The canonical contact structure on the space of oriented null geodesics of pseudospheres and products

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Abstract

Let N be a pseudo-Riemannian manifold such that $\mathcal{L}^0(N)$, the space of all its oriented null geodesics, is a manifold. B. Khesin and S. Tabachnikov introduce a canonical contact structure on $\mathcal{L}^0(N)$ (generalizing the definition given by R. Low in the Lorentz case), and study it for the pseudo-Euclidean space. We continue in that direction for other spaces.

Let $S^{k,m}$ be the pseudosphere of signature (k,m). We show that $\mathcal{L}^0(S^{k,m})$ is a manifold and describe geometrically its canonical contact distribution in terms of the space of oriented geodesics of certain totally geodesic degenerate hypersurfaces in $S^{k,m}$. Further, we find a contactomorphism with some standard contact manifold, namely, the unit tangent bundle of some pseudo-Riemannian manifold. Also, we express the null billiard operator on $\mathcal{L}^0(S^{k,m})$ associated with some simple regions in $S^{k,m}$ in terms of the geodesic flows of spheres.

For N the pseudo-Riemannian product of two complete Riemannian manifolds, we give geometrical conditions on the factors for $\mathcal{L}^0(N)$ to be a manifold and exhibit a contactomorphism with some standard contact manifold.

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1 Introduction

Let N be a complete pseudo-Riemannian manifold. Let γ_u denote the unique geodesic in N with initial velocity u. Two null geodesics γ_u and γ_v are said to be equivalent if there exist $\lambda > 0$ and $t \in \mathbb{R}$ such that $v = \lambda \dot{\gamma}_u(t)$. In particular, they have the same

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trajectory and orientation. We call $\mathcal{L}^0(N)$ the set of all equivalence classes of oriented null geodesics of N.

For $X \in T_pN$ we denote $\|X\| = \langle X, X\rangle$ and $|X| = \sqrt{|\langle X, X\rangle|}$. For $r = 0, 1$, let $T^rN = \{u \in TN \mid \|u\| = r, u \neq 0\}$.

By abuse of notation, we say that $\mathcal{L}^0(N)$ is a manifold if it admits a differentiable structure (not necessarily Hausdorff) such that the projection $\Pi : T^0N \to \mathcal{L}^0(N)$, $\Pi(u) = [\gamma_u]$, is a smooth submersion (throughout the paper, smooth means $\mathcal{C}^\infty$). This is not always the case, see for example the pseudo-Riemannian metric on the torus T^2 given in [8] such that the trajectory of each null geodesic is dense. Nevertheless, infinitesimal considerations at a fixed $[\gamma] \in \mathcal{L}^0(N)$ are always possible, for instance by means of Jacobi fields along γ.

B. Khesin and S. Tabachnikov introduce in [5] a canonical contact structure on $\mathcal{L}^0(N)$, provided that it is a manifold (generalizing the definition given in the Lorentz case by R. Low in [8]), and study it for the pseudo-Euclidean space. We continue in that direction for other spaces such as pseudospheres and some products.

Let $\mathbb{R}^{k+1,m}$ be the pseudo-Euclidean space of signature $(k+1, m)$, that is, $\mathbb{R}^{k+1} \times \mathbb{R}^m$ endowed with the inner product whose norm is given by $\|(u, v)\| = |u|^2 - |v|^2$ (here, $|\cdot|$ denotes the norm of the canonical inner product on the Euclidean space). The pseudosphere of radius 1 in $\mathbb{R}^{k+1,m}$ is the hyperquadric

$$S^{k,m} = \{p \in \mathbb{R}^{k+1,m} \mid \langle p, p\rangle = 1\} = \{(u, v) \in \mathbb{R}^{k+1,m} \mid |u|^2 - |v|^2 = 1\},$$

which is a hypersurface of $\mathbb{R}^{k+1,m}$ with induced metric of signature (k, m). Notice that the Lorentz pseudosphere $S^{k,1}$ is the de Sitter space. The null geodesics of $S^{k,m}$ are straight lines in $\mathbb{R}^{k+1,m}$ with initial velocity in $T^0S^{k,m}$. See other geometric properties of pseudospheres for example in [9].

In section 3 we show that $\mathcal{L}^0(S^{k,m})$ is a manifold and it is contactomorphic to the unit tangent bundle of a certain pseudo-Riemannian manifold. Besides, we describe geometrically its canonical contact distribution in terms of the space of oriented geodesics of some totally geodesic degenerate hypersurfaces in $S^{k,m}$. In this section we also express the null billiard operator on $\mathcal{L}^0(S^{k,m})$ associated with some simple regions in $S^{k,m}$ in terms of the geodesic flow of spheres.

