

UNIFORM STABILITY OF THE BALL WITH RESPECT TO THE FIRST DIRICHLET AND NEUMANN ∞ -EIGENVALUES

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ABSTRACT. In this note we analyze how perturbations of a ball $B_r \subset \mathbb{R}^n$ behaves in terms of their first (non-trivial) Neumann and Dirichlet ∞ -eigenvalues when a volume constraint $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r)$ is imposed. Our main result states that Ω is uniformly close to a ball when it has first Neumann and Dirichlet eigenvalues close to the ones for the ball of the same volume B_r . In fact, we show that, if

$$|\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| = \delta_2,$$

then there are two balls such that

$$B_{\frac{r}{\delta_1^{r+1}}} \subset \Omega \subset B_{\frac{r+\delta_2 r}{1-\delta_2 r}}.$$

In addition, we obtain a result concerning stability of the Dirichlet ∞ -eigenfunctions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain (connected open subset) with smooth boundary, $1 < p < \infty$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ (the standard p -Laplacian operator). Historically (cf. [13]), it well-known that the first eigenvalue (referred as *the principal frequency* in physical models) of the p -Laplacian Dirichlet eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda_{1,p}^D(\Omega) |u|^{p-2} u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

can be characterized variationally as the minimizer of the (normalized) problem

$$\lambda_{1,p}^D(\Omega) := \inf_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} |u|^p dx = 1 \right\}. \tag{1.2}$$

In the theory of shape optimization and non-linear eigenvalue problems obtaining (sharp) estimates for the eigenvalues in terms of geometric quantities of the domain (e.g. measure, perimeter, diameter, among others) plays a fundamental role due to several applications of these problems in pure and applied sciences. We recall that the explicit value to (1.2) is known only for some specific values of p or for very particular domains Ω . Notice that upper bounds for $\lambda_{1,p}^D(\Omega)$ are usually obtained by selecting particular test functions in (1.2). Nevertheless, lower bounds are a more

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challenging task. In this direction we have the remarkable *Faber-Krahn inequality*: Among all domains of prescribed volume the ball minimizes (1.2). More precisely,

$$\lambda_{1,p}^D(\Omega) \geq \lambda_{1,p}^D(B), \quad (1.3)$$

where B is the n -dimensional ball such that $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B)$ (along this paper $\mathcal{L}^n(\Omega)$ will denote the Lebesgue measure of Ω that is assumed to be fixed). Using isoperimetric or isodiametric inequality similar lower bounds for (1.2) in terms of the perimeter (resp. diameter) of Ω are also available (cf. [1] and [14, page 224], and the references therein). Recently, stability estimates for certain geometric inequalities were established in [10], thereby providing an improved version of (1.3) by adding a suitable remainder term, i.e.,

$$\lambda_{1,p}^D(\Omega) \geq \lambda_{1,p}^D(B)(1 + \gamma_{p,n}(\mathcal{S}(\Omega))^{2+p}),$$

where $\mathcal{S}(\Omega)$ is the so-called *Fraenkel asymmetry* of Ω , which is precisely defined as

$$\mathcal{S}(\Omega) := \inf_{x_0 \in \mathbb{R}^n} \left\{ \frac{\mathcal{L}^n(\Omega \Delta B_r(x_0))}{\mathcal{L}^n(\Omega)} : \mathcal{L}^n(B_r(x_0)) = \mathcal{L}^n(\Omega) \right\},$$

and $\gamma_{p,n}$ is a constant. Observe that \mathcal{S} measures the distance of a set Ω from being a ball. For such quantitative estimates and further related topics we quote [2, 4, 9] and references therein.

Our main goal here is to find stability results for the limit case $p = \infty$.

First, we introduce what is known for the limit as $p \rightarrow \infty$ in the eigenvalue problem for the p -Laplacian. When one takes the limit as $p \rightarrow \infty$ in the minimization problem (1.2), one obtains

$$\lambda_{1,\infty}^D(\Omega) := \lim_{p \rightarrow \infty} \sqrt[p]{\lambda_{1,p}^D(\Omega)} = \inf_{u \in W_0^{1,\infty}(\Omega) \setminus \{0\}} \|\nabla u\|_{L^\infty(\Omega)} > 0, \quad (1.4)$$

see [11]. Concerning the limit equation, also in [11] it is proved that any family of normalized eigenfunctions $\{u_p\}_{p>1}$ to (1.2) converges (up to a subsequence) locally uniformly to $u_\infty \in W_0^{1,\infty}(\Omega)$, a minimizer for 1.4 with $\|u_\infty\|_{L^\infty(\Omega)} = 1$. Moreover, the pair $(u_\infty, \lambda_{1,\infty}^D(\Omega))$ is a non-trivial solution to

$$\begin{aligned} \min \{ -\Delta_\infty v_\infty, |\nabla v_\infty| - \lambda_{1,\infty}^D(\Omega)v_\infty \} &= 0 \quad \text{in } \Omega \\ v_\infty &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.5)$$

