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Bispectrality and Time–Band Limiting: Matrix-valued Polynomials

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The subject of time–band limiting, originating in signal processing, is dominated by the miracle that a naturally appearing integral operator admits a commuting differential one allowing for a numerically efficient way to compute its eigenfunctions. Bispectrality is an effort to dig into the reasons behind this miracle and goes back to joint work with H. Duistermaat. This search has revealed unexpected connections with several parts of mathematics, including integrable systems. Here we consider a matrix-valued version of bispectrality and give a general condition under which we can display a constructive and simple way to obtain the commuting differential operator. Furthermore, we build an operator that commutes with both the time-limiting operator and the band-limiting operators.

1 Introduction

The problem of double concentration, that is, localizing a function both in physical and frequency space cuts across several areas of mathematics, physics, and engineering. This topic arises in harmonic analysis, signal processing, and quantum mechanics. Highly elaborate bodies of work, such as wavelet theory, spawn from efforts to find a good compromise between these two competing goals.

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In some instances this issue gives rise to a sharply posed question as was done (at least implicitly) by C. Shannon [29]; if you know the frequency components over a band $[-W, W]$ for an unknown signal of finite support in $[-T, T]$, what is the best use you can do of this (noisy) data? It is natural to look for the coefficients of an expansion of the unknown signal in terms of the singular functions of the problem. However, one faces a serious computational difficulty; these singular functions are the eigenfunctions of an integral operator with most of its eigenvalues crowded together.

In a remarkable series of papers written at Bell Labs in the 1960's, a mathematical miracle was uncovered, and exploited very successfully. We refer to it as the "time–band limiting phenomenon". We are alluding to the surprising fact that certain naturally appearing integral operators admit 2nd order commuting differential ones.

One of us has been looking for the reason that lies behind this miracle for quite a while and this search has given rise to what we refer to as the "bispectral problem". In our context this consists in the search for weights whose orthogonal polynomials are joint eigenfunctions of some differential operator.

There is a large number of papers dealing with the relations between these two issues. For a small sample, see [3, 12, 27]. We feel that the true reasons behind this remarkable algebraic "accident", see [34, 38], deserves further study.

The phenomenon of a pair of commuting integral and differential operators plays an important role in at least three areas of applied mathematics; the problem of time-and-band limiting studied by Slepian, Landau, and Pollak, see [19, 20, 32, 33, 35], nicely summarized in [18, 34], the problem of limited angle tomography, see [7], and finally in Random Matrix Theory, see [21, 36, 37]. For other applications of this work, see [13, 14, 30, 31]. For numerical aspects of this phenomenon, see [22]. All of the work mentioned above deals with scalar-valued functions.

A much more recent look at the relation between these two topics involves matrix-valued orthogonal polynomials, a subject started by M. G. Krein, see [16, 17]. Among the papers where this was explored we mention [2, 9].

Back in the scalar case, following [27] there is a short and elegant paper by Perline, see [26]. One of us was certainly aware of this paper back in the late 1980's, but somehow did not pay enough attention to it. It was A. Zhedanov who noticed this long forgotten paper and brought it to the attention of his coworkers. The very recent paper [11] shows that the ideas in [26] can be extended to other scenarios.

The aim of this paper is to give a general result on the relation between the bispectral property for matrix-valued orthogonal polynomials and the existence of a symmetric operator that commutes with the time-and-band limiting operator and can be used to yield their eigenfunctions. For any value of the relevant parameters we build explicitly a 2nd order differential operator T and a tridiagonal difference operator L that commute with **both** the time-limiting operator and the band-limiting operator. This proves, in a constructive way, the existence of commuting operators for the integral and the difference operators.

This general result, as well as those in [11], is inspired by the construction in [26].

Finally, in Section 4, after a brief mention of scalar cases, we use our general results to study some particular examples, all of them in the matrix-valued case.

In the 1st example we extend results previously obtained in [9]; in the 2nd one we verify a result that was conjectured in [2]; in the 3rd example we exploit the power of our construction to give a commuting differential operator for a case where the commuting operator problem was not studied before; the last example is included to indicate that bispectrality may not always guarantee the existence of a commuting differential operator.

In the scalar case treated in [26], the issue of the use of the commuting differential operator to obtain the eigenfunctions of the integral one was not dealt in detail. In this paper we take the same approach and intend to return to this point at a later time.

2 Preliminaries

Let $W(x)$ be an $R\times R$ matrix weight function in the open interval (a,b) and let $\{Q_n(x)\}_{n\in\mathbb{N}_0}$ be a sequence of matrix orthonormal polynomials with respect to the weight $W(x)$.

