Journal of Algebra and Its Applications
(2019) 1950036 (23 pages)

DOI: 10.1142/S0219498819500361

Irreducible continuous representations of the simple linearly compact n-Lie superalgebra of type S

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Received 21 September 2017
Accepted 29 January 2018
Published

Communicated by V. Futorny

In the present paper, we classify all irreducible continuous representations of the simple linearly compact n-Lie superalgebra of type S. The classification is based on a bijective correspondence between the continuous representations of the n-Lie algebras S^n and continuous representations of the Lie algebra of Cartan type S, on which some two-sided ideal acts trivially.

Keywords: Representation theory of n-Lie algebra; linearly compact n-Lie superalgebra.

Mathematics Subject Classification: 17B10

1. Introduction

A n-Lie algebra is mainly a vector space endowed with a n-Lie bracket which is taken between n elements instead of two. This new bracket is n-linear, anti-symmetric and satisfies a generalization of the Jacobi identity. It was introduced in 1985 by Filippov [6] as a generalization of a Lie algebra. In [6] and several subsequent papers, [7, 10–12] a structure theory of finite-dimensional n-Lie algebras over a field $\mathbb{F}$ of characteristic 0 was developed. In [12], Ling proved that for every $n \geq 3$ there is, up to isomorphism only one finite-dimensional simple n-Lie algebra, namely $\mathbb{C}^{n+1}$ where the n-ary operation is given by the generalized vector product, namely, if $e_1, \ldots, e_{n+1}$ is the standard basis of $\mathbb{C}^{n+1}$, the n-ary bracket is given by

$$[e_1, \ldots, \hat{e_i}, \ldots, e_{n+1}] = (-1)^{n+i-1} e_i,$$

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where i ranges from 1 to $n+1$ and the hat means that e_i does not appear in the bracket.

In 2010, Cantarini and Kac, [4], stated that there are no simple linearly compact n-Lie superalgebras over an algebraically closed field of characteristic zero, which are not n-Lie algebras and classified simple linearly compact n-Lie superalgebras with $n > 2$. A linearly compact algebra is a topological algebra, whose underlying vector space is linearly compact, namely is a topological product of finite-dimensional vector spaces, endowed with discrete topology (and it is assumed that the algebra product is continuous in this topology). They obtained a list consisting in four examples, one of them being the $n+1$-dimensional vector product n-Lie algebra presented above, and the remaining three are infinite-dimensional n-Lie algebras, (cf. Theorem 3 below, [4]). One of them is the simple linearly compact n-Lie algebra of type S we are interested in.

Also, representation theory for n-Lie algebras was developed. Dzhumadildaev studied in [5] the finite-dimensional irreducible representations of the simple n-Lie algebra $\mathbb{C}^{n+1}$. Balibanu and van de Leur in [1] classified both, finite and infinite-dimensional irreducible highest weight representations of this algebra.

In the present paper, we aim to classify all irreducible continuous representations of the simple linearly compact n-Lie algebra S^n. We tried to apply technics we also used in [3] to classify irreducible representation of the other infinite-dimensional simple linearly compact n-Lie algebra W^n. As in [1], the key idea is to reduce the problem to find irreducible continuous representations of simple linearly compact n-Lie algebra S^n to find irreducible continuous representations of its associated basic Lie algebra on which some two-sided ideal acts trivially.

The paper is organized as follow: In Sec. 2, we give the basic definitions and results related with n-Lie algebras and state the relationship between representations of n-Lie algebras and representations of its associated Lie algebra. In Sec. 3, we introduce the simple linearly compact n-Lie algebra S^n, we identify its associated Lie algebra with the Lie algebra of its inner derivations which is nothing but S_n, the Lie algebra of Cartan type S and finally we relate representations of the n-Lie algebra S^n with representations of S_n. In Sec. 4, we present some general results of the representation theory of S_n, prove some technical lemmas and we describe some generators of the two-sided ideal that must act trivially in our representations. Finally in Sec. 5, we state and prove the main result of the paper.

2. n-Lie Algebras and n-Lie Modules

For completeness and following the presentation of [3], we will give an introduction to basic definitions and notions related with n-Lie algebras and n-Lie modules. We will also introduce some useful results over the correspondence between representations of n-Lie algebra and representations of its basic associated Lie algebra.

As mentioned before, we are interested in studying irreducible representations of the linearly compact n-Lie algebra S^n. Cantarini and Kac stated in [4] that there are no simple linearly compact n-Lie superalgebras over an algebraically closed field of characteristic zero, which are not n-Lie algebras. Then we will use the representation theory of n-Lie algebras to give the representation theory of simple linearly compact n-Lie superalgebras. Given an integer $n \geq 2$, an *n-Lie algebra* V is a vector space over $\mathbb{C}$, the field of complex numbers, endowed with an n-ary anti-commutative product

$$\wedge^n V \to V$$
$$a_1 \wedge \cdots \wedge a_n \mapsto [a_1, \ldots, a_n],$$

subject to the following Filippov–Jacobi identity:

$$\begin{aligned}&[a_1, \ldots, a_{n-1}, [b_1, \ldots, b_n]]\\ &\quad = [[a_1, \ldots, a_{n-1}, b_1], b_2, \ldots, b_n] + [b_1, [a_1, \ldots, a_{n-1}, b_2],\\ &\qquad b_3, \ldots, b_n] + \cdots + [b_1, \ldots, b_{n-1}, [a_1, \ldots, a_{n-1}, b_n]].\end{aligned} \tag{1}$$

A *derivation* D of an n-Lie algebra V is an endomorphism of the vector space V such that:

$$D([a_1, \ldots, a_n]) = [D(a_1), a_2, \ldots, a_n] + [a_1, D(a_2), \ldots, a_n] + \cdots + [a_1, \ldots, D(a_n)].$$

As in the Lie algebra case ($n = 2$), the meaning of the Filippov–Jacobi identity is that all endomorphisms $D_{a_1,\ldots,a_{n-1}}$ of V $(a_1, \ldots, a_{n-1} \in V)$, defined by

$$D_{a_1,\ldots,a_{n-1}}(a) = [a_1, \ldots, a_{n-1}, a]$$

are derivations of V. These derivations are called *inner*.

A subspace $W \subset V$ is called an *n-Lie subalgebra* of the n-Lie algebra V if $[W, \ldots, W] \subset W$. An n-Lie subalgebra $I \subset V$ of an n-Lie algebra is called an *ideal* if $[I, V, \ldots, V] \subset I$. An n-Lie algebra is called *simple* if it has not proper ideal other than 0.

Let V be an n-Lie algebra, $n \geq 3$. We will associate to V a Lie algebra called *the basic Lie algebra*, following the presentation given in [1, 5]. Consider $\mathrm{ad} : \wedge^{n-1} V \to \mathrm{End}(V)$ given by $\mathrm{ad}(a_1 \wedge \cdots \wedge a_{n-1})(b) := D_{a_1,\ldots,a_{n-1}}(b) = [a_1, \ldots, a_{n-1}, b]$. One can easily see that we could have chosen the codomain of ad to be $\mathrm{Der}(V)$ (the set of derivations of V) instead of $\mathrm{End}(V)$. ad induces a map $\widetilde{\mathrm{ad}} : \wedge^{n-1} V \to \mathrm{End}(\wedge^\bullet V)$ defined as $\widetilde{\mathrm{ad}}(a_1 \wedge \cdots \wedge a_{n-1})(b_1 \wedge \cdots \wedge b_m) = \sum_{i=1}^m b_1 \wedge \cdots \wedge [a_1, \ldots, a_{n-1}, b_i] \wedge \cdots \wedge b_m$. Denote by $\mathrm{Inder}(V)$ the set of inner derivations of V, i.e. endomorphisms of the form $D_{a_1,\ldots,a_{n-1}} = \mathrm{ad}(a_1 \wedge \cdots \wedge a_{n-1})$.