Solutions to (1.5) must be understood in the viscosity sense (see [6] for a survey) and $\Delta_\infty u(x) := \nabla u(x)^T D^2 u(x) \cdot \nabla u(x)$ is the well-known ∞ -Laplace operator. In addition, in [11] it is given an interesting and useful geometrical characterization for (1.4):

$$\lambda_{1,\infty}^D(\Omega) = \left(\max_{x \in \Omega} \text{dist}(x, \partial\Omega) \right)^{-1}. \quad (1.6)$$

Such an information means that the ‘‘principal frequency’’ for the ∞ -eigenvalue problem can be detected from the geometry of the domain: it is precisely the reciprocal of radius $\tau_\Omega > 0$ of the largest ball inscribed in Ω . For more references concerning the first eigenvalue (1.5) we refer to [12], [15] and [18].

Now, let us turn our attention to Neumann boundary conditions and consider the eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda_{1,p}^N(\Omega)|u|^{p-2}u \quad \text{in } \Omega \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \quad (1.7)$$

As before, we stress that the first non-zero eigenvalue of (1.7) can also be characterized variationally as the minimizer of the normalized problem

$$\lambda_{1,p}^N(\Omega) := \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^p dx : \int_{\Omega} |u|^p dx = 1 \text{ and } \int_{\Omega} |u|^{p-2} u dx = 0 \right\}. \quad (1.8)$$

The celebrated *Payne-Weinberger inequality* provides a lower bound (on any convex domain $\Omega \subset \mathbb{R}^n$) for the first (non-trivial) Neumann p -eigenvalue (see [8, 17])

$$\lambda_{1,p}^N(\Omega) \geq (p-1) \left(\frac{2\pi}{p \operatorname{diam}(\Omega) \sin(\frac{\pi}{p})} \right)^p. \quad (1.9)$$

For a stability estimate for this problem with $p = 2$ we refer to [2].

When $p \rightarrow \infty$, the minimization problem (1.8) becomes

$$\lambda_{1,\infty}^N(\Omega) := \lim_{p \rightarrow \infty} \sqrt[p]{\lambda_{1,p}^N(\Omega)} = \inf_{\substack{u \in W^{1,\infty}(\Omega) \\ \max_{\Omega} u = -\min_{\Omega} u = 1}} \|\nabla u\|_{L^\infty(\Omega)}, \quad (1.10)$$

see [7, 16]. Concerning the limit equation, also in [7, 16], it is proved that any family of normalized eigenfunctions $\{u_p\}_{p>1}$ to (1.8) converges (up to subsequence) locally uniformly to a limit $u_\infty \in W_0^{1,\infty}(\Omega)$ with $\|u_\infty\|_{L^\infty(\Omega)} = 1$. Moreover, the pair $(u_\infty, \lambda_{1,\infty}^N(\Omega))$ is a non-trivial solution to

$$\begin{aligned} \min \{ -\Delta_\infty v_\infty, |\nabla v_\infty| - \lambda_{1,\infty}^N(\Omega) v_\infty \} &= 0 \quad \text{in } \Omega \cap \{v > 0\} \\ \max \{ -\Delta_\infty v_\infty, -|\nabla v_\infty| - \lambda_{1,\infty}^N(\Omega) v_\infty \} &= 0 \quad \text{in } \Omega \cap \{v < 0\} \\ -\Delta_\infty v_\infty &= 0 \quad \text{in } \Omega \cap \{v = 0\} \\ \frac{\partial v_\infty}{\partial \nu} &= 0 \quad \text{in } \partial\Omega. \end{aligned} \quad (1.11)$$

In addition, we have the following geometrical characterization for $\lambda_{1,\infty}^N(\Omega)$:

$$\lambda_{1,\infty}^N(\Omega) = \frac{2}{\operatorname{diam}(\Omega)}, \quad (1.12)$$

where the intrinsic diameter of Ω is defined as

$$\operatorname{diam}(\Omega) := \max_{\Omega \times \Omega} d_\Omega(x, y) = \max_{\partial\Omega \times \partial\Omega} d_\Omega(x, y),$$

where $d_\Omega(x, y)$ is the geodesic distance given by $d_\Omega(x, y) = \inf_\gamma \operatorname{Long}(\gamma)$, where the infimum is taken over all possible Lipschitz curves in $\bar{\Omega}$ connecting x and y .