Given M and N complete Riemannian manifolds, we consider on $M \times N$ the pseudo-Riemannian metric whose norm is defined by $\|(u, v)\| = |u|_M^2 - |v|_N^2$, for each $(u, v) \in T_{(p,q)}(M \times N)$ and $(p, q) \in M \times N$. We denote this pseudo-Riemannian manifold by $M_+ \times N_-$. In section 4 we prove that $\mathcal{L}^0(M_+ \times N_-)$ is a manifold if the geodesic flow of M is free and proper. We also find conditions on M for the existence of a contactomorphism between $\mathcal{L}^0(M_+ \times N_-)$ and $\mathcal{L}(M) \times T^1N$, where $\mathcal{L}(M)$ is the space of oriented geodesics of M.

Spaces of geodesics, their geometric structures and their applications have also been studied for instance in [1, 2, 4, 11, 12, 13].

2 Preliminaries

As in the introduction, let N be a complete pseudo-Riemannian manifold and $\mathcal{L}^0(N)$ the set of all equivalence classes of oriented null geodesic of N.

Let $\mathcal{A} = \mathrm{Aff}_+(\mathbb{R})$ be the Lie group of orientation preserving affine transformations of $\mathbb{R}$ and consider the right action from $\mathcal{A}$ on T^0N given as follows: if $u \in T^0N$ and $g \in \mathcal{A}$,

$$u \cdot g := \left.\frac{d}{dt}\right|_0 \gamma_u(g(t)). \tag{1}$$

If this action is free and proper, then $\mathcal{L}^0(N) \simeq T^0N/\mathcal{A}$ is a Hausdorff differentiable manifold such that the canonical projection $\Pi : T^0N \to \mathcal{L}^0(N)$ is a submersion (see for instance Proposition 2.3.8 of [10]).

Let $\pi : TN \to N$ be the canonical projection and for $r = 0, 1$ let $i : T^rN \hookrightarrow TN$ be the inclusion. Let θ and α be the canonical 1-forms on TN and T^rN respectively, that is, for $u \in TN$ and $\xi \in T_uTN$,

$$\theta_u(\xi) = \langle u, d\pi_u\xi\rangle \qquad \text{and} \qquad \alpha = i^*\theta. \tag{2}$$

Definition. [5, 8] *Let N be a pseudo-Riemannian manifold such that $\mathcal{L}^0(N)$ is a manifold. The canonical contact distribution $\mathcal{D}$ on $\mathcal{L}^0(N)$ is well defined by*

$$\mathcal{D}_{\Pi(u)} = d\,\Pi_u(\mathrm{Ker}\,\alpha_u), \tag{3}$$

for each $u \in T^0N$.

The canonical contact structure is presented here following the approach of [8], in a slightly different way as in the article [5] by Khesin and Tabachnikov (they define it in two steps via the space of scaled light-like geodesics, obtaining at the same time a symplectization of $\mathcal{L}^0(N)$).

3 The canonical contact structure on $\mathcal{L}^0(S^{k,m})$

The following theorem is motivated by the fact that unit tangent bundles of pseudo-Riemannian manifolds are among the standard examples of contact manifolds (with contact form as in (2)).

Let $S^k_+ \times S^{m-1}_-$ be the manifold $S^k \times S^{m-1}$ with the pseudo-Riemannian metric such that for each $(x, y) \in T_{(u,v)}(S^k \times S^{m-1})$, $\|(x, y)\| = |x|^2 - |y|^2$.

Theorem 1. *The set $\mathcal{L}^0(S^{k,m})$ is a manifold and if one considers on $\mathcal{L}^0(S^{k,m})$ and $T^1(S^k_+ \times S^{m-1}_-)$ the canonical contact structures, then the map*

$$F : T^1(S^k_+ \times S^{m-1}_-) \to \mathcal{L}^0(S^{k,m}), \;\; F((u, v), (x, y)) = [\gamma],$$

with $\gamma(t) = (x, y) + t(u, v)$, is a contactomorphism.