The set of derivations $\mathrm{Der}(V)$ of an n-Lie algebra V is a Lie algebra under the commutator and $\mathrm{Inder}(V)$ is a Lie ideal. Note the Lie bracket of $\mathrm{Inder}(V)$ can be given by

$$[\mathrm{ad}(a_1 \wedge \cdots \wedge a_{n-1}), \mathrm{ad}(b_1 \wedge \cdots \wedge b_{n-1})] = \mathrm{ad}(c_1 \wedge \cdots \wedge c_{n-1}),$$

where

$$c_1 \wedge \cdots \wedge c_{n-1} = \sum_{i=1}^{n-1} b_1 \wedge \cdots \wedge [a_1, \ldots, a_{n-1}, b_i] \wedge \cdots \wedge b_{n-1} = \widetilde{\mathrm{ad}}(a)(b). \quad (2)$$

By skew symmetric condition $c_1 \wedge \cdots \wedge c_{n-1}$ can be defined also by

$$c_1 \wedge \cdots \wedge c_{n-1} = -\sum_{i=1}^{n-1} a_1 \wedge \cdots \wedge [b_1, \ldots, b_{n-1}, a_i] \wedge \cdots \wedge a_{n-1} = -\widetilde{\mathrm{ad}}(b)(a). \quad (3)$$

Then $\widetilde{\mathrm{ad}}$ is skew-symmetric (cf. [2]). We give to $\wedge^{n-1}V$ a Lie algebra structure under the Lie bracket defined by

$$[a, b] = \widetilde{\mathrm{ad}}(a)(b). \quad (4)$$

Therefore, this proposition follows.

Proposition 1. $[\cdot,\cdot]$ *defines a Lie algebra structure on* $\wedge^{n-1}V$ *and* $\mathrm{ad} : \wedge^{n-1}V \to \mathrm{Inder}(V)$ *is a surjective Lie algebra homomorphism.*

Consider

$$\mathrm{Ker}(\mathrm{ad}) = \{a_1 \wedge \cdots \wedge a_{n-1} \in \wedge^{n-1}V : \mathrm{ad}(a_1 \wedge \cdots \wedge a_{n-1})(b) = 0 \text{ for all } b \in V\}$$

and

$$\mathrm{Ker}(\widetilde{\mathrm{ad}}) = \{a_1 \wedge \cdots \wedge a_{n-1} \in \wedge^{n-1}V : \widetilde{\mathrm{ad}}(a_1 \wedge \cdots \wedge a_{n-1})(b) = 0 \text{ for all } b \in \wedge^{\bullet}V\}.$$

It is straightforward to check that $\mathrm{Ker}(\mathrm{ad})$ is an abelian ideal of $\wedge^{n-1}V$ and $\mathrm{Ker}(\mathrm{ad}) \subseteq \mathrm{Ker}(\widetilde{\mathrm{ad}})$. Thus,

$$\wedge^{n-1} V/\mathrm{Ker}(\mathrm{ad}) \simeq \mathrm{Inder}(V), \quad (5)$$

as Lie algebras. Note that due to Eqs. (2) and (3), the $\mathrm{Ker}(\mathrm{ad})$ is a trivial submodule of $\mathrm{Inder}(V)$, thus

$$\wedge^{n-1} V \simeq \mathrm{Ker}(\mathrm{ad}) \rtimes \mathrm{Inder}(V). \quad (6)$$

Following the definition given by [1, 5], a vector space M is called an *n-Lie module* for the n-Lie algebra V, if on the direct sum $V \oplus M$ there is a structure of n-Lie algebra, such that the following conditions are satisfied:

- V is a subalgebra;
- M is an abelian ideal, i.e. when at least two slots of the n-bracket are occupied by elements in M, the result is 0.

We have the following results that establish some relations between representations of $\wedge^{n-1}V$ and n-Lie modules.

Theorem 1. (1) *Let M be an n-Lie module of the n-Lie algebra V and define $\rho : \wedge^{n-1}V \to \mathrm{End}(M)$ given by*

$$\rho(a_1 \wedge \cdots \wedge a_{n-1})(m) := [a_1, \ldots, a_{n-1}, m]$$

for all $m \in M$, where this n-Lie bracket corresponds to the n-Lie structure of $V \oplus M$. Then ρ is an homomorphism of Lie algebras.

(2) *Given (M, ρ) a representation of $\wedge^{n-1}V$ such that the two-sided ideal $Q(V)$ of the universal enveloping algebra of $\wedge^{n-1}V$, generated by the elements*

$$\begin{aligned} x_{a_1,\ldots,a_{2n-2}} &= [a_1,\ldots,a_n] \wedge a_{n+1} \wedge \cdots a_{2n-2} \\ &\quad - \sum_{i=1}^{n} (-1)^{i+n} (a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge a_n)(a_i \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}) \end{aligned} \tag{7}$$

acts trivially on M, then M is an n-Lie module.

Proof. Part (1) is direct from the definition of the Lie bracket in $\wedge^{n-1}V$ and the Filippov–Jacobi identity of the n-Lie bracket corresponding to the n-Lie structure of the semidirect product of V and M.

Let's prove part (2). Consider the n-ary map $[[\,,\,]] : \wedge^{n-1}(V \ltimes M) \to V \ltimes M$ such that M is an abelian ideal and V is a subalgebra with its own n-Lie bracket and define

$$[[a_1,\ldots,a_{n-1},m]] := \rho(a_1 \wedge \cdots \wedge a_{n-1})(m), \tag{8}$$

where $a_i \in V$, $m \in M$. We need to show that the Filippov–Jacobi identity holds for the n-ary bracket defined above. It is enough to show that

$$\begin{aligned} &[[a_1,\ldots,a_{n-1},[[b_1,\ldots,b_{n-1},m]]]] - [[b_1,\ldots,b_{n-1},[[a_1,\ldots,a_{n-1},m]]]] \\ &\quad = \sum_{i=1}^{n-1} [[b_1,\ldots,[a_1,\ldots,a_{n-1},b_i],\ldots,b_{n-1},m]] \end{aligned} \tag{9}$$

and

$$\begin{aligned} &[[[a_1,\ldots,a_n],a_{n+1},\ldots,a_{2n-2},m]] \\ &\quad = \sum_{i=1}^{n-1} (-1)^{n+i+1} [[a_1,\ldots,[a_{n+1},\ldots,a_{2n-2},a_i,m],\ldots,a_{2n-2}]] \end{aligned} \tag{10}$$

hold for a_i and $b_i \in V$ and $m \in M$.

Since ρ is a representation of $\wedge^{n-1}V$ and $\rho[a,b] = \rho(\widetilde{\mathrm{ad}}(a)(b))$ by definition of the Lie bracket, then the identity (9) holds.

Let's prove the identity (10). Writing the identity (10) using (8), we have that

$$\begin{aligned} &\rho([a_1,\ldots,a_n] \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2})(m) \\ &\quad = \sum_{i=1}^{n} (-1)^{i+n} \rho(a_1 \wedge \cdots \wedge \hat{a_i} \wedge \cdots \wedge a_n) \rho(a_i \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2})(m). \end{aligned} \tag{11}$$

Therefore, (11) is equivalent to the fact that the ideal $Q(V)$ acts trivially on M, finishing our proof. □

The following proposition was proven in [5].

Proposition 2. *Let M be a n-Lie module over an n-Lie algebra V. Then any submodule, any factor-module and dual module of M are also n-Lie modules. If M_1 and M_2 are n-Lie modules over V, then their direct sum $M_1 \oplus M_2$ is also n-Lie module.*

As in [5], we deduce the following Corollary.

Corollary 1. *Let M be an n-Lie module over n-Lie algebra V. Then*

(a) *M is irreducible if and only if M is irreducible as a Lie module over Lie algebra $\wedge^{n-1}V$.*
(b) *M is completely reducible, if only if M is completely reducible as a Lie module over Lie algebra $\wedge^{n-1}V$.*

Since we are aiming the study of the representation theory of V as an n-Lie algebra, Theorem 1 shows that it is closely related to the representation theory of the Lie algebra $\wedge^{n-1}V$. But first, due to (6), we need to characterize the ideal $\mathrm{Ker(ad)}$. We have the following lemma.

Lemma 1. *If $a \in \mathrm{Ker(ad)}$ and ρ is a representation of $\wedge^{n-1}V$, then $\rho(a)$ commutes with $\rho(b)$ for any $b \in \wedge^{n-1}V$.*

Proof. Consider $a \in \mathrm{Ker(ad)} \subseteq \mathrm{Ker}(\widetilde{\mathrm{ad}})$. By definition of Lie bracket in $\wedge^{n-1}V$ follows

$$\rho(a)\rho(b) - \rho(b)\rho(a) = \rho[a,b] = \rho(\widetilde{\mathrm{ad}}(a)(b)) = 0. \qquad \square$$

Thus, we have the following proposition.