We remark that in the limit case $p = \infty$, the geometrical characterization (1.12) of (1.10) yields several interesting consequences:

- ✓ If $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B)$, B being a ball, then $\lambda_{1,\infty}^N(\Omega) \leq \lambda_{1,\infty}^N(B)$, which establishes a *Szegő-Weinberger type inequality*: among all domains of prescribed volume the ball maximizes (1.10).
- ✓ $\lambda_{1,\infty}^N(\Omega) \leq \lambda_{1,\infty}^D(\Omega)$ for any convex Ω with equality if and only if Ω is a ball.
- ✓ The Payne-Weinberger inequality, (1.9), becomes an equality when $p = \infty$.

Taking into account the previous historic overview, we arrive to our main result, which establishes the stability of the ball with respect to small perturbations of their first Dirichlet and Neumann ∞ -eigenvalues. More precisely, if a domain $\Omega \subset \mathbb{R}^n$ has Dirichlet and Neumann ∞ -eigenvalues close enough to those of the ball B_r of the same Lebesgue measure, then Ω is uniformly “almost” ball-shaped.

Theorem 1.1. *Let Ω be an open domain satisfying $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r)$. If for some $\delta_i > 0$ ($i = 1, 2$) small enough it holds that*

$$|\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| = \delta_2,$$

then there are two balls such that

$$B_{\frac{r}{\delta_1 r + 1}} \subset \Omega \subset B_{\frac{r + \delta_2 r}{1 - \delta_2 r}}.$$

The previous theorem implies the following convergence result.

Theorem 1.2. *Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be a family of uniformly bounded domains satisfying $\mathcal{L}^n(\Omega_k) = \mathcal{L}^n(B_r)$. If*

$$|\lambda_{1,\infty}^D(\Omega_k) - \lambda_{1,\infty}^D(B_r)| = o(1) \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega_k) - \lambda_{1,\infty}^N(B_r)| = o(1) \quad \text{as } k \rightarrow \infty,$$

then $\Omega_k \rightarrow B_r$ in the sense that the Hausdorff distance between Ω and a ball B_r approaches zero, i.e.,

$$d_{\mathcal{H}}(\Omega_k, B_r) := \max \left\{ \sup_{x \in \Omega_k} \inf_{y \in B_r} d(x, y), \sup_{y \in B_r} \inf_{x \in \Omega_k} d(x, y) \right\} \rightarrow 0.$$

Note that our results imply

$$\max \left\{ \mathcal{L}^n \left(\Omega \Delta B_{\frac{r}{\delta_1 r + 1}} \right), \mathcal{L}^n \left(\Omega \Delta B_{\frac{r + \delta_2 r}{1 - \delta_2 r}} \right) \right\} \leq \mathfrak{C}(n, \delta_i, r) r^n. \quad (1.13)$$

where $\mathfrak{C}(n, \delta_i, r) = \omega_n \max\{(\delta_1 r + 1)^n - 1, (n - 1)\delta_2\} \rightarrow 0$ as $\delta_i \rightarrow 0$. Hence, we can control the Fraenkel asymmetry of the set, $S(\Omega)$. But our results give much more since we have a sort of uniform control on how far the set is from being a ball (for instance, we have convergence in Hausdorff distance in Theorem 1.2).

Another important question in this theory consists on how the corresponding ∞ -ground states (solutions to (1.5)) behave in relation to perturbations of the ∞ -eigenvalues of the ball. The next result provides an answer for this issue, showing that Dirichlet ∞ -eigenfunctions are uniformly close to a cone when the first Dirichlet and Neumann ∞ -eigenvalues are close to those for the ball. Note that, in general, the ∞ -eigenvalue problem (1.5) may have multiple solutions (the first eigenvalue may not be simple), see [5] and [18].

