ON DISTINGUISHED ORBITS OF REDUCTIVE REPRESENTATIONS

EDISON ALBERTO FERNÁNDEZ CULMA

ABSTRACT. Let G be a real reductive Lie group and let $\tau : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ be a real reductive representation of G with (restricted) moment map $m_{\mathfrak{g}} : V \smallsetminus \{0\} \longrightarrow \mathfrak{g}$. In this work, we introduce the notion of *nice space* of a real reductive representation to study the problem of how to determine if a G-orbit is *distinguished* (i.e. it contains a critical point of the norm squared of $m_{\mathfrak{g}}$). We give an elementary proof of the well-known convexity theorem of Atiyah-Guillemin-Sternberg in our particular case and we use it to give an easy-to-check sufficient condition for a G-orbit of a element in a nice space to be distinguished. In the case where G is algebraic and τ is a rational representation, the above condition is also necessary (making heavy use of recent results of M. Jablonski), obtaining a generalization of Nikolayevsky's nice basis criterium. We also provide useful characterizations of nice spaces in terms of the weights of τ. Finally, some applications to ternary forms and minimal metrics on nilmanifolds are presented.

1. INTRODUCTION

Let $\widehat{\mathrm{G}} = \mathrm{U}^{\mathbb{C}}$ be a complex reductive group, $\widehat{\mathfrak{g}} := \mathrm{Lie}(\widehat{\mathrm{G}})$ and $\widehat{\tau} : \widehat{\mathrm{G}} \longrightarrow \mathrm{GL}(\widehat{V})$ be a representation of $\widehat{\mathrm{G}}$. In [Nes], Linda Ness associates to $\widehat{\tau}$ a moment map (as in symplectic geometry) $m_{\widehat{\mathfrak{g}}} : \widehat{V} \smallsetminus \{0\} \longrightarrow \sqrt{-1}\mathfrak{u}$ to study the orbit space of $\widehat{\tau}$ (here $\mathfrak{u} := \mathrm{Lie}(\mathrm{U})$). By considering a (real) inner product on $\sqrt{-1}\mathfrak{u}$ such that U acts by isometries via the adjoint representation, Ness's studies of the functional $\|m_{\widehat{\mathfrak{g}}}\|^2$ not only recovered known results about the set of semi-stable vectors but she also proved that all critical points of $\|m_{\widehat{\mathfrak{g}}}\|^2$ in the *null cone* have similar properties to those of *minimal points* (which have closed $\widehat{\mathrm{G}}$-orbit). The gradient flow of $\|m_{\widehat{\mathfrak{g}}}\|^2$ was also studied independently by Francis Kirwan [Kir2] and the stratification given by such flow is often called the *Kirwan-Ness stratification*.

Numerous efforts have been made over the next 25 years to carry out Kirwan-Ness results to the context of real Lie groups. This has been successfully achieved by Peter Heinzner, Gerald Schwarz and collaborators within

---

their project of developing a geometric invariant theory for actions of real
Lie groups on complex spaces (see [HSchw, HSS]). At this point, we should
also mention the pioneering work of Roger Richardson and Peter Slodowy
[RS] on representations of real reductive algebraic groups, which was picked
up by Patrick Eberlein and Michael Jablonski to extend [Kir2, Nes] to this
particular case (see [EJ, Jab2]). The idea of all the mentioned works is to
use some connection with the complex case. In both cases, orbits of critical
points of $\|m_{\mathfrak{g}}\|^2$ play a distinguished role in the orbit space and it is for this
reason that such orbits are called *distinguished*.

Our purpose in this paper is to study the problem of how to determine
if a G-orbit is distinguished, where G is real reductive Lie group (in the
sense of Heinzner-Schwarz) and the action is via a real reductive represen-
tation of G (as it is defined in [Sto1]). In addition to the aforementioned
motivation to study distinguished orbits, a second reason comes from the
intriguing interplay between the Ricci flow on nilpotent Lie groups and the
gradient flow of the norm squared of the moment map associated to the
natural action of $\mathrm{GL}_n(\mathbb{R})$ on $V = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$; the distinguished orbits
of nilpotent Lie brackets are in 1-1 correspondence with *nilsoliton metrics*
on simply connected nilpotent Lie groups (see, for instance, [Lau2]). Niko-
layevsky proves in [Nik2] many theorems on *Einstein nilradicals* (nilpotent
Lie algebras admitting a nilsoliton metric) by using the results given in [RS].
Among these results, we can highlight the Nikolayevsky nice basis criterium,
which provides an easy-to-check convex geometry condition for a nilpotent
Lie algebra with a *nice basis* to admit a nilsoliton metric.

By following Nikolayevsky's ideas given in [Nik1, Nik2], we introduce the
notion of *nice space* of a real reductive representation and we show that such
criterium is a general fact of the theory of real reductive representations
which are *rational*. To do this, we give an elementary proof of the convexity
of $m_{\mathfrak{a}}(\mathrm{A} \cdot v)$ where A is a connected abelian Lie group with out compact
factor acting linearly and by symmetric operators on a finite dimensional real
vector space V. This result is related with the convexity theorem of Atiyah-
Guillemin-Sternberg ([Ati, Theorem 2] and [GS1, Theorem 5.2])and indeed
can be used to prove such theorem in the particular case of representations
of a complexified torus.

In Section 4, we generalize some of the results of [LW2] and give an
elementary characterization of a nice space in terms of weights of the repre-
sentation (which is very helpful, as we will see). In Section 5 we give some
applications of our results to the study of ternary forms and the existence
problem of minimal compatible metrics with geometric structures on nilpo-
tent Lie groups (as are defined by Lauret in [Lau1]). Theorem 5.5 provides
an elementary expression for the *stratifying set* associated with the natu-
ral action of $\mathrm{GL}_3(\mathbb{R})$ on ternary forms and Theorem 5.6 gives a complete
classification of distinguished orbits in the null cone of the $\mathrm{GL}_3(\mathbb{R})$-action
on $\mathbb{R}[x, y, z]_4$. Theorem 5.14 follows immediately from our main result and
provides a tool to find minimal compatible metrics for a very wide family

of class-γ nilpotent Lie groups; we use it to determine minimal compatible metrics on symplectic two-step Lie algebras of dimension 6 (see Theorem 5.17 and Table 2).

2. Preliminaries

Let G be a real reductive Lie group in the sense of Heinzner-Schwarz, i.e. there exists a complex reductive group $\widehat{G} = U^{\mathbb{C}}$ ($\widehat{G}$ is the *universal complexification* of a compact subgroup U) such that G is a closed subgroup of $\widehat{G}$ and is *compatible with the Cartan decomposition* $\widehat{G} = U \exp(\sqrt{-1}\mathfrak{u})$, in other words, the function

$$(2.1) \qquad \begin{aligned} \varphi : \ K \times \mathfrak{p} &\longrightarrow G \\ (k, X) &\longmapsto k \exp(X) \end{aligned}$$

is a diffeomorphism on G, where $K := G \cap U$ and $\mathfrak{p} := \mathfrak{g} \cap \sqrt{-1}\mathfrak{u}$.

Definition 2.1. [Sto1, Section 2] A representation $\tau : G \longrightarrow GL(V)$ of a real reductive Lie group is called *real reductive representation* if V is a G-invariant real subspace of a holomorphic $\widehat{G}$-representation space $\widehat{V}$.

It is fairly easy to see that given a real reductive representation, there exists a inner product on V, say $\langle \cdot, \cdot \rangle$, such that K acts by isometries and $\mathfrak{p}$ acts by symmetric operators. There also exists a inner product on $\mathfrak{g}$, say $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, such that K acts by isometries and $\mathfrak{p}$ acts by symmetric operators (via the adjoint representation) and $\mathfrak{g} = \mathfrak{k} \overset{\perp}{\oplus} \mathfrak{p}$ (it is a orthogonal decomposition). From now on, $\pi := d\tau|_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ and orthogonal complements and orthogonal projections on $\mathfrak{g}$ and V are considered with respect $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ and $\langle \cdot, \cdot \rangle$, respectively.

Let us fix a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ maximal in $\mathfrak{p}$. Since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$, it follows that $\mathfrak{a}$ is abelian. Thus, the families $\{ad(X)|X \in \mathfrak{a}\}$ and $\{\pi(X)|X \in \mathfrak{a}\}$ are families of commuting symmetric operators, which allow a weight space decomposition for $\mathfrak{g}$ and V; that is, there exist finite subsets $\Delta(\mathfrak{g})$ and $\Delta(V)$ of $\mathfrak{a}$, with $0 \notin \Delta(\mathfrak{g})$, such that

$$(2.2) \qquad \mathfrak{g} \ = \ \mathfrak{g}_0 \overset{\perp}{\oplus} \overset{\perp}{\underset{\gamma \in \Delta(\mathfrak{g})}{\bigoplus}} \mathfrak{g}_\gamma,$$

$$(2.3) \qquad V \ = \ \overset{\perp}{\underset{\alpha \in \Delta(V)}{\bigoplus}} V_\alpha$$

where

$$\begin{aligned} \mathfrak{g}_\gamma \ &= \ \{Y \in \mathfrak{g} : ad(X)Y = \langle\!\langle X, \gamma \rangle\!\rangle Y \text{ for all } X \in \mathfrak{a}\}, \\ V_\alpha \ &= \ \{v \in V : \pi(X)v = \langle\!\langle X, \alpha \rangle\!\rangle v \text{ for all } X \in \mathfrak{a}\}. \end{aligned}$$

The Decomposition (2.2) is often called *restricted-root space decomposition* of $\mathfrak{g}$ and the set $\Delta(\mathfrak{g})$ is called *set of roots* of $\mathfrak{g}$

Now, we consider the Cartan involution of $\mathfrak{g}$ given by

$$(2.4) \qquad \theta: \begin{array}{ccc} \mathfrak{k} \overset{\perp}{\oplus} \mathfrak{p} & \longrightarrow & \mathfrak{k} \overset{\perp}{\oplus} \mathfrak{p} \\ X + Y & \longmapsto & X - Y \end{array}, \quad \forall X \in \mathfrak{k}, Y \in \mathfrak{p}.$$

The following proposition summarizes the basic well-known properties of θ, $\langle \cdot, \cdot \rangle$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ so we omit the proof.

Proposition 2.2. *Let* G *be a real reductive Lie group, Let* $\tau : G \longrightarrow GL(V)$ *be a real reductive representation with* $\langle \cdot, \cdot \rangle$ *and* $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ *as above. Then, for all* X, Y *in* $\mathfrak{g}$,

   (1) $\pi(X)^{\mathrm{T}} = -\pi(\theta(X))$ *where* $\pi(X)^{\mathrm{T}}$ *is the transpose operator of* $\pi(X)$ *with respect to* $\langle \cdot, \cdot \rangle$.

   (2) $\mathrm{ad}(X)^{\mathrm{T}} = -\mathrm{ad}(\theta(X))$ *where* $\mathrm{ad}(X)^{\mathrm{T}}$ *is the transpose operator of* $\mathrm{ad}(X)$ *with respect to* $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

   (3) θ *is an isometry of* $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

   (4) $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{\theta(\lambda)} = \mathfrak{g}_{-\lambda}$. *Hence,* $\lambda \in \Delta(\mathfrak{g})$ *if and only if* $-\lambda \in \Delta(\mathfrak{g})$.

   (5) $[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}] \subseteq \mathfrak{g}_{\lambda_1 + \lambda_2}$. *Hence, if* $[\mathfrak{g}_{\lambda_1}, \mathfrak{g}_{\lambda_2}] \neq \{0\}$ *then* $\lambda_1 + \lambda_2 \in \Delta(\mathfrak{g}) \cup \{0\}$.

   (6) $\pi(\mathfrak{g}_\lambda) V_\alpha \subseteq V_{\lambda + \alpha}$. *Hence, if* $\pi(\mathfrak{g}_\lambda) V_\alpha \neq \{0\}$ *then* $\lambda + \alpha \in \Delta(V)$.

   (7) *For all* $Z \in \mathfrak{g}_\lambda$, $[\theta(Z), Z] \in \mathfrak{a}$, *thus* $[\theta(Z), Z] = \|Z\|^2 \lambda$.

By fixing a *regular element* $X_0 \in \mathfrak{a}$ (i.e. $\langle\!\langle \gamma, X_0 \rangle\!\rangle \neq 0$ for every $\gamma \in \Delta(\mathfrak{g})$), we obtain a set of positive roots

$$(2.5) \qquad \Delta(\mathfrak{g})^+ = \{\gamma \in \Delta(\mathfrak{g}) : \langle\!\langle \gamma, X_0 \rangle\!\rangle > 0\}$$

and a Weyl chamber

$$(2.6) \qquad \mathfrak{a}^+ = \{Y \in \mathfrak{a} : \langle\!\langle \gamma, Y \rangle\!\rangle > 0 \text{ for every } \gamma \in \Delta(\mathfrak{g})^+\}.$$

If $\Delta(\mathfrak{g})^- := -\Delta(\mathfrak{g})^+$, then $\Delta(\mathfrak{g}) = \Delta(\mathfrak{g})^+ \cup \Delta(\mathfrak{g})^-$ and $\theta(\Delta(\mathfrak{g})^+) = \Delta(\mathfrak{g})^-$.

We are now in a position to define the moment map of τ. This map is implicitly defined by

$$(2.7) \qquad \begin{array}{ccc} m_{\mathfrak{g}} : V \smallsetminus \{0\} & \longrightarrow & \mathfrak{g} \\ \langle\!\langle m_{\mathfrak{g}}(v), X \rangle\!\rangle & = & \frac{1}{\|v\|^2} \langle \pi(X) v, v \rangle, \end{array}$$

for all $X \in \mathfrak{p}$ and $v \in V \smallsetminus \{0\}$. It is easy to see that $m_{\mathfrak{g}}$ is K-equivariant (i.e. $m_{\mathfrak{g}}(k \cdot v) = \mathrm{Ad}(k)(m_{\mathfrak{g}}(v))$ for all $k \in K$ and $v \in V \smallsetminus \{0\}$) and $\mathbb{R}^*$-invariant (i.e. $m_{\mathfrak{g}}(\lambda v) = m_{\mathfrak{g}}(v)$ for all $\lambda \in \mathbb{R}^*$ and $v \in V \smallsetminus \{0\}$). Also, we note that the image of $m_{\mathfrak{g}}$ is contained in $\mathfrak{p}$, which is clear from the K-invariance of $\langle \cdot, \cdot \rangle$.