The Hilbert spaces $\ell^2(M_R,\mathbb{N}_0)$ and $L^2((a,b),W(t)\,\mathrm{d}t)$ are given by the real-valued $R\times R$ matrix sequences $\{C_n\}_{n\in\mathbb{N}_0}$ such that $\sum_{n=0}^{\infty}\operatorname{tr}\left(C_n C_n^*\right) < \infty$ and all measurable matrix-valued functions $f(x)$, $x \in (a,b)$, satisfying $\int_a^b \operatorname{tr}(f(x)W(x)f^*(x))\,\mathrm{d}x < \infty$, respectively. A natural analog of the Fourier transform is the isometry $F:\ell^2(M_R,\mathbb{N}_0)\longrightarrow L^2(W)$ given by

$$\{C_n\}_{n=0}^{\infty} \xmapsto{F} \sum_{n=0}^{\infty} C_n Q_n(x).$$

In the case when the matrix polynomials are dense in $L^2(W)$, this map is unitary with the inverse $F^{-1}:L^2(W)\longrightarrow \ell^2(M_R,\mathbb{N}_0)$ given by

$$f \xmapsto{F^{-1}} C_n = \int_a^b f(x)\,W(x)\,Q_n^*(x)\,\mathrm{d}x.$$

If we consider the problem of determining a function f from the following (typically noisy) data: f has support on the compact set $[0, N]$ and its Fourier transform Ff is known on a compact set $[a, \Omega]$, one concludes that we need to compute the singular vectors (and singular values) of the operator $E : \ell^2(M_R, \mathbb{N}_0) \longrightarrow L^2(W)$ given by

$$Ef = \chi_\Omega F \tilde{\chi}_N f,$$

where $\tilde{\chi}_N$ is the *time-limiting operator* on $\ell^2(M_R, \mathbb{N}_0)$ and χ_Ω is the *band-limiting operator* on $L^2(W)$. At level N, $\tilde{\chi}_N$ acts on $\ell^2(M_R, \mathbb{N}_0)$ by simply setting equal to 0 all the components with index larger than N. At level Ω, χ_Ω acts on $L^2(W)$ by multiplication by the characteristic function of the interval (a, Ω), $a < \Omega \leq b$.

We are thus lead to study the eigenvectors of the operators

$$E^*E = \tilde{\chi}_N F^{-1} \chi_\Omega F \tilde{\chi}_N \qquad \text{and} \qquad EE^* = \chi_\Omega F \tilde{\chi}_N F^{-1} \chi_\Omega.$$

The operator E^*E, acting on $\ell^2(M_R, \mathbb{N}_0)$, is just a finite-dimensional block matrix with each $R \times R$ block given by

$$(E^*E)_{m,n} = \int_a^\Omega Q_m(x) W(x) Q_n^*(x) \, \mathrm{d}x, \qquad 0 \leq m, n \leq N. \tag{1}$$

The operator EE^* acts on $L^2((a, \Omega), W(t)\, \mathrm{d}t)$ by means of the integral kernel

$$k(x, y) = \sum_{n=0}^{N} Q_n^*(x) Q_n(y). \tag{2}$$

The integral operator $S = EE^*$ with kernel k, defined in (2), acting on $L^2((a, \Omega), W)$ "from the right-hand side" is given by

$$(fS)(x) = \int_a^\Omega f(y) W(y) (k(x, y))^* \, \mathrm{d}y. \tag{3}$$

For general N and Ω there is no hope of finding the eigenfunctions of EE^* and E^*E analytically. However, there is a strategy to solve this typical inverse problem: finding an operator with simple spectrum that would have the same eigenfunctions as the operators EE^* or E^*E. This is exactly what Slepian, Landau, and Pollak did in the scalar case, when dealing with the unit circle and the usual Fourier analysis. They discovered the following properties:

- For each N, Ω there exists a symmetric tridiagonal matrix L, with simple spectrum, commuting with E^*E.
- For each N, Ω there exists a self-adjoint 2nd order differential operator T, with simple spectrum, commuting with the integral operator $S = EE^*$.

In this paper, which deals with a continuous-discrete version of the bispectral problem, we give an **explicit construction** of such symmetric operators L and T given certain hypothesis (which is automatically satisfied in the scalar case).

Symmetry for an operator T acting on functions defined in $[a, \Omega]$ means that

$$\langle PT, Q\rangle_\Omega = \langle P, QT\rangle_\Omega,$$

for every P, Q in an appropriate dense set of functions, where

$$\langle P, Q\rangle_\Omega = \int_a^\Omega P(x)W(x)Q^*(x)\,\mathrm{d}x. \tag{4}$$

From [9], given a symmetric differential operator T and an integral operator S, with kernel k, we have

$$TS = ST \qquad \text{if and only if} \qquad \left(k(x,y)^*\right) T_x = (k(x,y)T_y)^*. \tag{5}$$

(Here we use T_x to stress that T acts on the variable x.)

Notice that in principle there is no guarantee that we will find any such T except for a scalar multiple of the identity. For the problem at hand, namely the efficient computation of the eigenfunctions of S, we need to exhibit a differential operator T whose eigenfunctions are also eigenfunctions of the integral operator S. In the scalar case this is guaranteed by asking that T should have a simple spectrum. In the matrix-valued case the useful requirement on T is more subtle and will be analyzed in detail in a future publication.

3 The Symmetric Bispectral Problem

We start with a matrix weight W defined in the interval (a, b) and a 2nd order symmetric differential operator D with respect to W of the form

$$D = \partial^2 F_2 + \partial F_1 + F_0,$$

with F_j a polynomial of order less than or equal to j, for $j = 0, 1, 2$.