Proof. First we prove that $\mathcal{L}^0(S^{k,m})$ is a manifold. As explained above, since a straightforward computation yields that the action of $\mathcal{A}$ is clearly free, is suffices to check that the action is proper. In fact, let (p_n, u_n) be a sequence converging to (p, u) in $T^0S^{k,m}$ and let (s_n, λ_n) be a sequence in $\mathbb{R} \rtimes \mathbb{R}_+ \cong \mathcal{A}$ such that $(p_n, u_n) \cdot (s_n, \lambda_n)$ converges to (q, v) in $T^0S^{k,m}$. We have to show that there exists a convergent subsequence of (s_n, λ_n) in $\mathcal{A}$. The footpoints p_n converge to p in $S^{k,m}$ and as the null geodesics in $S^{k,m}$ are straight lines, for each $n \in \mathbb{N}$, $(p_n, u_n) \cdot (s_n, \lambda_n) = (p_n + s_n u_n, \lambda_n u_n)$. Hence, by hypothesis, $\lambda_n u_n \to v$ and $p_n + s_n u_n \to q$ as well. Considering the canonical inner product $\langle\, , \rangle$ on $\mathbb{R}^{k+1+m}$, since $u \neq 0$, we obtain that

$$\lambda_n \to \langle v, u\rangle/|u|^2 \quad \text{and} \quad s_n \to \langle q - p, u\rangle/|u|^2.$$

Next, we verify that F is a diffeomorphism. The map is well defined since given $(x, y) \in T^1_{(u,v)}(S^k_+ \times S^{m-1}_-)$, we have that

$$|u|^2 = 1 = |v|^2, \quad \langle u, x\rangle = 0 = \langle v, y\rangle \quad \text{and} \quad |x|^2 - |y|^2 = 1. \tag{4}$$

Then, $(x, y) \in S^{k,m}$, $(u, v) \in (x, y)^\perp = T_{(x,y)}S^{k,m}$, $\|(u, v)\| = 0$ and $t \mapsto (x, y) + t(u, v)$ is a null geodesic in $S^{k,m}$. Thus, $F((u, v), (x, y)) \in \mathcal{L}^0(S^{k,m})$.

Now, F is smooth since all the spaces involved are (quotients of) embedded submanifolds of $E = \mathbb{R}^{k+1+m} \times \mathbb{R}^{k+1+m}$ and $g : E \to E$, $g((u, v), (x, y)) = ((x, y), (u, v))$, is obviously smooth and descends to F.

On the other hand, if γ is a null geodesic in $S^{k,m}$, then $\gamma(t) = (x, y) + t(u, v)$ with $(x, y) \in S^{k,m}$, $0 \neq (u, v) \perp (x, y)$ in $\mathbb{R}^{k+1,m}$ and $|u|^2 - |v|^2 = 0$. So, we have that

$$F^{-1}([\gamma]) = \big(|u|^{-1}(u, v), (x, y) - |u|^{-2}\langle\, x, u\rangle(u, v)\big) \tag{5}$$

and this is a smooth map.

Finally, we check that F is a contactomorphism, that is $dF(\operatorname{Ker}\omega) = \mathcal{D}$, where $\mathcal{D}$ is defined in (3) and ω is the canonical contact form on $T^1(S^k_+ \times S^{m-1}_-)$ as in (2).

Let $p : T^1(S^k_+ \times S^{m-1}_-) \to S^k_+ \times S^{m-1}_-$ be the canonical projection and let $f : T^1(S^k_+ \times S^{m-1}_-) \to T^0S^{k,m}$ be the restriction of g defined above. Let $U = ((u, v), (x, y)) \in T^1(S^k_+ \times S^{m-1}_-)$ and let $\xi \in \operatorname{Ker}\omega_U$. Since $F = \Pi \circ f$, we only have to verify that $df_U\, \xi \in \operatorname{Ker}\alpha_{f(U)}$. For this, let $t \mapsto (c(t), z(t))$ be a curve in $T^1(S^k_+ \times S^{m-1}_-)$ such that $c(0) = (u, v)$, $z(0) = (x, y)$ and with initial velocity ξ.

By definition of ω, we have that

$$0 = \omega_U(\xi) = \langle dp_U\, \xi, z(0)\rangle = \langle c'(0), z(0)\rangle.$$

Since $(z(t), c(t)) = f(c(t), z(t)) \in T^0S^{k,m}$, it follows that $c(t) \perp z(t)$ in $\mathbb{R}^{k+1,m}$ for all t. Then,

$$0 = \frac{d}{dt}\bigg|_0 \langle c(t), z(t)\rangle = \langle c'(0), z(0)\rangle + \langle c(0), z'(0)\rangle.$$

Therefore,

$$\alpha_{f(U)}\,(df_U\, \xi) = \langle d\pi_{f(U)}(df_U\, \xi), c(0)\rangle = \langle d(\pi \circ f)_U\, \xi, c(0)\rangle = \langle z'(0), c(0)\rangle = 0.$$

Consequently, $dF_U\,\xi \in \mathcal{D}_{F(U)}$, and since both contact distributions have the same dimension, they are equal. $\square$

The following is an analogue of Proposition 2.6 (1) of [5].

