}\|u_l\|^2\} + \frac{1}{2c}(\max(0, p^i + cI(w_{k+N}, \{u_l\}_{k\le l\le k+N-1}))^2 - (p^i)^2) \tag{24}$$

with i the dual iteration, and under:

$$z_{l+1} = F_P(z_l, u_l),\ w_l = F_W(w_{l+1}, u_l),\ w_{k+N} = 0 \tag{25}$$

z_k given by P_k at the beginning of the window.

To solve this optimization, we use a Lagrangian function, and Kuhn–Tucker conditions. The Lagrangian can read:

$$L(\{u_l\}_{l=k,k+N-1}, \{z_l\}_{l=k,k+N-1}, \{w_l\}_{l=k,k+N-1}, \{\lambda_l^1\}_{l=k,k+N-1}, \{\lambda_l^2\}_{l=k,k+N-1}, p) = qT(z_{k+N}) + \sum_{l=k}^{k+N-1} T(z_l) + \frac{r}{2}\|u_l\|^2 + \lambda_{l+1}^{1T}(F_P(z_l, u_l) - z_{l+1}) + \lambda_{l+1}^{2T}(F_W(w_{l+1}, u_l) - w_l) + \frac{1}{2c}\{\max(0, p^i + cI(w_k, \{u_l\}_{k\le l\le k+N-1}))^2 - (p^i)^2\} \tag{26}$$

and Kuhn–Tucker conditions on the input are ($\frac{\partial L}{\partial u_l} = 0$):

$$0 = ru_l + \frac{\partial F_P^T}{\partial u_l}(z_l, u_l)\lambda_{l+1}^1 + \frac{\partial F_W}{\partial u_l}^T(w_{l+1}, u_l)\lambda_{l+1}^2 \ldots + \frac{\partial I^T}{\partial u_l}(w_k, \{u_l\}_{k\le l\le k+N-1})\max(0, p^i + cI(w_k, \{u_l\}_{k\le l\le k+N-1})) \tag{27}$$

with $l = k, \ldots, k+N-1$.

Then, the conditions on state z lead to:

$$\frac{\partial L}{\partial z_l} = 0 \Leftrightarrow \frac{\partial T}{\partial z_l}^T(z_l) + \frac{\partial F_P}{\partial z_l}^T(z_l, u_l)\lambda_{l+1}^1 - \lambda_l^1 = 0 \quad \text{with } l = k+1, \ldots, k+N-1 \tag{28a}$$

$$\frac{\partial L}{\partial z_{k+N}} = 0 \Leftrightarrow \lambda_{k+N}^1 - q\frac{\partial T^T}{\partial z_{k+N}}(z_{k+N}) = 0 \tag{28b}$$

while the conditions on state w can be written as:

$$\frac{\partial L}{\partial w_l} = 0 \Leftrightarrow -\lambda_{l+1}^2 + \frac{\partial F_W^T}{\partial w_l}(w_l, u_{l-1})\lambda_l^2 = 0 \tag{29a}$$

with $l = k+1, \ldots, k+N-1$,

$$\frac{\partial L}{\partial w_k} = 0 \Leftrightarrow \lambda_{k+1}^2 = \frac{\partial I^T}{\partial w_k}(w_k, \{u_l\}_{k\le l\le k+N-1}) \max(0, p^i + cI(w_k, \{u_l\}_{k\le l\le k+N-1})). \tag{29b}$$

Finally, the problem is to find u_l for $l = k, \ldots, k+N-1$ satisfying Eqs. (27)–(29b). More precisely, this amounts to solving (from Eq. (27)) $f(u_l) = 0$ with:

$$f(u_l) := ru_l + \frac{\partial F_P}{\partial u}(z_l, u_l)^T\lambda_{l+1}^1 + \frac{\partial F_W}{\partial u}(w_{l+1}, u_l)^T\lambda_{l+1}^2 + \frac{\partial I^T}{\partial u_l}(w_k, \{u_l\}_{k\le l\le k+N-1}) \max(0, p^i + cI(w_k, \{u_l\}_{k\le l\le k+N-1})) \tag{30}$$

for λ_l^1, λ_{l+1}^2 as in (28a)–(28b) and (29a)–(29b) respectively.

