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Numerical approximation of equations involving minimal/maximal operators by successive solution of obstacle problems

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ABSTRACT

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let L_i, $i = 1, 2$, be two elliptic operators of the form

$$L_i u(x) := -\operatorname{div}(A_i(x)\nabla u(x)) + c_i(x)\, u(x) - f_i(x)\,.$$

Motivated by the results in Blanc et al. (2016), we propose a numerical iterative method to compute the numerical approximation to the solution of the minimal problem

$$\begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The convergence of the method is proved, and numerical examples illustrating our results are included.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a polygon with largest interior angle less than or equal to $\pi/2$, and let L_i, $i = 1, 2$, be two elliptic operators of the form

$$L_i u(x) := -\operatorname{div}(A_i(x)\nabla u(x)) + c_i(x)\, u(x) - f_i(x)\,,$$

where $A_i := \left[a_{jk}\right]_{2\times 2}$ with $a_{jk} \in C^1(\overline{\Omega})$, $0 \le c_i \in L^\infty(\Omega)$ and $f_i \in L^p(\Omega)$ for some $p > 2$. Assume also that the operators are uniformly elliptic, that is, there exist $\Lambda, \lambda > 0$ such that $\Lambda|\xi|^2 \ge \langle A_i(x)\,\xi, \xi\rangle \ge \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^2$.

Although for simplicity we confine our analysis to two-dimensional polygons, one should be able to obtain similar results for $C^{1,1}$ domains Ω in $\mathbb{R}^3$, approximating Ω with a sequence of polyhedrons Ω_h, proceeding as in [1].

Our interest here is to find a numerical approximation for the problem

$$(P) \qquad \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Analogous results can be obtained for

$$\begin{cases} \max\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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but we concentrate on (P). Notice that, since we assumed uniform ellipticity on the operators L_1 and L_2, then also $\min\{L_1u, L_2u\}$ and $\max\{L_1u, L_2u\}$ are uniformly elliptic and hence we have existence and uniqueness of viscosity solutions to our problems, see [2]. Also remark that, in spite of the fact that L_1 and L_2 are assumed to be in divergence form, $\min\{L_1u, L_2u\}$ and $\max\{L_1u, L_2u\}$ are not in general in divergence form.

Maximal and minimal operators appear naturally in the literature as prototypes of fully nonlinear second order PDEs. For example, when one considers the family of uniformly elliptic second order operators of the form $-tr(AD^2u)$ and looks for maximal operators, one finds the so-called Pucci maximal operators, $P^{+}_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} - tr(AD^2u)$ and $P^{-}_{\lambda,\Lambda}(D^2u) = \min_{A\in\mathcal{A}} - tr(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ. This maximal operator plays a crucial role in the regularity theory for uniformly elliptic second order operators, see [2].

In [3], the authors show that one can obtain the solution to (P) by taking the limit of a sequence constructed iterating obstacle problems alternating the involved operators L_1 and L_2 with the previous term in the sequence as obstacle. More precisely, let u_1 be the unique solution of

$$\begin{cases} L_1u_1 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and let $u_2 := \mathcal{O}(L_2, u_1)$ be the unique solution of the obstacle problem with L_2 as operator and u_1 as obstacle, that is,

$$(P_{L_2,u_1}) := \begin{cases} u_2 \geq u_1 & \text{in } \Omega, \\ L_2u_2 \geq 0 & \text{in } \Omega, \\ L_2u_2 = 0 & \text{in } \{u_2 > u_1\}, \\ u_2 = 0 & \text{on } \partial\Omega; \end{cases}$$

or equivalently,

$$\begin{cases} \min\{L_2u_2, u_2 - u_1\} = 0 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Inductively, let us define u_n, $n \geq 2$, to be the solution of the obstacle problem

$$u_n := \begin{cases} \mathcal{O}(L_1, u_{n-1}) & \text{if } n \text{ is odd}, \\ \mathcal{O}(L_2, u_{n-1}) & \text{if } n \text{ is even}. \end{cases}$$

It was proved in [3, Theorem 1.1] that u_n is an increasing sequence that converges uniformly to the viscosity solution u of the problem (P).

In this work, inspired by the ideas in [3], we propose a numerical iterative method to compute an approximation to the solution to (P). Moreover, we prove that the proposed numerical solution converges to the solution of (P). More precisely, given some partition $\mathcal{T}_h$ of Ω, let us denote by S^h the standard piecewise linear finite element space, and let $u_1^h \in S^h$ be the approximation of the exact solution u_1, that is,

$$\begin{cases} L_1u_1^h = 0 & \text{in } \Omega, \\ u_1^h = 0 & \text{on } \partial\Omega, \end{cases}$$

where the solution is understood in a suitable weak sense (see Section 2.2). Analogously, we set

$$u_n^h := \begin{cases} \mathcal{O}^h\left(L_1, u_{n-1}^h\right) & \text{if } n \text{ is odd}, \\ \mathcal{O}^h\left(L_2, u_{n-1}^h\right) & \text{if } n \text{ is even}, \end{cases} \tag{1.2}$$

where by $\mathcal{O}^h\left(L, \phi^h\right)$ we denote the discretization of $\mathcal{O}(L, \phi)$. We remark that $u_n^h \in S^h$ and the condition $u_n^h \geq u_{n-1}^h$ is imposed only at the nodes of the triangulation. For the precise definitions and more details see Section 2.2.

We will show in Corollary 4.2 that if u is the solution of problem (P) and u_n^h is given by (1.2), then there exists $h_n > 0$ with $h_n \to 0$ such that

$$\lim_{n\to\infty} \left\| u_n^{h_n} - u \right\|_{L^\infty(\Omega)} = 0.$$

Let us mention that recently, in [4], the authors study the numerical analysis of second order elliptic Hamilton–Jacobi–Bellman (HJB) equations that include, as a particular case, the problem (P). We note however that in [4] it is required that all the coefficients of L_i belong to $C(\overline{\Omega})$, while here we only impose that $c_i \in L^\infty(\Omega)$ and $f_i \in L^p(\Omega)$ with $p > 2$. In fact, in the examples that we include here to illustrate our results, the functions c_i and f_i are chosen in a way such that $c_i, f_i \notin C(\overline{\Omega})$. Moreover, in Example 3 we present a problem in which our algorithm converges even when $f_i \notin L^2(\Omega)$ and with the exact solution of (P) not lying in $W^{2,2}(\Omega)$. For further references regarding the numerical analysis of (HJB) equations we refer to [5,6] and references therein.

To finish this introduction we remark that there is a large number of references dealing with numerical approximations of obstacle problems, we quote the recent papers [7–10] and references therein. Observe that any numerical scheme that approximates solutions to obstacle problems (including finite elements) can be iterated to obtain a numerical method for (P). Therefore, the idea presented here is quite flexible. As we have already mentioned, note that, in general, maximal or minimal operators are fully nonlinear ones (due to the presence of the max or min) and hence they are not in divergence form. This makes that classical second order finite element methods are not directly applicable to approximate (P) (instead one has to use finite differences to approximate this problem directly).

The rest of this article is organized as follows. In Section 2 we give the precise formulations for the discrete and continuous problems. In Section 3 we collect some necessary L^∞-error estimates, and we establish a key lemma concerning the stability of the discrete obstacle problem. In Section 4 we prove our main results, and in the last section we present three numerical examples illustrating the behavior of our iterative process.