Theorem 1.3. *Let Ω be an open domain satisfying $\mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r)$. Given $\varepsilon > 0$ there are $\delta_i(\varepsilon) > 0$ ($i = 1, 2$) small enough such that: if*

$$|\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| < \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| < \delta_2,$$

then

$$|u(x) - v_\infty(x)| < \varepsilon \quad \text{in } \Omega \cap B_r,$$

where $v_\infty(x) = 1 - \frac{|x|}{r}$ is the normalized ∞ -ground state to (1.5) in B_r .

Theorem 1.3 can be rewritten as follows:

Corollary 1.4. *Let $\{u_k\}_{k \in \mathbb{N}}$ be a family of normalized solutions to (1.5) in Ω_k such that*

$$|\lambda_{1,\infty}^D(\Omega_k) - \lambda_{1,\infty}^D(B_r)| = o(1) \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega_k) - \lambda_{1,\infty}^N(B_r)| = o(1) \quad \text{as } k \rightarrow \infty.$$

Then $u_k \rightarrow v_\infty$ locally uniformly in B_r , where

$$v_\infty(x) = 1 - \frac{|x|}{r}$$

is the normalized ∞ -ground state to (1.5) in B_r .

Our approach can be applied for other classes of operators with p -Laplacian type structure. We can deal with p -Laplace type problems involving an *anisotropic p -Laplace operator*

$$-\mathcal{Q}_p u := -\operatorname{div}(\mathbb{F}^{p-1}(\nabla u)\mathbb{F}_\xi(\nabla u)),$$

where \mathbb{F} is an appropriate (smooth) norm of \mathbb{R}^n and $1 < p < \infty$. The necessary tools for studying the anisotropic Dirichlet eigenvalue problem, as well as its limit as $p \rightarrow \infty$ can be found in [3]. Here, to obtain results similar to ours, one has to replace Euclidean balls with balls in the norm \mathbb{F} .

This article is organized as follows: In Section 2 we prove our main stability results including the behaviour of the corresponding ∞ -eigenfunctions. In Section 3 we collect several examples that illustrate our results.

2. PROOF OF MAIN RESULTS

Before proving our main result we introduce some notation which will be used throughout this section. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a ball $B_r \subset \mathbb{R}^n$ of radius $r > 0$ we denote $\lambda_{1,\infty}^D(\Omega)$ and $\lambda_{1,\infty}^D(B_r)$ the first Dirichlet eigenvalues (1.6) in Ω and in B_r , respectively; analogously, $\lambda_{1,\infty}^N(\Omega)$ and $\lambda_{1,\infty}^N(B_r)$ stand for the first non-trivial Neumann eigenvalues (1.12) in Ω and in B_r .

We introduce the following class of sets which will play an important role in our approach. For non-negative constants δ_1 and δ_2 we define the class

$$\begin{aligned} \Xi_{\delta_1,\delta_2}(B_r) := & \left\{ \Omega \subset \mathbb{R}^n \text{ bounded domain with } \mathcal{L}^n(\Omega) = \mathcal{L}^n(B_r) \right. \\ & \left. : |\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| = \delta_1, |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| = \delta_2 \right\}. \end{aligned}$$

Notice that $\Xi_{0,0}(B_r)$ consists of the family of all balls with radius $r > 0$. Another important remark is that the elements of $\Xi_{\delta_1,\delta_2}(B_r)$ are invariant by rigid transformations (rotations, translations, etc).

Similarly, we can define the class $\Xi_{\delta_1}^D(B_r)$ (resp. $\Xi_{\delta_2}^N(B_r)$) as being $\Xi_{\delta_1,\delta_2}(B_r)$ with the restriction on the Dirichlet (resp. Neumann) eigenvalues only.

In the next lemma we show that a control on the difference of the first Dirichlet eigenvalue implies that Ω contains a large ball.

Lemma 2.1. *If $\Omega \in \Xi_{\delta_1}^D(B_r)$ then there exists a ball such that $B_{\frac{r}{\delta_1 r + 1}} \subset \Omega$. Moreover,*

$$\mathcal{L}^n\left(\Omega \Delta B_{\frac{r}{\delta_1 r + 1}}\right) \leq c(n, \delta_1, r)r^n.$$

where $c = o(1)$ as $\delta_1 \rightarrow 0$.