Set $F_{\mathfrak{g}} = \|m_{\mathfrak{g}}\|^2$, which we shall call the *norm squared of the moment map*. As we said above, the critical points of $F_{\mathfrak{g}}$ play a important role in the study of the orbit space. We thus have the following definition.

Definition 2.3. [Jab1, Definition 2.6] An orbit $G \cdot v$ is said to be G-*distinguished* if $G \cdot v$ has a critical point of $F_{\mathfrak{g}}$.

Theorem 2.4.

   (1) [HSS, Corollaries 6.10, 6.11] *For each* $v \in V \smallsetminus \{0\}$, *the following conditions are equivalent:*

(a) v is a critical point of $\mathrm{F}_{\mathfrak{g}}$.

(b) $\pi(m_{\mathfrak{g}}(v))v = \lambda v$ for some $\lambda \in \mathbb{R}$.

(c) v is a global minimum of $\mathrm{F}_{\mathfrak{g}}|_{\mathrm{G}\cdot v}$.

Under these conditions, the set of critical points of $\mathrm{F}_{\mathfrak{g}}$ lying in $\mathrm{G}\cdot v$ equals $\mathrm{K}\cdot v$ (up to scaling).

(2) [HSS, Theorem 7.3 y Corollary 7.6] **(Kirwan-Ness Stratification)** The gradient flow of $\mathrm{F}_{\mathfrak{g}}$ determines a stratification of $V \smallsetminus \{0\}$, it is to say that

$$V \smallsetminus \{0\} = \bigcup_{\beta \in \overline{\mathfrak{a}^+}} \mathscr{S}_{\beta},$$

where a stratum $\mathscr{S}_{\beta}$ is given by

$$\mathscr{S}_{\beta} = \left\{ v \in V \smallsetminus \{0\} : \lim_{t \to -\infty} s_v(t) \in \mathscr{C}(\beta) \right\}.$$

Here, $s_v(t)$ is the integral curve through v of the gradient field of $\mathrm{F}_{\mathfrak{g}}$ and $\mathscr{C}(\beta)$ is the set

$$\mathscr{C}(\beta) = \{v \in V \smallsetminus \{0\} : v \text{ is a critical point of } \mathrm{F}_{\mathfrak{g}} \text{ and } m_{\mathfrak{g}}(v) \in \mathrm{Ad}(\mathrm{K})\beta\}.$$

The strata satisfy

$$\overline{\mathscr{S}_{\beta}} \subseteq \mathscr{S}_{\beta} \cup \{\mathscr{S}_{\beta'} : \|\beta'\| > \|\beta\|\}.$$

(3) [HSS, Theorem 5.4] For all $v \in \mathscr{S}_{\beta}$, $\overline{\mathrm{G}\cdot v} \cap \mathscr{S}_{\beta}$ contains a unique distinguished orbit.

Let us denote by $\mathscr{B}$ the set of all $\beta \in \overline{\mathfrak{a}^+}$ such that $\mathscr{S}_{\beta} \neq \phi$ or, what is the same thing,

$$\mathscr{B} = \{\beta \in \overline{\mathfrak{a}^+} : \mathscr{C}(\beta) \neq \phi\}.$$

We call $\mathscr{B}$ the *stratifying set* of V.

Notation 2.5. Let Φ be a finite subset of $\mathfrak{a}$. The convex hull of Φ will be denoted by $\mathrm{CH}(\Phi)$ while by $\mathrm{Aff}(\Phi)$ we denote the affine space generated by Φ. $\mathrm{mcc}(\Phi)$ denotes the *minimal convex combination* of Φ; i.e. the unique vector closest to the origin in $\mathrm{CH}(\Phi)$. The notation $\mathrm{int}(\mathrm{CH}(\Phi))$ represents the interior of $\mathrm{CH}(\Phi)$ relative to the usual topology of $\mathrm{Aff}(\Phi)$.

Fix $v \in V \smallsetminus \{0\}$, say $v = v_1 + \ldots + v_s$ with each $v_i \in V_{\alpha_i}$ (by Equation (2.3)). Let $\mathfrak{R}(v)$ denote the ordered set of weights related with v:

(2.8) $$\mathfrak{R}(v) = \{\alpha_i \in \Delta(V) : v_i \neq 0\}$$

and set $\beta_v = \mathrm{mcc}(\mathfrak{R}(v))$

We note that the set $\{\beta_v : v \in V \smallsetminus \{0\}\}$ is a finite set.

Proposition 2.6.

(1) Let H be a subgroup of G compatible whit the Cartan decomposition of G given by Equation (2.1). Then $m_{\mathfrak{h}} = \mathrm{Proy}_{\mathfrak{h}} \circ m_{\mathfrak{g}}$, where $\mathrm{Lie}(\mathrm{H}) = \mathfrak{h}$ and $\mathrm{Proy}_{\mathfrak{h}} : \mathfrak{g} \longrightarrow \mathfrak{h}$ is the orthogonal projection of $\mathfrak{g}$ on $\mathfrak{h}$ with respect to $\langle\!\langle \cdot, \cdot \rangle\!\rangle$.

(2) Let $v \in V \smallsetminus \{0\}$ such that $m_{\mathfrak{g}}(v) = \beta \in \mathfrak{a}$. Then

(a) $m_{\mathfrak{g}}(v) \in \operatorname{int}(\operatorname{CC}(\mathfrak{R}(v)))$.
(b) v is a critical point of $\mathrm{F}_{\mathfrak{g}}$ if and only if $\beta = \operatorname{mcc}(\mathfrak{R}(v))$.
(3) $\mathscr{B}$ is a finite subset of $\mathfrak{a}$.

3. Generalization of Nikolayevsky's nice basis criterium

In this section, we formulate and prove a generalization of Nikolayevsky's nice basis criterium ([Nik2, Theorem 3.]). We begin with an elementary proof of the convexity of $m_{\mathfrak{a}}(\mathrm{A} \cdot v)$ where A is a connected abelian Lie group without compact factor acting linearly by symmetric operators on a real vector space V (with respect some inner product $\langle \cdot, \cdot \rangle$ on V).

Proposition 3.1. *For every $v \in V \smallsetminus \{0\}$, $v = v_1 + \ldots + v_s$ (as in Decomposition (2.3))*
$$m_{\mathfrak{a}}(\mathrm{A} \cdot v) = \operatorname{int}(\operatorname{CH}(\mathfrak{R}(v))).$$

Proof. Fix $X \in \mathfrak{a}$. Let us first find an expression for $m_{\mathfrak{a}}(\exp(X) \cdot v)$ in terms of weights. Let $Y \in \mathfrak{a}$

$$
\begin{aligned}
\langle\!\langle m_{\mathfrak{a}}(\exp(X) \cdot v), Y \rangle\!\rangle &= \frac{1}{\|\exp(X) \cdot v\|^2} \langle \pi(Y) \exp(X) \cdot v, \exp(X) \cdot v \rangle \\
&= \frac{1}{\|\exp(X) \cdot v\|^2} \langle \sum \mathrm{e}^{\langle\!\langle X, \alpha_i \rangle\!\rangle} \pi(Y) v_i, \sum \mathrm{e}^{\langle\!\langle X, \alpha_j \rangle\!\rangle} v_j \rangle \\
&= \frac{1}{\|\exp(X) \cdot v\|^2} \sum \mathrm{e}^{2\langle\!\langle X, \alpha_i \rangle\!\rangle} \|v_i\|^2 \langle\!\langle Y, \alpha_i \rangle\!\rangle \\
&= \langle\!\langle \frac{1}{\|\exp(X) \cdot v\|^2} \sum \mathrm{e}^{2\langle\!\langle X, \alpha_i \rangle\!\rangle} \|v_i\|^2 \alpha_i, Y \rangle\!\rangle.
\end{aligned}
$$

This gives

$$(3.1) \qquad m_{\mathfrak{a}}(\exp(X)v) = \frac{1}{\sum \mathrm{e}^{2\langle\!\langle X, \alpha_i \rangle\!\rangle} \|v_i\|^2} \sum \mathrm{e}^{2\langle\!\langle X, \alpha_i \rangle\!\rangle} \|v_i\|^2 \alpha_i$$

and so $m_{\mathfrak{a}}(\mathrm{A} \cdot v) \subseteq \operatorname{int}(\operatorname{CH}(\mathfrak{R}(v)))$.

We next prove that $m_{\mathfrak{a}}(\mathrm{A} \cdot v)$ is a open convex set in $\operatorname{Aff}(\mathfrak{R}(v))$. Let $\mathfrak{b} \subseteq \mathfrak{a}$ be a subspace of $\mathfrak{a}$ and let $\rho_{\mathfrak{b}}$ be given by

$$
\begin{aligned}
\rho_{\mathfrak{b}} : \mathfrak{b} &\longrightarrow \mathbb{R} \\
X &\longmapsto \ln \|\exp(X) \cdot v\|^2
\end{aligned} \quad .
$$

By using that $\mathfrak{b}$ is abelian, a straightforward computation gives

$$m_{\mathfrak{b}}(\exp(Y) \cdot v) = \frac{1}{2}(\mathrm{d}\rho_{\mathfrak{b}})_Y.$$

Therefore the image of $\mathrm{d}\rho_{\mathfrak{b}}$ equals $2m_{\mathfrak{b}}(\mathrm{B} \cdot v)$ ($\mathrm{B} = \exp(\mathfrak{b})$). The basic idea of the proof is to apply the Fenchel's Convexity Theorem (see Appendix) to certain $\rho_{\widetilde{\mathfrak{b}}}$ and then to relate $m_{\mathfrak{a}}(\mathrm{A} \cdot v)$ with $\mathrm{d}\rho_{\widetilde{\mathfrak{b}}}$. We need to calculate $\mathrm{d}^2 \rho_{\mathfrak{b}}$

for any $\mathfrak{b} \subseteq \mathfrak{a}$. Let $X, Y \in \mathfrak{b}$ and $\widetilde{v} = \exp(X) \cdot v$

$$
\begin{aligned}
\langle\!\langle \mathrm{d}^2 \rho_{\mathfrak{b}}(X)Y, Y \rangle\!\rangle \;&=\; 2 \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_0 \langle\!\langle m_{\mathfrak{b}}(\exp(X+tY)\cdot v), Y \rangle\!\rangle \\
&=\; 2 \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_0 \frac{1}{\|\exp(tY)\cdot\widetilde{v}\|^2} \langle \pi(Y)\exp(tY)\cdot\widetilde{v}, \exp(tY)\cdot\widetilde{v} \rangle \\
&=\; \frac{2}{\|\widetilde{v}\|^4}\big[(\langle \pi(Y)\pi(Y)\widetilde{v},\widetilde{v}\rangle + \langle\pi(Y)\widetilde{v},\pi(Y)\widetilde{v}\rangle)\|\widetilde{v}\|^2 \\
&\qquad\qquad -2\langle\pi(Y)\widetilde{v},\widetilde{v}\rangle\langle\pi(Y)\widetilde{v},\widetilde{v}\rangle \big] \\
&=\; \frac{4}{\|\widetilde{v}\|^4}\big[\|\pi(Y)\widetilde{v}\|^2\|\widetilde{v}\|^2 - \langle\pi(Y)\widetilde{v},\widetilde{v}\rangle^2\big].
\end{aligned}
$$

By Cauchy-Schwartz Inequality, $\langle\!\langle \mathrm{d}^2\rho_{\mathfrak{b}}(X)Y, Y\rangle\!\rangle \geq 0$ and the equality holds if and only if $\pi(y)\widetilde{v} = \lambda\widetilde{v}$ for some $\lambda \in \mathbb{R}$, which is equivalent to $\pi(y)v = \lambda v$ for some $\lambda \in \mathbb{R}$. Thus $\mathrm{d}^2\rho_{\mathfrak{b}}(X)$ is positive definite for any $X \in \mathfrak{b} \smallsetminus \{0\}$ if and only if $\mathfrak{b} \cap \mathfrak{d} = \{0\}$ where

$$
\mathfrak{d} := \mathfrak{d}_v(\mathfrak{a}) \;=\; \{Y \in \mathfrak{a} : \pi(Y)v = \lambda v \text{ for some } \lambda \in \mathbb{R}\}.
$$

Thus, let $\widetilde{\mathfrak{b}} := \mathfrak{d}^\perp$ where $\mathfrak{d}^\perp$ is the orthogonal complement of $\mathfrak{d}$ in $\mathfrak{a}$.

We consider

$$
\mathfrak{a}_v \;=\; \{Y \in \mathfrak{a} : \pi(Y)v = 0\}.
$$

It is easy to see that $\mathfrak{a}_v$ is an ideal of $\mathfrak{d}$ of codimension 1 or 0; we have two cases

$$
\tag{3.2} \mathfrak{d} \;=\; \mathfrak{a}_v,
$$

$$
or
$$

$$
\tag{3.3} \mathfrak{d} \;=\; \mathfrak{a}_v \overset{\perp}{\oplus} \mathbb{R}Z.
$$

with $Z \in \mathfrak{a}$ such that $\pi(Z)v = v$.