Proposition 3. *Let ρ be an irreducible representation of $\wedge^{n-1}V$ in M with countable dimension. Then $\mathrm{Ker(ad)}$ acts by scalars in M.*

Proof. Immediate from the lemma above and Schur Lemma. $\square$

Theorem 2. *Let (M,ρ) be an irreducible representation of $\wedge^{n-1}V$ such that the ideal $Q(V)$ acts trivially on M. Then*

(a) *$\rho|_{\mathrm{Ker(ad)}} := \lambda\,\mathrm{Id}$ with Id the identity map in $\mathrm{End}(M)$ and $\lambda \in (\mathrm{Ker(ad)})^*$ is an $\mathrm{Inder}(V)$-module homomorphism (where $\mathbb{C}$ is thought as a trivial $\mathrm{Inder}(V)$-module),*
(b) *$\rho|_{\mathrm{Inder}(V)}$ is an irreducible representation of $\mathrm{Inder}(V)$ such that the ideal $Q(V)$ acts trivially on M.*
(c) *$\rho = \rho|_{\mathrm{Inder}(V)} \oplus \lambda\,\mathrm{Id}$.*

Proof. Let's prove part (a). If $l \in \mathrm{Inder}(V)$ and $a \in \mathrm{Ker(ad)}$, since $\mathrm{Ker(ad)}$ is an abelian ideal, by Lemma 1, we have $0 = \rho([l,a])(m) = \lambda([l,a])\,\mathrm{Id}(m)$. Thus λ is an $\mathrm{Inder}(V)$-module homomorphism.

Let's prove part (b). Consider $N \subsetneq M$ a nontrivial Inder(V)-subrepresentation of M and take $0 \neq m \in M$ such that $0 \neq \widetilde{N} := \rho(\mathrm{Inder}(V))(m) \subseteq N$. Note if $a \in \mathrm{Ker}(\mathrm{ad})$, due to Lemma 1 and Proposition 3, $\rho(a)\widetilde{N} = \rho(a)\rho(\mathrm{Inder}(\mathfrak{g}))(m) = \rho(\mathrm{Inder}(V)\rho(a)(m) = \lambda(a)\rho(\mathrm{Inder}(V))(m) = \widetilde{N}$. Using (6), we can conclude that $0 \neq \widetilde{N}$ is a subrepresentation of M as a $\wedge^{n-1}V$-module but M was irreducible by hypothesis which is a contradiction. Part (c) is an immediate consequence of (6) and Lemma 1. □

3. The Simple Linearly Compact n-Lie Algebra S^n

As mentioned in the introduction, Cantarini and Kac proved the following classification theorem.

Theorem 3 ([4]). (a) *Any simple linearly compact n-Lie algebra with $n > 2$, over an algebraically closed field $\mathbb{F}$ of characteristic 0, is isomorphic to one of the following four examples*:

(i) *the $(n+1)$-dimensional vector product n-Lie algebra $\mathbb{C}^{n+1}$*;

(ii) *the n-Lie algebra, denoted by S^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \ldots, x_n]]$, endowed with the n-ary bracket*

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} D_1(f_1) \ \ldots \ D_1(f_n) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ D_n(f_1) \ \ldots \ D_n(f_n) \end{pmatrix}.$$

Where $D_i = \frac{\partial}{\partial x_i}$;

(iii) *the n-Lie algebra, denoted by W^n, which is the linearly compact vector space of formal power series $\mathbb{F}[[x_1, \ldots, x_{n-1}]]$, endowed with the n-ary bracket,*

$$[f_1, \ldots, f_n] = \det \begin{pmatrix} f_1 & \ldots & f_n \\ D_1(f_1) & \ldots & D_1(f_n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots \\ D_{n-1}(f_1) & \ldots & D_{n-1}(f_n) \end{pmatrix}.$$

Where $D_i = \frac{\partial}{\partial x_i}$;

(iv) *the n-Lie algebra, denoted by SW^n, which is the direct sum of $n-1$ copies of $\mathbb{F}[[x]]$, endowed with the following n-ary bracket, where $f^{\langle j \rangle}$ is an element of the jth copy and $f' = \frac{df}{dx}$*:

$$[f_1^{\langle j_1 \rangle}, \ldots, f_n^{\langle j_n \rangle}] = 0, \quad \textit{unless } \{j_1, \ldots, j_n\} \supset \{1, \ldots, n-1\},$$

$$[f_1^{\langle 1 \rangle}, \ldots, f_{k-1}^{\langle k-1 \rangle}, f_k^{\langle k \rangle}, f_{k+1}^{\langle k \rangle}, f_{k+2}^{\langle k+1 \rangle}, \ldots, f_n^{\langle n-1 \rangle}]$$
$$= (-1)^{k+n}(f_1 \ldots f_{k-1}(f_k' f_{k+1} - f_{k+1}' f_k) f_{k+2} \ldots f_n)^{\langle k \rangle}.$$

(b) *There are no simple linearly compact n-Lie superalgebras over $\mathbb{F}$, which are not n-Lie algebras, if $n > 2$.*

Throughout the rest of this paper, we will consider $\mathbb{F} = \mathbb{C}$ and the simple infinite-dimensional linearly compact n-Lie algebra S^n.

Remark 1. (a) The map $\mathrm{ad} : \wedge^{n-1} S^n \to \mathrm{Inder}(S^n)$, which sends $f_1 \wedge \cdots \wedge f_{n-1} \to \mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1})$. By Proposition 1, it is an epimorphism of Lie algebras and we will show that in this case,

$$\ker(\mathrm{ad}) = \mathrm{span}\{f_1 \wedge \cdots \wedge f_{n-1} : f_i \in \mathbb{C},\ \text{for some } 1 \leq i \leq n-1\}.$$

Note that any $f_1 \wedge \cdots \wedge f_{n-1}$ such that $f_i \in \mathbb{C}$ for some $1 \leq i \leq n-1$, clearly is in $\ker(\mathrm{ad})$. On the other hand, if we assume that $f_1 \wedge \cdots \wedge f_{n-1} \in \ker(\mathrm{ad})$, we have

$$\mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1})(f) = \det \begin{pmatrix} D_1(f_1) & \cdots & D_1(f) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ D_n(f_1) & \cdots & D_n(f) \end{pmatrix} = 0, \tag{12}$$

for any $f \in S^n$. Since f is arbitrary, we have that at least two of the first $n-1$ columns of this matrix should be linearly dependent, in other words, there exist $i < j \in \{1, \ldots, n-1\}$ such that $\bigtriangledown f_i = c \bigtriangledown f_j$ for some $c \in \mathbb{C}$. Since $\mathbb{C}$ is a field of characteristic zero, we can deduce that $f_i = c f_j + k$, for some $k \in \mathbb{C}$. Thus, $f_1 \wedge \cdots \wedge f_{n-1} = k\,(f_1 \wedge \cdots \wedge 1 \wedge \cdots \wedge f_j \wedge \cdots \wedge f_{n-1})$.

(b) Let (M, ρ) be an irreducible representation of $\wedge^{n-1} S^n$ such that the ideal $Q(S^n)$ acts trivially on M. By Theorems 2(a) and 2(b), we have that $\rho|_{\mathrm{Ker(ad)}} := \lambda\,\mathrm{Id}$, with Id the identity map in $\mathrm{End}(M)$ and $\lambda \in (\mathrm{Ker(ad)})^*$, and $\rho|_{\mathrm{Inder}(S^n)}$ is an irreducible representation of $\mathrm{Inder}(S^n)$ such that the ideal $Q(S^n)$ acts trivially on M. We will show that $\lambda = 0$. Consider $x_{f_1,\ldots,f_{2n-2}}$ an element of $Q(S^n)$ such that $f_i \notin \mathbb{C}$ for all $i = 1, \ldots, 2n-2$ and $[f_1, \ldots, f_n] \in \mathbb{C}$, then

$$\begin{aligned} x_{f_1,\ldots,f_{2n-2}} &= [f_1, \ldots, f_n] \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2} \\ &\quad - \sum_{i=1}^{n} (-1)^{i+n} (f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n)(f_i \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2}) \\ &\in \mathrm{Ker(ad)} + U(\mathrm{Inder}(S^n)). \end{aligned}$$

Fix $m \in M$. We have,

$$\begin{aligned} 0 = \rho(x_{f_1,\ldots,f_{2n-2}})(m) &= \lambda(1 \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2})(m) \\ &- \sum_{i=1}^{n} (-1)^{i+n} \rho(\mathrm{ad}(f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n)) \rho(\mathrm{ad}(f_i \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2}))(m). \end{aligned} \tag{13}$$

Note that (13) is in the image of the ideal $Q(S^n)$ by the ad map acting on m. Thus, by Theorem 2(b), $\sum_{i=1}^{n} (-1)^{i+n} \rho(\mathrm{ad}(f_1 \wedge \cdots \wedge \hat{f}_i \wedge \cdots \wedge f_n)) \rho(\mathrm{ad}(f_i \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2})) \cdot m = 0$, from where we deduce that $\lambda(1 \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2}) \cdot m = 0$.