Proof. According to (1.6) we have

$$\delta_1 = |\lambda_{1,\infty}^D(\Omega) - \lambda_{1,\infty}^D(B_r)| = \left| \frac{1}{r_\Omega} - \frac{1}{r} \right|.$$

It follows that

$$r_\Omega \geq \frac{r}{\delta_1 r + 1}.$$

and then there is ball such that $B_{\frac{r}{\delta r + 1}} \subset \Omega$. Finally,

$$\begin{aligned} \mathcal{L}^n(\Omega \Delta B_{\frac{r}{\delta r + 1}}) &= \mathcal{L}^n(\Omega) - \mathcal{L}^n(B_{\frac{r}{\delta r + 1}}) \\ &= \omega_n r^n \left(1 - \frac{1}{(\delta r + 1)^n} \right) \end{aligned}$$

$$\begin{aligned} &\leq \omega_n r^n ((\delta r + 1)^n - 1) \\ &= \mathfrak{c}(n, \delta, r) r^n \end{aligned}$$

and the lemma follows. \square

Now, we show that a control on the difference of the first Neumann eigenvalue implies that Ω is contained in a small ball.

Lemma 2.2. *If $\Omega \in \Xi_{\delta_2}^N(B_r)$ then there is a ball such that $\Omega \subset B_{\frac{r}{1-\delta_2 r}}$. Moreover,*

$$\mathcal{L}^n\left(\Omega \Delta B_{\frac{r}{1-\delta_2 r}}\right) \leq (n-1)\omega_n r^n \delta_2.$$

Proof. Using (1.12) we have

$$\delta_2 = |\lambda_{1,\infty}^N(\Omega) - \lambda_{1,\infty}^N(B_r)| = \left| \frac{2}{\text{diam}(\Omega)} - \frac{1}{r} \right|.$$

It follows that

$$\text{diam}(\Omega) \leq \frac{2r}{1-\delta_2 r} = r + \frac{r(1+\delta r)}{1-\delta_2 r}$$

and then there exists a ball such that

$$\Omega \subset B_{\frac{\text{diam}(\Omega)}{2}} = B_{\frac{r}{1-\delta_2 r}}.$$

Moreover,

$$\begin{aligned} \mathcal{L}^n\left(\Omega \Delta B_{\frac{\text{diam}(\Omega)}{2}}\right) &= \mathcal{L}^n\left(B_{\frac{\text{diam}(\Omega)}{2}}\right) - \mathcal{L}^n(\Omega) \\ &= \omega_n r^n \left(\left(1 + \frac{\delta_2}{1-\delta_2 r}\right)^n - 1 \right) \\ &= \omega_n r^n \delta_2 \sum_{k=2}^n \left(\frac{\delta_2}{1-\delta_2 r} \right)^k \\ &\leq (n-1)\omega_n \delta_2 r^n \end{aligned}$$

and the lemma follows. \square

Proof of Theorem 1.1. The theorem follows as an immediate consequence of Lemmas 2.1 and 2.2. \square

Proof of Theorem 1.2. The hypothesis implies that $\Omega_k \in \Xi_{\delta_k, \varepsilon_k}(B_r)$ for $\delta_k, \varepsilon_k = o(1)$ as $k \rightarrow \infty$. For this reason, by Theorem 1.1 there are two balls such that

$$B_{\frac{r}{\delta_k r + 1}} \subset \Omega_k \subset B_{\frac{r + \varepsilon_k r}{1 - \varepsilon_k r}}.$$

Now, using that all these balls are centered at points that are bounded (since we assumed that the family Ω_k is uniformly bounded), we can extract a subsequence such that the centers converge and therefore we conclude that there is a ball B_r such that $\Omega_k \rightarrow B_r$ as $k \rightarrow \infty$. \square

Proof of Theorem 1.3. The proof follows by contradiction. Let us suppose that there exists an $\varepsilon_0 > 0$ such that the thesis of theorem fails to hold. This means that for each $k \in \mathbb{N}$ we might find a domain Ω_k and u_k , a normalized ∞ -ground state to (1.5) in Ω_k , such that $\Omega_k \in \Xi_{\gamma_k, \zeta_k}(B_r)$ with $\gamma_k, \zeta_k = o(1)$ as $k \rightarrow \infty$, that is,

$$|\lambda_{1,\infty}^D(\Omega_k) - \lambda_{1,\infty}^D(B_r)| < \gamma_k \quad \text{and} \quad |\lambda_{1,\infty}^N(\Omega_k) - \lambda_{1,\infty}^N(B_r)| < \zeta_k,$$

with $\gamma_k, \zeta_k = o(1)$ as $k \rightarrow \infty$, together with

$$|u_k(x) - v_\infty(x)| > \varepsilon_0 \quad \text{in } \Omega_k \cap B_r, \tag{2.1}$$

for every $k \in \mathbb{N}$.