Let $\{R_n\}_{n\geq 0}$ be the monic matrix orthogonal polynomials with respect to W and $\{Q_n\}_{n\geq 0}$ the sequence of orthonormal polynomials defined by $Q_n = S_n R_n$, with $S_n = \|R_n\|^{-1}$ the inverse of the matrix-valued norm of R_n.

We have that these polynomials are eigenfunctions of D, with matrix-valued eigenvalues,

$$R_n D = \Lambda_n R_n, \qquad Q_n D = \tilde{\Lambda}_n Q_n, \quad \text{for all } n \geq 0, \tag{6}$$

with $\tilde{\Lambda}_n = S_n \Lambda_n S_n^{-1}$.

They also satisfy the three-term recursion relations

$$\begin{aligned} xR_n(x) &= R_{n+1} + B_n R_n + A_n R_{n-1}, \\ xQ_n(x) &= \tilde{A}^*_{n+1} Q_{n+1} + \tilde{B}_n Q_n + \tilde{A}_n Q_{n-1}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} A_n &= \|R_n\|^2 \|R_{n-1}\|^{-2}, & (B_n S_n) &= (B_n S_n)^*, \\ \tilde{A}_n &= S_n A_n S_{n-1}^{-1} = \|R_n\|\|R_{n-1}\|^{-1}, & \tilde{B}_n &= S_n B_n S_n^{-1}, \end{aligned}$$

here we adopt the convention that $P_{-1} = Q_{-1} = 0$.

The fact that the symmetry of D implies that we have a bispectral situation as above has been established in [4, 10], where the pairs (W, D) are called "classical pairs".

Recall the setup in the section on Preliminaries.

We fix a natural number N and $\Omega \in (a, b)$ and consider the following operators χ_Ω and χ_N in $L^2(W)$: χ_Ω acts on $L^2(W)$ by multiplication by the characteristic function of the interval (a, Ω) and $\chi_N = \mathcal{F}\tilde{\chi}_N\mathcal{F}^{-1}$ is the "projection" on the (left) module (over the ring of matrices) spanned by $\{Q_0, Q_1, \ldots, Q_N\}$. Explicitly,

$$\chi_N(f) = \sum_{n=0}^{N} \langle f, Q_n \rangle Q_n. \tag{8}$$

Hence, the band-time-band limiting operator EE^*, that now can be rewritten as $EE^* = \chi_\Omega \chi_N \chi_\Omega$, is an integral operator acting from the right-hand side as in (3), with kernel

$$k(x, y) = \sum_{n=0}^{N} Q_n^*(x) Q_n(y).$$

The operator E^*E is the finite-dimensional block matrix given in (1). Also, now we have that the action of the time-band-time limiting operator $\mathcal{F}E^*E\mathcal{F}^{-1} = \chi_N\chi_\Omega\chi_N$ is given by

$$\chi_N\chi_\Omega\chi_N(f) = \sum_{i=0}^{N}\left(\int_a^\Omega f(x)W(x)Q_i^*(x)\,\mathrm{d}x\right)Q_i,$$

for $f \in L^2(W)$.

The main result of this section is a simple proof of the existence of a commuting symmetric operator for both of these time and band limiting operators EE^* and $\mathcal{F}E^*E\mathcal{F}^{-1}$. For this purpose, we will construct an operator T that commutes with **each of** χ_N and χ_Ω. This important idea already appears in [26]. It is also used in the later paper [38].

While this will clearly imply the commutativity with both EE^* and $\mathcal{F}E^*E\mathcal{F}^{-1}$ we do not look into the possibility of finding a local operator that commutes with these ones but fails to commute with both χ_N and χ_Ω.

We **assume the following hypothesis** on the weight W and the differential operator D; there exists a matrix M, independent of the variables x, n, and the parameter Ω, but possibly dependent on N, such that

$$\Big(M - x(\Lambda_{N+1}+\Lambda_N)\Big)W(x) - W(x)\Big(M - x(\Lambda_{N+1}+\Lambda_N)\Big)^* = 0. \tag{9}$$

In the expression above the dependence on the differential operator D is hidden in the eigenvalues Λ_N of the monic orthogonal polynomials. Explicitly if the differential operator D is of the form $D = \partial^2 F_2 + \partial F_1 + F_0$ and we write $F_2 = F_{22}x^2 + F_{21}x + F_{20}$, $F_1 = F_{11}x + F_{10}$, we have that

$$\Lambda_n = \Lambda_n(D) = n(n-1)F_{22} + nF_{11} + F_0. \tag{10}$$

From the symmetric differential operator D, the eigenvalues of the monic orthogonal polynomials, and this matrix M, we build the following differential operator, acting on the "right-hand side"

$$T = xD + Dx - 2\Omega D - (\Lambda_{N+1}+\Lambda_N)x + M. \tag{11a}$$

Let us observe that if $D = \partial^2 F_2 + \partial F_1 + F_0$ then $xD = Dx + 2\partial F_2 + F_1$. Therefore,

$$\tfrac{1}{2}T = D(x-\Omega) + \partial F_2(x) + \tfrac{1}{2}\Big(F_1(x) - x(\Lambda_{N+1}+\Lambda_N) + M\Big). \tag{11b}$$

Proposition 3.1. The differential operator T is a symmetric operator with respect to W, in $[a,b]$ and also in $[a,\Omega]$.