Now, suppose $\lambda \neq 0$ with $\lambda : \ker(\mathrm{ad}) \to \mathbb{C}$ and let $\beta = \{1, \alpha_1, \alpha_2, \ldots, \}$ a basis of $\mathbb{C}[[x_1, \ldots, x_n]]$. Then $\alpha = \{1 \wedge \alpha_{i_1} \cdots \wedge \alpha_{i_{n-2}} : i_1 < \cdots < i_{n-2}\}$ is a basis of Ker(ad). Then there exists $\alpha_{i_1} \cdots \alpha_{i_{n-2}} \in \beta$ such that $\lambda(1 \wedge \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_{n-2}}) \neq 0$. Choosing $f_{n+1}, \ldots, f_{2n-2}$ as $\alpha_{i_1} \cdots \alpha_{i_{n-2}}$ for $x_{f_1,\ldots,f_{2n-2}}$, we have that $\lambda(1 \wedge f_{n+1} \wedge \cdots \wedge f_{2n-2}) \cdot m \neq 0$, which is a contradiction. Then it follows that $\lambda = 0$.

Denote $W(m,n)$ the Lie superalgebra of continuous derivations of the tensor product $\mathbb{C}(m,n)$ of the algebra of formal power series in m even commuting variables $x_1, \ldots, x_m$ and the Grassmann algebra in n anti-commuting odd variables $\xi_1, \ldots, \xi_n$. Elements of $W(m,n)$ can be viewed as linear differential operators of the form

$$X = \sum_{i=1}^{m} P_i(x,\xi)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi)\frac{\partial}{\partial \xi_j}, \quad P_i, Q_j \in \mathbb{C}(m,n).$$

The Lie superalgebra $W(m,n)$ is simple linearly compact (and it is finite-dimensional if and only if $m = 0$).

Now, we shall describe $S(m,n)$ a linearly compact subalgebras of $W(m,n)$.

First, given a subalgebra L of $W(m,n)$, a continuous linear map $\mathrm{Div} : L \to \mathbb{C}(m,n)$ is called a divergence if the action π_λ of L on $\mathbb{C}(m,n)$, given by

$$\pi_\lambda(X)f = Xf + (-1)^{p(X)p(f)}\lambda f \,\mathrm{Div}\, X, \quad X \in L,$$

is a representation of L in $\mathbb{C}(m,n)$ for any $\lambda \in \mathbb{C}$. Note that

$$S'_{\mathrm{Div}}(L) := \{X \in L \mid \mathrm{Div}\, X = 0\}$$

is a closed subalgebra of L. We denote by $S_{\mathrm{Div}}(L)$ its derived subalgebra.

An example of a divergence on $L = W(m,n)$ is the following, denoted by div:

$$\mathrm{div}\left(\sum_{i=1}^{m} P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j \frac{\partial}{\partial \xi_j}\right) = \sum_{i=1}^{m} \frac{\partial P_i}{\partial x_i} + \sum_{j=1}^{n} (-1)^{p(Q_j)} \frac{\partial Q_j}{\partial \xi_j}.$$

Hence for any $\lambda \in \mathbb{C}$, we get the representation π_λ of $W(m,n)$ in $\mathbb{C}(m,n)$. Also, we get closed subalgebras $S'_{\mathrm{div}}(W(m,n)) \supset S_{\mathrm{div}}(W(m,n))$ denoted by $S'(m,n) \supset S(m,n)$. Observe that $S'(m,n) = S(m,n)$ is simple if $m > 1$. From now on, we will denoted the Lie algebras $S(n,0)$ by S_n.

Proposition 5.1 in [4] gives the description of the Lie algebra of continuous derivation of each simple linearly compact n-Lie algebra. Moreover, they state in particular, that the Lie algebra of continuous derivations of the n-Lie algebra S^n is isomorphic to S_n and in the proof of this proposition, they show that the Lie algebra of continuous derivations of the n-Lie algebra S^n coincides with the Lie algebra of its inner derivations. Thus,

$$\mathrm{Inder}(S^n) \simeq S_n. \tag{14}$$

Therefore, Theorems 1, 2 and Remark 1 give us the following.

Theorem 4. *Irreducible representations of the n-Lie algebra S^n are in $1-1$ correspondence with irreducible representations of the universal enveloping algebra $U(S_n)$, on which the two-sided ideal $Q(S^n)$, generated by the elements*

$$\begin{aligned} x_{f_1,\dots,f_{2n-2}} &= \mathrm{ad}([f_1,\dots,f_n]\wedge f_{n+1}\wedge\cdots\wedge f_{2n-2}) \\ &\quad -\sum_{i=1}^{n}(-1)^{i+n}\mathrm{ad}(f_1\wedge\cdots\wedge\widehat{f_i}\cdots\wedge f_n)\mathrm{ad}(f_i\wedge f_{n+1}\wedge\cdots\wedge f_{2n-2}), \end{aligned}$$

where $f_i\in\mathbb{C}[[x_1,\dots,x_n]]$ and $f_i\neq 1$ for all $i=1,\dots,n$, acts trivially.

4. Representations of Simple Linearly Compact Lie Algebra S_n

In this section, we present the approach given by Rudakov in [13] for the representation theory of the infinite-dimensional simple linearly compact Lie algebra S_n. The algebra S_n is a subalgebra of the algebra W_n of all derivations of the ring $\mathbb{C}$ of formal power series in n variables. The elements $D\in W_n$ has the form $D=\sum_{i=1}^n f_i\partial/\partial x_i$ with $f_i\in\mathbb{C}[[x_1,\dots,x_n]]$. The algebra W_n is endowed with the filtration

$$(W_n)_{(j)}=\{D,\ \deg f_i\geq j+1, i=1,\dots,n\}$$

and a compatible gradation

$$(W_n)_j=\{D,\ \deg f_i=j+1, i=1,\dots,n\}.$$

The subalgebra S_n is defined by the condition

$$\sum_{i=1}^n\frac{\partial f_i}{\partial x_i}=0.$$

The filtration and gradation of W_n induce a filtration and gradation in S_n. The gradation of S_n gives a triangular decomposition

$$S_n=(S_n)_-\oplus(S_n)_0\oplus(S_n)_+,$$

with $(S_n)_\pm=\oplus_{\pm m>0}(S_n)_m$. We shall consider continuous representations in spaces with discrete topology. The continuity of a representation of a linearly compact Lie superalgebra S_n in a vector space V with discrete topology means that the stabilizer $(S_n)_v=\{g\in S_n\,|\,gv=0\}$ of any $v\in V$ is an open (hence of finite codimension) subalgebra of S_n. Let $(S_n)_{\geq 0}=(S_n)_{>0}\oplus(S_n)_0$. Denote by $P(S_n,(S_n)_{\geq 0})$ the category of all continuous S_n-modules V, where V is a vector space with discrete topology, that are $(S_n)_0$-locally finite, that is, any $v\in V$ is contained in a finite-dimensional $(S_n)_0$-invariant subspace. Given an $(S_n)_{\geq 0}$-module F, we may consider the associated induced S_n-module

$$M(F)=\mathrm{Ind}_{(S_n)_{\geq 0}}^{S_n}F=U(S_n)\otimes_{U((S_n)_{\geq 0})}F$$

called the *generalized Verma module* associated to F.

Let V be an S_n-module. The elements of the subspace

$$\mathrm{Sing}(V) := \{v \in V \mid (S_n)_{>0}v = 0\}$$

are called *singular vectors*. When $V = M(F)$, the $(S_n)_{\geq 0}$-module F is canonically an $(S_n)_{\geq 0}$-submodule of $M(F)$, and $\mathrm{Sing}(F)$ is a subspace of $\mathrm{Sing}(M(F))$, called the *subspace of trivial singular vectors*. Observe that $M(F) = F \oplus F_+$, where $F_+ = U_+((S_n)_-) \otimes F$ and $U_+((S_n)_-)$ is the augmentation ideal in the symmetric algebra $U((S_n)_-)$. Then

$$\mathrm{Sing}_+(M(F)) := \mathrm{Sing}(M(F)) \cap F_+$$

are called the *nontrivial singular vectors*.