Using our previous results, we can suppose that every $\Omega_k \subset B_{2r}$. Then, by extending u_k to zero outside of Ω_k , we may assume that $\{u_k\}_{k \in \mathbb{N}} \subset W_0^{1,\infty}(B_{2r})$. In this context, standard arguments using viscosity theory show that, up to a subsequence, $u_k \rightarrow u_\infty$ uniformly in $\overline{B_{2r}}$, being the limit u_∞ a normalized eigenfunction for some domain $\hat{\Omega}$ with $\hat{\Omega} \Subset B_{2r}$. Moreover, we have that $\lambda_{1,\infty}^D(\Omega_k) \rightarrow \lambda_{1,\infty}^D(\hat{\Omega})$.

According to Theorem 1.2, $\Omega_k \rightarrow B_r$ as $k \rightarrow \infty$. By the previous sentences we conclude that $\hat{\Omega} = B_r$. Now, by uniqueness of solutions to (1.5) in B_r we conclude that $u_\infty = v_\infty$. However, this contradicts (2.1) for $k \gg 1$ (large enough). Such a contradiction proves the theorem. \square

3. EXAMPLES

Given a fixed ball B and a domain Ω having both of them the same volume, Theorem 1.1 says that if the ∞ -eigenvalues are close each other then Ω is almost ball-shaped uniformly. The following examples illustrate Theorems 1.1 and 1.2.

Example 3.1. The converse of Theorem 1.1 (and Theorem 1.2) is not true: given a fixed ball B , clearly, there are domains Ω fulfilling (1.13) such that the difference between the Neumann (and Dirichlet) eigenvalues in Ω and in B is not small. Let us present some illustrative examples.

- (1) A stadium. Let B be the unit ball in \mathbb{R}^2 and Ω the stadium domain given in Figure 1 (a) with $\ell = \frac{\pi(1-\varepsilon^2)}{2\varepsilon}$. In this case $\mathcal{L}^n(B) = \mathcal{L}^n(\Omega) = \pi$ for any $0 < \varepsilon < 1$. However,

$$\lambda_{1,\infty}^N(B) = 1, \quad \lambda_{1,\infty}^N(\Omega) = \frac{2}{\text{diam}(\Omega)} = \frac{4\varepsilon}{\pi + \varepsilon^2(4 - \pi)} < \frac{1}{3} \quad \text{if } \varepsilon < \frac{1}{4}.$$

- (2) A ball with holes. If $\Omega = B(0, \sqrt{1 + \varepsilon^2}) \setminus B(0, \varepsilon)$ is the domain given in Figure 1 (b), then $\mathcal{L}^n(B) = \mathcal{L}^n(\Omega) = \pi$, however

$$\lambda_{1,\infty}^D(B) = 1, \quad \lambda_{1,\infty}^D(\Omega) = \frac{1}{\sqrt{1 + \varepsilon^2}} > \frac{3}{2} \quad \text{if } \frac{3}{4} < \varepsilon < 1.$$

- (3) A ball with thin tubular branches. If Ω is the domain given in Figure 1 (c), the condition $\mathcal{L}^n(B) = \mathcal{L}^n(\Omega)$ gives the relation

$$r(r + \varepsilon) + \varepsilon\left(\frac{1}{\pi} + \frac{\varepsilon}{2}\right) = 1, \quad \text{diam}(\Omega) = 1 + r + \pi(1 + r).$$

For instance, if we take $\varepsilon = 10^{-3}$ it follows that $r \sim 0.999465$ and then

$$\lambda_{1,\infty}^N(B) = \frac{2}{\text{diam}(B)} = 1, \quad \lambda_{1,\infty}^N(\Omega) = \frac{2}{\text{diam}(\Omega)} \sim 0.2415.$$

Hence, in view of these examples we conclude that a domain that has Dirichlet and Neumann ∞ -eigenvalues close to the ones for the ball is close to a ball not only in the sense that $\mathcal{L}^n(\Omega \Delta B_r)$ is small but it can not contain holes deep inside (small holes near the boundary are allowed) and can not have thin tubular branches.