This can be done using a gradient-based descent method:

- Start with some initial guess for all the u_l's,
- Compute $f(u_l)$,
- Update u_l via $u_l^+ = u_l + grad\,[f(u_l)]$,

[2] Notice that even nonsmooth mixed optimization problems (i.e. with both integer and real decision variables) have been successfully solved by using such approaches (see e.g. Georges, 1994).

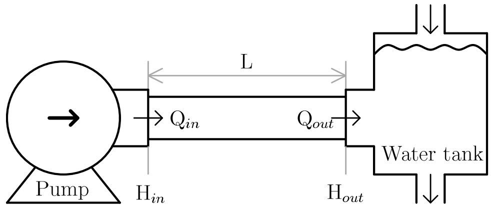

Fig. 1. Pipeline system.

• Until $\|u_l^+ - u_l\|$ becomes small enough.

Numerically, to compute the $f(u_l)$'s from a vector $u = [u_k, \ldots, u_{k+N-1}]^T$, one gets first in direct time all z_l's over the horizon, from Eq. (25), by:

$$z_{l+1} = F_P(z_l, u_l), \text{ knowing } z_k \text{ from } P_k \tag{31}$$

then in retrograde time all w_l's, from Eq. (25):

$$w_l = F_W(w_{l+1}, u_l), \text{ knowing } w_{k+N} = 0 \tag{32}$$

as well as all λ_l^1's, using Eqs. (28a)–(28b), by:

$$\lambda_l^1 = \frac{\partial F_P^T}{\partial z_l}(z_l, u_l)\lambda_{l+1}^1 + \frac{\partial T^T}{\partial z_l}(z_l)$$
$$\text{from } \lambda_{k+N}^1 = q\frac{\partial T^T}{z_{k+N}}(z_{k+N}) \tag{34}$$

and in direct time all λ_l^2's, from Eqs. (29a)–(29b):

$$\lambda_{l+1}^2 = \frac{\partial F_W^T}{\partial w_l}(w_l, u_{l-1})\lambda_l^2 \text{ from}$$
$$\lambda_{k+1}^2 = \frac{\partial I^T}{\partial w_k}(w_k, \{u_l\}_{k\leq l\leq k+N-1}) \tag{35}$$
$$\max(0, p^i + cI(w_k, \{u_l\}_{k\leq l\leq k+N-1})).$$

Next section proposes an illustrative example of this approach.

5. Application example: fault detection in a pipeline

Let us consider a system as depicted in Fig. 1, made of a pipeline fed via a pump, and with an outflow in a tank. Monitoring leaks in such a system motivated a lot of work (as reported e.g. in Verde & Torres, 2017).

Considering here a possible leak in the pipe (of magnitude F), and the pipe dynamics driven by output pressure $u = H_{out}$, with sampled measurements of output flow Q_{out}, a discrete-time model of the form (1) can be obtained, with output flow Q_{out} and (normalized) input pressure $\frac{ga}{c}H_{in}$ as state variables x_1, x_2 (a, g, c are constant parameters), together with:

$$A = \begin{pmatrix} 1 & \frac{cT_s}{L} \\ -\frac{cT_s}{L} & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -\frac{agT_s}{L}u - \frac{fT_s}{2Da}y|y| \\ -\frac{cT_s}{L}\sqrt{u}F + \frac{cT_s}{L}Q_{in} \end{pmatrix}.$$

In order to monitor the input flow rate Q_{in}, fixed by the pump, and the leak parameter F, one can just extend the model with state variables $x_3 := F$ and $x_4 := Q_{in}$, assuming they are constant ($x_3(k+1) = x_3(k)$, $x_4(k+1) = x_4(k)$). Some limitations are here taken into account according to experimental prototype of Begovich, Pizano, and Besançon (2012), $H_{out_{min}} = 2.8$ m, $H_{out_{max}} = 3.3$ m, $dH_{out}/dt_{min} = -0.3$ m/s and $dH_{out}/dt_{max} = +0.3$ m/s for a maximum frequency of H_{out} equal to 1 Hz, while state and measurement noises are added in simulations with covariance matrices given by $R = 1\cdot 10^{-5}$ and $Q = \mathtt{diag}([1\cdot 10^{-6}, 1\cdot 10^{-7}, 1\cdot 10^{-12}, 1\cdot 10^{-12}])$.