Proof. Since D is symmetric with respect to W in $[a,b]$ it is clear that $xD+Dx$ and $2\Omega D$ are also symmetric operators in $[a,b]$. Hence, from (11a), for any smooth enough functions $f,g \in L^2(W)$ we have

$$\langle f\,T,g\rangle - \langle f,gT\rangle = \int_a^b f(x)\,(M - x(\Lambda_{N+1}+\Lambda_N))\,W(x) - W(x)\,(M - x(\Lambda_{N+1}+\Lambda_N))^*\,g(x)\,\mathrm{d}x.$$

Thus, we have that T is a symmetric operator in $[a,b]$ if and only if the operator of order zero $M - x(\Lambda_{N+1}+\Lambda_N)$ satisfies (9).

Now we will prove that T is symmetric with respect to W in $[a,\Omega]$.

From [10] or [4] we have that a differential operator $D = \frac{\mathrm{d}^2}{\mathrm{d}x^2}F_2(x) + \frac{\mathrm{d}}{\mathrm{d}x}F_1(x) + F_0$ is symmetric with respect to a weight W defined in (a,b) if and only if it satisfies, for $a < x < b$, the symmetry equations

$$\begin{aligned} F_2W &= WF_2^*,\\ 2(F_2W)' - F_1W &= WF_1^*, \\ (F_2W)'' - (F_1W)' + F_0W &= WF_0^*, \end{aligned} \tag{12}$$

and the boundary conditions

$$\lim_{x\to a,b} F_2(x)W(x) = 0, \quad \lim_{x\to a,b} \left(F_1(x)W(x) - WF_1^*(x)\right) = 0. \tag{13}$$

We have the following relations among the coefficients of the differential operators $D = \partial^2 F_2 + \partial F_1 + F_0$ and $T = \partial^2\tilde{F}_2 + \partial\tilde{F}_1 + \tilde{F}_0$,

$$\begin{aligned} \tilde{F}_2 &= (x-\Omega)F_2,\\ \tilde{F}_1 &= (x-\Omega)F_1 + F_2,\\ \tilde{F}_0 &= (x-\Omega)F_0 + \tfrac{1}{2}\left(F_1(x) - x(\Lambda_{N+1}+\Lambda_N) + M\right). \end{aligned}$$

Since T is a symmetric operator with respect to the weight W in the interval (a,b) we have that $\{\tilde{F}_0,\tilde{F}_1,\tilde{F}_2\}$ satisfy (12) and (13). Then, to prove that T is symmetric in (a,Ω) it suffices to prove that

$$\lim_{x\to\Omega} \tilde{F}_2(x)W(x) = 0, \quad \lim_{x\to\Omega} \left(\tilde{F}_1(x)W(x) - W\tilde{F}_1^*(x)\right) = 0. \tag{14}$$

Since D is symmetric with respect to the weight W in the interval (a, b) we have that $\{F_0, F_1, F_2\}$ also satisfy (12), thus

$$\lim_{x\to\Omega} \tilde{F}_2 W = \lim_{x\to\Omega} (x-\Omega) F_2 W = 0,$$

and

$$\lim_{x\to\Omega} \left(\tilde{F}_1(x)W(x) - W\tilde{F}_1^*(x)\right) = \lim_{x\to\Omega} \left((x-\Omega)(F_1 W - WF_1^*) + F_2 W - WF_2^*\right) = 0,$$

completing the proof. ■

Proposition 3.2. The differential operator T commutes with the band-limiting operator χ_Ω.

Proof. Let us observe that $T\chi_\Omega = \chi_\Omega T$ if and only if $(f\,T)\chi_\Omega = (f\chi_\Omega)T$, for all smooth enough $f \in L^2(W)$. Since the operator T is symmetric with respect to W in $[a, b]$ and also in $[a, \Omega]$ we have

$$\begin{aligned}
\langle(\chi_\Omega f)T, g\rangle &= \langle \chi_\Omega f, gT\rangle = \int_a^b \chi_\Omega(x) f(x) W(x) (gT)^*(x)\,\mathrm{d}x = \int_a^\Omega f(x)W(x)(gT)^*(x)\,\mathrm{d}x \\
&= \langle f, gT\rangle_\Omega = \langle f\,T, g\rangle_\Omega = \int_a^\Omega (f\,T)(x)W(x)g^*(x)\,\mathrm{d}x = \int_a^b (f\,T)(x)\chi_\Omega(x)W(x)g^*(x)\,\mathrm{d}x \\
&= \langle f\,T\chi_\Omega, g\rangle,
\end{aligned}$$

for all smooth enough f and g. Hence, T commutes with χ_Ω. ■

Remark 3.3. We observe that if T is a symmetric operator with respect to W in $[a, b]$ then T commutes with χ_Ω if and only if T is symmetric with respect to W in $[a, \Omega]$.