Example 3.2. The regular polygon \mathbb{P}_k of k -sides ($k \geq 3$) centered at the origin such that $\mathcal{L}^n(\mathbb{P}_k) = \mathcal{L}^n(B_r)$ satisfies

$$|\lambda_{1,\infty}^D(\mathbb{P}_k) - \lambda_{1,\infty}^D(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(B_r) - \lambda_{1,\infty}^N(\mathbb{P}_k)| = \delta_2,$$

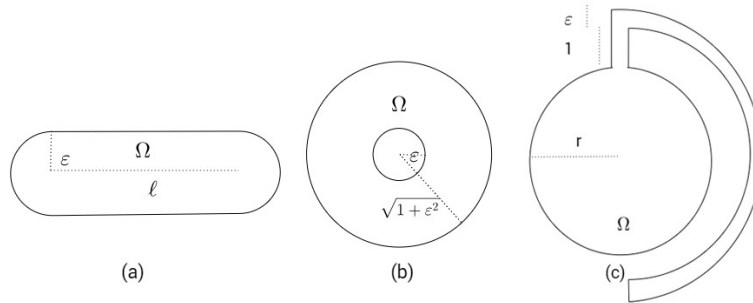


FIGURE 1. Three examples of domains

where

$$\delta_1 = \frac{1}{r \sqrt{\frac{\pi}{k \tan(\frac{\pi}{k})}}} - \frac{1}{r} \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{r \sqrt{\frac{2\pi}{k \sin(\frac{2\pi}{k})}}}.$$

Therefore, we can recover the well known convergence $\mathbb{P}_k \rightarrow B_r$ as $k \rightarrow \infty$.

Example 3.3. Given $k \in \mathbb{N}$ and positive constants $\mathbf{a}_1^k, \dots, \mathbf{a}_n^k$, the n -dimensional ellipsoid given by

$$\mathcal{E}_k := \left\{ (x_1, \dots, x_n) : \sum_{i=1}^n \left(\frac{x_i}{\mathbf{a}_i^k} \right)^2 < 1 \right\}$$

such that $\mathcal{L}^n(\mathcal{E}_k) = \mathcal{L}^n(B_r)$ satisfies

$$|\lambda_{1,\infty}^D(\mathcal{E}_k) - \lambda_{1,\infty}^D(B_r)| = \delta_1 \quad \text{and} \quad |\lambda_{1,\infty}^N(B_r) - \lambda_{1,\infty}^N(\mathcal{E}_k)| = \delta_2,$$

where

$$\delta_1 = \frac{1}{\min_i \{\mathbf{a}_i^k\}} - \frac{1}{r}, \quad \text{and} \quad \delta_2 = \frac{1}{r} - \frac{1}{\max_i \{\mathbf{a}_i^k\}}.$$

Therefore, we recover the fact that if $\min_i \mathbf{a}_i^k \rightarrow r$ and $\max_i \mathbf{a}_i^k \rightarrow r$ as $k \rightarrow \infty$, then $\mathcal{E}_k \rightarrow B_r$.

Example 3.4. Given $r > 0$ let $k_0 \in \mathbb{N}$ such that $\frac{1}{2\pi} \sqrt{\frac{4}{k^2} + 4\pi^2 r^2} > \frac{1}{k\pi}$ for all $k \geq k_0$. For each $k \in \mathbb{N}$ let Ω_k be the planar stadium domain from Figure 1 (a) with $l_k = \frac{1}{k}$ and $\varepsilon_k = \frac{1}{2\pi} \sqrt{\frac{4}{k^2} + 4\pi^2 r^2} - \frac{1}{k\pi}$. It is easy to check that $\Omega_k \in \Xi_{\frac{1}{\varepsilon_k} - \frac{1}{r}, \frac{2}{2\varepsilon_k + \frac{1}{k}} - \frac{1}{r}}(B_r)$. Furthermore, in this case we have that the eigenfunctions are explicit and given by

$$u_k(x) = \frac{1}{\varepsilon_k} \text{dist}(x, \partial\Omega_k).$$

Finally, from Corollary 1.4

$$u_k(x) \rightarrow v_\infty(x) = \frac{1}{r} \text{dist}(x, \partial B_r) \quad \text{locally uniformly in } B_r \text{ as } k \rightarrow \infty.$$

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