Now, let $X \in \mathfrak{a}$. We note that $m_{\mathfrak{a}}(\exp(X)v) = m_{\mathfrak{a}}(\exp(X')v)$ where X' is the component of X in $\widetilde{\mathfrak{b}}$. In fact, say $X = X' + X''$ with $X' \in \widetilde{\mathfrak{b}}$ and $X'' \in \mathfrak{d}$,

$$
\begin{aligned}
m_{\mathfrak{a}}(\exp(X)v) \;&=\; m_{\mathfrak{a}}(\exp(X')\exp(X'')v) \\
&=\; m_{\mathfrak{a}}(t\exp(X')v) \text{ for some } t \in \mathbb{R} \\
\tag{3.4} &=\; m_{\mathfrak{a}}(\exp(X')v).
\end{aligned}
$$

Here, we have used again that $\mathfrak{a}$ is abelian and $m_{\mathfrak{a}}$ is invariant under a rescaling. Also, it is easy to see that for any $X \in \mathfrak{a}$

$$
\tag{3.5} m_{\mathfrak{a}}(\exp(X)v) \in \mathfrak{a} \overset{\perp}{\ominus} \mathfrak{a}_v.
$$

We are in a position to describe $m_{\mathfrak{a}}(\mathrm{A}\cdot v)$ in terms of $\mathrm{d}\,\rho_{\widetilde{\mathfrak{b}}}$. By the Equation (3.5), $m_{\mathfrak{a}}$ could have component in $\mathbb{R}Z$ if the Case (3.3) holds. In this case,

such component is always $\frac{1}{\|Z\|^2}Z$:

$$
\begin{aligned}
\langle\!\langle m_{\mathfrak{a}}(\exp(X)v), Z\rangle\!\rangle &= \frac{1}{\|\exp(X)v\|^2}\langle\pi(Z)\exp(X)v, \exp(X)v\rangle \\
&= \frac{1}{\|\exp(X)v\|^2}\langle\exp(X)\pi(Z)v, \exp(X)v\rangle \\
&= \frac{1}{\|\exp(X)v\|^2}\langle\exp(X)v, \exp(X)v\rangle \\
&= 1.
\end{aligned}
$$

We recall that $m_{\widetilde{\mathfrak{b}}}$ is the projection of $m_{\mathfrak{a}}$ on $\widetilde{\mathfrak{b}}$ (Proposition 2.6, item 1), from this, if $X' = \mathrm{Proj}_{\widetilde{\mathfrak{b}}}X$ for $X \in \mathfrak{a}$ we have

$$
\begin{aligned}
m_{\mathfrak{a}}(\exp(X)v) &= m_{\mathfrak{a}}(\exp(X')v) \text{ (by Equation (3.4))} \\
&= \mathrm{Proy}_{\widetilde{\mathfrak{b}}}\, m_{\mathfrak{a}}(\exp(X')v) + \mathrm{Proy}_{\mathfrak{d}}\, m_{\mathfrak{a}}(\exp(X')v) \\
&= m_{\widetilde{\mathfrak{b}}}(\exp(X')v) + \mathrm{Proy}_{\mathfrak{d}}\, m_{\mathfrak{a}}(\exp(X')v) \\
\text{(3.6)} \qquad &= \begin{cases} m_{\widetilde{\mathfrak{b}}}(\exp(X')\cdot v), \text{ if } \mathfrak{d} = \mathfrak{a}_v \\ m_{\widetilde{\mathfrak{b}}}(\exp(X')\cdot v) + \frac{1}{\|Z\|^2}Z, \text{ if } \mathfrak{d} = \mathfrak{a}_v \overset{\perp}{\oplus} \mathbb{R}Z \end{cases}
\end{aligned}
$$

Now (3.6) becomes $m_{\mathfrak{a}}(\mathrm{A}\cdot v) = m_{\widetilde{\mathfrak{b}}}(\widetilde{\mathrm{B}}\cdot v)$ or $m_{\mathfrak{a}}(\mathrm{A}\cdot v) = m_{\widetilde{\mathfrak{b}}}(\widetilde{\mathrm{B}}\cdot v) + \frac{1}{\|Z\|^2}Z$ (here, $\widetilde{\mathrm{B}} = \exp(\widetilde{\mathfrak{b}})$), and since $m_{\widetilde{\mathfrak{b}}}(\widetilde{\mathrm{B}}\cdot v)$ is a convex set ($\rho_{\widetilde{\mathfrak{b}}}$ satisfies the hypotheses of the Fenchel's Convexity Theorem), whatever the case, we conclude that $m_{\mathfrak{a}}(\mathrm{A}\cdot v)$ is a convex set.

To finish the proof, it is sufficient to prove that $\overline{m_{\mathfrak{a}}(\mathrm{A}\cdot v)} = \mathrm{CC}(\mathfrak{R}(v))$ (by [Gru, Theorem 2.1.7], we have for any convex subset Ω of $\mathbb{R}^n$, $\mathrm{int}(\overline{\Omega}) = \mathrm{int}(\Omega)$). By Theorem 6.4 item 3 we could prove that each exposed point of $\mathrm{CC}(\mathfrak{R}(v))$ belongs to $\overline{m_{\mathfrak{a}}(\mathrm{A}\cdot v)}$. Let α_i be an exposed point of $\mathrm{CC}(\mathfrak{R}(v))$. By definition, there exist $H_i \in \mathfrak{a}$ and $h_i \in \mathbb{R}$ such that for each $X \in \mathrm{CC}(\mathfrak{R}(v))$, $\langle\!\langle X, H_i\rangle\!\rangle \geq h_i$ and the equality holds if and only if $X = \alpha_i$. By using this inequality in the Equation (3.1) it follows that

$$
\lim_{t\to-\infty} m_{\mathfrak{a}}(\exp(tH_i)\cdot v) = \alpha_i.
$$

On account of the above, we have $m_{\mathfrak{a}}(\mathrm{A}\cdot v)$ is a convex set such that $\overline{m_{\mathfrak{a}}(\mathrm{A}\cdot v)} = \mathrm{CC}(\mathfrak{R}(v))$, and, in consequence, $m_{\mathfrak{a}}(\mathrm{A}\cdot v) = \mathrm{int}(\mathrm{CC}(\mathfrak{R}(v)))$. $\square$

Remark 3.2. The above proposition is related with well-known results on convexity properties of the moment map (see, for instance [HSto, Proposition 3]). The argument used in the proof shows that it is independent of complex case (which is a difference with analogous results). In fact, by a similar argument, we can prove the respective result in the complex setting, this is, if T is a torus and $\mathrm{T}^{\mathbb{C}}$ its universal complexification which acts linearly on complex vector space $\widehat{V}$ then $m_{\mathfrak{t}^{\mathbb{C}}}(\mathrm{T}^{\mathbb{C}}\cdot v) = \mathrm{int}(\mathrm{CC}(\mathfrak{R}(v)))$ for any $v \in \widehat{V}\smallsetminus\{0\}$ ($\mathfrak{R}(v)$ is defined analogously as above). Convexity properties

of moment maps is currently a well-developed theory and has a rich history which includes results due to Schur, Horn, Kostant, Mumford, Atiyah, Guillemin and Sternberg, Kirwan, etc. (see [HSchu]).

Definition 3.3. Let $\tau : G \longrightarrow GL(V)$ be a real reductive representation and let W be a vector subspace of V. W is called a *nice space* if it is A-invariant and $m_{\mathfrak{g}}(w) \in \mathfrak{a}$ for any $w \in W \smallsetminus \{0\}$ (here, $A = \exp(\mathfrak{a})$). Elements of W are called *nice elements*.

Remark 3.4. Given that A is compatible with the Cartan decomposition of G, if W is a nice space, then $m_{\mathfrak{a}}(w) = m_{\mathfrak{g}}(w)$ for any $w \in W \smallsetminus \{0\}$.

Remark 3.5. One could define a nice element $w \in V$ as that which satisfies $m_{\mathfrak{g}}(A \cdot w) \subseteq \mathfrak{a}$. In the Proposition 4.8, we prove that this notion coincides with that given in Definition 3.3.

Remark 3.6. In [Jab1] there is not a name to the notion of nice space but certainly this should be A-*detectable*. Since our motivation comes from the Nikolayevsky's nice basis criterium, we prefer to use name "nice" in this definition.

Now, we are ready to give a first step in the proof of our generalization of Nikolayevsky's nice basis criterium.

Proposition 3.7. *Let W be a nice space and let $w \in W \smallsetminus \{0\}$. If $\mathrm{mcc}(\mathfrak{R}(w)) \in \mathrm{int}(\mathrm{CC}(\mathfrak{R}(w)))$, then the G-orbit of w is distinguished. Furthermore, there exists $X \in \mathfrak{a}$ such that $\exp(X)w$ is a critical point of $F_{\mathfrak{g}}$.*

Proof. We suppose that $\mathrm{mcc}(\mathfrak{R}(w)) \in \mathrm{int}(\mathrm{CH}(\mathfrak{R}(w)))$. By Proposition 3.1 there exists $X \in \mathfrak{a}$ such that $m_{\mathfrak{a}}(\exp(X) \cdot w) = \mathrm{mcc}(\mathfrak{R}(w))$ and moreover, $F_{\mathfrak{a}}|_{A \cdot w}$ has a minimum value at $\widetilde{w} = \exp(X) \cdot w$. By Theorem 2.4 item 1, $\widetilde{w}$ is a critical point of $F_{\mathfrak{a}}$ and so $\pi(m_{\mathfrak{a}}(\widetilde{w}))\widetilde{w} = \lambda \widetilde{w}$ for some $\lambda \in \mathbb{R}$. Since w is nice and so is $\widetilde{w}$, $m_{\mathfrak{a}}(\widetilde{w}) = m_{\mathfrak{g}}(\widetilde{w})$. From this and, again, from Theorem 2.4 item 1, $\widetilde{w}$ is a critical point of $F_{\mathfrak{g}}$, and so $G \cdot w$ is a distinguished orbit. $\square$

Notation 3.8. Let w be a nice element and consider the ordered set $\mathfrak{R}(w)$. We denote by $\boldsymbol{U}_w$ the Gram matrix of $(\mathfrak{R}(w), \langle\langle \cdot, \cdot \rangle\rangle)$, i.e.

$$(3.7) \qquad \boldsymbol{U}_w(p, q) = \langle\langle \mathfrak{R}(w)_p, \mathfrak{R}(w)_q \rangle\rangle$$

with $1 \le p, q \le \sharp \mathfrak{R}(w)$

We omit the proof of the following corollaries, which follows easily from the method of Lagrange multipliers.

Corollary 3.9. *Let w be a nice element. If the equation*

$$(3.8) \qquad \boldsymbol{U}_w[x_i] = \lambda[1]$$

has a solution $[x_i]$ for some $\lambda \in \mathbb{R}$ such that each x_i is positive (positive solution) then $G \cdot w$ is a distinguished orbit.

Corollary 3.10. *Let w be a nice element and suppose that $0 \notin \mathrm{CH}(\mathfrak{R}(w))$. If the equation*

$$(3.9) \qquad\qquad \boldsymbol{U}_w[x_i] = [1]$$

has a positive solution $[x_i]$ then $\mathrm{G} \cdot w$ is a distinguished orbit.

To obtain the proof of the converse of Proposition 3.7 we need to restrict ourselves to the algebraic case. The idea is to apply Jablonski's results given in [Jab1]. Let us recall some notions and results of [Jab1] which we need.

Definition 3.11. [Jab1, Definition 2.9] Let G be a (real or complex) linear reductive algebraic group acting linearly and rationally on a finite dimensional vector space V. Let H be a compatible subgroup of G and W be an H-stable smooth subvariety of V. We say that the G-action is H-*detectable along W* if $m_{\mathfrak{g}}(w) \in \mathfrak{h}$ for any $w \in W$.

Theorem 3.12. [Jab1, Corollary 3.4] *Let* G, H, V *and* W *be as in the above definition and assume that* W *is a closed subset of* V. *If the action is* H-*detectable along* W, *then for any* $w \in W$, $\mathrm{G} \cdot w$ *is* G-*distinguished if and only if* $\mathrm{H} \cdot w$ *is* H-*distinguished.*

Proposition 3.13. *Let* G *and* V *as in the above definition and* W *a nice space, and let* $w \in W \smallsetminus \{0\}$ *such that* $\mathrm{G} \cdot w$ *is a distinguished orbit. Then*

$$\mathrm{mcc}(\mathfrak{R}(w)) \in \mathrm{int}(\mathrm{CC}(\mathfrak{R}(w))).$$

Proof. Since $\mathrm{G} \cdot w$ is a distinguished, $\mathrm{A} \cdot w$ is a distinguished orbit of the A-action (by Theorem 3.12). Thus, there exist $\widetilde{w} \in \mathrm{A} \cdot w$ which is a critical point of $\mathrm{F}_{\mathfrak{a}}$. By Theorem 2.4 item 1 we have $\|m_{\mathfrak{a}}(\widetilde{w})\|^2 \leq \|m_{\mathfrak{a}}(w')\|^2$ for any $w' \in \mathrm{A} \cdot w$. It follows from Proposition 3.1 that $m_{\mathfrak{a}}(\widetilde{w})$ is a vector of minimal norm in the convex set $\mathrm{int}(\mathrm{CC}(\mathfrak{R}(w)))$ and by a continuity argument, $m_{\mathfrak{a}}(\widetilde{w})$ is also a vector of minimum norm in the set $\mathrm{CC}(\mathfrak{R}(w))$; hence $\mathrm{mcc}(\mathfrak{R}(w)) \in \mathrm{int}(\mathrm{CC}(\mathfrak{R}(w)))$. $\qquad\square$

We have from Propositions 3.7 and 3.13 our main result and equivalences in the Corollaries 3.9 and 3.10 to the algebraic case.

Theorem 3.14. *Let* G *be a real linear reductive algebraic group and* $\tau : \mathrm{G} \longrightarrow \mathrm{GL}(V)$ *be a rational representation of* G, *and let* w *be a nice element. The orbit* $\mathrm{G} \cdot w$ *is distinguished if and only if*

$$\mathrm{mcc}(\mathfrak{R}(w)) \in \mathrm{int}(\mathrm{CC}(\mathfrak{R}(w))).$$

Moreover, in such case, there exists $X \in \mathfrak{a}$ *such that* $\exp(X)w$ *is a critical point of* $\mathrm{F}_{\mathfrak{g}}$.

Remark 3.15. It is easy to see that the above results are valid in the complex case.