Theorem 5 ([9, 13]). (a) *If F is a finite-dimensional $(S_n)_{\geq 0}$-module, then $M(F)$ is in $P(S_n, (S_n)_{\geq 0})$.*

(b) *In any irreducible finite-dimensional $(S_n)_{\geq 0}$-module F, the subalgebra $(S_n)_+$ acts trivially.*

(c) *If F is an irreducible finite-dimensional $(S_n)_{\geq 0}$-module, then $M(F)$ has a unique maximal submodule.*

(d) *Denote by $I(F)$ the quotient by the unique maximal submodule of $M(F)$. Then the map $F \mapsto I(F)$ defines a bijective correspondence between irreducible finite-dimensional $(S_n)_{\geq 0}$-modules and irreducible (S_n)-modules in $P((S_n), (S_n)_{\geq 0})$, the inverse map being $V \mapsto \mathrm{Sing}(V)$.*

(e) *An $(S_n)_{\geq 0}$-module $M(F)$ is irreducible if and only if the $(S_n)_{\geq 0}$-module F is irreducible and $M(F)$ has no nontrivial singular vectors.*

Remark 2. (a) Note that

$$(S_n)_0 \cong \mathfrak{sl}_n(\mathbb{C}), \tag{15}$$

the isomorphism is given by the map that sends $x_i\frac{\partial}{\partial x_j} \to E_{i,j}$ for $(i \neq j)$ and $x_i\frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}} \to E_{i,i} - E_{i+1,i+1}$, where $E_{i,j}$ denote as usual the matrix whose (i,j) entry is 1 and all the other entries are 0 for $i,j = 1,\ldots,n$.

(b) Due to Theorem 5(b), any irreducible finite-dimensional $(S_n)_{\geq 0}$-module F will be obtained extending by zero the irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module.

In the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, we choose the Borel subalgebra $\mathfrak{b} = \{x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}, x_i\frac{\partial}{\partial x_j} : i < j, i,j = 1,\ldots,n\}$. We denote by

$$\mathfrak{h} = \mathrm{span}\left\{h_i = x_i\frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}},\ i = 1,\ldots,n-1\right\}$$

the corresponding Cartan subalgebra.

Let $F_0,\ldots,F_{n-1}$ be the irreducible $(S_n)_{\geq 0}$-modules got by extending trivially the irreducible $\mathfrak{sl}_n(\mathbb{C})$-modules with highest weight $\lambda_0 = (0,0,\ldots,0)$,

$\lambda_1 = (1,0,\ldots,0)$, $\lambda_2 = (0,1,\ldots,0),\ldots,\lambda_{n-1} = (0,0,\ldots,1)$, respectively. We will call them *exceptional* $(S_n)_{\geq 0}$-modules.

Theorem 6 ([13]). *Let F be an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module. If F is not isomorphic to one of the exceptional modules $F_0,\ldots,F_{n-1}$, then the S_n-module $M(F)$ is irreducible. Each module $N_p := M(F_p)$ contains a unique irreducible submodule K_p which is generated by all its nontrivial singular vectors.*

Corollary 2 ([13]). *If the S_n-module E is irreducible, then the $\mathfrak{sl}_n(\mathbb{C})$-module $F := \mathrm{Sing}(E)$ is also irreducible. If F coincides with none of the modules $F_0,\ldots,F_{n-1}$, then $E = M(F)$. If $F = F_p$, then E is isomorphic to $J(F_p) := N_p/K_p$.*

4.1. *Some useful lemmas*

Let F be an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module with highest weight vector v_λ and highest weight λ. Let $J(F)$ be the irreducible module $M(F)$ if F is an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module which coincides with none of the exceptional modules $F_0,\ldots,F_{n-1}$ and N_p/K_p otherwise. Note that if F is an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module which coincides with none of the exceptional modules $F_0,\ldots,F_{n-1}$, $\mathrm{Sing}_+(M(F)) = \{0\}$, (cf. Theorem 5(e)).

Our main goal is to find those irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-modules F for which $J(F)$ is an irreducible module over the n-Lie algebra S^n, more precisely, we are looking for those $J(F)$ where the ideal $Q(S^n)$ acts trivially.

Lemma 2. (1) *If F is an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module which coincides with none of the exceptional modules $F_0,\ldots,F_{n-1}$, then $Q(S^n) \otimes_{U(S_n)_{\geq 0}} F$ is equal to the trivial submodule if and only if $Q(S^n)$ acts trivially on $M(F)$.*

(2) *If $F = F_p$, then $Q(S^n) \otimes_{U(Sn)_{\geq 0}} F \subset K_p$ if and only if $Q(S^n)$ acts trivially on $J(F_p)$.*

Proof. It follows from the definitions of a factor module and a two-sided ideal. □

Lemma 3. (1) *If F is an irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module which coincides with none of the exceptional modules $F_0,\ldots,F_{n-1}$, then $Q(S^n) \otimes v_\lambda = 0$ if and only if $Q(S^n)$ acts trivially on $M(F)$.*

(2) *If $F = F_p$, then $Q(S^n) \otimes_{U(S_n)_{\geq 0}} v_\lambda \subset K_p$ if and only if $Q(S^n)$ acts trivially on $J(F_p)$.*

Proof. Due to Lemma 2, it is immediate from the definition of generalized Verma module and the facts that F is a highest weight $\mathfrak{sl}_n(\mathbb{C})$-module and $\mathfrak{sl}_n(\mathbb{C}) \subseteq U(S_n)_{\geq 0}$. □

4.2. *Description of the ideal $Q(S^n)$*

$\mathrm{Inder}(S^n) \simeq S_n$, where the isomorphism is given explicitly by

$$\mathrm{ad}(f_1 \wedge \cdots \wedge f_{n-1}) \to \sum_{i=1}^{n} (-1)^{n+i} \det \begin{pmatrix} D_1(f_1) & \cdots & D_1(f_{n-1}) \\ \cdots & \cdots & \cdots \\ \hat{D}_i(f_1) & \cdots & \hat{D}_i(f_{n-1}) \\ \cdots & \cdots & \cdots \\ D_n(f_1) & \cdots & D_n(f_{n-1}) \end{pmatrix} D_i, \tag{16}$$

for any $f_1, \ldots, f_{n-1} \in \mathbb{C}[[x_1, \ldots, x_n]], D_j = \frac{\partial}{\partial x_j}$ and the hat means that the ith row does not appear in the matrix. Consider the subset

$$A = \left\{ D = \sum_{i=1}^{n} f_i D_i \in S_n : f_i \in \mathbb{C}[x_1, \ldots, x_n] \right\}.$$

It is dense in S_n. Since we are classifying continuous representations, it is enough to characterize a set of generator of $Q_A(S^n) := Q(S^n) \bigcap A$. Take $f_1, \ldots, f_{2n-2} \in \mathbb{C}[x_1, \ldots, x_n]$, where $f_l = X^{I_l}$ with

$$X^{I_l} := x_1^{i_1^l} x_2^{i_2^l} \cdots x_n^{i_n^l},$$

where $I_l := (i_1^l, \ldots, i_n^l)$ with $i_1^l, \ldots, i_n^l \in \mathbb{Z}_{\geq 0}$ and $l \in \{1, \ldots, 2n-2\}$. Then the generators of $Q_A(S^n)$ are given by

$$x_{f_1,\ldots,f_{2n-2}} = \left(\sum_{k=1}^{n} \widetilde{\alpha}(k) D_k \right) - \sum_{i=1}^{n} (-1)^{i+n} \left(\sum_{q=1}^{n} \widetilde{\beta}(i,q) D_q \right) \left(\sum_{s=1}^{n} \widetilde{\gamma}(i,s) D_s \right), \tag{17}$$

where

$$\widetilde{\alpha}(k) = (-1)^{n+k} \frac{f_1 \cdots f_{2n-2}}{x_1^2 \cdots x_k \cdots x_n^2} \det \widetilde{A} \det \widetilde{B}_k, \quad k = 1, \ldots, n,$$

$$\widetilde{\beta}(i,q) = (-1)^{n+q} \frac{f_1 \cdots \hat{f}_i \cdots f_n}{x_1 \cdots \hat{x}_q \cdots x_n} \det \widetilde{A}_{q,i}, \quad q = 1, \ldots, n,$$

$$\widetilde{\gamma}(i,s) = (-1)^{n+s} \frac{f_i f_{n+1} \cdots \cdots f_{2n-2}}{x_1 \cdots \hat{x}_s \cdots x_n} \det \widetilde{C}_s^{(i)}, \quad s = 1, \ldots, n,$$

with $i = 1, \ldots, n$ and the matrices $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$'s are defined as follows:

$$\widetilde{A} = \begin{pmatrix} i_1^1 & \cdots & i_1^n \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ i_n^1 & \cdots & i_n^n \end{pmatrix},$$

$\widetilde{A}_{q,i}$ is the matrix $\widetilde{A}$ with the q-row and the i-column removed,

$$\widetilde{B}_k = \begin{pmatrix} \sum_{r=1}^{n}(i_1^r - 1) & i_1^{n+1} & \cdots & i_1^{2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{\sum_{r=1}^{n}(i_k^r - 1)} & \widehat{i_k^{n+1}} & \cdots & \widehat{i_k^{2n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{r=1}^{n}(i_n^r - 1) & i_n^{n+1} & \cdots & i_n^{2n-2} \end{pmatrix}$$

and

$$\widetilde{C}_s^{(i)} = \begin{pmatrix} i_1^i & i_1^{n+1} & \cdots & i_1^{2n-2} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{i_s^i} & \widehat{i_s^{n+1}} & \cdots & \widehat{i_s^{2n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ i_n^i & i_n^{n+1} & \cdots & i_n^{2n-2} \end{pmatrix},$$

where the hats mean that the corresponding row is removed.