Proposition 2. *Let $\gamma(t) = p + tu$ be a null geodesic in $S^{k,m}$. Let H be the totally geodesic degenerate hypersurface of $S^{k,m}$ containing the image of γ, given by $H = u^\perp \cap S^{k,m}$ and let $\mathcal{L}(H)$ be the space of all oriented geodesics of H. If $\mathcal{D}$ is the canonical contact distribution on $\mathcal{L}^0(S^{k,m})$, then, at the infinitesimal level,*

$$\mathcal{D}_{[\gamma]} = T_{[\gamma]}\,\mathcal{L}(H).$$

Proof. The statement is meant in the following sense (we do not address the question whether $\mathcal{L}(H)$ is a manifold): Given $X = d\,\Pi_{[\gamma]}(\xi) \in \mathcal{D}_{[\gamma]}$ (we recall that $\mathcal{D}$ is defined in (3)), there exists a variation by geodesics *contained in* H whose associated Jacobi field along γ satisfies $J(0) = d\pi_u\xi$ and $J'(0) = K_u(\xi)$ (here $K_u : T_uT^0S^{k,m} \to T_{\pi(u)}S^{k,m}$ is the connection operator).

Specifically, since $\xi \in \operatorname{Ker}\alpha_u \subset T_uT^0S^{k,m}$ we have that $\langle d\pi_u\xi, u\rangle = 0 = \langle K_u(\xi), u\rangle$ and this implies that $d\pi_u\xi,\ K_u(\xi) \in T_{\pi(u)}H$. Let c be a curve in H such that $c(0) = \pi(u)$ and $c'(0) = d\pi_u\xi$ and consider

$$s \mapsto v(s) = \tau_0^s(u + sK_u(\xi)),$$

where τ_0^s denotes the parallel transport along c from 0 to s. Since H is totally geodesic and $u + sK_u(\xi) \in T_{\pi(u)}H$ for all $s \in \mathbb{R}$, we obtain that $v(s) \in T_{c(s)}H$ and the image of $\gamma_{v(s)}$ is contained in H for any s (see for instance [9, page 125]). Besides, since

$$v(0) = u \quad \text{and} \quad \left.\frac{D}{ds}\right|_0 v(s) = K_u(\xi),$$

then the Jacobi field $J(s) = \left.\dfrac{d}{dt}\right|_0 \gamma_{v(s)}(t)$ along γ has the desired properties. $\square$

3.1 Billiards

We recall the definition of the null billiard map (see Section 3 of [5]) in a special case. Let N be a complete pseudo-Riemannian manifold and let R be a region in N with smooth nondegenerate boundary M. We require additionally that any null geodesic γ intersecting the interior of R satisfies that $\gamma(\mathbb{R}) \cap R = \gamma([t_0, t_1])$. We call $\mathfrak{L} \subset \mathcal{L}^0(N)$ the set of all oriented null geodesics intersecting the interior of R.

Let γ be a null geodesic of N such that $[\gamma] \in \mathfrak{L}$. Decompose $\dot\gamma(t_1)$ into its tangential and normal components, that is, $\dot\gamma(t_1) = u^T + u^\perp$ with $u^T \in T_{\gamma(t_1)}M$ and $u^\perp \in (T_{\gamma(t_1)}M)^\perp$. The null billiard operator B is well defined in the following way:

$$B : \mathfrak{L} \to \mathfrak{L}, \quad B([\gamma]) = [\gamma_w], \text{ with } w = u^T - u^\perp.$$

As in the pseudo-Euclidean case [5], the null billiard operator preserves the contact structure on $\mathcal{L}^0(N)$. For the sake of completeness, we include this fact as a proposition.