4. How to know if your A-space is nice

In this section, we consider an A-invariant subspace W of V and its decomposition in weight spaces

$$(4.1) \qquad W = W_{\alpha_1} \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} W_{\alpha_r}$$

with weight set $\Delta(W) = \{\alpha_1, \ldots, \alpha_r\}$. We follow ideas go back at least as far as [Sja, Lemma 7.1] and results presented in [LW2].

Remark 4.1. It is easily seen that the representation $\pi|_{\mathfrak{a}} : \mathfrak{a} \longrightarrow \mathfrak{gl}(V)$ is a completely reducible representation of $\mathfrak{a}$. Hence, $\Delta(W)$ is a subset of $\Delta(V)$ and each W_{α_i} is a subspace of some V_{α_j}. For this reason, there is no problem in considering $\mathfrak{R}(w)$ with respect to the decomposition (4.1) for all $w \in W$.

Lemma 4.2. *Let* $\gamma \in \Delta(\mathfrak{g})$, $Y \in \mathfrak{g}_\gamma$, $\alpha_i \in \Delta(W)$ *and* $w_i \in W_{\alpha_i}$ *such that* $\pi(Y)w_i \neq 0$, *then* $\pi(Y)w_i$ *is a simultaneous eigenvector for the family* $\{\pi(X) : X \in \mathfrak{a}\}$. *Moreover, if* $\mathrm{Proj}_W \pi(Y)w_i \neq 0$ *then* $\mathrm{Proj}_W \pi(Y)w_i$ *is a weight vector with corresponding weight* $\gamma + \alpha_i$, *and in consequence,* $\gamma + \alpha_i = \alpha_j$ *for some* $\alpha_j \in \Delta(W)$ *with* $\alpha_i \neq \alpha_j$.

Proof. Since $Y \in \mathfrak{g}_\gamma$, $[X,Y] = \langle\!\langle \gamma, X \rangle\!\rangle Y$ for all $X \in \mathfrak{a}$ and thus

$$\langle\!\langle \gamma, X \rangle\!\rangle \pi(Y) = \pi([X,Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

and finally

$$\begin{aligned}
\pi(X)(\pi(Y)w_i) &= \langle\!\langle \gamma, X \rangle\!\rangle \pi(Y)w_i + \pi(Y)\pi(X)w_i \\
&= \langle\!\langle \gamma, X \rangle\!\rangle \pi(Y)w_i + \langle\!\langle \alpha_i, X \rangle\!\rangle \pi(Y)w_i \\
&= \langle\!\langle \gamma + \alpha_i, X \rangle\!\rangle \pi(Y)w_i.
\end{aligned}$$

To see the second part, let $\pi(Y)w_i = w + w'$ with $w \in W$ ($w = \mathrm{Proj}_W \pi(Y)w_i$) and $w' \in W^\perp$.

$$\begin{aligned}
\pi(X)w + \pi(X)w' &= \pi(X)\pi(Y)w_i \\
&= \langle\!\langle \gamma + \alpha_i, X \rangle\!\rangle \pi(Y)w_i \\
&= \langle\!\langle \gamma + \alpha_i, X \rangle\!\rangle w + \langle\!\langle \gamma + \alpha_i, X \rangle\!\rangle w'.
\end{aligned}$$

By using that $W^\perp$ is also A-invariant as W, we have $\pi(X)w' \in W^\perp$, we thus get $\pi(X)w = \langle\!\langle \gamma + \alpha_i, X \rangle\!\rangle w$. $\qquad\square$

Corollary 4.3. *Fix a weight space* W_{α_i} *of the decomposition (4.1).* W_{α_i} *is a nice space where all its points are critical points of* $\mathrm{F}_{\mathfrak{g}}$ *with* $m_{\mathfrak{g}}(w_i) = \alpha_i$ *for all* $w_i \in W_{\alpha_i}$.

Proof. We must prove that $m_{\mathfrak{g}}(w_i) \in \mathfrak{a}$. By Equation (2.7) and the Decomposition (2.2), this is equivalent to show $\langle \pi(Y)w_i, w_i \rangle = 0$ for all $\gamma \in \Delta(\mathfrak{g})$ and $Y \in \mathfrak{g}_\gamma$.

Let $w = \mathrm{Proj}_W \pi(Y)w_i$, so $\langle \pi(Y)w_i, w_i \rangle = \langle w, w_i \rangle$. If $w = 0$, there is nothing to prove. In other case, by the previous lemma, w is a weight vector of weight $\gamma + \alpha_i$. Since $\gamma \neq 0$, $\gamma + \alpha_i \neq \alpha_i$ which gives $w \perp w_i$.

From Expression (3.1), we have $m_{\mathfrak{g}}(w_i) = \alpha_i$ and so $\pi(m_{\mathfrak{g}}(w_i))w_i = \pi(\alpha_i)w_i = \langle\!\langle \alpha_i, \alpha_i \rangle\!\rangle w_i$; w_i is a critical point of $F_{\mathfrak{g}}$ $\qquad\qquad$ □

Lemma 4.4. *Suppose that for all $\alpha_i, \alpha_j \in \Delta(W)$ such that $\alpha_j - \alpha_i \in \Delta(\mathfrak{g})$ one has that $\pi(\mathfrak{g}_\gamma)W_{\alpha_i} \perp W$, where $\gamma = \alpha_j - \alpha_i$. Then,*

$$\pi(\mathfrak{g}_\gamma)W_{\alpha_i} \perp W,$$

for any $\gamma \in \Delta(\mathfrak{g})$, and $\alpha_i \in \Delta(W)$.

Proof. Suppose the assertion of the lemma is false. From Lemma 4.2 we could find $\gamma \in \Delta(\mathfrak{g})$, $Y \in \mathfrak{g}_\gamma$, $\alpha_i, \alpha_j \in \Delta(W)$ and $w_i \in W_{\alpha_i}$ such that $\mathrm{Proj}_W \pi(Y)w_i$ is a weight vector of weight $\gamma + \alpha_i = \alpha_j$. This gives $\alpha_j - \alpha_i \in \Delta(\mathfrak{g})$ and the hypothesis now becomes $\mathrm{Proj}_W \pi(Y)w_i = 0$; this is a contradiction. $\qquad\qquad$ □

Theorem 4.5. *An A-invariant subspace of W of V is nice if and only if for all $\alpha_i, \alpha_j \in \Delta(W)$ such that $\alpha_j - \alpha_i \in \Delta(\mathfrak{g})$,*

$$\pi(\mathfrak{g}_\gamma)W_{\alpha_i} \perp W$$

where $\gamma = \alpha_j - \alpha_i$

Proof. We suppose that W is nice and let $\alpha_i, \alpha_j \in \Delta(W)$ be such that $\alpha_j - \alpha_i \in \Delta(\mathfrak{g})$, say $\alpha_j - \alpha_i = \gamma$ with $\gamma \in \Delta(\mathfrak{g})$. We must prove that $\mathrm{Proj}_W \pi(Y)w_i = 0$ with $Y \in \mathfrak{g}_\gamma$ and $w_i \in W_{\alpha_i}$. Since $\mathrm{Proj}_W \pi(Y)w_i \in W_j$ (from Lemma 4.2), it will thus be sufficient to prove that

$$\langle \mathrm{Proj}_W \pi(Y)w_i, w_j \rangle = 0$$

for any $w_j \in W_j$. Consider a $w_j \in W_j$ and set $w = w_i + w_j$. Since W is nice, $m_{\mathfrak{g}}(w) \in \mathfrak{a}$ and using the Decomposition (2.2) we have

$$
\begin{aligned}
0 &= \langle m_{\mathfrak{g}}(w), Y \rangle \\
&= \langle \pi(Y)(w_i + w_j), w_i + w_j \rangle \\
&= \langle \pi(Y)w_i, w_i \rangle + \langle \pi(Y)w_i, w_j \rangle \\
&\quad + \langle \pi(Y)w_j, w_i \rangle + \langle \pi(Y)w_j, w_j \rangle \\
&= \langle \pi(Y)w_i, w_j \rangle + \langle \pi(Y)w_j, w_i \rangle
\end{aligned}
$$

(4.2)

where we have used that $\langle \pi(Y)w_i, w_i \rangle = \langle \pi(Y)w_j, w_j \rangle = 0$ from Corollary 4.3.

Note that $\langle \pi(Y)w_j, w_i \rangle = 0$. Indeed, suppose, contrary to our claim, that $\langle \pi(Y)w_j, w_i \rangle \neq 0$. As in the proof of the Lemma 4.4, we have $\mathrm{Proj}_W \pi(Y)w_j$ is a weight vector of weight $\gamma + \alpha_j = \alpha_i$. But, $\gamma + \alpha_i = \alpha_j$ since $\alpha_j - \alpha_i = \gamma$, we now have a contradiction to that $\gamma \neq 0$.

According to the above, we have $0 = \langle \pi(Y)w_i, w_j \rangle = \langle \mathrm{Proj}_W \pi(Y)w_i, w_j \rangle$

We now proceed to show necessary part. Let $w \in W$, say $w = \sum w_i$ and let $Y \in \mathfrak{g}_\gamma$ with $\gamma \in \Delta(\mathfrak{g})$. We must see that $\langle m_\mathfrak{g}(w), Y \rangle = 0$

$$\begin{aligned}
\langle m_\mathfrak{g}(w), Y \rangle &= \frac{1}{\|w\|^2} \langle \pi(Y)w, w \rangle \\
&= \frac{1}{\|w\|^2} \sum_{i,j} \langle \pi(Y)w_i, w_j \rangle.
\end{aligned}$$

From Lemma 4.4, we have the proof. $\qquad\qquad\qquad\qquad\qquad\square$

By the θ-invariance of $\langle\!\langle \cdot, \cdot \rangle\!\rangle$, the fact that $\theta(\mathfrak{g}_\gamma) = \mathfrak{g}_{-\gamma}$ for each $\gamma \in \Delta(\mathfrak{g})$ and recalling that the image of the moment map is contained in $\mathfrak{p}$, we have:

Corollary 4.6. *An* A-*invariant subspace W of V is nice if and only if for all $\gamma \in \Delta(\mathfrak{g})^+$*

$$\pi(\mathfrak{g}_\gamma)W \perp W$$

An obvious but very useful corollary is the following

Corollary 4.7. *Let W be a* A-*invariant subspace of V. If for all α_i and α_j in $\Delta(W)$, $\alpha_i - \alpha_j \notin \Delta(\mathfrak{g})$, then W is nice.*

We can now prove the equivalence between the two possible notions of nice element.

Proposition 4.8. *Let $w = w_{i_1} + \ldots + w_{i_s}$ in V with each w_{i_j} a non-null vector in $V_{\alpha_{i_j}}$ (by Equation (2.3)) and such that $m_\mathfrak{g}(A \cdot w) \subset \mathfrak{a}$. Then $W = \mathbb{R}w_{i_1} \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathbb{R}w_{i_s}$ is a nice space.*

Proof. From Corollary 4.6, we need to prove that $\langle \pi(\mathfrak{g}_\lambda)w_{i_j}, w_{i_k} \rangle = 0$, for all $\lambda \in \Delta(\mathfrak{g})$ and $1 \leq j \leq k \leq s$. Suppose the proposition was false. Then we could find $\lambda \in \Delta(\mathfrak{g})$, $Y \in \mathfrak{g}_\lambda$ and j and k such that

$$\langle \pi(Y)w_{i_j}, w_{i_k} \rangle \neq 0.$$

Reasoning as in the above results, we have $\lambda + \alpha_{i_j} = \alpha_{i_k}$, and so, the set

$$\Omega = \{\alpha_{i_j} : \lambda + \alpha_{i_j} = \alpha_{i_k} \in \Delta(W) \text{ and } \langle \pi(Y)w_{i_j}, w_{i_k} \rangle \neq 0\}$$

is nonempty. As $\langle\!\langle m_\mathfrak{g}(A \cdot w), Y \rangle\!\rangle = 0$, we have for all $X \in \mathfrak{a}$

$$\begin{aligned}
0 &= \langle \pi(Y)\exp(X)w, \exp(X)w \rangle \\
&= \sum_{j,k} e^{\langle\!\langle X, \alpha_{i_j} + \alpha_{i_k} \rangle\!\rangle} \langle \pi(Y)w_{i_j}, w_{i_k} \rangle \\
&= \sum_{\alpha_{i_j} \in \Omega} e^{\langle\!\langle X, \lambda + 2\alpha_{i_j} \rangle\!\rangle} \langle \pi(Y)w_{i_j}, w_{i_k} \rangle \\
(4.3) \qquad &= \sum_{\alpha_{i_j} \in \Omega} e^{\langle\!\langle X, 2\alpha_{i_j} \rangle\!\rangle} \langle \pi(Y)w_{i_j}, w_{i_k} \rangle
\end{aligned}$$

We consider the convex hull of Ω and let α_{i_m} be an exposed point of such convex set. So, there exist $H \in \mathfrak{a}$ and $h \in \mathbb{R}$ such that $\langle\!\langle X, H \rangle\!\rangle \geq h$ for all $X \in \mathrm{CH}(\Omega)$ and the equality holds if and only if $X = \alpha_{i_m}$.

From Equation 4.3 follows that

$$0 \;=\; \frac{1}{\mathrm{e}^{\langle\!\langle tH, 2\alpha_{i_m}\rangle\!\rangle}} \sum_{\alpha_{i_j}\in\Omega} \mathrm{e}^{\langle\!\langle tH, 2\alpha_{i_j}\rangle\!\rangle} \langle \pi(Y)w_{i_j}, w_{i_k}\rangle,$$

and by letting $t \to -\infty$, we obtain

$$0 \;=\; \langle \pi(Y)w_{i_m}, w_{i_n}\rangle$$

which is contrary to $\alpha_{i_m} \in \Omega$. $\qquad\square$

Corollary 4.9. *Let α_i and α_j be such that $W = W_{\alpha_i}\overset{\perp}{\oplus}W_{\alpha_j}$ is a nice space. Then $\beta := \mathrm{mcc}(\{\alpha_i, \alpha_j\})$ defines a stratum, i.e. $\mathscr{S}_\beta \neq \phi$.*

Proof. If $\mathrm{mcc}(\{\alpha_i, \alpha_j\})$ is α_i or α_j, we have from Corollary 4.3 the conclusion for this case. In the other case, $\mathrm{mcc}(\{\alpha_i, \alpha_j\})$ is in the interior of $\mathrm{CH}\{\alpha_i, \alpha_j\}$ and so, Proposition 3.7 and Theorem 2.4 complete the proof. $\qquad\square$

5. Applications

5.1. **Ternary forms.** In this part, we want to discuss some applications of the above results to *classical invariant theory*. In classical invariant theory one studies polynomials and their intrinsic and geometrical properties; we mean those properties which are unaffected by a change of variables and are purely geometric (for instance, multiplicities of roots). Such theory was a focal and major topic in the 19th century and in the beginning of the 20th century, and was strongly impacted by Hilbert's contributions (which were used by David Mumford to develop the modern geometric invariant theory).