Proposition 3.4. For any $n \geq 0$, there exist matrices X_n, Y_n, and Z_n such that

$$Q_n T = X_n Q_{n+1} + Y_n Q_n + Z_n Q_{n-1}.$$

Moreover, $X_n^* = Z_{n+1}$ and $Y_n^* = Y_n$, with the convention $Q_{-1} = 0$.

Proof. For any n, Q_nT is a polynomial of degree $n+1$ or less, since M is a matrix independent of x. Hence, $Q_nT = \sum_{j=0}^{n+1} K_{n,j}Q_j$, for some matrices $\{K_{n,j}\}$.

It is easy to see that, since T is symmetric, we have

$$\langle Q_nT, Q_j\rangle = \langle Q_n, Q_jT\rangle = 0, \text{ for all } j < n-1.$$

hence

$$Q_nT = \sum_{j=n-1}^{n+1} K_{n,j}Q_j = X_nQ_{n+1} + Y_nQ_n + Z_nQ_{n-1}.$$

Now we observe that $X_n = \langle Q_nT, Q_{n+1}\rangle = \langle Q_n, Q_{n+1}T\rangle = Z_{n+1}^*$ and that $Y_n = \langle Q_nT, Q_n\rangle = \langle Q_n, Q_nT\rangle = Y_n^*$. This concludes the proof. ■

Corollary 3.5. We have

$$X_n = \|R_n\|^{-1}\left(\Lambda_{n+1} + \Lambda_n - \Lambda_{N+1} - \Lambda_N\right)\|R_{n+1}\|,$$

where $\{R_n\}_n$ is the sequence of monic orthogonal polynomials.

In particular $X_N = Z_{N+1} = 0$.

Proof. From (11a), by using the three-term recursion relation (7) and (6), we have

$$\begin{aligned}
\langle R_nT, R_{n+1}\rangle &= \langle (R_n)(xD + Dx - 2\Omega D - (\Lambda_{N+1} + \Lambda_N)x + M), R_{n+1}\rangle \\
&= \langle (R_n)(xD + Dx - (\Lambda_{N+1} + \Lambda_N)x), R_{n+1}\rangle \\
&= \langle R_{n+1}D + R_n\Lambda_nx - R_{n+1}(\Lambda_{N+1} + \Lambda_N), R_{n+1}\rangle
\end{aligned}$$

(and since $\{R_n\}$ is the monic sequence of orthogonal polynomials)

$$\begin{aligned}
&= \langle R_{n+1}\Lambda_{n+1} + R_{n+1}\Lambda_n - (\Lambda_{N+1} + \Lambda_N)R_{n+1}, R_{n+1}\rangle \\
&= (\Lambda_{n+1} + \Lambda_n - \Lambda_{N+1} - \Lambda_N)\langle R_{n+1}, R_{n+1}\rangle.
\end{aligned}$$

Hence, by using that $Q_n = \|R_n\|^{-1}R_n$ we get

$$\begin{aligned}
\langle Q_nT, Q_{n+1}\rangle &= \|R_n\|^{-1}(\Lambda_{n+1} + \Lambda_n - \Lambda_{N+1} - \Lambda_N)\langle R_{n+1}, R_{n+1}\rangle\|R_{n+1}\|^{-1} \\
&= \|R_n\|^{-1}(\Lambda_{n+1} + \Lambda_n - \Lambda_{N+1} - \Lambda_N)\|R_{n+1}\|.
\end{aligned}$$

By Proposition 3.4 we know that $\langle Q_n T, Q_{n+1}\rangle = \langle X_n Q_{n+1}, Q_{n+1}\rangle = X_n$. Thus, the proof is complete. ■

Proposition 3.6. The differential operator T commutes with the time-limiting operator χ_N.

Proof. Let f be a smooth enough function in $L^2(W)$, by using Proposition 3.4, the fact that T is symmetric and the explicit expression in (8) we have

$$f\,T\chi_N = \sum_{n=0}^{N}\langle f\,T, Q_n\rangle Q_n = \sum_{n=0}^{N}\Big(\langle f, Q_{n+1}\rangle X_n^* + \langle f, Q_n\rangle Y_n^* + \langle f, Q_{n-1}\rangle Z_n^*\Big)Q_n.$$

On the other hand,

$$\begin{aligned}(f\chi_N)T &= \sum_{n=0}^{N}\langle f, Q_n\rangle Q_n T = \sum_{n=0}^{N}\langle f, Q_n\rangle\left(X_n Q_{n+1} + Y_n Q_n + Z_n Q_{n-1}\right)\\ &= \sum_{n=1}^{N+1}\langle f, Q_{n-1}\rangle X_{n-1} Q_n + \sum_{n=0}^{N}\langle f, Q_n\rangle Y_n Q_n + \sum_{n=0}^{N-1}\langle f, Q_{n+1}\rangle Z_{n+1} Q_n,\end{aligned}$$

by Corollary 3.5 $X_N = 0$, thus

$$(f\chi_N)T = \sum_{n=0}^{N}\Big(\langle f, Q_{n-1}\rangle X_{n-1} + \langle f, Q_n\rangle Y_n + \langle f, Q_{n+1}\rangle Z_{n+1}\Big)Q_n.$$

Now the proposition follows from the fact that $X_n^* = Z_{n+1}$ and $Y_n^* = Y_n$, see Corollary 3.5. ■

Theorem 3.7. The 2nd order differential operator T is symmetric and commutes with the time-band-limiting operators EE^* and $\mathcal{F}E^*E\mathcal{F}^{-1}$.