Proposition 3. *Let N be a complete pseudo-Riemannian manifold and let R be a region in N as above. Then the canonical contact structure on $\mathcal{L}^0(N)$ is preserved by B.*

Proof. Let $\ell \in \mathfrak{L}$ and $X \in \mathcal{D}_\ell$. By the definition of $\mathcal{L}^0(N)$ we can take $u \in T^0N$ such that $\Pi(u) = \ell$ and $\pi(u) \in M$. There exists $\eta \in \operatorname{Ker} \alpha_u$ such that $d\Pi_u \eta = X$. Since $T_{\pi(u)}N = \mathbb{R}u + T_{\pi(u)}M$, then $d\pi_u\eta = \lambda u + v$, with $v \in T_{\pi(u)}M$ and $\lambda \in \mathbb{R}$. Let $\tau : T_uTN \to T_{\pi(u)}N \times T_{\pi(u)}N$ be the isomorphism given by $\tau(\xi) = (d\pi_u\xi, K_u(\xi))$. Thus, $\xi = \tau^{-1}(v, K_u(\eta))$ satisfies that $\xi \in \operatorname{Ker}\alpha_u$, $d\Pi_u\xi = X$ and $d\pi_u\xi \in T_{\pi(u)}M$. Let c be a curve in M with initial velocity $d\pi_u\xi$. Since $\pi|_{T^0N}$ is a submersion, there exists a curve $t \mapsto u(t)$ in T^0N such that $u(0) = u$, $u'(0) = \xi$ and $\pi(u(t)) = c(t)$. So,

$$0 = \alpha_u(\xi) = \langle u(0), d\pi_{u(0)}u'(0)\rangle = \langle u(0), c'(0)\rangle. \tag{6}$$

We decompose $u(t) = u^T(t) + u^\perp(t)$, where $u^T(t) \in T_{c(t)}M$ and $u^\perp(t) \in (T_{c(t)}M)^\perp$ (we recall that M is supposed to be nondegenerate). Taking $\ell(t) = \Pi(u(t))$, we have

$$dB_\ell X = \left.\frac{d}{dt}\right|_0 B(\ell(t)) = \left.\frac{d}{dt}\right|_0 \Pi(u^T(t) - u^\perp(t)).$$

We observe that $\pi(u^T(t) - u^\perp(t)) = c(t)$. Thus, to see that $dB_\ell X \in \mathcal{D}_{B(\ell)}$ we only have to show that

$$\langle u^T(0) - u^\perp(0), c'(0)\rangle = 0. \tag{7}$$

But, by (6) and the fact that $c'(0) \in T_{c(0)}M$, we obtain that $\langle u^T(0), c'(0)\rangle = 0$, and this implies that (7) holds.

Finally, since $\mathcal{D}$ has constant dimension and dB_ℓ is nonsingular, it follows that $dB_\ell\,\mathcal{D}_\ell = \mathcal{D}_{B(\ell)}$. □

For $c > 0$, let R_c be the region in $S^{k,m}$ given by

$$R_c = \{(u, v) \in S^{k,m} \mid |v| \le c\},$$

with boundary $M_c = \{(u, v) \in S^{k,m} \mid |v| = c\}$, which is nondegenerate since $V(u, v) = (c^2u, (1 + c^2)v)$ is an outside pointing normal light-like vector field.

We write the null billiard operator B via F of Theorem 1, in terms of the geodesic flow of spheres. For this, we consider the map

$$i : T^1(S^k_+ \times S^{m-1}_-) \to TS^k \times TS^{m-1}, \quad i((u, v), (x, y)) = ((u, x), (v, y)).$$

As before, we call $\mathfrak{L}$ the set of all oriented null geodesics in $S^{k,m}$ that intersect the interior of R_c and denote $L = i \circ F^{-1}(\mathfrak{L}) \subset TS^k \times TS^{m-1}$.

We call φ and ψ the geodesic flows of S^k and S^{m-1}, respectively.

Proposition 4. *Let $\tilde{B} : L \to L$ be the conjugate of the null billiard operator on $\mathfrak{L}$ by the map $i \circ F^{-1}$. Then,*

$$\tilde{B}((u, x), (v, y)) = (\,|x|\,\varphi_{2\theta_x}(u, x/|x|),\ |y|\,\psi_{2\theta_y}(v, y/|y|)), \tag{8}$$

where θ_x, $\theta_y \in (-\frac{\pi}{2}, 0]$ are such that $|x| \tan\theta_x = -\sqrt{c^2 - |y|^2} = |y| \tan\theta_y$.