We prove a result that describes the stratifying set for the natural action of $\mathrm{GL}_3(\mathbb{R})$ on ternary forms in a very simple way (see Theorem 5.5) and later we give a classification of distinguished orbits in the null cone of real ternary quartics (the results can easily be extended to the complex case).

Along this section, let $\mathrm{G} := \mathrm{GL}_n(\mathbb{R})$, $\mathfrak{g} := \mathrm{Lie}(G) = \mathfrak{gl}_n(\mathbb{R})$ and we consider the Cartan decomposition of $\mathfrak{g}$ given by $\mathfrak{so}(n)\overset{\perp}{\oplus}\mathrm{sym}(n)$ ($\mathfrak{k} := \mathfrak{so}(n)$ and $\mathfrak{p} := \mathrm{sym}(n)$). Let us denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the usual inner product on $\mathfrak{g}$

$$\langle\!\langle X, Y\rangle\!\rangle = \mathrm{tr}(XY^{\mathrm{T}}), \; \forall X, Y \in \mathfrak{g}.$$

Let $\mathfrak{a} = \{(a_1 \ldots a_n) := \mathrm{Diag}(a_1 \ldots a_n) : a_i \in \mathbb{R}\}$ and let $\mathfrak{a}^+$ denote the usual Weyl chamber of $\mathfrak{gl}_n(\mathbb{R})$

$$\mathfrak{a}^+ = \{(a_1, \ldots, a_n) \in \mathfrak{a} : a_1 < \ldots < a_n\}.$$

Thus, a restricted-root space decomposition for $\mathfrak{g}$ is given for

$$\mathfrak{g} = \mathfrak{a}\overset{\perp}{\oplus}\overset{\perp}{\bigoplus_{i\neq j}}\mathbb{R}E_{ij}$$

with $\Delta(\mathfrak{g}) = \{\gamma_{ij} := E_{ii} - E_{jj} : i \neq j\}$ ($\{E_{ij}\}_{1\leq i,j\leq n}$ is the canonical basis of $\mathfrak{gl}_n(\mathbb{R})$).

Let $V := \mathbb{R}[x_1, \ldots, x_n]_d$, the vector space of all homogeneous polynomials of degree d on n variables (*n-ary forms of degree d*) and we set $\langle \cdot, \cdot \rangle$ the

inner product such that monomials of V are orthogonal and $\|x_1^{d_1} \ldots x_n^{d_n}\|^2 = d_1! \ldots d_n!$ ($\sum d_i = d$). $\langle \cdot, \cdot \rangle$ is easily seen to have the required conditions.

The action of G on V is given by linear change of variables,

$$g \cdot p(x_1, \ldots, x_n) = p\left(g^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right),$$

for all $g \in$ G and $p \in V$. It follows easily that

$$\pi(E_{ij})p = \tfrac{\mathrm{d}}{\mathrm{d}t}|_0 p(\mathrm{e}^{-tE_{ij}} \bullet) = -x_j \tfrac{\partial p}{\partial x_i},$$

for all $p \in V$. Hence, we obtain that the moment map to the action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}[x_1, \ldots, x_n]_d$ is given by

$$m_{\mathfrak{gl}_n(\mathbb{R})}(p) = \frac{-1}{\|p\|^2}\left(\langle x_j \frac{\partial p}{\partial x_i}, p \rangle\right), \ \forall p \in \mathbb{R}[x_1, \ldots, x_n]_d \smallsetminus \{0\}.$$

The basis of weight vectors of V are monomials and we have

$$\pi(Y)x_1^{d_1} \ldots x_n^{d_n} = -\left(\sum_{i=1}^n y_i d_i\right)x_1^{d_1} \ldots x_n^{d_n}$$

for all $Y = (y_1, \ldots, y_n) \in \mathfrak{a}$. Consequently, $\Delta(V) = \{\alpha = -(d_1, \ldots, d_n) \in \mathfrak{a} : \sum d_i = d\}$. In the sequel, given $\alpha \in \Delta(V)$, say $\alpha = -(d_1, \ldots, d_n)$ with $\sum d_i = d$, we will denote by $\mathbf{x}^\alpha$ the monomial $x_1^{d_1}, \ldots, x_n^{d_n}$.

Remark 5.1. Since the identity matrix, Id, which is in $\mathfrak{a}$, acts as $\pi(\mathrm{Id})p = -p$ for any $p \in V$, from Theorem 2.4 (1), it follows that critical points of $\mathrm{F}_{\mathfrak{gl}(n,\mathbb{R})}$ and $F_{\mathfrak{sl}(n,\mathbb{R})}$ are the same. Moreover, the respective stratifications coincide up to translation in $\mathscr{B}$ sets.

We now consider the case of ternary forms (i.e. $n = 3$). Binary forms (i.e. $n = 2$) were studied by Ness in [Nes, Lemma 10.4], where she gives a description of non-minimal critical points of $\mathrm{F}_\mathfrak{g}$.

First observe that all weights are in the equilateral triangle with vertices $-(d, 0, 0)$, $-(0, d, 0)$ and $-(0, 0, d)$. It follows that if $\Phi \subseteq \Delta(V)$, the $\mathrm{CH}(\Phi)$ is contained in such triangle. In the sequel, β_0 will denote the barycenter of triangle; $\beta_0 := -(\frac{d}{3}, \frac{d}{3}, \frac{d}{3})$. Thus, we have that $\beta = \mathrm{mcc}(\Phi)$ if and only if β is the unique vector in $\mathrm{CH}(\Phi)$ closest to β_0. Accordingly, $\mathrm{mcc}(\Phi)$ is either equal to β_0, or there are $\alpha_i, \alpha_j \in \Phi$ such that $\mathrm{mcc}(\Phi) = \mathrm{mcc}(\{\alpha_i, \alpha_j\})$. We can also realize the barycenter as the minimal convex combination of two weights as follows:

$$\beta_0 = \mathrm{mcc}(\{\alpha_i, \alpha_j\}) \text{ with } \begin{cases} \alpha_i = -(k, k, 0) \text{ and } \alpha_j = -(0, 0, 2k) \text{ if } d = 2k, \\ \alpha_i = -(k, k, 1) \text{ and } \alpha_j = -(0, 0, 2k+1) \text{ if } d = 2k+1, \end{cases}$$

with the property that $W = \mathbb{R}\mathbf{x}^{\alpha_i} \overset{\perp}{\oplus} \mathbb{R}\mathbf{x}^{\alpha_j}$ is a nice space. We will see that this holds for every β such that $\mathcal{S}_\beta \neq \phi$.

Lemma 5.2. *Let α_i, α_j in $\Delta(V)$ and $W = \mathrm{span}_\mathbb{R}\{\mathbf{x}^{\alpha_i}, \mathbf{x}^{\alpha_j}\}$. Then W is a nice space if and only if $\alpha_j - \alpha_i \notin \Delta(\mathfrak{g})$.*

Proof. From Corollary 4.7, it remains to prove the if part. Let W be a nice space and contrary to our claim, that $\alpha_j - \alpha_i \in \Delta(\mathfrak{g})$. Let $\alpha_i = -(m_1, m_2, m_3)$ and $\alpha_j = -(n_1, n_2, n_3)$ and without loss of generality we can assume $\alpha_j - \alpha_i = \gamma_{12} = E_{11} - E_{22} = (1, -1, 0)$. So, $m_1 = 1 + n_1$, $m_2 = n_2 - 1$ and $m_3 = n_3$. Let $Y = E_{12}$, $Y \in \mathfrak{g}_{\gamma_{12}}$

$$
\begin{aligned}
\pi(Y)\mathbf{x}^{\alpha_i} &= \pi(E_{12})x^{1+n_1}y^{n_2-1}z^{n_3} \\
&= (1+n_1)x^{n_1}y^{n_2}z^{n_3} \\
&= (1+n_1)\mathbf{x}^{\alpha_j}
\end{aligned}
$$

Therefore, $\mathrm{Proj}_W \pi(Y)\mathbf{x}^{\alpha_j} \neq 0$, which contradicts the Theorem 4.5. $\qquad\square$

Lemma 5.3. *Let $\beta \in \mathfrak{a}$ such that $\mathscr{S}_\beta \neq \phi$. Then there exist α_i and α_j in $\Delta(V)$ such that $W = \mathrm{span}_\mathbb{R}\{\mathbf{x}^{\alpha_i}, \mathbf{x}^{\alpha_j}\}$ is a nice space and $\beta = \mathrm{mcc}(\{\alpha_i, \alpha_j\})$. In particular, there exists $q \in W$ such that q is a critical point of $\mathrm{F}_\mathfrak{g}$ and $q \in \mathscr{S}_\beta$*

Proof. According to the above remark, there exist $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_j$ in $\Delta(V)$ such that $\beta = \mathrm{mcc}(\{\widetilde{\alpha}_i, \widetilde{\alpha}_j\})$. If $\widetilde{\alpha}_i$ and $\widetilde{\alpha}_j$ satisfy the conclusion of the lemma, then we are done. If they do not, then from the previous lemma, $\widetilde{\alpha}_j - \widetilde{\alpha}_i \in \Delta(\mathfrak{g})$, say $\widetilde{\alpha}_j - \widetilde{\alpha}_i = (1, -1, 0)$; this involves no loss of generality. As in the previous lemma, $\widetilde{\alpha}_i = -(1+n_1, n_2-1, n_3)$ and $\widetilde{\alpha}_j = -(n_1, n_2, n_3)$. It is easy to check that $\alpha_i = -(0, n_1+n_2, n_3)$ and $\alpha_j = -(n_1+n_2, 0, n_3)$ are in $\mathrm{Aff}(\{\widetilde{\alpha}_i, \widetilde{\alpha}_j\})$, and $\beta = \mathrm{mcc}(\{\alpha_i, \alpha_j\})$. In this case, the pair α_i and α_j proves the existence part, because we must have $n_1 + n_2 > 1$; on the contrary, $\beta = -(\frac{1}{2}, \frac{1}{2}, n_3)$ and $\mathscr{S}_\beta = \phi$. $\qquad\square$

Lemma 5.4. *If $\beta \in \{-(\frac{1}{2}, \frac{1}{2}, d-1), -(\frac{1}{2}, d-1, \frac{1}{2}), -(d-1, \frac{1}{2}, \frac{1}{2})\}$, then $\mathscr{S}_\beta$ is not a stratum.*

Proof. It suffices for our purposes to prove that $\mathscr{S}_\beta$ has not critical points of $\mathrm{F}_\mathfrak{g}$. From Theorem 2.4 (1), we can prove that there exists no $q \in V$ such that $m_\mathfrak{g}(q) = \beta$ and $\pi(\beta)q = \|\beta\|^2 q$. If $q = \sum_{\alpha \in \Delta(V)} a_\alpha \mathbf{x}^\alpha$, $\pi(\beta)q = \|\beta\|^2 q$ if and only if $q \in \mathbb{R}\mathbf{x}^{\alpha_{i_1}} \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathbb{R}\mathbf{x}^{\alpha_{i_s}}$ with $\langle\!\langle \alpha_{i_j}, \beta \rangle\!\rangle = \|\beta\|^2$; i.e. α_{i_j}'s are in the affine space

$$
\Omega = \{X \in \mathfrak{a} : \langle\!\langle X, \beta \rangle\!\rangle = \|\beta\|^2\}.
$$

Ω meets the equilateral triangle (that contains all weights) at 1-dimensional affine space, which has only two weights of $\Delta(V)$; namely $\alpha_i = (1, 0, d-1)$ and $\alpha_j = (0, 1, d-1)$. Thus, one such q is of the form $axz^{d-1} + byz^{d-1}$. It is easily seen that q is in the orbit of $p = cxz^{d-1}$ and since $p \in \mathscr{S}_{\beta'}$ with $\beta' = -(1, 0, d-1)$, it follows that $\mathscr{S}_\beta = \phi$ by definition of stratum. $\qquad\square$

By Corollary 4.9 and Lemma 5.3 we deduce the main result of this section.