Proof. The symmetry of T is proved in Proposition 3.1. Recalling that $EE^* = \chi_\Omega \chi_N \chi_\Omega$ and $\mathcal{F}E^*E\mathcal{F}^{-1} = \chi_N\chi_\Omega\chi_N$, the proof follows from Proposition 3.2 and Proposition 3.6. ■

So far the operators D, S, T act in $L^2(W)$. Conjugating with $\mathcal{F}$ you get difference operators acting in $\ell^2(M_R, \mathbb{N}_0)$. If we define

$$L = \mathcal{F}^{-1}T\mathcal{F},$$

the following result is straightforward.

Corollary 3.8. The difference operator L is given by a tridiagonal hermitian semi-infinite matrix, with $R \times R$-block entries, and it commutes with the time-band-limiting operators $\mathcal{F}^{-1}EE^*\mathcal{F}$ and E^*E. The operator L, in the standard basis of $\ell^2(M_R, \mathbb{N}_0)$, is explicitly given by

$$L = \begin{pmatrix} Y_0 & X_0^* & 0 & 0 & 0 & \cdots \\ X_0 & Y_1 & X_1^* & 0 & 0 & \cdots \\ 0 & X_1 & Y_2 & X_2^* & 0 & \cdots \\ 0 & 0 & X_2 & Y_3 & X_3^* & \cdots \\ 0 & 0 & 0 & X_2 & Y_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with X_j and Y_j given in Proposition 3.4 and Corollary 3.5.

Remark 3.9. From Corollary 3.5 it is clear that L breaks into two blocks, an upper-left block of size $(N+1) \times (N+1)$ yielding a matrix such as the one displayed in [8] and a lower-right block that is semi-infinite.

4 Examples

4.1 Scalar cases

In the scalar case, condition (9) is automatically satisfied. For several examples of a commuting differential operator given by (11a) one can see [11].

4.2 Matrix Gegenbauer weight

In [23] we study 2×2 matrix-valued orthogonal polynomials associated with spherical functions in the q-dimensional sphere S^q (originally q was a natural number, but these results were later extended to any real positive number). The weight matrix, depending on parameters $0 < p < q$, is given by

$$W(x) = (1-x^2)^{\frac{q}{2}-1} \begin{pmatrix} px^2 + q - p & -qx \\ -qx & (q-p)x^2 + p \end{pmatrix}, \qquad x \in [-1, 1].$$

In this case there exist four linearly independent symmetric differential operators of degree 2 in the algebra $D(W)$, namely D_1, D_2, E_3, and E_4. See Section 5 in [23], and the last paragraph in this example.

In [9] we considered the time-band limiting operators E^*E and EE^* for this example. For given N and Ω, we found a symmetric tridiagonal matrix L, with simple spectrum, commuting with the block matrix E^*E and a self-adjoint differential operator $\tilde{D}$ commuting with the integral operator EE^*.

The results on the present paper give a unified way to obtain such a commuting operators in both situations; starting with a symmetric differential operator of order two, we search for a matrix M such that condition (9) is satisfied and we build up the operator T by the formula (11a).

The monic orthogonal polynomials $\{R_n\}$ are eigenfunctions of the differential operators D_1 and D_2, whose eigenvalues are respectively

$$\Lambda_n(D_1) = \begin{pmatrix} (n+p)(n+q-p+1) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda_n(D_2) = \begin{pmatrix} 0 & 0 \\ 0 & (n+p+1)(n+q-p) \end{pmatrix}.$$

For the differential operators D_1 and D_2, the matrices M_1 and M_2 given by

$$M_1 = \frac{(-2N(N+p+1)(N+q-p+1)+q-2p)}{q-2p} \begin{pmatrix} 0 & q-p \\ p & 0 \end{pmatrix},$$

$$M_2 = \frac{(2N(N+p+1)(N+q-p+1)+q-2p)}{q-2p} \begin{pmatrix} 0 & q-p \\ p & 0 \end{pmatrix},$$

satisfy the requirement that

$$\Big(M_1 - x(\Lambda_{N+1}(D_1) + \Lambda_N(D_1))\Big) W(x) \quad \text{and} \quad \Big(M_2 - x(\Lambda_{N+1}(D_2) + \Lambda_N(D_2))\Big) W(x)$$

are symmetric matrices and therefore they give two differential operators, T_1 and T_2, commuting with the time and band-limiting operators.