2. Preliminaries

2.1. Weak formulation of the problems

Throughout the paper, we shall denote by $\|\cdot\|_p$ and $\|\cdot\|_{k,p}$ the usual norms in the spaces $L^p(\Omega)$ and $W^{k,p}(\Omega)$ respectively. For $i = 1, 2$, let $\mathcal{B}_i : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and $F_i : H^1(\Omega) \to \mathbb{R}$ be given by

$$\mathcal{B}_i(u, v) := \int_\Omega \langle A_i \nabla u, \nabla v\rangle + c_i uv \quad \text{and} \quad F_i(v) := \int_\Omega f_i v.$$

As usual, a function $u \in H_0^1(\Omega)$ is called a weak solution of (1.1) if

$$\mathcal{B}_1(u, v) = F_1(v), \qquad \text{for every } v \in H_0^1(\Omega). \tag{2.1}$$

The assumptions on the coefficients of the matrix A_1 and on c_1 guarantee the continuity and coercivity of the bilinear form $\mathcal{B}_1$ in $H_0^1(\Omega) \times H_0^1(\Omega)$ and therefore this elliptic problem admits a unique weak solution u. Moreover, when Ω is a polygon with largest interior angle α and $2 \le p < (1 - \pi/(2\alpha))^{-1}$, then there exists a constant $C_E = C_E(\Omega, p, L_1) > 0$ such that

$$\|u\|_{2,p} \le C_E \|f_1\|_p \tag{2.2}$$

(see [11, Theorem 5.2.7]).

On the other hand, given $\phi \in H_0^1(\Omega) \cap C(\overline{\Omega})$, we call a function $u := \mathcal{O}(L_i, \phi) \in K_\phi := \{w \in H_0^1(\Omega) : w \ge \phi\}$ a weak solution of the obstacle problem $\left(P_{L_i,\phi}\right)$ if

$$\mathcal{B}_i(u, u - v) \le F_i(u - v), \qquad \text{for every } v \in K_\phi. \tag{2.3}$$

It is well known that the obstacle problem admits a unique solution u, see e.g. [12, Chapter II]. Furthermore, assume that the largest interior angle of Ω is less than or equal to $\pi/2$. Then, for any $p \ge 2$, if the source $f_i \in L^p(\Omega)$ and the obstacle $\phi \in W^{2,p}(\Omega)$, arguing as in [13, Theorem 6.3], we have that $u \in W^{2,p}(\Omega)$ and there exists $C_O = C_O(\Omega, p, L_i) > 0$ such that

$$\|u\|_{2,p} \le C_O \left(\|f_i\|_p + \|\phi\|_{2,p}\right). \tag{2.4}$$

2.2. Finite element discretization and formulation of the discrete problems

Let $\{\mathcal{T}_h\}_{0<h<1}$, be a conforming family of triangulations of the domain $\Omega \subset \mathbb{R}^2$, that is, a family of partitions of Ω into triangles $T \in \mathcal{T}_h$, such that if two triangles intersect, they do so at a full vertex/edge of both of them. For each element $T \in \mathcal{T}_h$, let $h_T := \operatorname{diam}(T)$. We shall assume that $h := \max_{T\in\mathcal{T}_h} h_T$ for each mesh $\mathcal{T}_h$; and that the family of triangulations $\{\mathcal{T}_h\}_{0<h<1}$ is shape regular, that is,

$$\sup_{h>0} \sup_{T\in\mathcal{T}_h} \frac{\operatorname{diam}(T)}{\rho_T} < \infty,$$

where ρ_T is the radius of the largest ball contained in T.

The standard piecewise linear finite element space $S^h \subset H^1(\Omega)$ is defined by

$$S^h := \{v \in C(\overline{\Omega}) : v_{|_T} \text{ is linear } \forall\, T \in \mathcal{T}_h\}.$$

For the discretization of the continuous problems we consider the space

$$S_0^h := \{v \in S^h : v = 0 \text{ on } \partial\Omega\}.$$

Observe that $S_0^h \subset H_0^1(\Omega)$.

The discrete counterpart of (2.1) reads:

$$\text{Find } u^h \in S_0^h \text{ such that } \quad \mathcal{B}_1\left(u^h, v^h\right) = F_1\left(v^h\right), \quad \text{for every } v^h \in S_0^h. \tag{2.5}$$

Clearly, this discrete problem has a unique solution for each mesh; the system matrix is not affected by the right-hand side and is invertible because the assumptions on the coefficients guarantee the coercivity of the bilinear form $\mathcal{B}_1(\cdot,\cdot)$ in $S_0^h \times S_0^h$.

Now, let $\mathcal{I}_h : C(\overline{\Omega}) \to S^h$ be the Lagrange interpolation operator. In the case of the obstacle problem $\left(P_{L_i,\phi}\right)$(i.e., (2.3)), the discrete formulation is the following:

$$\text{Find } u^h \in K^h_\phi \text{ such that } \quad \mathcal{B}_i\left(u^h, u^h - v^h\right) \leq F_i\left(u^h - v^h\right), \quad \text{for every } v^h \in K^h_\phi, \tag{2.6}$$

where $K^h_\phi := \{w^h \in S^h_0 : w^h \geq \phi^h\}$ and $\phi^h := \mathcal{I}_h\phi$. It is also well known that the problem (2.6) admits a unique solution u^h (see e.g. [14]), which we denote by $\mathcal{O}^h\left(L_i, \phi^h\right)$.

3. Stability and error analysis for the discrete problems

In this section we establish some pointwise a priori error estimates for both the elliptic and the obstacle problem, and, under an additional condition on $\mathcal{T}_h$, we prove a key stability result for the discrete obstacle problem with respect to the obstacle.

In the sequel, we shall denote by C (or C_k) positive constants which are independent of h (but which may depend on the data of the given problems).

3.1. L^∞-error estimates for the elliptic problem

We start with the following lemma concerning the elliptic problem. The proof can be found for instance in [15, Remark 3.25] or [16, Remark 6.2.3].

Lemma 3.1. *Let $u_1 \in W^{2,2}(\Omega)$ be the solution of (2.1) and $u^h_1 \in \mathcal{S}^h_0$ be the solution of (2.5). Then, there exists $C_1 > 0$ such that*

$$\left\|u_1 - u^h_1\right\|_\infty \leq C_1 h \left\|u_1\right\|_{2,2}. \tag{3.1}$$

3.2. Stability and L^∞-error estimates for the obstacle problem

The goal of this subsection is to prove a stability result and give an analogue pointwise a priori error estimate as the one given in (3.1) for the discretized obstacle problem. To obtain these results, we have to restrict our analysis to triangulations of a special kind.

Given a fixed triangulation $\mathcal{T}_h$ of the domain Ω, denote by $x_1, \ldots, x_{n+m}$ its vertices, where

$$x_l \in \partial\Omega \quad \Leftrightarrow \quad n+1 \leq l \leq n+m.$$

Let $\varphi_1, \ldots, \varphi_{n+m}$ be the nodal basis of the space S^h, i.e., the unique basis with

$$\varphi_j(x_l) = \delta_{l,j}, \qquad 1 \leq l, j \leq n+m.$$

With respect to the nodal basis, a function $v^h \in S^h$ can be written as

$$v^h = \sum_{j=1}^{n+m} v_j\varphi_j, \quad \text{with } v_j = v^h(x_j) \qquad \text{for all } j \in \{1, \ldots, n+m\}.$$

Therefore, if v^h and w^h are functions in S^h,

$$\mathcal{B}_i\left(w^h, v^h\right) = \sum_{l=1}^{n+m}\sum_{j=1}^{n+m} w_l v_j \mathcal{B}_i\left(\varphi_l, \varphi_j\right).$$

Definition 3.2. Let $i = 1, 2$. A triangulation $\mathcal{T}_h$ of the domain Ω is said to satisfy the condition (M) if for all $j \neq l$ with $j = 1, \ldots, n$ and $l = 1, \ldots, n+m$ it holds that

$$\mathcal{B}_i\left(\varphi_l, \varphi_j\right) = \int_\Omega \left\langle A_i\nabla\varphi_l, \nabla\varphi_j\right\rangle + c_i\varphi_l\varphi_j \leq 0. \tag{3.2}$$