Theorem 5.5. *Let $\mathscr{B}$ be as in the Theorem 2.4 for the natural action of $\mathrm{GL}_3(\mathbb{R})$ on $\mathbb{R}[x, y, z]_d$. Then*

$$(5.1) \qquad \mathscr{B} = \{\beta = \mathrm{mcc}(\{\alpha_i, \alpha_j\}) : \alpha_i, \alpha_j \in \Delta(V) \text{ with } \alpha_j - \alpha_i \notin \Delta(\mathfrak{g})\} \cap \overline{\mathfrak{a}^+}$$
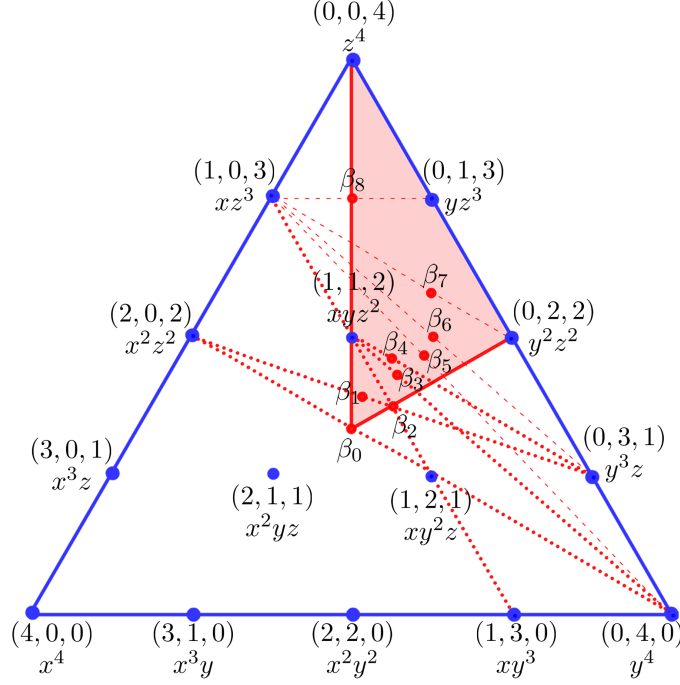
FIGURE 1. Equilateral triangle, $n = 3$ and $d = 4$

Although the Theorem 5.5 makes very easy to find the set $\mathscr{B}$, the problem of finding the critical points of $F_{\mathfrak{g}}$ in each stratum is a very difficult task. For instance, the critical points of $F_{\mathfrak{g}}$ in $\mathscr{S}_{\beta_0}$ (recall that $\beta_0 = -(\frac{d}{3}, \frac{d}{3}, \frac{d}{3})$) bring us to the classification of closed $\mathrm{SL}_3(\mathbb{R})$-orbits in V (which is a very well-known open problem). Now, we explore such problem in ternary quartics ($n = 3$ and $d = 4$, Figure 1).

By using the Theorem 5.5, we can find the set $\mathscr{B}$ in the following straightforward way. For each $\beta \in \mathscr{B}$, we calculate the vector subspace of V, Z_β given by

$$Z_\beta := \left\{ p = \sum_{\alpha \in \Delta(V)} a_\alpha \mathbf{x}^\alpha : \langle\!\langle \alpha, \beta \rangle\!\rangle = \|\beta\|^2 \text{ with } a_\alpha \neq 0 \right\}.$$

This subspace is associated to the affine space

$$\Omega(\beta) = \{ X \in \mathfrak{a} : \langle\!\langle X, \beta \rangle\!\rangle = \|\beta\|^2 \}$$

and we must find the weights in the intersection of $\Omega(\beta)$ with the equilateral triangle, which is a k-dimensional convex set with $0 \leq k \leq 2$. Finally, we find

(5.2) $\{ p \in Z_\beta : m_{\mathfrak{g}}(p) = \beta \text{ with } \|p\| = 1 \}$,

which gives the set $\mathscr{C}(\beta)$ (as in Theorem 2.4 (2)).

The case β_0, as we said above, the critical points in $\mathscr{S}_{\beta_0}$ correspond to $\mathrm{SL}_3(\mathbb{R})$-closed orbits and are also called minimal (they are global minima

of $F_\mathfrak{g}$). In Table 1, we give the strata of ternary quartic forms. The third column in the table shows the form that a critical point should take according the β given in the type column. For $\beta_2 = \left(1, \frac{3}{2}, \frac{3}{2}\right)$, a critical point must be of the form $p = x(az^3 + byz^2 + cy^2z + dy^3)$. In this case, it is easy to see that p is a critical point if and only if the $\mathrm{SL}_2(\mathbb{R})$-orbit of the binary form $\frac{p}{x} = az^3 + byz^2 + cy^2z + dy^3$ is closed (for instance, by comparing $m_{\mathfrak{sl}_3(\mathbb{R})}(p)$ with β_2). The same holds for critical points of type $(1, 1, 2)$ and $(0, 2, 2)$. In the first case, a critical point must be of the form $q = z^2(ax^2 + bxy + cy^2)$ and so the binary form $\frac{q}{z^2} = ax^2 + bxy + cy^2$ must has a $\mathrm{SL}_2(\mathbb{R})$-closed orbit. In the last case, we have a binary quartic form, as in the mentioned cases above, must has a $\mathrm{SL}_2(\mathbb{R})$-closed orbit. To study such cases, we use the well-known canonical forms for binary forms of low degree (see, [Gur, Chapter V: 22.4, 23.2, §25-Exercises 13 and 14] or [Olv, Pag. 9, Pag. 28, Pag.30-Exercise 2.25]); by using such classification, it is fairly easy to see those with $\mathrm{SL}_2(\mathbb{R})$-closed orbit. For the convenience of the reader, we recall in the appendix (Tables 3, 4 and 5), the canonical forms for the real or complex binary forms of degree 4,3 y 2 respectively.

For the remaining cases, however, it is very simple to find all distinguished orbits by explicit calculation of the set in Equation (5.2).

Theorem 5.6. *The classification of distinguished orbits in the null cone of* $\mathbb{R}[x, y, z]_4$ *for the natural action of* $\mathrm{SL}_3(\mathbb{R})$ *is given in Table 1.*

Remark 5.7. Although we did not classify closed $\mathrm{SL}_3(\mathbb{R})$-orbits, our approach provides different families of closed $\mathrm{SL}_3(\mathbb{R})$-orbits. For instance, we consider a maximal family Φ of weights which are not *neighbors*, (by neighbors we mean those pairs of weights that have a rest in $\Delta(\mathfrak{gl}_n(\mathbb{R}))$) and such that $\beta_0 \in \mathrm{CH}(\Phi)$. From Corollary 4.7 the generated space by Φ is nice and a polynomial p such that $\mathfrak{R}(p) = \Phi$ must have closed $\mathrm{SL}_3(\mathbb{R})$-orbit (see Proposition 3.7). One can prove that the family $p_{a,\ldots,f}(x, y, z) = ax^4 + by^4 + cz^4 + dx^2y^2 + ex^2z^2 + fy^2z^2$ contains three-parametric families of closed $\mathrm{SL}_3(\mathbb{R})$-orbits.

5.2. **Canonical compatible metrics for geometric structures on nilmanifolds.** In [Lau1], Jorge Lauret noted that there is an intriguing relationship between the geometry of nilpotent Lie groups and the geometric invariant theory (GIT) applied to actions of reductive subgroups of $\mathrm{GL}_m(\mathbb{R})$ on $\Lambda^2(\mathbb{R}^m)^* \otimes \mathbb{R}^m$. For instance, the recent advances in the study of *Einstein solvmanifolds* and, more generally, *Solvsolitons*, have come from using powerful tools that are given by GIT (see [Lau2, Nik2]). By using this fact, it has been proposed in [Lau1] a way to study the problem of finding "the best metric" which is compatible with a fixed geometric structure on a simply connected nilpotent Lie group. In this approach, it is defined the notion of *minimal compatible metric* and the properties that make a minimal metric "special" are proved; a minimal metric is unique (up to isometry and scaling) when it exists and it can be characterized as a *soliton solution* of the

Nt.	Type	Critical point	
β_0	$(\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$	$\displaystyle\sum_{\alpha\in\Delta(V)} a_\alpha \mathbf{x}^\alpha$	minimal
β_1	$(\frac{8}{7}, \frac{9}{7}, \frac{11}{7})$	$ax^2z^2 + by^3z$	$a = \pm\sqrt{\frac{1}{7}},\ b = \sqrt{\frac{1}{14}}$
β_2	$(1, \frac{3}{2}, \frac{3}{2})$	$x\left(az^3 + bz^2y + czy^2 + dy^3\right)$	$x[y^3 + yz^2]$ $x[y^3 - yz^2]$
β_3	$(\frac{6}{7}, \frac{10}{7}, \frac{12}{7})$	$axyz^2 + by^4$	$a = \sqrt{\frac{3}{7}},\ b = \pm\sqrt{\frac{1}{168}}$
β_4	$(\frac{5}{6}, \frac{8}{6}, \frac{11}{6})$	$axyz^2 + by^3z$	$a = \sqrt{\frac{5}{12}},\ b = \sqrt{\frac{1}{36}}$
α_{xyz^2}	$(1, 1, 2)$	$z^2\left(ax^2 + bxy + cy^2\right)$	$z^2[x^2 + y^2]$ $-z^2[x^2 + y^2]$ $z^2[x^2 - y^2]$
β_5	$(\frac{8}{13}, \frac{20}{13}, \frac{24}{13})$	$axz^3 + by^4$	$a = \sqrt{\frac{4}{39}},\ b = \pm\sqrt{\frac{5}{312}}$
β_6	$(\frac{1}{2}, \frac{3}{2}, 2)$	$axz^3 + by^3z$	$a = \sqrt{\frac{1}{12}},\ b = \sqrt{\frac{1}{12}}$
β_7	$(\frac{1}{3}, \frac{4}{3}, \frac{7}{3})$	$axz^3 + by^2z^2$	$a = \sqrt{\frac{1}{18}},\ b = \pm\sqrt{\frac{1}{6}}$
$\alpha_{y^2z^2}$	$(0, 2, 2)$	$\sum_{i=0}^{4} a_i y^{4-i} z^i$	$y^4 + 2ty^2z^2 + z^4\ (t\in\mathbb{R})$ $-[y^4 + 2ty^2z^2 + z^4]\ (-1\le t)$ $y^4 + 2ty^2z^2 - z^4\ (t\in\mathbb{R})$
β_8	$(\frac{1}{2}, \frac{1}{2}, 3)$	$axz^3 + byz^3$	`empty`
α_{yz^3}	$(0, 1, 3)$	ayz^3	$a = \sqrt{\frac{1}{6}}$
α_{z^4}	$(0, 0, 4)$	az^4	$a = \pm\sqrt{\frac{1}{24}}$

TABLE 1. Classification of distinguished orbits in the null cone of $\mathbb{R}[x,y,z]_4$ for the natural action of $\mathrm{GL}_3(\mathbb{R})$.

invariant Ricci flow (see [Lau1, Theorem 4.4]). A priori, such properties are far from obvious, however, such facts follow of the last-mentioned connection. We refer the interested reader to [Lau1] for a complete discussion on this approach.

It is our purpose in this section to study the natural action of $\mathrm{GL}_n(\mathbb{R})$ and its reductive subgroups on $V := \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ given by change of basis:

$$g \cdot \mu(X, Y) = g\mu(g^{-1}X, g^{-1}Y),\ X, Y \in \mathbb{R}^n,\ g \in \mathrm{GL}_n(\mathbb{R}),\ \mu \in V.$$

The corresponding representation of $\mathfrak{gl}_n(\mathbb{R})$ on V is given by

$$A \cdot \mu(X, Y) = A\mu(X, Y) - \mu(AX, Y) - \mu(X, AY),\ A \in \mathfrak{gl}_n(\mathbb{R})\ \mu \in V,$$

The inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathfrak{gl}_n(\mathbb{R})$ and a Cartan decomposition of $\mathrm{GL}_n(\mathbb{R})$ is given in the above application and the inner product $\langle \cdot, \cdot \rangle$ on V is the induced inner product by the canonical inner product of $\mathbb{R}^n$ (denoted also by $\langle \cdot, \cdot \rangle$) as follows:

$$\langle \mu, \lambda \rangle = \sum_{ijk}\langle \mu(e_i, e_j), e_k \rangle \langle \lambda(e_i, e_j), e_k \rangle,\ \forall \mu, \lambda \in V.$$

An easy computation shows that the inner product $\langle \cdot, \cdot \rangle$ on V satisfies the required conditions.

Let (N_μ, γ) be a *class-γ nilpotent Lie group*, i.e. N_μ is a simply connected nilpotent Lie group with Lie algebra $\mathfrak{n} = (\mathbb{R}^n, \mu)$ and γ is an invariant geometric structure on N satisfying [Lau1, Defintion 2.1]. The main result of this section is a result of existence of minimal compatible metrics for a very wide family of class-γ nilpotent Lie groups (see Theorem 5.14). Let us recall basic definitions and results of [Lau1] and then to give the mentioned application.

Definition 5.8. [Lau1, Definition 2.2] Let $(\cdot, \cdot)$ be a *compatible metric* with a class-γ nilpotent Lie group (N_μ, γ) (as in the [Lau1, Definition 2.1 (iii)]). We consider the orthogonal projection $Ric^\gamma_{(\cdot, \cdot)}$ of the Ricci operator $Ric_{(\cdot, \cdot)}$ on $\mathfrak{g}_\gamma = \text{Lie}(G_\gamma)$ relative to the inner product $\langle\langle \cdot, \cdot \rangle\rangle$. $Ric^\gamma_{(\cdot, \cdot)}$ is said to be *invariant Ricci operator*, and the corresponding *invariant Ricci tensor* is given by $\text{ric}^\gamma = \langle Ric^\gamma \cdot, \cdot \rangle$.

Definition 5.9 (Minimal compatible metric). [Lau1, Definition 2.3] A left invariant metric $(\cdot, \cdot)$ compatible with a class-γ nilpotent Lie group (N_μ, γ) is called *minimal* if

$$\|Ric^\gamma_{(\cdot, \cdot)}\|^2 = \min \left\{ \|Ric^\gamma_{((\cdot, \cdot))}\|^2 : \begin{array}{l} ((\cdot, \cdot)) \text{ is a compatible metric with } (N_\mu, \gamma) \\ \text{and } \text{sc}(((\cdot, \cdot))) = \text{sc}((\cdot, \cdot)) \end{array} \right\}$$

Without loss of generality, from now on we can assume that the canonical inner product of $\mathbb{R}^n$ (which we denoted also by $\langle \cdot, \cdot \rangle$) is compatible with γ and that G_γ is self-adjoint with respect to this one. Therefore G_γ is compatible with the usual Cartan decomposition of $GL_n(\mathbb{R})$: $G_\gamma = K_\gamma \exp(\mathfrak{p}_\gamma)$ where $K_\gamma \subseteq O(n)$ is a maximal compact subgroup of G_γ and $\mathfrak{p}_\gamma \subseteq \text{sym}(n)$ is a vector subspace of symmetric matrices. It follows that $m_{\mathfrak{g}_\gamma}(v)$ is the orthogonal projection of $m_{\mathfrak{g}_\gamma}(v)$ onto $\mathfrak{p}_\gamma$ for all $v \in V \smallsetminus \{0\}$.