Proof. Let $((u,x),(v,y)) \in L$. Using (4), we find that $t_1 = \sqrt{c^2 - |y|^2}$ is as in the definition of the null billiard operator. So, we have that $F((u,v),(x,y)) = [\gamma]$ with $\gamma(t) = (x,y) + t_1(u,v) + t(u,v)$ and we can decompose the vector (u,v) into its tangential and normal parts at $\gamma(0)$. Indeed,

$$(u,v)^T = \left(\frac{1}{1+c^2}(|x|^2 u - t_1 x), \frac{1}{c^2}(|y|^2 v - t_1 y) \right)$$

$$\text{and} \qquad (u,v)^\perp = \left(\frac{t_1}{1+c^2}(t_1 u + x), \frac{t_1}{c^2}(t_1 v + y) \right).$$

Then, by definition of B and using the expression for the inverse of F given in (5), we obtain that $\tilde{B}((u,x),(v,y)) = ((u',x'),(v',y'))$, where

$$\begin{aligned} (u',x') &= \left(\frac{|x|^2 - t_1^2}{1+c^2} u - \frac{2t_1|x|}{1+c^2} \frac{x}{|x|}, |x| \left(\frac{2t_1|x|}{1+c^2} u + \frac{|x|^2 - t_1^2}{1+c^2} \frac{x}{|x|} \right) \right) \\ &= |x|\, \varphi_{2\theta_x}(u, x/|x|), \end{aligned}$$

with θ_x such that $\tan\theta_x = -t_1/|x|$, and

$$\begin{aligned} (v',y') &= \left(\frac{|y|^2 - t_1^2}{c^2} v - \frac{2t_1|y|}{c^2} \frac{y}{|y|}, |y| \left(\frac{2t_1|y|}{c^2} v + \frac{|y|^2 - t_1^2}{c^2} \frac{y}{|y|} \right) \right) \\ &= |y|\, \psi_{2\theta_y}(v, y/|y|), \end{aligned}$$

with θ_y such that $\tan\theta_y = -t_1/|y|$. □

Corollary 5. (Lorentz case) *Let $\tilde{B}$ be the conjugate of the null billiard operator on $\mathcal{L}^0(S^{k,1})$ by the identifications $\mathcal{L}^0(S^{k,1}) \simeq T^1(S^k_+ \times S^0_-) \simeq T^1S^k \times \{-1,1\}$, then*

$$\tilde{B}((u,x),\varepsilon) = (\varphi_{-2\arctan(c)}(u,x), -\varepsilon),$$

where $u \in S^k$, $x \perp u$ and $\varepsilon = \pm 1$.

4 The canonical contact structure on $\mathcal{L}^0(M_+ \times N_-)$

Let M and N be complete Riemannian manifolds. Let $M_+ \times N_-$ be the manifold $M \times N$ with the pseudo-Riemannian metric whose norm is defined by $\|(u,v)\| = |u|^2_M - |v|^2_N$, for each $(u,v) \in T_{(p,q)}(M \times N)$ and $(p,q) \in M \times N$.

Let $\mathcal{L}(M)$ be the space of oriented geodesics of M, that is, the quotient of T^1M by the action of $\mathbb{R}$ on it determined by the geodesic flow of M.

We call p_1, p_2 the projections of $\mathcal{L}(M) \times T^1N$ onto the first and second factors, respectively, and let α_1 and α_2 be the canonical 1-forms on T^1M and T^1N, respectively, defined as in (2).

Theorem 6. *Let M and N be complete Riemannian manifolds such that the geodesic flow of M is free and proper. Then, $\mathcal{L}^0(M_+ \times N_-)$ is a manifold. Suppose additionally that there exists a smooth global section $S : \mathcal{L}(M) \to T^1M$. Then $\theta_S = p_1^* S^* \alpha_1 - p_2^* \alpha_2$ is a contact 1-form on $\mathcal{L}(M) \times T^1N$ and the map*

$$G : \mathcal{L}(M) \times T^1N \to \mathcal{L}^0(M_+ \times N_-), \quad G([\sigma], v) = [(\gamma_{S([\sigma])}, \gamma_v)]$$

is a contactomorphism, where $\mathcal{L}^0(M_+ \times N_-)$ is endowed with its canonical contact structure.

Proof. First, notice that $\mathcal{L}(M) = T^1M/\mathbb{R}$ is a manifold since the geodesic flow of M is free and proper. Now, $\mathcal{L}^0(M_+ \times N_-)$ is also a manifold since the right action from $\mathcal{A}$ on $T^0(M_+ \times N_-)$ defined in (1) turns out to be proper and free. Indeed, the action is free due to the fact that the geodesics have constant speed and the geodesic flow of M is free. On the other hand, given a sequence (u_n, v_n) converging to (u, v) in $T^0(M_+ \times N_-)$ and a sequence (s_n, λ_n) in $\mathbb{R} \rtimes \mathbb{R}_+ \cong \mathcal{A}$ such that the sequence $(u_n, v_n) \cdot (s_n, \lambda_n) = (\lambda_n \dot\gamma_{u_n}(s_n), \lambda_n \dot\gamma_{v_n}(s_n))$ converges to (z, w) in $T^0(M_+ \times N_-)$, then we have that