5. Main Theorems and Their Proofs

In this section, we will state the main result of this paper. We applied successfully the same technics used to classify irreducible continuos representations of the simple linearly compact n-Lie algebra of type W, (cf. [3]). Recall that the inner derivations of the simple linearly compact n-Lie algebra S^n are isomorphic to S_n and denote by $\mathfrak{h}$ the Cartan subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ chosen above Theorem 6. Let F be a finite-dimensional irreducible highest weight $\mathfrak{sl}_n(\mathbb{C})$-module, with highest weight $\lambda \in \mathfrak{h}^*$ and highest weight vector v_λ. Recall that our goal is to determine for which $\lambda \in \mathfrak{h}^*$ the two-sided ideal $Q(S^n)$ acts trivially on the irreducible highest weight module $J(F)$ (see Sec. 4.1). This will ensure us that $J(F)$ is an n-Lie module of S^n. Let us denote by $\lambda_i = \lambda(E_{i,i} - E_{i+1,i+1})$ for $i = 1, \ldots, n-1$ and introduce the following useful notation for the proof of the theorem,

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{otherwise,} \end{cases} \tag{18}$$

with $i, j \in \{1, \ldots, n\}$.

Theorem 7. *Let $n \geq 3$ and F be a finite-dimensional irreducible highest weight $\mathfrak{sl}_n(\mathbb{C})$-module, then the irreducible continuous representation $J(F)$ of S_n is an irreducible continuous representation of the simple linearly compact n-Lie algebra S^n if and only if $\lambda \in \mathfrak{h}^*$ is such that $\lambda = (0, 0, \ldots, 0)$.*

Proof. Let F be a highest weight irreducible finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-module, with highest weigh $\lambda \in \mathfrak{h}^*$ and highest weigh vector v_λ. Recall that $\mathfrak{h} := \oplus_{i=1}^{n-1}\mathbb{C}\,(E_{i,i} - E_{i+1,i+1})$ is the chosen Cartan subalgebra of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$.

Here, we are identifying the subalgebra $\mathfrak{h}$ with the subalgebra of S_n generated by the elements $x_i\frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}}, i = 1, \ldots, n-1$. Consider F as a $(S_n)_{\geq 0}$-module and take the induced module $M(F) = U(S_n) \otimes_{U((S_n)_{\geq 0})} F$. We will use Lemma 3 and the general look of the generators of $Q_A(S^n)$ to find out for which λ's, $Q_A(S^n)$ acts trivially in $J(F)$. Let $w_\lambda = 1 \otimes_{U((S_n)_{\geq 0})} v_\lambda = 1 \otimes v_\lambda$.

According to the description of the generators given in (17) and taking into account that $(S_n)_+$ acts by zero on w_λ, it is enough to consider the subset of generators $Q_A(S^n)$ and ask them to either act trivially w_λ if F is nonexceptional or $Q_A(S^n) \otimes v_\lambda \subseteq \mathrm{Sing}_+(M(F))$ otherwise. It is enough to consider $x_{f_1,\ldots,f_{2n-2}}$ with monomials $f_i \in \mathbb{C}[x_1, \ldots, x_n]$ as in (17) such that,

(1) $\deg(f_1 \cdots f_{2n-2}) = 2n-1$ and there exist $i \in \{1, \ldots, n\}$ such that
  (a) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n-1$ or
  (b) $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n$,

(2) $\deg(f_1 f_2 \cdots f_{2n-2}) = 2n$, and there exist $i \in \{1, \ldots, n\}$ such that $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n$,

(3) $\deg(f_1 f_2 \cdots f_{2n-2}) = 2n+1$, and there exist $i \in \{1, \ldots, n\}$ such that $\deg(f_i f_{n+1} \cdots f_{2n-2}) = n$,

since the remaining ones are either zero or act trivially any way. Here, we are assuming by simplicity that $i = n$ and $f_i \notin \mathbb{C}$ for all $i = 1, \ldots, 2n-2$. Let us analyze each possible case.

Case 1(a): Here, $\deg(f_1 \cdots f_{2n-2}) = 2n-1$, $\deg(f_1 \cdots f_{n-1}) = n$ and $\deg(f_n \cdots f_{2n-2}) = n-1$. We have two possible expressions for $f_1 \cdots f_{n-1}$ such that $x_{f_1,\ldots,f_{2n-2}} \neq 0$ and two expression for $f_n f_{n+1} \cdots f_{2n-2}$. Namely, there exist $q, l, j, k, m \in \{1, \ldots, n\}$, such that

$$f_1 \cdots f_{n-1} = x_1 \cdots x_n \tag{19}$$

or

$$f_1 \cdots f_{n-1} = x_1 \cdots \hat{x}_l \cdots x_m^2 \cdots x_n, \tag{20}$$

and

$$f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_k} \cdots x_n \tag{21}$$

or

$$f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_q} \cdots \hat{x_j} \cdots x_k^2 \cdots x_n. \tag{22}$$

Suppose we have (19) and (21), namely $f_1 \cdots f_{n-1} = x_1 \cdots x_n$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_k} \cdots x_n$ for some $k \in \{1, \ldots, n\}$. Therefore, we can consider the monomials as follows.

(i) Let $n \geq 3$ and $l, j, k \in \{1, \ldots, n\}$. Note that to define the monomials $f_{n+1}, \ldots, f_{2n-2}$, we are assuming that $j < k$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \ldots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \ldots, l-1, \\
&f_s = x_{s+1}, && s = l, \ldots, n-1, \quad s \neq j - \delta_{j,l}, \\
&f_{j-\delta_{j,l}} = x_j x_l, && f_n = x_j, \\
&f_{n+s} = x_s, && s = 1, \ldots, j-1, \\
&f_{n+s} = x_{s+1}, && s = j, \ldots, k-2, \\
&f_{n+s} = x_{s+2}, && s = k-1, \ldots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that

$$\begin{aligned}
x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{j+l+k+\delta_{k,l}} (&D_l \otimes E_{l,k} v_\lambda - D_j \otimes E_{j,k} v_\lambda \\
&- D_k \otimes (E_{l,l} - E_{j,j}) v_\lambda). \qquad (23)
\end{aligned}$$

Now, suppose (19) and (22), namely $f_1 \cdots f_{n-1} = x_1 \cdots x_n$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_q} \cdots \hat{x_j} \cdots x_k^2 \cdots x_n$ for some $j, q, k \in \{1, \ldots, n\}$. Therefore, we have the following possibilities.

(ii) Let $n \geq 4$ and $q, l, j, k \in \{1, \ldots, n-1\}$. Note that to define the monomials $f_{n+1}, \ldots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \ldots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \ldots, l-1, \quad s \neq k - \delta_{k,l}, \\
&f_s = x_{s+1}, && s = l, \ldots, n-1, \\
&f_{k-\delta_{k,l}} = x_k x_l, && f_n = x_k, \\
&f_{n+s} = x_s, && s = 1, \ldots, q-1, \\
&f_{n+s} = x_{s+1}, && s = q, \ldots, j-2, \\
&f_{n+s} = x_{s+2}, && s = j-1, \ldots, n-2.
\end{aligned}$$

By (17), we have

$$x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{l+q+j+\delta_{q,j}} (D_j \otimes E_{k,q} v_\lambda - D_q \otimes E_{k,j} v_\lambda). \qquad (24)$$

Equation (20) combined with Eqs. (21) and (22) does not give us new results.

Case 1(b): Do not provide new information.