Remark 3.3. It is worth mentioning that condition (M) is strongly related to the discrete maximum principle. It is well known that this is a sufficient condition for the validity of the discrete maximum principle for a fully discrete linear simplicial finite element discretization of a reaction–diffusion problem, see [17,18]. The validity of the condition (M) is connected with the dihedral angles of the used simplices and hence it translates into geometric issues. Let us be more precise. Suppose $A_i(x) = a_i(x)\,Id$, where Id denotes the identity matrix. For a given triangle $T \in \mathcal{T}_h$, define the set of indices of basis functions whose support contains T,

$$I(T) := \left\{j \in \mathbb{N} : 1 \leq j \leq n+m,\ T \subset \operatorname{supp}\varphi_j\right\},$$

and let $\iota_T : \{1, 2, 3\} \to I(T)$ denote a bijective local numbering map. We also write the vertices of T as $x_{\iota_T(s)}$, $s = 1, 2, 3$, and by $\varphi_{\iota_T(s)}$ we denote the associated basis functions. We denote by F_s and F_t the two edges of the triangle T opposite to the vertices $x_{\iota_T(s)}$ and $x_{\iota_T(t)}$. The interior dihedral angle α_{st} between F_s and F_t is defined as $\alpha_{st} = \pi - \gamma_{st}$, where $\gamma_{st} \in [0, \pi]$ is the angle between outward normals η_s and η_t to F_s and F_t, respectively. To stress the dependence on the edges, we will write $cos(F_s, F_t)$ for $cos(\alpha_{st})$. Finally, denote the proper lengths/areas by $|F_s|$, $|F_t|$ and $|T|$, and write σ_s for the (positive) height of T above F_s, which satisfies $\sigma_s = \frac{2|T|}{|F_s|}$, relating the area of T to the length of its edges. With this notation, for $s, t \in \{1, 2, 3\}$ with $s \neq t$, we can express the key integrals as follows:

$$\int_T \varphi_{\iota_T(s)}\varphi_{\iota_T(t)} = \frac{|T|}{12} \quad \text{and} \quad \int_T a_i \left\langle \nabla\varphi_{\iota_T(s)}, \nabla\varphi_{\iota_T(t)} \right\rangle = \frac{-cos(F_s, F_t)}{\sigma_s\sigma_t} \int_T a_i.$$

Using the above notation and writing $a_i^T := \int_T a_i$, we have that a triangulation $\mathcal{T}_h$ satisfies condition (M) if for each $T \in \mathcal{T}_h$,

$$-a_i^T \frac{cos(F_s, F_t)}{\sigma_s\sigma_t} + \|c_i\|_\infty \frac{|T|}{12} \leq 0 \quad \text{for } s, t \in \{1, 2, 3\} \text{ with } s \neq t. \tag{3.3}$$

In general, condition (3.3) is satisfied provided all dihedral angles are acute and the mesh is sufficiently fine. In the case of the Poisson problem or pure diffusion problem ($c_i \equiv 0$), the crucial condition (3.3) reduces to

$$cos(F_s, F_t) \geq 0. \tag{3.4}$$

This corresponds to the well-known requirement of nonobtuseness of all dihedral angles in the triangulation $\mathcal{T}_h$. In [18], a condition sharper than (3.3) is given in terms of the stiffness matrices.

In order to prove the stability of the discrete obstacle problem with respect to the obstacle, we need to introduce the concept of discrete supersolutions for problem (2.6). We note that the following definition extends the notion of supersolutions utilized in [12] to the discrete setting.

Definition 3.4. A function $g^h \in S^h$ is a discrete supersolution of problem (2.6) if it holds:

(i) $\mathcal{B}_i\left(g^h, v^h\right) \leq F_i\left(v^h\right)$, for every $v^h \in S_0^h$ with $v^h \leq 0$,
(ii) $g^h \geq \phi^h$ in Ω,
(iii) $g^h \geq 0$ on $\partial\Omega$.

The next two lemmas are adaptations of [1, Theorems 8 and 9], where similar results are proved in the case of the Laplacian operator. Let us point out that the continuous counterpart of Lemma 3.5 can be found in [12, Theorem 6.4, Chapter II].

Lemma 3.5. *Assume that $\mathcal{T}_h$ satisfies the condition (M). Let u^h be the solution of (2.6) with obstacle $\phi^h \in S_0^h$. Then, for every discrete supersolution g^h of (2.6) it holds that $u_h \leq g^h$ in Ω.*

Proof. Let $v_h \in S_0^h$ be defined by

$$v_h(x_l) := \min(u^h(x_l), g^h(x_l)), \qquad \text{for every } l \in \{1, \ldots, n+m\},$$

where $\{x_l\}$ denotes the set of all vertices of the triangulation $\mathcal{T}_h$. It is clear from the construction that $\phi^h \leq v^h \leq u^h$, and therefore $v^h \in K_\phi^h$.

Now, since u^h is the solution of problem (2.6), it satisfies

$$\mathcal{B}_i\left(u^h, u^h - v^h\right) \leq F_i\left(u^h - v^h\right), \tag{3.5}$$

and on the other hand, from the first property in Definition 3.4 we have that

$$\mathcal{B}_i\left(g^h, u^h - v^h\right) \geq F_i\left(u^h - v^h\right). \tag{3.6}$$

Then, subtracting (3.6) from (3.5) we obtain

$$\mathcal{B}_i\left(u^h - g^h, u^h - v^h\right) \leq 0.$$

Let $y_l := u^h(x_l) - g^h(x_l)$ for $l = 1, \ldots, n+m$. Then,

$$\begin{aligned} 0 &\geq \mathcal{B}_i\left(u^h - g^h, u^h - v^h\right) \\ &= \sum_{l=1}^{n+m} y_l \max(0, y_l)\mathcal{B}_i(\varphi_l, \varphi_l) + \sum_{l\neq j}^{n+m} y_l \max(0, y_j)\mathcal{B}_i\left(\varphi_l, \varphi_j\right) \\ &= \sum_{l=1}^{n+m} \max(0, y_l)^2\mathcal{B}_i(\varphi_l, \varphi_l) + \sum_{\substack{l\neq j,\, j=1,\ldots,n, \\ l=1,\ldots,n+m}} y_l \max(0, y_j)\mathcal{B}_i\left(\varphi_l, \varphi_j\right). \end{aligned} \tag{3.7}$$

Now, from the condition (M) we know that for all $l \neq j$ with $j = 1, \dots, n$ and $l = 1, \dots, n+m$ it holds that

$$y_l \max(0, y_j)\mathcal{B}_i\left(\varphi_l, \varphi_j\right) \geq \max(0, y_l)\max(0, y_j)\mathcal{B}_i\left(\varphi_l, \varphi_j\right).$$

Thus, (3.7) implies

$$0 \geq \sum_{l=1}^{n}\sum_{j=1}^{n} \max(0, y_l)\max(0, y_j)\mathcal{B}_i\left(\varphi_l, \varphi_j\right) = \mathcal{B}_i\left(u^h - v^h, u^h - v^h\right) \geq 0$$

and consequently

$$u^h(x_l) - v^h(x_l) = \max(0, u^h(x_l) - g^h(x_l)) = 0 \qquad \forall\, l \in \{1, \dots, n+m\}.$$

Using again the piecewise linearity of the involved functions, we deduce $u^h \leq g^h$ in Ω and this ends the proof. ■

Now, we prove a key stability result for the discrete obstacle problem with respect to the obstacle. This lemma will be useful in the next section.

Lemma 3.6. *Assume $\mathcal{T}_h$ is a triangulation satisfying condition (M). Let ψ, ϕ be two obstacles in S_0^h, and let $u_\psi^h := \mathcal{O}^h\left(L_i, \psi\right)$ and $u_\phi^h := \mathcal{O}^h\left(L_i, \phi\right)$. Then,*

$$\left\|u_\psi^h - u_\phi^h\right\|_\infty \leq \|\psi - \phi\|_\infty.$$

Proof. Let $g^h := u_\phi^h + \|\psi - \phi\|_\infty$. Then, it clearly holds that

$$g^h \in S^h, \quad g^h \geq 0 \text{ on } \partial\Omega, \quad \text{and} \quad g^h \geq u_\phi^h + \psi - \phi \geq \psi.$$

From the definition of the bilinear form $\mathcal{B}_i$ and the variational inequality (2.6), for all $v^h \in S_0^h$ with $v^h \leq 0$ in Ω, we have that

$$\mathcal{B}_i\left(g^h, v^h\right) \leq \mathcal{B}_i\left(u_\phi^h, v^h\right) = \mathcal{B}_i\left(u_\phi^h, u_\phi^h - \left(u_\phi^h - v^h\right)\right) \leq F_i\left(v^h\right).$$

Thus, g^h is a discrete supersolution for the discrete obstacle problem with obstacle ψ. Hence, by Lemma 3.5, we obtain $u_\psi^h \leq g^h = u_\phi^h + \|\psi - \phi\|_\infty$ in Ω and therefore,

$$u_\psi^h - u_\phi^h \leq \|\psi - \phi\|_\infty \quad \text{in } \Omega.$$

Since interchanging the roles of ψ and ϕ yields

$$u_\phi^h - u_\psi^h \leq \|\psi - \phi\|_\infty \quad \text{in } \Omega,$$

the lemma follows. ■

Let us observe that the estimate in the above lemma holds also in the continuous setting for similar obstacle problems, see [12, Theorem 8.5, Chapter 4].