The differential commuting operator $\widetilde{D}$, given in [9], is a scalar combination of $T_1 + T_2$ and the identity, namely $T_1 + T_2 = -2\widetilde{D} + 2\Omega(q-p)$. Notice that for $T_1 + T_2$ the expression (11a) involves a matrix M that does not depend on N, namely

$$M = M_1 + M_2 = -2 \begin{pmatrix} 0 & q-p \\ p & 0 \end{pmatrix}.$$

On the other hand, the matrices L_1, L_2, L_3 given in [8] are in the span of $\{\mathcal{F}^{-1}T_1\mathcal{F}, \mathcal{F}^{-1}T_2\mathcal{F}, I\}$. Furthermore, L_1 and L_2 scalar multiples of $\mathcal{F}^{-1}T_2\mathcal{F}$ and $\mathcal{F}^{-1}T_1\mathcal{F}$, respectively, and

$$L_3 = \frac{p(q+p+1)}{q+2}(L_1 + L_2).$$

In [8] we proved that L_1 and L_2 have a simple spectrum.

It is worth to notice that for the symmetric differential operators E_3 and E_4 in [9] there is no matrix M satisfying condition (9). This phenomenon, namely that given a weight W one should look at the algebra $D(W)$ introduced in [1], and for each differential operator in it see if a matrix M satisfying condition (9) exist will reappear later in example (4.4). When M exists our general result yields a commuting operator T.

4.3 Completing the proof of the result stated in [2]

In [2] one looks at matrix-valued polynomials that are orthogonal in the interval $[0, 1]$ with respect to the weight density matrix originating in [24, 25] and given by

$$W(x) = (1-x)^{\alpha} x^{\beta} \begin{pmatrix} \beta+1-kx & (\beta+1-k)x \\ (\beta+1-k)x & (\beta+1-k)x^2 \end{pmatrix}.$$

The monic orthogonal polynomials $R_n(x)$, associated to this weight W are eigenfunctions of the symmetric differential operator

$$D = \partial^2 x(1-x) + \partial(C - xU) - V$$

with

$$C = \begin{pmatrix} \beta+1 & 1 \\ 0 & \beta+3 \end{pmatrix}, \; U = \begin{pmatrix} \alpha+\beta+3 & 0 \\ 0 & \alpha+\beta+4 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 \\ k-\beta-1 & \alpha+\beta+2-k \end{pmatrix}.$$

This operator acts on the right and we have $R_n D = \Lambda_n R_n$, with

$$\Lambda_n = \begin{pmatrix} -n(\alpha+\beta+n+2) & 0 \\ 1+\beta-k & -(n+1)(\alpha+\beta+n+2)+k \end{pmatrix}.$$

The main take-home message in [2] is that the differential operator $\widetilde{D}$ given by

$$\widetilde{D} = (x-\Omega)D - \frac{\mathrm{d}}{\mathrm{d}x} x(1-x) + \mathcal{P}_N(x),$$

with

$$\mathcal{P}_N(x) = \begin{pmatrix} x(N^2 + (\alpha+\beta+3)(N+1)) & \alpha+\beta+N+2 \\ x(k-\beta-1) & x(N^2+(\alpha+\beta+4)N+2\alpha+2\beta-k+6)+\beta \end{pmatrix},$$

commutes with the integral operator S given by (3).

The argument given in [2] consists in verifying certain identities depending on an index n. These have been checked with the use of the computer algebra package Maxima up to very large values of n, but no analytical proof is given. We will see below that the results above complete the arguments in [2].

One can see that the matrix

$$M = \begin{pmatrix} 1+\beta & 2(\alpha+\beta)+2N+5 \\ 0 & 3(1+\beta) \end{pmatrix},$$

is such that

$$(M - x(\Lambda_{n+1} + \Lambda_n))W(x)$$

is a symmetric matrix. Therefore, the assumption in (9) is verified and one can check that the commuting operator in [2] is given according to the recipe in (11b)

$$\widetilde{D} = D(x-\Omega) + \partial F_2(x) + \tfrac{1}{2}\left(F_1(x) - x(\Lambda_{N+1}+\Lambda_N) + M\right).$$

4.4 An example violating condition (9)

Consider the matrix-valued polynomials that are orthogonal in the interval $[0, 1]$ with respect to the weight density matrix originating in [6], Section 3.3, with parameters $\alpha = \beta = 0$, $\kappa = 1/2$, $t_0 = 0$, and given by

$$W(x) = \begin{pmatrix} 1+x^2 & 1-x \\ 1-x & (1-x)^2 \end{pmatrix}.$$

We have that

$$D_+ = \partial^2 \begin{pmatrix} 2(x^2-x) & 2x \\ 0 & 0 \end{pmatrix} + \partial \begin{pmatrix} 8x-7 & 7-x \\ x-1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 3 & -5 \\ 1 & 3 \end{pmatrix}$$

is a symmetric differential operator with respect to $W(x)$, ($\phi^+ = 1$ in the notation of [6]).

The monic orthogonal polynomials $R_n(x)$ satisfy $R_n(x)D_+ = \Lambda_n(D_+)R_n(x)$, where the eigenvalues are given by

$$\Lambda_n(D_+) = \begin{pmatrix} 2n^2+6n+3/2 & -n-5/2 \\ n+1/2 & -3/2 \end{pmatrix},$$

see (10). One can check that in this case condition (9) is not satisfied.