Case 2: Here, $\deg(f_1 \cdots f_{2n-2}) = 2n$, $\deg(f_1 \cdots f_{n-1}) = n = \deg(f_n \cdots f_{2n-2})$ The two possible expressions for $\deg(f_1 \cdots f_{n-1})$ such that $x_{f_1,\ldots,f_{2n-2}} \neq 0$ are the same that (19) and (20). We have three expression for $f_n \cdots f_{2n-2}$. Namely, there exist $q, j, k, r \in \{1, \ldots, n\}$, such that

$$f_n \cdots f_{2n-2} = x_1 \cdots x_n \qquad (25)$$

or

$$f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_j} \cdots x_k^2 \cdots x_n \tag{26}$$

or

$$f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_q} \cdots \hat{x_r} \cdots {x_j}^2 \cdots x_k^2 \cdots x_n. \tag{27}$$

Consider (19) and (25), namely $f_1 \cdots f_{n-1} = x_1 \cdots x_n = f_n \cdots f_{2n-2}$. Therefore, we can consider the monomials as follows:

(i) Let $n \geq 4$ and $l, m, j, k \in \{1, \ldots, n\}$. Note that to define the monomials $f_{n+1}, \ldots, f_{2n-2}$, we are assuming that $m < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \ldots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \ldots, j-1, \;\; s \neq l - \delta_{l,j}, \\
&f_s = x_{s+1}, && s = j, \ldots, n-1, \\
&f_{l-\delta_{l,j}} = x_l x_j, && f_n = x_j, \\
&f_{n+s} = x_s, && s = 1, \ldots, m-1, \;\; s \neq k - \delta_{k,m} - \delta_{k,j}, \\
&f_{k-\delta_{k,m}-\delta_{k,j}} = x_k x_m, && \\
&f_{n+s} = x_{s+1}, && s = m, \ldots, j-2, \\
&f_{n+s} = x_{s+2}, && s = j-1, \ldots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that,

$$\begin{aligned}
x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{m+\delta_{m,j}} (&1 \otimes E_{j,m} E_{m,j} v_\lambda - 1 \otimes E_{jk} E_{kj} v_\lambda \\
&+ 1 \otimes (E_{m,m} - E_{k,k})(1 - (E_{j,j} - E_{l,l})) v_\lambda).
\end{aligned} \tag{28}$$

Suppose in the definitions of $f_1, \ldots, f_{2n-2}$ in (i), we take $m := l$, then we have

(ii) Let $n \geq 3$ and $j, k, l \in \{1, \ldots, n\}$,

$$\begin{aligned}
x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{l+\delta_{j,l}} (&1 \otimes E_{jk} E_{kj} v_\lambda \\
&- 1 \otimes (E_{l,l} - E_{k,k})(1 - (E_{j,j} - E_{l,l})) v_\lambda).
\end{aligned} \tag{29}$$

Now, suppose in the definitions of $f_1, \ldots, f_{2n-2}$ in (i), we take $l := k$, then we have the following.

(iii) Let $n \geq 3$ and $m, j, k \in \{1, \ldots, n\}$,

$$\begin{aligned}
&x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) \\
&\quad = (-1)^{m+\delta_{m,j}} (1 \otimes (E_{j,j} - E_{k,k})(1 - (E_{m,m} - E_{k,k})) v_\lambda).
\end{aligned} \tag{30}$$

Suppose in the definitions of $f_1, \ldots, f_{2n-2}$ in (iii), we take $l := j$, $j := m$ and $m := q$ then we have

(iv) Let $n \geq 4$ and $q, m, j, k \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) \\ = (-1)^{q+m+j+\delta_{j,q}}(1 \otimes (E_{m,m} - E_{j,j})(E_{q,q} - E_{k,k})v_\lambda). \quad (31)$$

Now, consider (19) and (26), namely $f_1 \cdots f_{n-1} = x_1 \cdots x_n$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_j} \cdots x_k^2 \cdots x_n$.

(v) Let $n \geq 5$ and $q, m, j, k, l \in \{1, \dots, n\}$. Note that to define the monomials $f_{n+1}, \dots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \dots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \dots, k-1, \\
&f_s = x_{s+1}, && s = k, \dots, n-1,\ \ s \neq l - \delta_{l,j}, \\
&f_{l-\delta l,j} = x_l x_k, && f_n = x_k, \\
&f_{n+s} = x_s, && s = 1, \dots, q-1, \quad s \neq m - \delta_{m,q} - \delta_{m,j}, \\
&f_{m-\delta_{m,q}-\delta_{m,j}} = x_m x_q, && \\
&f_{n+s} = x_{s+1}, && s = q, \dots, j-2, \\
&f_{n+s} = x_{s+2}, && s = j-1, \dots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(j+q+k+\delta_{q,k})}(1 \otimes E_{k,q}E_{q,j}v_\lambda - 1 \otimes E_{k,m}E_{m,j}v_\lambda \\ - 1 \otimes E_{kj}(E_{q,q} - E_{m,m})v_\lambda). \quad (32)$$

Suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (v), we take $m := l$, then we have

(vi) Let $n \geq 4$ and $m, j, k, l \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(j+m+k+\delta_{m,k})}(1 \otimes E_{k,q}E_{q,j}v_\lambda \\ - 1 \otimes E_{kj}(E_{q,q} - E_{l,l})v_\lambda). \quad (33)$$

Now, suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (v), we take $l := j$, then we have the following.

(vii) Let $n \geq 4$ and $m, j, k, l \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) \\ = (-1)^{(l+m+k+\delta_{m,l})}(1 \otimes E_{k,q}E_{q,j}v_\lambda - 1 \otimes E_{km}E_{m,j}v_\lambda). \quad (34)$$

(viii) Let $n \geq 5$ and $m, q, j, k, l \in \{1, \dots, n\}$. Note that to define the monomials $f_{n+1}, \dots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \dots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \dots, m-1, \ \ s \neq l - \delta_{l,m},\\
&f_s = x_{s+1}, && s = m, \dots, n-1,\\
&f_{l-\delta_{l,m}} = x_l x_m, && f_n = x_k x_q,\\
&f_{n+s} = x_s, && s = 1, \dots, q-1,\\
&f_{n+s} = x_{s+1}, && s = q, \dots, j-2,\\
&f_{n+s} = x_{s+2}, && s = j-1, \dots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(m+j+q+\delta_{q,j})}(1 \otimes E_{k,j}(E_{m,m} - E_{l,l})v_\lambda). \qquad (35)$$

Suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (viii), we take $l := k$, then we have the following.

(ix) Let $n \geq 4$ and $m, j, k, l \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(m+j+l+\delta_{l,j})}(1 \otimes E_{k,j}(E_{m,m} - E_{k,k} - 1)v_\lambda). \qquad (36)$$

(x) Let $n \geq 3$ and $l, j, k \in \{1, \dots, n\}$. Note that to define the monomials $f_{n+1}, \dots, f_{2n-2}$, we are assuming that $l < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \dots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \dots, l-1, \ \ s \neq j - \delta_{j,l},\\
&f_s = x_{s+1}, && s = l, \dots, n-1,\\
&f_{j-\delta_{j,l}} = x_l x_j, && f_n = x_l,\\
&f_{n+s} = x_s, && s = 1, \dots, l-1, \ \ s \neq k - \delta_{k,l} - \delta_{k,j},\\
&f_{n+k-\delta_{k,l}-\delta_{k,j}} = x_k^2, && \\
&f_{n+s} = x_{s+1}, && s = l, \dots, j-2,\\
&f_{n+s} = x_{s+2}, && s = j-1, \dots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(j+\delta_{j,l})}(1 \otimes E_{k,j}(E_{l,l} - E_{j,j})v_\lambda). \qquad (37)$$

Suppose (19) and (27), namely $f_1 \cdots f_{n-1} = x_1 \cdots x_n$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_q} \cdots \hat{x_r} \cdots {x_j}^2 \cdots x_k^2 \cdots x_n$, for some $q, r, j, k \in \{1, \dots, n\}$.

(xi) Let $n \geq 5$ and $r, q, j, k, l \in \{1, \dots, n\}$. Note that to define the monomials $f_{n+1}, \dots, f_{2n-2}$, we are assuming that $r < q$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \ldots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \ldots, l-1,\\
&f_s = x_{s+1}, && s = l, \ldots, n-1, \;\; s \neq j - \delta_{j,l},\\
&f_{j-\delta_{j,l}} = x_j x_l, && f_n = x_j,\\
&f_{n+s} = x_s, && s = 1, \ldots, r-1, \;\; s \neq k - \delta_{k,r} - \delta_{k,q},\\
&f_{n+k-\delta_{k,r}-\delta_{k,q}} = x_k^2, &&\\
&f_{n+s} = x_{s+1}, && s = r, \ldots, q-2,\\
&f_{n+s} = x_{s+2}, && s = q-1, \ldots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that

$$x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(l+q+\delta_{q,r})} 2(1 \otimes E_{j,r} E_{k,q} v_\lambda - 1 \otimes E_{j,q} E_{k,r} v_\lambda). \tag{38}$$

Equations (20) and (25) do not give us new information. Now, consider (20) and (26), namely $f_1 \cdots f_{n-1} = x_1 \cdots \hat{x_l} \cdots x_m^2 \cdots x_n$ and $f_n \cdots f_{2n-2} = x_1 \cdots \hat{x_j} \cdots x_k^2 \cdots x_n$, for some $l, m, j, k \in \{1, \ldots, n\}$.