We conclude this section with the following pointwise a priori error estimate for the obstacle problem, for a proof see e.g. [13,14].

Lemma 3.7. *Let $\mathcal{T}_h$ be a triangulation satisfying condition (M) and an obstacle $\phi \in W^{2,p}(\Omega)$, $p > 2$. Let $u \in W^{2,p}(\Omega)$ be the solution of (2.3), and let $u^h \in S_0^h$ be the solution of (2.6). Then there exists a constant $C_D > 0$ such that*

$$\left\|u - u^h\right\|_\infty \leq C_D h^{2-2/p}\left|\log h\right|\left(\|u\|_{2,p} + \|\phi\|_{2,p}\right).$$

4. Convergence of the discrete iteration

We are now in position to prove our main results. Recall that u_1 and u_1^h are the solutions of (2.1) and (2.5) respectively, and that for $n \geq 2$,

$$u_n := \begin{cases} \mathcal{O}\left(L_1, u_{n-1}\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}\left(L_2, u_{n-1}\right) & \text{if } n \text{ is even,} \end{cases}$$
$$u_n^h := \begin{cases} \mathcal{O}^h\left(L_1, u_{n-1}^h\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}^h\left(L_2, u_{n-1}^h\right) & \text{if } n \text{ is even.} \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $\{\mathcal{T}_h\}_{0<h<1}$ be a family of triangulations satisfying condition (M). Let $u_n \in W^{2,p}(\Omega)$, $p \geq 2$, and $u_n^h \in S_0^h$ be as in (4.1). Then, for all fixed $n \geq 2$,*

$$\lim_{h \to 0^+} \left\|u_n^h - u_n\right\|_\infty = 0.$$

Proof. For $n \geq 2$, let $\widetilde{u}_n^h \in S_0^h$ be defined by

$$\widetilde{u}_n^h := \begin{cases} \mathcal{O}^h\left(L_1, \mathcal{I}_h u_{n-1}\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}^h\left(L_2, \mathcal{I}_h u_{n-1}\right) & \text{if } n \text{ is even.} \end{cases} \tag{4.2}$$

That is, $\widetilde{u}_n^h$ is the solution of the discrete obstacle problem with obstacle $\mathcal{I}_h u_{n-1}$. By Lemma 3.7 we have that

$$\left\|\widetilde{u}_n^h - u_n\right\|_\infty \leq C_D h^{2-2/p} \left|\log h\right| \left(\|u_n\|_{2,p} + \|u_{n-1}\|_{2,p}\right). \tag{4.3}$$

Taking into account Lemma 3.6 and (4.3) we deduce that

$$\begin{aligned} \left\|u_n^h - u_n\right\|_\infty &\leq \left\|u_n^h - \widetilde{u}_n^h\right\|_\infty + \left\|\widetilde{u}_n^h - u_n\right\|_\infty \\ &\leq \left\|u_{n-1}^h - \mathcal{I}_h u_{n-1}\right\|_\infty \\ &\quad + C_D h^{2-2/p} \left|\log h\right| \left(\|u_n\|_{2,p} + \|u_{n-1}\|_{2,p}\right) \\ &\leq \left\|u_{n-1}^h - u_{n-1}\right\|_\infty + \|u_{n-1} - \mathcal{I}_h u_{n-1}\|_\infty \\ &\quad + C_D h^{2-2/p} \left|\log h\right| \left(\|u_n\|_{2,p} + \|u_{n-1}\|_{2,p}\right). \end{aligned}$$

Now, for any $v \in W^{2,p}(\Omega)$, $p \geq 2$, there exists a constant $C_L > 0$ such that the Lagrange interpolation satisfies the following estimate (see [19, Remark 4.4.27]):

$$\|v - \mathcal{I}_h v\|_\infty \leq C_L h^{2-2/p} \|v\|_{2,p}.$$

Let us now set $C := \max\left(2, C_D, C_L, C_O, C_E, C_1\right)$. We have

$$\begin{aligned} \left\|u_n^h - u_n\right\|_\infty &\leq \left\|u_{n-1}^h - u_{n-1}\right\|_\infty \\ &\quad + C h^{2-2/p} \left[\|u_{n-1}\|_{2,p} + \left|\log h\right| \left(\|u_n\|_{2,p} + \|u_{n-1}\|_{2,p}\right)\right]. \end{aligned}$$

Repeating this $n-1$ times and applying Lemma 3.1 we arrive at

$$\begin{aligned} \left\|u_n^h - u_n\right\|_\infty &\leq \left\|u_1^h - u_1\right\|_\infty \\ &\quad + C h^{2-2/p} \sum_{j=1}^{n-1} \left[\left\|u_j\right\|_{2,p} + \left|\log h\right| \left(\left\|u_{j+1}\right\|_{2,p} + \left\|u_j\right\|_{2,p}\right)\right] \\ &\leq \left\|u_1^h - u_1\right\|_\infty + 3C h^{2-2/p} \left|\log h\right| \sum_{j=1}^{n} \left\|u_j\right\|_{2,p}. \end{aligned}$$

Also, from (3.1) and (2.2),

$$\begin{aligned} \left\|u_1^h - u_1\right\|_\infty &\leq C_1 h \|u_1\|_{2,2} \\ &\leq C_E C_1 h \|f_1\|_2. \end{aligned}$$

On the other hand, calling $f := \max\left(|f_1|, |f_2|\right)$ and using (2.4) we have

$$\begin{aligned} \sum_{j=1}^{n} \left\|u_j\right\|_{2,p} &\leq \|u_1\|_{2,p} \sum_{j=0}^{n-1} C_O^j + \|f\|_p \sum_{j=1}^{n-1} (n-j)\, C_O^j \\ &\leq \|f\|_p \sum_{j=1}^{n} (n+1-j)\, C^j. \end{aligned}$$

Therefore, since $\|f\|_2 \leq C_\Omega \|f\|_p$ with $C_\Omega := |\Omega|^{\frac{1}{2}-\frac{1}{p}}$ and $C \geq 2$, for all $h > 0$ small enough it holds that

$$\begin{aligned} \left\|u_n^h - u_n\right\|_\infty &\leq C^2 h \|f\|_2 + 3h^{2-2/p} \left|\log h\right| \|f\|_p \sum_{j=1}^{n} (n+1-j)\, C^{j+1} \\ &\leq h \|f\|_p \left[C_\Omega C^2 + 3\left|\log h\right| \sum_{j=1}^{n} (n+1-j)\, C^{j+1}\right] \\ &\leq 4h \|f\|_p \left|\log h\right| \sum_{j=1}^{n} (n+1-j)\, C^{j+1} \\ &= 4h \|f\|_p \left|\log h\right| \frac{C^{n+2} - nC^2}{C-1} \\ &\leq 8h \|f\|_p \left|\log h\right| C^{n+1}. \end{aligned} \tag{4.4}$$

Finally, letting $h \to 0^+$ the theorem follows. ■

As a direct consequence of the above theorem and the convergence result in [3], we have the following corollary. Let us point out that in [3] the solutions are considered in the viscosity sense. However, since our weak solutions u_n lie in $W^{2,p}(\Omega)$, $p \geq 2$, an immediate application of the strong maximum principle for strong solutions (e.g. [20, Theorem 9.6]) shows that they are also viscosity solutions (for general theory of viscosity solutions we refer the reader to [21,22]).