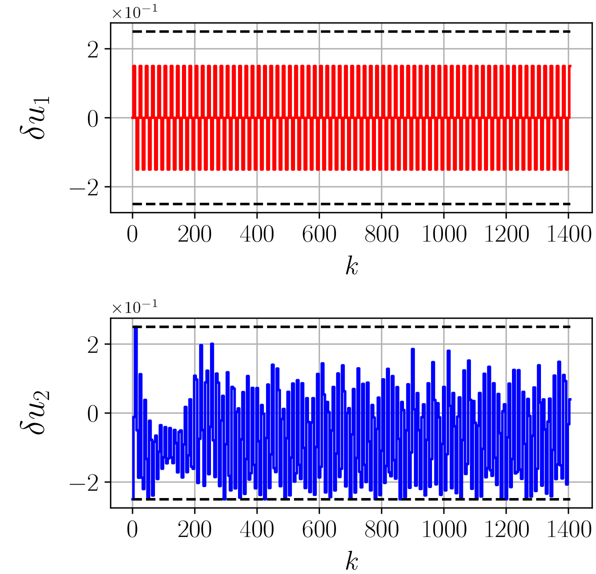

Fig. 2. Persistent inputs (δu_1 in red and δu_2 in blue).

Finally, the input is set to $H_{out} = U + \delta u$, where δu is a variation to be chosen following the procedure described before, to guarantee observer convergence and filtering enhancement (while respecting above limitations).

5.1. Simulation results

Optimization problem (15) is solved using the function `root` of Python-based open-source software SciPy (Jones, Oliphant, Peterson, et al., 2001–), and all the simulations were performed within the same software. Two input sequences are computed:

The first one (referred to as "δu_1") only guarantees the lower bound of the Gramian and it is computed with the algorithm formerly presented in Rubio Scola et al. (2013a), for the sake of comparison (it does not optimize the noise attenuation in the estimation error). The bound on the minimal eigenvalue of the Gramian is $\alpha = 10^{-3}$.

The second one, denoted by "δu_2", is obtained by the algorithm of the present paper, with $q = 10^3$, $r = 0.01$, $c = 100$ and $\rho = 25$. Those parameters put emphasis on the first objective, i.e. the minimization of the trace of P at the end of the time window, with respect to that of minimizing the input energy.

The size σ of the receding-window for the Gramian is set to 5 steps for both algorithms, and the initial condition for Riccati matrix in both algorithms are the same.

In all figures below, red and blue colors respectively refer to "δu_1" and "δu_2", and both of them are first displayed in Fig. 2 while variation rate of "δu_2" is shown in Fig. 3.

Fig. 4 shows the respective minimal eigenvalue achieved for Gramian W_k over a σ-step in each case, confirming that observability is achieved in both cases.

Fig. 5 then presents the evolution of the trace of Riccati matrix $\mathrm{Tr}(P_k)$, solution of Eq. (16), when each input sequence is applied. Clearly the input obtained from algorithm (15) reduces the trace over the time.

Since observability is achieved in both cases, the observer can be implemented for each of them. Some comparison results can be seen on Fig. 6 for x_3 and x_4.

In all cases, the initial conditions are $x_0 = [4.3, 0.53, 0, 4]^T \cdot 10^{-3}$ and $\hat{x}_0 = x_0$ and the random white Gaussian noise sequences are the same.

Table 1
Performances both algorithm solutions.

	δu_1	δu_2
Error norm ($\|x - x_e\|$)	$1.775 \cdot 10^{-3}$	$1.094 \cdot 10^{-3}$
Sum of the trace of Riccati matrix for $0 \geq k \geq 1408$	1.2378	0.4841
Trace of Riccati matrix for $k = 1408$	$8.835 \cdot 10^{-3}$	$2.903 \cdot 10^{-3}$
Input Energy ($\|u\|$)	3.977	5.473
Cost function (15) for $0 \geq k \geq 1408$	2.674	1.822

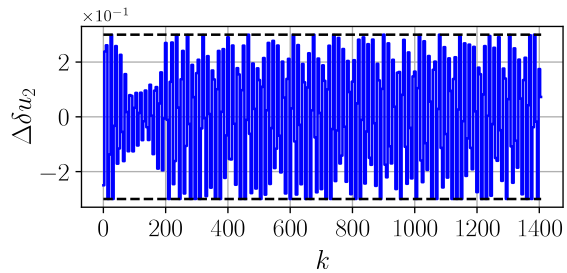

Fig. 3. Variation rate of δu_1.