Proposition 5.10. [Lau1, Propositon 4.2] *Let (N_μ, γ) be a class-γ nilpotent Lie group. Then*

$$(5.3) \qquad\qquad m_{\mathfrak{gl}_n(\mathbb{R})}(\mu) \;\; = \;\; 4Ric_\mu,$$
$$(5.4) \qquad\qquad m_{\mathfrak{g}_\gamma}(\mu) \;\; = \;\; 4Ric^\gamma_\mu,$$

where Ric_μ is the Ricci operator of the Riemannian manifold $(N_\mu, \langle \cdot, \cdot \rangle)$ and Ric^γ_μ is the invariant Ricci operator of $(N_\mu, \gamma, \langle \cdot, \cdot \rangle)$.

Theorem 5.11. [Lau1, Proposition 4.3 and 4.4] *Let (N_μ, γ) be a class-γ nilpotent Lie group. (N_μ, γ) admits a minimal compatible metric if and only if the G_γ-orbit of μ is distinguished for the natural action of G_γ on V. Moreover, there is at most one minimal compatible metric on (N, γ) up to isometry (and scaling)*

We are now in a position to apply previous results to study the existence of minimal metrics on a special family of nilpotent Lie algebras. We denote

by $\mathfrak{a}_\gamma := \mathfrak{a} \cap \mathfrak{p}_\gamma$ an abelian subalgebra of $\mathfrak{g}_\gamma$ maximal in $\mathfrak{p}_\gamma$ and $A_\gamma := \exp(\mathfrak{a}_\gamma)$. Since we are considering many group actions and their respective moment maps, it is convenient to introduce the following definition.

Definition 5.12. Let G_γ be a reductive subgroup of $GL_n(\mathbb{R})$ compatible with the usual Cartan decomposition of $GL_n(\mathbb{R})$ and let W be an A_γ-invariant vector subspace of $V := \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We call W A_γ-nice if W is nice with respect to the action of G_γ on V.

Notation 5.13. Let us denote by $\mathfrak{R}_\gamma(\mu)$ denote the ordered set of weights related with μ to the action of G_γ on V. It is clear that $\mathfrak{R}_\gamma(\mu) = \mathrm{Proj}_{\mathfrak{a}_\gamma} \mathfrak{R}(\mu)$ where $\mathfrak{R}(\mu)$ are weights related with μ to the action of $GL_n(\mathbb{R})$ on V.

Theorem 5.14. *Let W be a A_γ-nice space and let (N_μ, γ) be a class-γ nilpotent Lie group with $\mu \in W$. (N_μ, γ) admits a compatible minimal metric if and only if the equation*

$$\boldsymbol{U}_\mu^\gamma[x_i] = \lambda[1]$$

has a positive solution $[x_i]$ for some $\lambda \in \mathbb{R}$. Here, $\boldsymbol{U}_\mu^\gamma$ is the Gram matrix of $(\mathfrak{R}_\gamma(\mu), \langle\!\langle \cdot, \cdot \rangle\!\rangle)$

Example 5.15. We consider the 6-dimensional nilpotent Lie algebra $\mathfrak{n} := (\mu, \mathbb{R}^n)$ given by the direct sum of two 3-dimensional Heisenberg Lie algebra, i.e.

$$\mu := \{[e_1, e_4] = e_6, \ [e_2, e_3] = e5$$

$\mathfrak{n}$ admits only two symplectic structures up to symplecto-isomorphism (see, for instance, [KGM, Theorem 5. 24])

$$\omega(\pm) = \pm(e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + e_3^* \wedge e_4^*)$$

We will now prove that $(N_\mu, \omega(\pm))$ admit a compatible minimal metric.

It is easy to see that the canonical metric of $\mathbb{R}^6$, $\langle \cdot, \cdot \rangle$, defines a compatible metric with $\omega(\pm)$ and $G_{\omega(\pm)} = Sp(3, \mathbb{R})$. Any compatible metric with $\omega(\pm)$ is in the $Sp(3, \mathbb{R})$-orbit of $\langle \cdot, \cdot \rangle$, hence we must minimize the function $\|Ric_{\langle \cdot, \cdot \rangle}^\omega\|^2$ over all $Sp(3, \mathbb{R}) \cdot \langle \cdot, \cdot \rangle$, which is equivalent to minimize $\|m_{\mathfrak{sp}(6, \mathbb{R})}\|^2$ over the $Sp(3, \mathbb{R})$-orbit of μ. By Theorem 2.4 item 1, we must show that $Sp(3, \mathbb{R}) \cdot \mu$ is $Sp(3, \mathbb{R})$-distinguished.

Let

$$W = \mathrm{span}_\mathbb{R}\{\mu_{14}^6, \mu_{23}^5\}$$

where μ_{ij}^k is the bracket defined as $\mu_{ij}^k(e_i, e_j) = e_k = -\mu_{ij}^k(e_j, e_i)$ and zero in otherwise. Let us first see that W is a A_ω-nice. A way of making this is to note that W is a nice space and since $\mathrm{Proj}_{\mathfrak{p}_\omega} \mathfrak{a} \subseteq \mathfrak{a}_\omega$, it follows that W is $\mathfrak{a}_\omega$-nice.

Other way of proving the same is using Corollary 4.7. The root set $\Delta(\mathfrak{sp}(6, \mathbb{R}))$ of is given by

$$\left\{ \begin{array}{lll} \pm \mathrm{Diag}(1,0,0,0,0,-1), & \pm\frac{1}{2}\mathrm{Diag}(1,1,0,0,-1,-1), & \pm\frac{1}{2}\mathrm{Diag}(1,-1,0,0,1,-1), \\ \pm \mathrm{Diag}(0,1,0,0,-1,0), & \pm\frac{1}{2}\mathrm{Diag}(1,0,1,-1,0,-1), & \pm\frac{1}{2}\mathrm{Diag}(1,0,-1,1,0,-1), \\ \pm \mathrm{Diag}(0,0,1,-1,0,0), & \pm\frac{1}{2}\mathrm{Diag}(0,1,1,-1,-1,0), & \pm\frac{1}{2}\mathrm{Diag}(0,1,-1,1,-1,0) \end{array} \right\}$$

The Weights of W with respect to the action of $\mathrm{Sp}(3,\mathbb{R})$ are

$$\{\alpha_1 := \mathrm{Diag}(-1, 0, \tfrac{1}{2}, -\tfrac{1}{2}, 0, 1), \alpha_2 := \mathrm{Diag}(0, -1, -\tfrac{1}{2}, \tfrac{1}{2}, 1, 0)\}$$

Since $\alpha_1 - \alpha_2 \notin \Delta(\mathfrak{sp}(3,\mathbb{R}))$, it follows that W is $\mathfrak{a}_\omega$-nice.

In the light of the Theorem 3.7, we need to show that $\mathrm{mcc}(\alpha_1, \alpha_2) \in \mathrm{int}(\mathrm{CH}(\{\alpha_1, \alpha_2\}))$.

$$\boldsymbol{U}_\mu^\gamma = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}$$

Since $X = (\tfrac{1}{2}, \tfrac{1}{2})$ is a positive solution to the problem $\boldsymbol{U}_\mu^\gamma = [1]_2$, it follows that $(\mathrm{N}_\mu, \omega(\pm))$ admit a minimal metric.

To find such metric, we solve the problem

(5.5) $m_{\mathfrak{sp}(3,\mathbb{R})}(\exp(X) \cdot \mu) = \mathrm{mcc}(\{\alpha_1, \alpha_2\}) = \mathrm{Diag}(-\tfrac{1}{2}, -\tfrac{1}{2}, 0, 0, \tfrac{1}{2}, \tfrac{1}{2})$

with $X \in \mathfrak{a}_\omega$; $X = \mathrm{Diag}(a, b, c, -c, -b, -a)$. Let us denote by $\widetilde{\mu}$ to $\exp(X) \cdot \mu$

$$\widetilde{\mu} = \left\{ [e_1, e_4] = \frac{\mathrm{e}^c}{\mathrm{e}^{2a}} e_6, \ [e_2, e_3] = \frac{1}{\mathrm{e}^{2b+c}} e_5 \right.$$

We have $m_{\mathfrak{sp}(3,\mathbb{R})}(\widetilde{\mu})$ is given by

$$\mathrm{Diag}(-2e^{2c-4a}, -2e^{-4b-2c}, e^{2c-4a} - e^{-4b-2c}, e^{-4b-2c} - e^{2c-4a}, 2e^{-4b-2c}, 2e^{2c-4a})$$

Setting $X = (\ln(2), 0, \ln(2), -\ln(2), 0, -\ln(2))$, we can solve the equation (5.5), hence $\widetilde{\mu} = \tfrac{1}{2}\mu$ is a critical point of $\mathrm{F}_{\mathfrak{sp}(3,\mathbb{R})}$ which defines a minimal metric of $(\mathrm{N}_\mu, \omega(\pm))$.

Remark 5.16. The first infinite families of symplectic Lie algebras appear in dimension 6. In a forthcoming paper, we develop analogous results to those in [Nik2] for the symplectic case and these are used to classify minimal metrics on 6-dimensional symplectic Lie algebras.

Proceeding in an entirely analogous way, we can study the remainder symplectic two-step Lie algebras given in [KGM, Theorem 5. 24] (up to 18.(b_t) and 18.(c), which were studied with similar methods used in [Fer1, Example 1.]).

Theorem 5.17. *All symplectic two-step Lie algebras of dimension* 6 *admit a minimal compatible metric.*

Remark 5.18. We must say that we have found several mistakes in the classification given in [KGM]. For instance, 16.(b) does not define a symplectic structure. Some errors have already been corrected by personal communication with authors; as the symplectic structure given in 23.(c).

In the Table 2, each Lie algebra defines a symplectic two-step Lie algebra given by $(\mathbb{R}^6, \mu, \omega_{cn})$ where

$$\omega_{cn} := e_1^* \wedge e_6^* + e_2^* \wedge e_5^* + e_3^* \wedge e_4^*,$$

and is such that the canonical inner product on $\mathbb{R}^6$ defines a minimal metric for every $(\mathbb{R}^6, \mu, \omega_{cn})$. In the column $\|\beta\|^2$ we give the norm squared of the

stratum associated to the minimal metric and in Derivation column, we give the derivation of $(\mathbb{R}^6, \widetilde{\mu})$ such that

$$m_{\mathfrak{sp}_6(\mathbb{R})}(\widetilde{\mu}) = -\|\beta\|^2 \operatorname{Id} + \text{Derivation}.$$

In the last column, We give the dimension of automorphism group of the symplectic two-step Lie algebra $(\mathbb{R}^6, \widetilde{\mu}, \omega_{cn})$.

Not.	Critical point	Derivation	$\|\beta\|^2$	dim Aut
16.(a)	$[e_1, e_2] = \frac{\sqrt{2}}{4} e_3,\ [e_1, e_5] = \frac{\sqrt{2}}{4} e_6,$ $[e_2, e_4] = \frac{\sqrt{2}}{4} e_6,\ [e_4, e_5] = \frac{\sqrt{2}}{4} e_3$	$\frac{1}{2} \operatorname{Diag}(1, 2, 3, 1, 2, 3)$	1	6
17.	$[e_1, e_3] = \frac{\sqrt{6}}{6} e_5,\ [e_1, e_4] = \frac{\sqrt{6}}{6} e_6,$ $[e_2, e_3] = \frac{\sqrt{6}}{6} e_6$	$\frac{1}{6} \operatorname{Diag}(3, 5, 6, 8, 9, 11)$	$\frac{7}{6}$	7
18.(a_t)	$[e_1, e_2] = \frac{1}{2} \sqrt{\frac{(t-1)^2}{t^2 - t + 1}} e_4,$ $[e_1, e_3] = \frac{t}{2(t-1)} \sqrt{\frac{(t-1)^2}{t^2 - t + 1}} e_5,$ $[e_2, e_3] = \frac{1}{2(t-1)} \sqrt{\frac{(t-1)^2}{t^2 - t + 1}} e_6$	$\operatorname{Diag}(1, 1, 1, 2, 2, 2)$	$\frac{3}{2}$	8
18.(b_t)	$[e_1, e_2] = \frac{t}{\sqrt{3t^2 + 1}} e_4,$ $[e_1, e_3] = \frac{t}{2\sqrt{3t^2 + 1}} e_5 + \frac{1}{2\sqrt{3t^2 + 1}} e_6,$ $[e_2, e_3] = \frac{1}{2\sqrt{3t^2 + 1}} e_5 - \frac{t}{2\sqrt{3t^2 + 1}} e_6$	$\operatorname{Diag}(1, 1, 1, 2, 2, 2)$	$\frac{3}{2}$	8
18.(c)	$[e_1, e_2] = -\frac{\sqrt{3}}{12} e_4 - \frac{\sqrt{3}}{4} e_5,$ $[e_1, e_3] = \frac{\sqrt{3}}{4} e_4 + \frac{\sqrt{3}}{12} e_5,$ $[e_2, e_3] = \frac{\sqrt{3}}{6} e_6$	$\operatorname{Diag}(1, 1, 1, 2, 2, 2)$	$\frac{3}{2}$	10
23.(a)	$[e_1, e_2] = \frac{1}{2} e_5,\ [e_1, e_3] = \frac{1}{2} e_6$	$\frac{1}{4} \operatorname{Diag}(4, 5, 6, 8, 9, 10)$	$\frac{7}{4}$	9
23.(b)	$[e_1, e_2] = -\frac{1}{2} e_4,\ [e_2, e_3] = \frac{1}{2} e_6$	$\operatorname{Diag}(1, 1, 1, 2, 2, 2)$	$\frac{3}{2}$	8
23.(c)	$[e_1, e_2] = \frac{1}{2} e_5,\ [e_1, e_4] = \frac{1}{2} e_3$	$\operatorname{Diag}(1, 1, 2, 1, 2, 2)$	$\frac{3}{2}$	8
24.(a)	$[e_1, e_4] = \frac{1}{2} e_6,\ [e_2, e_3] = \frac{1}{2} e_5$	$2 \operatorname{Diag}(1, 1, 2, 2, 3, 3)$	1	6
24.(b)	$[e_3, e_6] = -\frac{1}{2} e_1,\ [e_4, e_5] = -\frac{1}{2} e_2$	$\frac{1}{2} \operatorname{Diag}(3, 3, 2, 2, 1, 1)$	1	6
25.	$[e_1, e_2] = \frac{\sqrt{2}}{2} e_6$	$\frac{1}{2} \operatorname{Diag}(3, 4, 5, 5, 6, 7)$	$\frac{5}{2}$	12

TABLE 2. Classification of minimal compatible metrics on symplectic two-step Lie algebras of dimension 6

Remark 5.19. The above minimal metrics define *soliton solutions* to the *Street-Tian symplectic curvature flow*, which was recently introduced in [ST]. To be more precise, following Vezzoni [Vez], in [Poo] has been noted that, in two-step nilpotent Lie groups, the *symplectic curvature flow* reduces to the *anti-complexified Ricci flow*, which was given by Hông Vân Lê and Guofang Wang in [LeW]. Given that the invariant Ricci tensor $\operatorname{ric}^\omega_{\langle \cdot, \cdot \rangle}$ coincides with

the *anti-complexified Ricci tensor* $\mathrm{ric}^{ac}_{\langle\cdot,\cdot\rangle}$ (see [Lau1, Equation (23)]), the affirmation follows from [Lau1, Proposition 2.7].