Corollary 4.2. *Let u be the solution of (P) and let u_n^h be as in (4.1). Then, there exists $h_n > 0$ with $h_n \to 0$ such that*

$$\lim_{n\to\infty} \left\| u_n^{h_n} - u \right\|_\infty = 0.$$

Proof. We observe that

$$\left\| u_n^{h_n} - u \right\|_\infty \leq \left\| u_n^{h_n} - u_n \right\|_\infty + \| u_n - u \|_\infty .$$

Let $\varepsilon > 0$. By the convergence result Theorem 1.1 in [3], there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\| u_n - u \|_\infty \leq \frac{\varepsilon}{2}.$$

Taking into account (4.4), it is enough to choose h_n such that

$$h_n \left| \log h_n \right| \leq \frac{\varepsilon}{16 \, \|f\|_p \, C^{n+1}} \quad \text{for all } n \geq n_0$$

and the corollary is proved. ■

5. Numerical experiments

In this section we consider three different numerical examples that document the behavior of the iterative process. We point out that we shall consider simple problems in which we know the exact solution of (P), in order to be able to compare such solution with the numerical approximation.

In the first part we shall test the performance of our algorithm when we vary h and fix n, and vice versa; and in the second part we consider in all the examples the same sequence h_n and we analyze the asymptotic behavior of $\left\| u_n^{h_n} - u \right\|_\infty$.

Let us add that in order to solve each obstacle problem during the iterative process we followed the augmented Lagrangian method proposed in [23, p. 466–467].

In order to avoid repetitions, for the rest of the section we fix

$$\omega := (0, 1) \quad \text{and} \quad \Omega := \omega \times \omega.$$

Example 1. We consider the following operators,

$$L_1 u := -\Delta u + f_1(x, y), \qquad L_2 u := -\operatorname{div}(A(x, y) \nabla u) + f_2(x, y),$$

where

$$f_1(x, y) := \begin{cases} 20\,(1 - 2xy) & \text{if } x \in \left(0, \frac{1}{2}\right) \times \omega, \\ -54\,(yr(y) + r(x)(3y - 1)) & \text{if } x \in \left[\frac{1}{2}, 1\right) \times \omega, \end{cases}$$

$$f_2(x, y) := \begin{cases} 27\,(g(x, y) + h(x, y)) & \text{if } x \in \left(0, \frac{1}{2}\right] \times \omega, \\ 27\,(g(x, y) + h(x, y)) + 10 & \text{if } x \in \left(\frac{1}{2}, 1\right) \times \omega, \end{cases}$$

$$r(t) := t\,(1 - t),$$

$$g(x, y) := r(x)\left(2 - 6y + 4xy - 9xy^2\right),$$

$$h(x, y) := yr(y)\,(y - 2 - 4xy),$$

$$A(x, y) := \begin{pmatrix} 1 + xy & 0 \\ 0 & 1 + xy \end{pmatrix}.$$

One can see that the function

$$u(x, y) := 27r(x)\, yr(y)$$

satisfies that $L_1 u = 0 \leq L_2 u$ if $x \geq 1/2$ and $L_2 u = 0 \leq L_1 u$ if $x < 1/2$, and thus it is the solution of the problem

$$(P_1) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

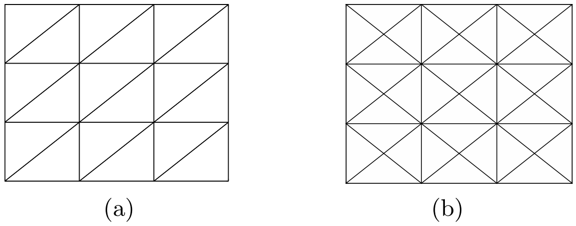

Fig. 1. (a) Triangulation $\mathcal{T}_h$ considered in Examples 1 and 3. (b) Triangulation $\mathcal{T}_h$ considered in Example 2.

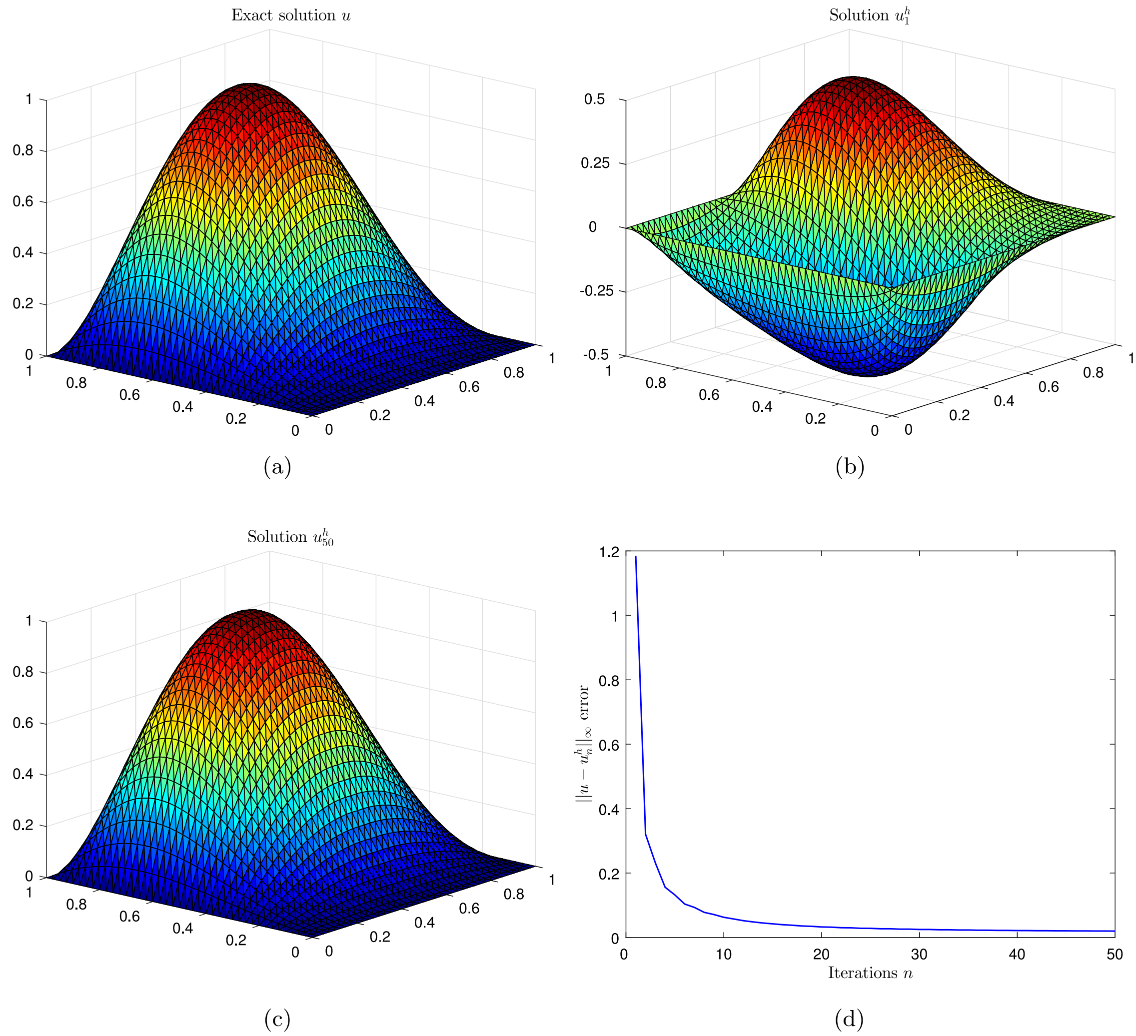

Fig. 2. (a)–(d) Iterative process considering $h = \frac{\sqrt{2}}{40}$ fixed (41 nodes at each boundary side) and varying n from 1 to 50. (a) Exact solution $u(x, y) = 27xy^2\,(1-x)\,(1-y)$. (b) Approximated solution u_1^h. (c) Approximated solution u_{50}^h. In spite of starting with poor initial data, the algorithm is able to give a good approximation of the exact solution. (d) Error $\left\|u - u_n^h\right\|_\infty$ versus the number of iterations n.