$$\lambda_n \dot\gamma_{u_n}(s_n) \to z \quad \text{and} \quad u_n \to u$$

in TM. So,

$$\lambda_n |\dot\gamma_{u_n}(s_n)| \to |z| \quad \text{and} \quad |\dot\gamma_{u_n}(s_n)| \to |u| \neq 0,$$

and then $\lambda_n \to |z|/|u|$. Furthermore, since

$$\dot\gamma_{u_n/|u_n|}(|u_n| s_n) = |u_n|^{-1}\, \dot\gamma_{u_n}(s_n) \quad \text{and} \quad \dot\gamma_{u_n}(s_n) = \lambda_n^{-1}(\lambda_n\, \dot\gamma_{u_n}(s_n)) \to |u| z/|z|,$$

we obtain that

$$\dot\gamma_{u_n/|u_n|}(|u_n| s_n) \to z/|z|$$

in T^1M. Since the sequence $u_n/|u_n|$ converges to $u/|u|$ in T^1M and the geodesic flow of M is proper, there exits a subsequence $|u_{n_j}| s_{n_j}$ converging to some s in $\mathbb{R}$. Therefore, $(s_{n_j}, \lambda_{n_j}) \to (s/|u|, |z|/|u|)$ in $\mathcal{A}$, and so the action is proper.

To verify that $(\mathcal{L}(M) \times T^1N, \theta_S)$ is a contact manifold we show that G is a diffeomorphism such that $dG(\operatorname{Ker} \theta_S) = \mathcal{D}$, where $\mathcal{D}$ is the contact distribution as in (3).

Let $h : T^1M \times T^1N \to T^0(M_+ \times N_-)$ be the canonical inclusion. Since $G = \Pi \circ h \circ (S \times \mathrm{id})$ and any of these maps is smooth, we obtain that G is smooth.

Let $\pi_M : T^1M \to \mathcal{L}(M)$ be the canonical projection. Under the hypothesis on the geodesic flow of M, $(T^1M, \pi_M, \mathcal{L}(M))$ is an $\mathbb{R}$-principal bundle (see for instance [10, Proposition 2.3.8 (iii)]). So, there exists a smooth map $x : T^1M \to \mathbb{R}$ such that $S(\pi_M(u)) = \dot\gamma_u(x(u))$. Then, if γ and σ are geodesics in M and N, respectively, such that $[(\gamma, \sigma)] \in \mathcal{L}^0(M_+ \times N_-)$, we have that

$$G^{-1} : \mathcal{L}^0(M_+ \times N_-) \to \mathcal{L}(M) \times T^1N, \quad G^{-1}([(\gamma, \sigma)]) = ([\gamma_u], \dot\gamma_v(x(u))),$$

where $u = \dot\gamma(0)/|\dot\gamma(0)| \in T^1M$ and $v = \dot\sigma(0)/|\dot\sigma(0)| \in T^1N$. Since $G^{-1}\circ\pi_M$ is smooth and π_M is a submersion, it follows that G^{-1} is a smooth map. Therefore, G is a diffeomorphism.

Finally, we check that $dG(\mathrm{Ker}\,\theta_S) = \mathcal{D}$. For this, let $p = ([\sigma], v) \in \mathcal{L}(M)\times T^1N$ and take $(\xi,\eta) \in \mathrm{Ker}\,(\theta_S)_p$. Let $t \mapsto (\ell_t, v_t)$ be a curve in $\mathcal{L}(M)\times T^1N$ such that $(\ell_0, v_0) = p$ and $(\ell_0', v_0') = (\xi,\eta)$. Since $G(\ell_t, v_t) = \Pi(S(\ell_t), v_t)$, then

$$dG_p(\xi,\eta) = \frac{d}{dt}\bigg|_0 G(\ell_t, v_t) = d\,\Pi_{(S([\sigma]),v)}\,\frac{d}{dt}\bigg|_0 (S(\ell_t), v_t).$$

By definition of $\mathcal{D}$, we only have to verify that $X = \frac{d}{dt}\big|_0 (S(\ell_t), v_t)$ is in $\mathrm{Ker}\,\alpha_{(S([\sigma]),v)}$. If we call $\pi^1 : T^1M \to M$ and $\pi^2 : T^1N \to N$ the canonical projections, we have that

$$d\pi_{(S([\sigma]),v)}X = (d\pi^1_{S([\sigma])}(dS_{[\sigma]}\xi), d\pi^2_v(\eta)).$$