We have ample evidence that, for a given N and Ω, the corresponding integral operator commutes with the differential one given by

$$\widetilde{D} = \partial^2 x(x-1)(x-\Omega) + \partial X + Y$$

with

$$X = \begin{pmatrix} 5x^2-4\Omega x-4x+3\Omega & 2(x-\Omega) \\ 0 & 5x^2-4\Omega x-2x+\Omega \end{pmatrix},$$

$$Y = \begin{pmatrix} \Omega/2-3-N(N+4)x & (\Omega+5)/2 \\ (\Omega-1)/2 & -N(N+4)x-\Omega/2 \end{pmatrix}.$$

Clearly this differential operator does not have the form advertised in (11a). We will see below that our explicit construction yields an interesting result.

The weight matrix $W(x)$ admits another symmetric differential operator (with $\phi^- = 1/3$)

$$D_- = \partial^2 \begin{pmatrix} 0 & 2x \\ 0 & 2x(1-x) \end{pmatrix} + \partial \begin{pmatrix} -1 & 3-x \\ x-1 & -8x+3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 5 & -3 \\ 3 & -5 \end{pmatrix}.$$

For D_- condition (9) is, once again, not satisfied.

Nevertheless for the symmetric differential operator

$$\tfrac{1}{2}(D_+ - D_-) = \partial^2 x(x-1) + \partial \begin{pmatrix} 4x-3 & 2 \\ 0 & 4x+1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix},$$

condition (9) is satisfied with $M = \begin{pmatrix} 3 & -3 \\ 1 & -1 \end{pmatrix}$.

We observe that in this case, the eigenvalues of the monic polynomials R_n are given by

$$\Lambda_n = n(n+3) + \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Now it easy to verify that the differential operator $\tilde{D}$ above is exactly the differential operator T given in (11a) for $D = \frac{1}{2}(D_+ - D_-)$, therefore it commutes with the integral operator EE^* (see Theorem 3.7).

At the end of example (4.2), we alluded to the phenomenon seen above; our method can be applied to some of the operators in the algebra $D(W)$ but not necessarily to all of them. When the algebra has several generators this increases our chances of being able to use our construction. The next example features a case when there is only one generator of order two.

4.5 An example showing that bispectrality may not be enough to produce a commuting differential operator

In this section we discuss an example with a behavior quite different from the ones seen so far. This example has appeared in [1].

The weight density on the real line is given by

$$W(x) = e^{-x^2-2x}\begin{pmatrix} e^{4x}+x^2 & x \\ x & 1 \end{pmatrix}.$$

This weight gives rise to a bispectral family of polynomials and as observed in [1] the algebra of differential operators going with this weight has just one generator of order two. See also [5].

One can easily check that condition (9) does not hold in this case for the operator of order two that generates the algebra.

One could still be able to produce, for each value of the parameters N, Ω, a (nontrivial) symmetric 2nd order differential operator that would commute with the kernel

$$k_N(x,y) = \sum_{n=0}^{N} Q_n^*(x)Q_n(y),$$

acting on $(-\infty, \Omega]$, even if this operator is not given by the nice prescription for T above.

We have plenty of evidence that such an operator **does not exist**, at least if we insist that our operator should have polynomial coefficients (this is the case of all known examples so far). Some of this evidence is described below.

We postulate a commuting symmetric 2nd order differential operator of the form

$$D = \partial^2 F_2 + \partial F_1 + F_0,$$

where we allow F_0, F_1, F_2 to be polynomials of degree not higher than 6.

By imposing the necessary condition

$$k_N(x,y)^* D_x = (k_N(x,y) D_y)^*,$$

see (5), we deduce that with arbitrary constants r_1, r_2, r_3 one has

$$F_2(x) = \begin{pmatrix} (Nr_2 - r_1)/(2N) & r_1 x/(2N) \\ 0 & r_2/2 \end{pmatrix},$$

as well as

$$F_1(x) = \begin{pmatrix} (Nr_2 - r_1)(1-x)/N & -(r_1 x^2 + 2Nr_2 x - r_1 x - Nr_2)/N \\ 0 & -r_2(x+1) \end{pmatrix},$$

and finally

$$F_0(x) = \begin{pmatrix} -r_1 + r_3 & r_1 x - r_2 \\ 0 & r_2 + r_3 \end{pmatrix}.$$

When we look at one of the boundary conditions, we get that up to a nonzero scalar the value of

$$F_2(\Omega) W(\Omega)$$

is given by

$$\begin{pmatrix} ((Nr_2 - r_1)e^{4\Omega} + Nr_2\Omega^2)/(2N) & r_2\Omega/2 \\ r_2\Omega/2 & r_2/2 \end{pmatrix},$$

and from here it follows that r_1, r_2 both vanish. This implies that D is a scalar multiple of the identity.

We have not given a proof that a nontrivial commuting differential operator with more complicated coefficients may not exist. However, we are confident that this is the case, since looking at the finite-dimensional block matrix given by E^*E we can verify that the only block-tridiagonal matrix that commutes with it is the identity matrix.

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