For this first example, we consider a uniform fixed mesh with $N+1$ nodes at each boundary, dividing the unit square into N^2 subsquares and then each subsquare is divided into two triangles. Therefore, we have a triangulation $\mathcal{T}_h$ with $h = \frac{\sqrt{2}}{N}$. See Fig. 1(a).

Let us observe that here, since L_1 and L_2 are pure diffusion problems (i.e., $c_1 = c_2 \equiv 0$), the triangulation $\mathcal{T}_h$ satisfies (3.4) and the condition (M) holds.

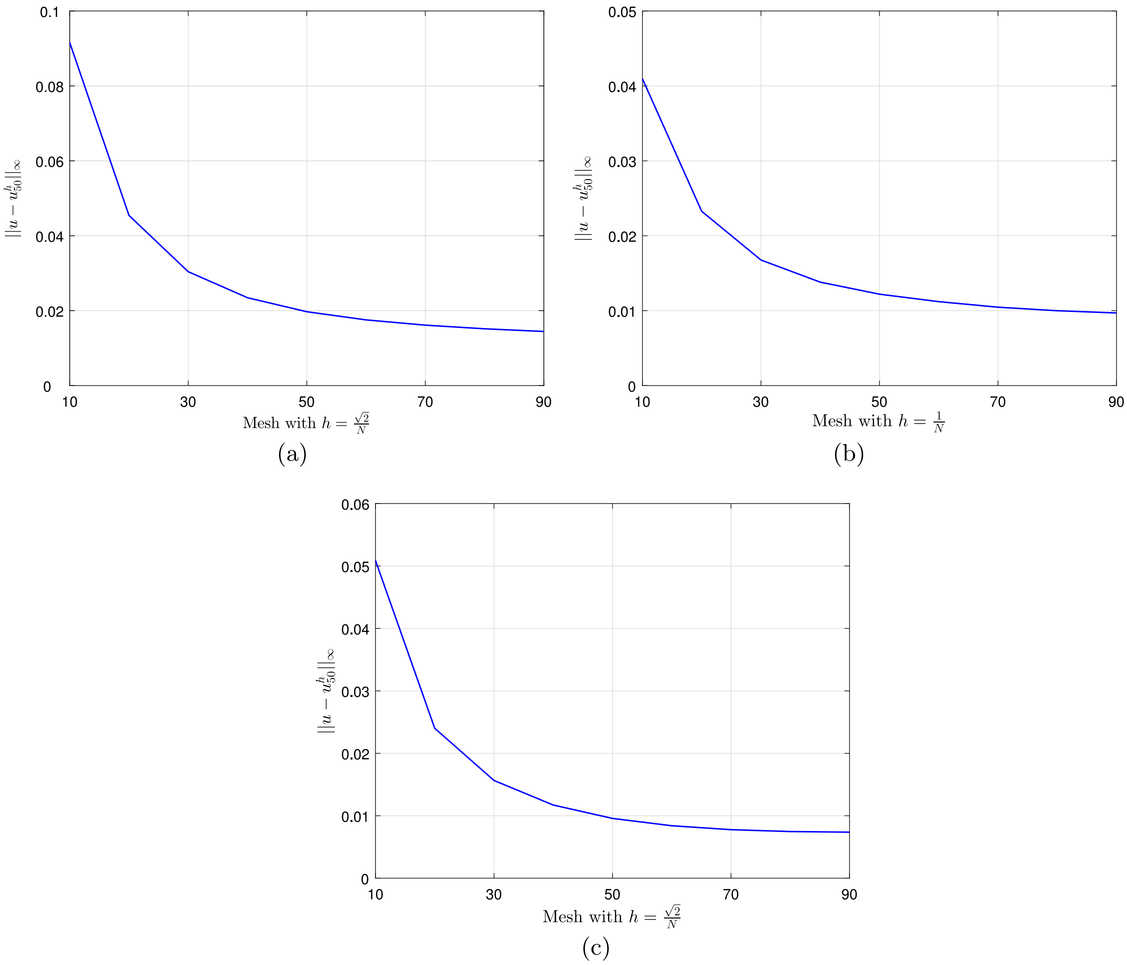

Fig. 3. Plot of the errors $\left\|u - u^h_n\right\|_\infty$. We considered $n = 50$ fixed and varied the mesh diameter h. In Examples 1 and 3 (Figs. (a) and (c)) we used $h = \frac{\sqrt{2}}{N}$ from $N = 10$ to $N = 90$; in Example 2 (Fig. (b)) we used $h = \frac{1}{N}$ from $N = 10$ to $N = 90$.

Next we examine the performance of the iterative process for different values of h and n.

In first place, we ran the algorithm in order to get the numerical solution u^h_n and we compared it with the known exact solution u. In Fig. 2 we show $N = 40$ and $n = 50$: at the top left the exact solution u, at the right the approximate solution u^h_1, and at the bottom left u^h_{50}. One can observe that, in spite of starting with poor initial data, the algorithm is able to give a good approximation of the exact solution. Moreover, at the bottom right we plot the $\left\|u - u^h_n\right\|_\infty$ error versus the number of iterations, and we can see how this error decreases when n increases.

Finally, in Fig. 3(a), we plot $\left\|u - u^h_{50}\right\|_\infty$ for several choices of h. One can also observe, as expected from the theoretical results, that this error gets smaller as h decreases.

Example 2. We consider the operators

$$L_1 u := -\Delta u + c_1(x, y)\, u + f_1(x, y), \qquad L_2 u := -\Delta u + c_2(x, y)\, u + f_2(x, y),$$

where

$$c_1(x, y) := \begin{cases} \pi^2 & \text{if } (x, y) \in \left(0, \frac{3}{10}\right] \times \omega, \\ 2\pi^2 (1 - x) & \text{if } (x, y) \in \left(\frac{3}{10}, 1\right) \times \omega, \end{cases}$$