Then,

$$\begin{aligned}
\alpha_{(S([\sigma]),v)}(X) &= \langle (S([\sigma]), v), d\pi_{(S([\sigma]),v)}X\rangle \\
&= \langle S([\sigma]), d\pi^1_{S([\sigma])}(dS_{[\sigma]}\xi)\rangle_M - \langle v, d\pi^2_v(\eta)\rangle_N \\
&= (S^*\alpha_1)_{[\sigma]}(\xi) - (\alpha_2)_v(\eta) \\
&= (p_1^*S^*\alpha_1 - p_2^*\alpha_2)_{(S([\sigma]),v)}(\xi,\eta) \\
&= (\theta_S)_p(\xi,\eta) = 0.
\end{aligned}$$

Hence, $dG_p(\xi,\eta) \in \mathcal{D}_{G(p)}$. Since $dG(\mathrm{Ker}\,\theta_S)$ and $\mathcal{D}$ have the same dimension, we obtain their equality. Consequently, since $\mathcal{D}$ is a contact distribution, θ_S is a contact 1-form on $\mathcal{L}(M)\times T^1N$ and G is a contactomorphism. □

Example 1. Writing $\mathbb{R}^{n,k} = \mathbb{R}^n_+ \times \mathbb{R}^k_-$ one has $\mathcal{L}^0(\mathbb{R}^{n,k}) \simeq \mathcal{L}(\mathbb{R}^n)\times T^1\mathbb{R}^k \simeq TS^{n-1}\times \mathbb{R}^k \times S^{k-1}$. Proposition 2.6 (2) in [5] gives another presentation of $\mathcal{L}^0(\mathbb{R}^{n,k})$, in terms of 1-jets, which has the advantage of being natural.

Example 2. If M is either a Hadamard manifold or the paraboloid of revolution $\{(x, y, x^2+y^2) \mid x, y \in \mathbb{R}\}$, then $\mathcal{L}(M)$ is a manifold and has a smooth section into T^1M, and hence it satisfies the hypotheses of Theorem 6.

Suppose first that M is a Hadamard manifold. The geodesic flow of M is free since the exponential map is a diffeomorphism at every point. Besides, given a sequence (p_n, v_n) converging to (p, v) in T^1M and a sequence t_n in $\mathbb{R}$ such that $(\gamma_{v_n}(t_n), \dot\gamma_{v_n}(t_n))$ converges to (q, u), we have that $d(p_n, \gamma_{v_n}(t_n)) = |t_n|$, because geodesics in M minimize the distance. Since the distance is a continuous map, it follows that $|t_n| \to d(p, q)$. Then the sequence t_n has a convergent subsequence and the geodesic flow of M is proper. Therefore, $\mathcal{L}(M)$ is a manifold.

Fixing $p \in M$, let $H : T^1_pM \to \mathcal{L}(M)$ be the map defined as follows: Let $X \in T^1_pM$ and $Y \in T_pM$ with $X \perp Y$, then $H(X, Y)$ is the oriented geodesic with initial point $\exp_p(Y)$ and initial velocity the parallel transport of X along the geodesic $t \mapsto$

$\exp_p(tX)$. Proposition 4.14 of [3] asserts that H is a diffeomorphism. Thus, there exists a global section from $\mathcal{L}(M)$ into T^1M, namely, S assigns to each oriented unit speed geodesic of M its velocity at the closest point to p.

Now, let M be the paraboloid of revolution. The geodesic flow φ_t is free since M has no periodic geodesics (see [6, Example 2.9.2]). Next, we show that it is proper. Suppose that $u_n \to u$ and $\varphi_{t_n}(u_n) \to z$ in T^1M. Let $c > 0$ such that the footpoints of u and z belong to the interior of $C = \{p \in M \mid z \leq c\}$. Hence, for $n \geq N$ the footpoints of u_n and $\varphi_{t_n}(u_n)$ also belong to the interior of C. Now, again by [6, Example 2.9.2], C is totally convex. Hence, by Proposition 2.9.14 in [6], there exists $L > 0$ such that every geodesic segment in C has length $\leq L$. In particular, $|t_n| \leq L$, since $|t_n|$ is the length of the geodesic segment $\gamma_{u_n}|_{I_n}$, where $I_n = [0, t_n]$ for $t_n > 0$ and $I_n = [t_n, 0]$ for $t_n < 0$. Therefore, t_n has a convergent subsequence.

The existence of a smooth global section is proved in an analogous way as for a Hadamard manifold. Notice that each geodesic in the paraboloid which is not a meridian has an infinite number of self-intersections.

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