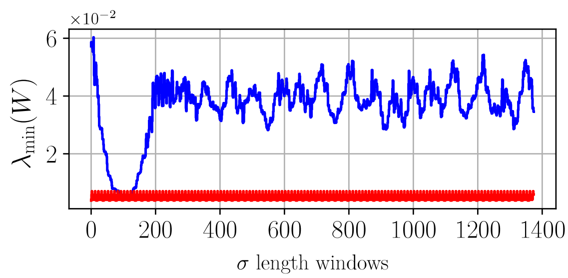

Fig. 4. Gramian minimum eigenvalues for both inputs.

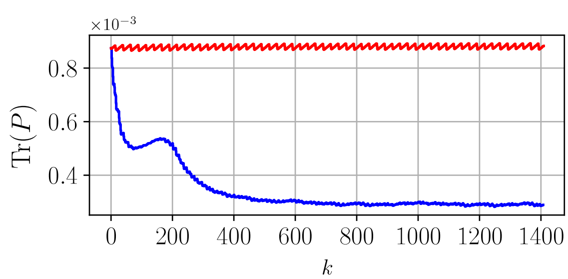

Fig. 5. Trace of the matrix P under both inputs.

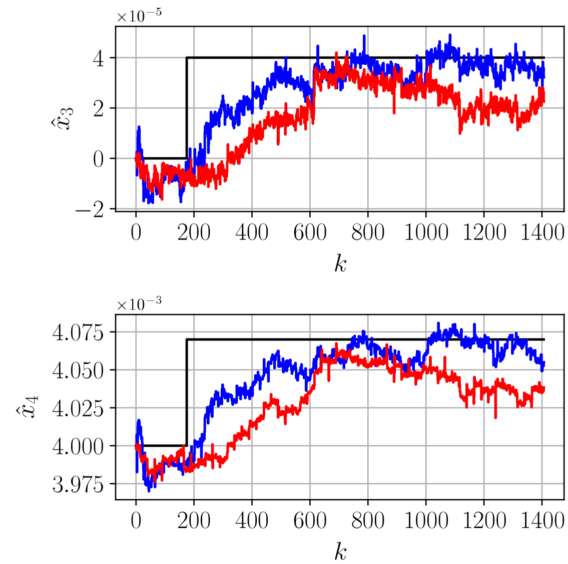

Fig. 6. Comparison of estimate x_3, x_4 for both inputs, with real values in black.

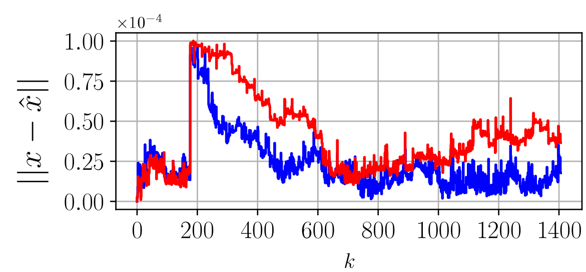

Fig. 7. Norm of the estimation error under both inputs.

At time $k = 176$, F is changed from 0 to $4 \cdot 10^{-5}$, simulating a leak in the pipeline.

Finally Fig. 7 compares the norm of the estimation error corresponding to each input case, both of them converging to some region near zero (depending on the noise magnitude).

Table 1 summarizes the performances of each input with respect to its corresponding energy, trace of Riccati matrix, and norm estimation error.

6. Conclusions

In this paper, we have proposed a methodology for input selection so as to guarantee observability (and related observer convergence) as well as filtering enhancement for a class of state affine systems. This input design is based on some optimal control approach, for which a numerical implementation method has been described. The method has been illustrated with an example of application in pipeline monitoring.

Various possible extensions have also been mentioned and will be part of future studies, as well as the problem of coupling with control purposes.

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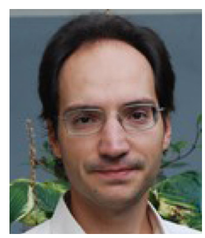

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