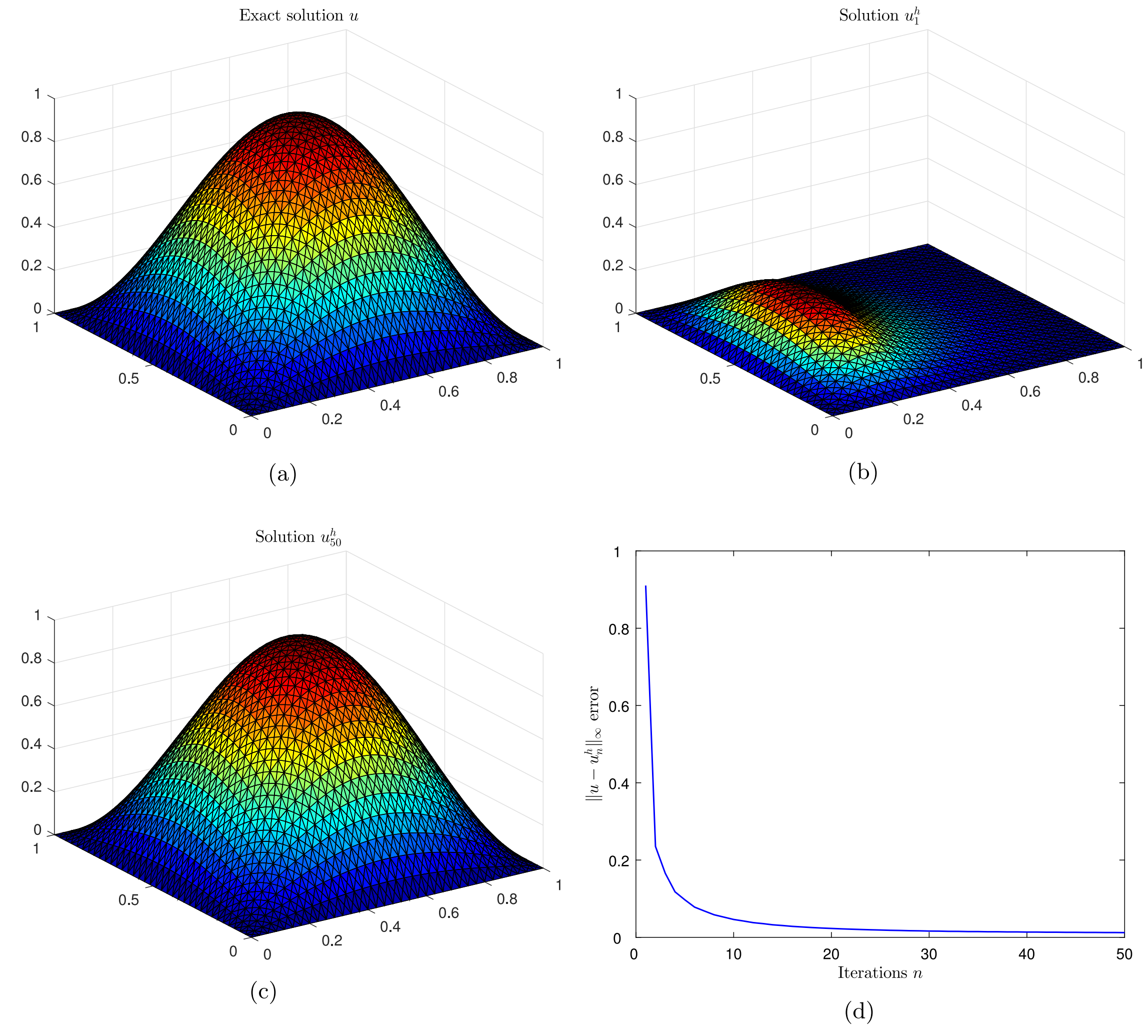

Fig. 4. (a)–(d) Iterative process considering $h=\frac{1}{40}$ fixed and varying n from 1 to 50. (a) Exact solution $u(x,y)=sin(\pi x)sin(\pi y)$. (b) Approximated solution u_1^h. (c) Approximated solution u_{50}^h. In spite of starting with a poor initial data, the algorithm is able to give a good approximation of the exact solution. (d) Error $\|u-u_n^h\|_\infty$ versus the number of iterations n.

$$c_2(x,y) := \begin{cases} 2\pi^2 x & \text{if } (x,y)\in\left(0,\frac{3}{10}\right)\times\omega, \\ \pi^2 & \text{if } (x,y)\in\left[\frac{3}{10},1\right)\times\omega, \end{cases}$$

$$f_1(x,y) := \begin{cases} -3\pi^2\sin(\pi x)\sin(\pi y) & \text{if } (x,y)\in\left(0,\frac{3}{10}\right]\times\omega, \\ 0 & \text{if } (x,y)\in\left(\frac{3}{10},1\right)\times\omega, \end{cases}$$

$$f_2(x,y) := \begin{cases} 0 & \text{if } (x,y)\in\left(0,\frac{3}{10}\right)\times\omega, \\ -3\pi^2\sin(\pi x)\sin(\pi y) & \text{if } (x,y)\in\left[\frac{3}{10},1\right)\times\omega. \end{cases}$$

One can check that the function

$$u(x,y) := \sin(\pi x)\sin(\pi y)$$

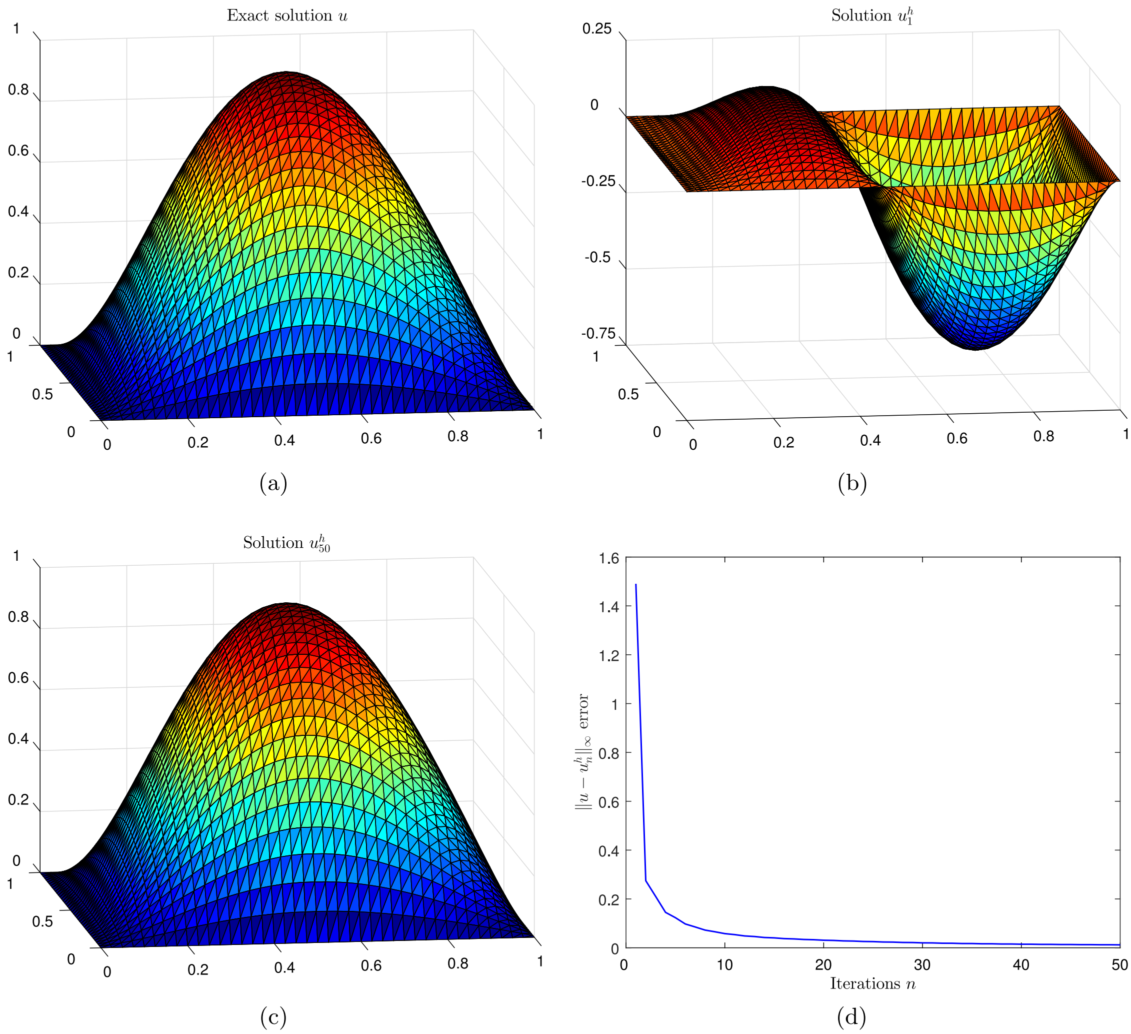

Fig. 5. (a)–(d) Iterative process considering $h = \frac{\sqrt{2}}{40}$ fixed and varying n from 1 to 50. (a) Exact solution $u(x, y) = \widetilde{u}\,(x)\, y\,(1 - y)$. (b) Approximated solution u_1^h. (c) Approximated solution u_{50}^h. In spite of starting with a poor initial data (and although $u \notin W^{2,2}\,(\Omega)$ and $f_1, f_2 \notin L^2\,(\Omega)$), the algorithm is able to give a good approximation of the exact solution. (d) Error $\left\|u - u_n^h\right\|_\infty$ versus the number of iterations n.

satisfies that $L_1 u = 0 \leq L_2 u$ if $x \leq 3/10$ and $L_2 u = 0 \leq L_1 u$ if $x > 3/10$, and therefore u is the solution of the problem

$$(P_2) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For this second example, we built a triangulation in which for every T the interior angles are acute. We consider a uniform fixed mesh with $N+1$ nodes at each boundary, dividing the unit square into N^2 subsquares and then each subsquare is divided into four triangles. Therefore, we have a triangulation $\mathcal{T}_h$ with $h = \frac{1}{N}$. See Fig. 1(b). We point out that a simple computation shows that for all $h > 0$ small enough, (3.3) holds and therefore the triangulation $\mathcal{T}_h$ satisfies the condition (M).