6. Appendix

6.1. **Convex Functions and Convex Geometry.** In this part, let us recall some notions and basic results of convex functions and convex geometry which were needed for the proof of Propositions 3.1 and 4.8.

Definition 6.1. [Nee, Definitions in V.3] Let V be a real vector space and let $f : V \longrightarrow \mathbb{R}_\infty$ be a convex function. The set $D_f := f^{-1}(\mathbb{R})$ is called the *domain* of f.

The function
$$f^* : V^* \longrightarrow \mathbb{R}_\infty$$
$$\xi \longmapsto \sup_V(\xi - f)$$
is called the conjugate of f.

Theorem 6.2. [Nee, Corollary V.3.32] *(**Fenchel's Convexity Theorem**) Let $f \in \mathscr{C}^2(V)$ be such that $\mathrm{d}^2 f(X)$ is positive definite for all $X \in V$. Then $\mathrm{d}f$ maps V diffeomorphically onto the open convex set $\mathrm{int}\, D_{f^*}$ and $\mathrm{d}f^* : \mathrm{int}\, D_{f^*} \longrightarrow V$ is the inverse of $\mathrm{d}f$.*

Definition 6.3. [Gru, Definitions in 2.4] Let Ω be a convex subset of $\mathbb{R}^n$. A point $X \in \Omega$ is an *extreme point* of Ω if it does not belong to the relative interior of any segment contained in Ω, e.d. X is an extreme point if $X = tY + (1-t)Z$ with $0 < t < 1$ and Y and Z in Ω implies $X = Y = Z$. Intuitively, an extreme point is a "corner" of Ω. The set of all extreme points of Ω is denoted by $\mathrm{xt}(\Omega)$.

A point $X \in \Omega$ is an *exposed point* of Ω if there exists a supporting hyperplane of Ω whose intersection with Ω is X alone (Fig. 2); we mean, there exists a hyperplane of $\mathbb{R}^n$, namely $\Pi = \{Y \in \mathbb{R}^n : \langle\langle Y, H \rangle\rangle = h\}$ with $H \in \mathbb{R}^n$, $h \in \mathbb{R}$ and $\langle\langle \cdot, \cdot \rangle\rangle$ is a inner product in $\mathbb{R}^n$, such that $\langle\langle Y, H \rangle\rangle \geq h$ for all $Y \in \Omega$ and the equality holds if and only if $X = Y$. The set of all exposed points of Ω is denoted by $\mathrm{xp}(\Omega)$.

It is easily seen that $\mathrm{xp}(\Omega) \subseteq \mathrm{xt}(\Omega)$ for all convex set Ω.

Theorem 6.4. [Gru, Theorems 2.4.5 and 2.4.9]
  (1) *(**Minkowski-Krein-Milman Theorem**) Let Ω be a compact convex subset of $\mathbb{R}^n$. Then $\Omega = \mathrm{CH}(\mathrm{xt}(\Omega))$. Moreover, if $\Omega = \mathrm{CH}(\Phi)$ then $\Phi \supseteq \mathrm{xt}(\Omega)$.*
  (2) *(**Straszewicz's Theorem**) If Ω is a closed convex set then $\mathrm{xt}(\Omega) \subseteq \overline{\mathrm{xp}(\Omega)}$.*
  (3) *If $\Omega = \mathrm{CH}(\Phi)$ with Φ a finite subset of $\mathbb{R}^n$ (Ω is a polytope) then $\Omega = \mathrm{CH}(\mathrm{xp}(\Omega))$ and $\Phi \supseteq \mathrm{xp}(\Omega)$.*

6.2. **Canonical forms for binary forms of low degree.** We recall canonical forms for binary forms of degree 2, 3 and 4.
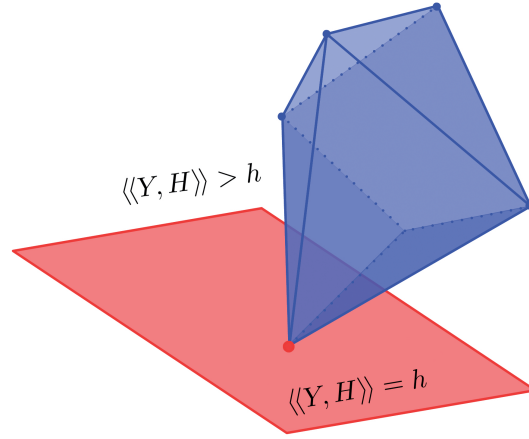
$\langle\!\langle Y, H \rangle\!\rangle > h$

$\langle\!\langle Y, H \rangle\!\rangle = h$

FIGURE 2. Exposed point

Canonical Form Binary quartic forms				
Not.	Over $\mathbb{C}$	Not.	Over $\mathbb{R}$	
I	$x^4 + 2tx^2y^2 + y^4$ $(t \neq 1)$	a.	$x^4 + 2tx^2y^2 + y^4$ $(-1 < t \neq 1)$	
		b.	$-[x^4 + 2tx^2y^2 + y^4]$ $(-1 < t \neq 1)$	
		c.	$x^4 + 2tx^2y^2 - y^4$ $(t \in \mathbb{R})$	
		d.	$x^4 + 2tx^2y^2 + y^4$ $(-1 > t)$	
II	$x^2y^2 + y^4$	a.	$x^2y^2 + y^4$	
		b.	$-[x^2y^2 + y^4]$	
		c.	$x^2y^2 - y^4$	
		d.	$-[x^2y^2 - y^4]$	
III	x^2y^2	a.	x^2y^2	
		b.	$-[x^2y^2]$	
		c.	$x^4 + 2x^2y^2 + y^4$	
		d.	$-[x^4 + 2x^2y^2 + y^4]$	
IV	x^3y	a.	x^3y	
V	x^4	a.	x^4	
		b.	$-x^4$	
VI	0	a.	0	

TABLE 3. Canonical form for binary quartic forms

Canonical form Binary cubic forms			
Tipo	Sobre $\mathbb{C}$	Tipo	Sobre $\mathbb{R}$
I	$y^3 + yx^2$	a.	$y^3 + yx^2$
		b.	$y^3 - yx^2$
II	$y^2 x$	a.	$y^2 x$
III	y^3	a.	y^3
IV	0	a.	0

Table 4. Canonical forms for binary cubic forms

Canonical form Binary quadratic forms			
Tipo	Sobre $\mathbb{C}$	Tipo	Sobre $\mathbb{R}$
I	$x^2 + y^2$	a.	$x^2 + y^2$
		b.	$-\left[x^2 + y^2\right]$
		c.	$x^2 - y^2$
II	x^2	a.	x^2
		b.	$-x^2$
III	0	a.	0

Table 5. Canonical forms for binary quadratic forms

References

[Ati]   M. F. Atiyah: *Convexity and commuting Hamiltonians.* Bull. London Math. Soc., **14** (1982) 1–15.

[EJ]   P. Eberlein, and M. Jablonski: *Closed orbits of semisimple group actions and the real Hilbert-Mumford function.* New Developments in Lie Theory and Geometry, Contemp. Math., **491** (2009), 283–321.

[Fer1]   E. A. Fernández-Culma: *Classification of 7-dimensional Einstein Nilradicals.* Transformation Groups, **17** (2012) 639–656.

[Fer2]   E. A. Fernández-Culma: *Anti-complexified Ricci flow on 6-dimensional Symplectic Nilpotent Lie Algebras and its soliton solutions.* **Paper in preparation**

[Gur]   G. B Gurevich: *Foundations of the theory of algebraic invariants.* P. Noordhoff, Groningen, The Netherlands (1964). Translated by J.R.M. Radok, A.J.M. Spencer.

[Gru]   B. Grünbaum: *Convex Polytopes.* Graduate Texts in Math. **221**, Springer-Verlag, New York (2003). Second edition prepared by V. Kaibel, V. Klee and G. M. Ziegler

[GS1]   V. Guillemin, and S. Sternberg: *Convexity properties of the moment map.* Invent. Math., **67** (1982) 491–513.

[GS2]   V. Guillemin, and S. Sternberg: *Convexity properties of the moment map. II.* Invent. Math., **77** (1984) 533–546.

[HSchw]   P. Heinzner, G. W. Schwarz: *Cartan decomposition of the moment map.* Math. Ann. **337** (2007) 197–232

[HSto]   P. Heinzner, H. Stötzel: *Semistable points with respect to real forms.* Math. Ann. **338** (2007) 1–9

[HSS]   P. Heinzner, G. W. Schwarz, H. Stötzel: *Stratifications with respect to actions of real reductive groups.* Compositio Math., **144** (2008) 163–185

[HSchu] P. Heinzner, P. Schützdeller: *Convexity properties of gradient maps.* Adv. Math., **225** (2010) 1119–1133.

[Jab1] M. Jablonski: *Detecting orbits along subvarieties via the moment map.* Münster J. Math., **3** (2010), 67–88.

[Jab2] M. Jablonski: *Distinguished orbits of Reductive groups.* Rocky Mountain J. Math., Volume **42**, Number 5 (2012), 1521–1549.

[KGM] Yu. Khakimdjanov, M. Goze, and A. Medina: *Symplectic or contact structures on Lie groups.* Differential Geometry and its Applications, **21** (2004) 41–54.

[Kir1] F. Kirwan: *Convexity properties of the moment mapping. III.* Invent. Math., **77** (1984) 547–552

[Kir2] F. Kirwan: *Cohomology of Quotients in Symplectic and Algebraic Geometry.* Mathematical Notes **31**, Princeton Univ. Press, Princeton, NJ, 1984.

[Lau1] J. Lauret: *A canonical compatible metric for geometric structures on nilmanifolds.* Ann. Global Anal. Geom., **30** (2006) 107–138.

[Lau2] J. Lauret: *Ricci soliton solvmanifolds.* J. Reine Angew. Math. **650** (2011) 1–21.

[LW2] J. Lauret, C. Will: *On the diagonalization of the Ricci flow on Lie groups.* **To appear in** Proceedings of the American Mathematical Society. (2011).

[LeW] H.V. Le, G. Wang: *Anti-complexified Ricci flow on compact symplectic manifolds.* J. Reine Angew. Math. **530** (2001) 17–31.

[MFK] D. Mumford, J. Fogarty, F.Kirwan: *Geometric invariant theory.* Ergebnisse der Mathematik und ihrer Grenzgebiete **34**. Third Enlarged Edition, Springer-Verlag, New York (1994). 2nd printing edition (2003).

[Nee] K. H. Neeb: *Holomorphy and Convexity in Lie Theory.* De Gruyter expositions in mathematics **28**, Walter de Gruyter, Berlin (2000).

[Nes] L. Ness: *A stratification of the null cone via the moment map.* Amer. J. Math. **106** (1984). 1281–1329 (con un apendice por D. Mumford)

[Nik1] Y. Nikolayevsky: *Einstein solvmanifolds with a simple Einstein derivation.* Geom. Dedicata **135** (2008), 87–102.

[Nik2] Y. Nikolayevsky: *Einstein solvmanifolds and the pre-Einstein derivation.* Trans. Amer. Math. Soc. **363** (2011), 3935–3958.

[Olv] P. J. Olver: *Classical Invariant Theory.* London Mathematical Society Student Texts **44**, Cambridge University Press, London, United Kingdom (1999).

[Poo] J. Pook: *Homogeneous and locally homogeneous solutions to symplectic curvature flow.* arXiv.org (2012).

[RS] R. W. Richardson, and P. J. Slodowy: *Minimum vectors for real reductive algebraic groups.* J. London Math. Soc., **s2-42** (1990) 409–429.

[Sal] S. M. Salamon: *Complex structures on nilpotent Lie algebras.* Journal of Pure and Applied Algebra, **157** (2001) 311–333.

[Sja] R. Sjamaar: *Convexity properties of the moment mapping re-examined.* Adv. Math., **138** (1998) 46–91.

[Sto1] H. Stötzel: *Closed orbits of Real reductive representations.* arXiv.org (2009).

[ST] J. Streets, G. Tian: *Symplectic curvature flow.* **To appear in** J. Reine Angew. Math. (2010).

[Vez] L. Vezzoni: *A note on Canonical Ricci forms on 2-step nilmanifolds.* Proc. Amer. Math. Soc., Volume **141**, Number 1 (2013) Pages 325-333.

[Woo] C. Woodward: *Moment maps and geometric invariant theory.* Les cours du CIRM. Actions hamiltoniennes: Invariants et Classification. **1** (2010) 55–98.

Current affiliation: CIEM, FaMAF, Universidad Nacional de Córdoba, Ciudad Universitaria,
(5000) Córdoba,
Argentina
E-mail address: efernandez@famaf.unc.edu.ar