Here we examined the performance of the iterative process for different values of h and n, doing a similar analysis to the one made for Example 1. The results are shown in Figs. 4 and 3(b).

Example 3. We shall present a last example in which the exact solution $u \notin W^{2,2}\,(\Omega)$ (and thus, $u \notin W^{2,p}\,(\Omega)$ for any $p > 2$) and the coefficients $f_1, f_2 \notin L^2\,(\Omega)$. Let us first define

$$\widetilde{u}\,(x) := \begin{cases} 32(x\,(1-x))^{\frac{3}{2}} & \text{if } x \in \left[0, \dfrac{1}{2}\right], \\ 4(4(x - \dfrac{1}{2})^3 - 6(x - \dfrac{1}{2})^2 + 1) & \text{if } x \in \left[\dfrac{1}{2}, 1\right]. \end{cases}$$

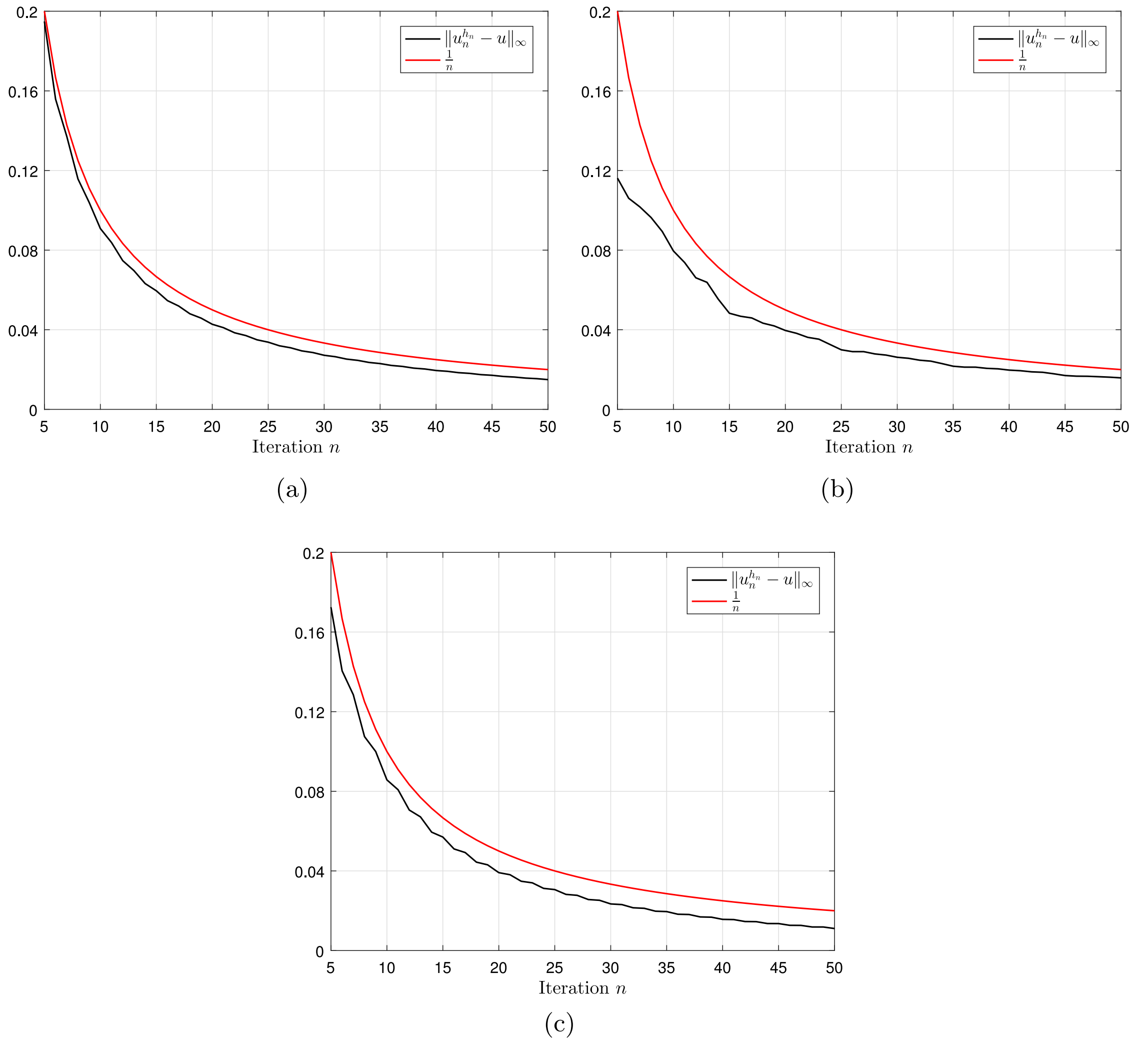

Fig. 6. Plot of the errors $\left\|u - u_n^{h_n}\right\|_\infty$ considering $h_n := \frac{\sqrt{2}}{2n}$. One can observe that for all the previous examples it holds that $\left\|u_n^{h_n} - u\right\|_\infty < \frac{1}{n}$ for all n.

A few computations yield the following facts: $\widetilde{u} > 0$ in ω, $\widetilde{u} = 0$ on $\partial\omega$, $\widetilde{u} \in C^2(\omega) \cap C^1(\overline{\omega})$ and $\widetilde{u} \notin W^{2,2}(\omega)$ (and so, $\widetilde{u} \notin W^{2,p}(\omega)$ for any $p > 2$). We now consider the operators

$$L_1 u := -\Delta u + f_1(x, y), \qquad L_2 u := -\Delta u + f_2(x, y),$$

where

$$f_1 := \begin{cases} 24\dfrac{\left(8x^2 - 8x + 1\right) r(y)}{\sqrt{r(x)}} - 64 r(x)^{\frac{3}{2}} & \text{if } (x, y) \in (0, \frac{1}{2}] \times \omega, \\ 20 & \text{if } (x, y) \in (\frac{1}{2}, 1) \times \omega, \end{cases}$$

$$f_2 := \begin{cases} 24\dfrac{\left(8x^2 - 8x + 1\right) r(y)}{\sqrt{r(x)}} & \text{if } (x, y) \in (0, \frac{1}{2}) \times \omega, \\ 96(x-1) r(y) - 8(4(x - \frac{1}{2})^3 - 6(x - \frac{1}{2})^2 + 1) & \text{if } (x, y) \in [\frac{1}{2}, 1) \times \omega, \end{cases}$$

$$r(t) := t(1 - t).$$

It can be verified that the function

$$u(x, y) := \widetilde{u}(x)\, y(1 - y)$$

satisfies that $L_1 u = 0 \le L_2 u$ if $x \le 1/2$ and $L_2 u = 0 \le L_1 u$ if $x > 1/2$, and so u is the solution of the problem

$$(P_3) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this example we consider a uniform fixed mesh as in Example 1. As there, the condition (M) holds. We examined the performance of our iterative process for different values of h and n, doing an analysis similar to the one made in the previous examples. The results are shown in Figs. 5 and 3(c).

To conclude this section, we recall that Corollary 4.2 yields the existence of some $h_n \to 0^+$ such that $\lim_{n\to\infty} \left\| u_n^{h_n} - u \right\|_\infty = 0$. Utilizing in the three examples the mesh in Fig. 1(a), and choosing $h_n := \frac{\sqrt{2}}{2n}$, we see that

$$\left\| u_n^{h_n} - u \right\|_\infty < \frac{1}{n} \quad \text{for all } n,$$

see Fig. 6. In particular, the asymptotic behavior of the iterative process is, at least, $O(\frac{1}{n})$. Let us also note that in Example 2, although the mesh utilized here does not fulfill (3.3), the algorithm performs as in the other two examples.

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