Suppose $m := k$ and $l := q$ in the Eq. (20), then we have

(xii) Let $n \geq 4$ and $r, q, j, k \in \{1, \ldots, n\}$. Note that to define the monomials $f_{n+1}, \ldots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \ldots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \ldots, k-1,\\
&f_s = x_{s+1}, && s = k, \ldots, n-1, \;\; s \neq q - \delta_{q,k},\\
&f_{q-\delta_{q,k}} = x_q^2, && f_n = x_k,\\
&f_{n+s} = x_s, && s = 1, \ldots, q-1, \;\; s \neq r - \delta_{r,q} - \delta_{r,j},\\
&f_{n+r-\delta_{r,q}-\delta_{r,j}} = x_r x_q, &&\\
&f_{n+s} = x_{s+1}, && s = q, \ldots, j-2,\\
&f_{n+s} = x_{s+2}, && s = j-1, \ldots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that,

$$\begin{aligned} x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(k+j+q+\delta_{j,q})} (1 \otimes E_{q,r} E_{r,j} v_\lambda \\ + 1 \otimes E_{q,j}(E_{q,q} - E_{r,r} + 1) v_\lambda). \end{aligned} \tag{39}$$

Suppose in the definitions of $f_1, \ldots, f_{2n-2}$ in the Eq. (19), we take $m := k$ and $l := j$, then we have the following.

(xiii) Let $n \geq 4$ and $r, q, j, k \in \{1, \ldots, n\}$. Note that to define the monomials $f_{n+1}, \ldots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \dots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \dots, k-1,\\
&f_s = x_{s+1}, && s = k, \dots, n-1, \ s \neq j - \delta_{j,k},\\
&f_{j-\delta_{j,k}} = x_j^2, && f_n = x_k,\\
&f_{n+s} = x_s, && s = 1, \dots, q-1, \ s \neq r - \delta_{r,q} - \delta_{r,j},\\
&f_{n+r-\delta_{r,q}-\delta_{r,j}} = x_r x_q, &&\\
&f_{n+s} = x_{s+1}, && s = q, \dots, j-2,\\
&f_{n+s} = x_{s+2}, && s = j-1, \dots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that

$$\begin{aligned}
&x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda)\\
&\quad = (-1)^{(k+j+q+\delta_{j,q})}(1 \otimes E_{q,j}E_{j,q}v_\lambda - 1 \otimes E_{r,j}E_{j,r}v_\lambda\\
&\qquad + 1 \otimes (E_{q,q} - E_{r,r})v_\lambda). \qquad (40)
\end{aligned}$$

(xiv) Let $n \geq 5$ and $m, q, j, k, l \in \{1, \dots, n\}$. Note that to define the monomials $f_{n+1}, \dots, f_{2n-2}$, we are assuming that $q < j$. Otherwise, we can interchange those indexes in the definition of $f_{n+1}, \dots, f_{2n-2}$.

$$\begin{aligned}
&f_s = x_s, && s = 1, \dots, m-1,\\
&f_s = x_{s+1}, && s = m, \dots, n-1, \ s \neq l - \delta_{l,m},\\
&f_{l-\delta_{l,m}} = x_l^2, && f_n = x_k x_q,\\
&f_{n+s} = x_s, && s = 1, \dots, q-1,\\
&f_{n+s} = x_{s+1}, && s = q, \dots, j-2,\\
&f_{n+s} = x_{s+2}, && s = j-1, \dots, n-2.
\end{aligned}$$

Thus, using (17) for these f_i's, it follows that,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(l+q+\delta_{r,q})}(1 \otimes E_{l,m}E_{k,j}v_\lambda). \qquad (41)$$

Suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (xiv), we take $l := k$, then we have the following.

(xv) Let $n \geq 4$ and $m, q, j, k \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(q+\delta_{q,k})}(1 \otimes E_{k,m}E_{k,j}v_\lambda). \qquad (42)$$

Now, suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (xiv), we take $l := k$ and $m := j$, then we have the following.

(xvi) Let $n \geq 3$ and, $q, j, k \in \{1, \dots, n\}$,

$$x_{f_1,\dots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(q+\delta_{q,k})}(1 \otimes E_{k,j}E_{k,j}v_\lambda). \qquad (43)$$

Suppose in the definitions of $f_1, \dots, f_{2n-2}$ in (xiv), we take $m := k$ then we have the following.

(xvii) Let $n \geq 4$ and, $q, j, k, l \in \{1, \ldots, n\}$,

$$x_{f_1,\ldots,f_{2n-2}} \cdot (1 \otimes v_\lambda) = (-1)^{(j+k+q+\delta_{j,q})}(1 \otimes E_{l,j}v_\lambda - 1 \otimes E_{k,j}E_{l,k}v_\lambda). \tag{44}$$

Equations (20) and (27) do not give us new information.

Case (3): Do not give us new equations.
Observe that the right-hand side of all the equations from (28) to (44) belongs to $1\otimes_{U((S_n)_{\geq 0})}$, therefore they are trivial singular vectors. Due to Lemma 3 and the fact that $\mathrm{Sing}_+(M(F))$ does not contain trivial singular vectors, we need to ensure that all the equations from (28) to (44) are equal to zero. Since different equations hold for $n = 3$ and $n \geq 4$, we will study these cases separately.

If $n = 3$, Eqs. (29), (30), (37) and (43) hold and they have to be zero. Note that Eq. (30) is equivalent to

$$-(\lambda_1 + \lambda_2)(1 - \lambda_2)(1 \otimes v_\lambda) = 0, \tag{45}$$

$$\lambda_1(1 + \lambda_2)(1 \otimes v_\lambda) = 0, \tag{46}$$

$$\lambda_1(1 + \lambda_1 + \lambda_2)(1 \otimes v_\lambda) = 0. \tag{47}$$

Thus, Eqs. (45)–(47) implies $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = 0$, $\lambda_2 = 1$.

Now, if $n \geq 4$, Eqs. (30) and (32) equate to zero implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$ or $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = 0$ and $\lambda_{n-1} = 1$.

Then we will apply the Freudenthal's formula to calculate the dimensions of the weight spaces and check whether the remaining equations are satisfied.

We will need the following notation to apply Freudental's formula to $\mathfrak{sl}_n(\mathbb{C})$ (cf. [8, Sec. 22.3]):

Let $\mathfrak{h}$ be our chosen Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ and ϵ_j be defined by $\epsilon_j\left(\sum_{i=1}^n a_i E_{i,i}\right) = a_j$. We will consider the roots

$$\phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\},$$

where the root space associated to $(\epsilon_i - \epsilon_j)$ is generated by $E_{i,j}$ and simple roots are

$$\Delta = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n\}.$$

Let Λ^+ be the set of all dominant weights and $\delta = \frac{1}{2}\sum_{\alpha\succ 0}\alpha$. If $\alpha_i := \epsilon_i - \epsilon_{i+1}$, the fundamental dominant weights relatives to Δ of $\mathfrak{sl}_n(\mathbb{C})$ are given by

$$\pi_i = \frac{1}{n}[(n-i)\alpha_1 + 2(n-i)\alpha_2 + \cdots (i-1)(n-i)\alpha_{i-1} + i(n-i)\alpha_i + i(n-1-i)\alpha_{i+1} + \cdots + i\alpha_{n-1}]. \tag{48}$$

Therefore, Λ is a lattice with basis $\pi_i, i = 1, \ldots, n-1$.

Let $n \geq 3$. Require that $(\alpha_i, \alpha_i) = 1$, $(\alpha_i, \alpha_j) = -1/2$ if $|i - j| = 1$ and $(\alpha_i, \alpha_j) = 0$ if $|i - j| \geq 2$.

First, we will consider $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) = (0, \ldots, 0)$. Since $(\lambda + \delta, \lambda + \delta) - (\mu + \delta, \mu + \delta) = 0$ for $\mu = -\alpha_{k-1}$ with $k \in \{1, \ldots, n\}$ it follows from [8, Proposition 21.3 and Lemma C of (13.4)], that μ is not a weight, therefore multiplicities $\mu = -\alpha_{k-1}$ are equal to zero. Also, it follows from Freudenthals formula, that the multiplicities for $\mu = -\sum_{k=j}^{i-1} \alpha_k$ are also equal to zero for all $i, j \in \{1, \ldots, n\}, i > j$. Thus, $E_{i,j} v_\lambda = 0$, for all $i > j$. In particular, all the equations from (23) to (44) are equal to zero. Observe that if $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) = (0, \ldots, 0)$, for $n \geq 3$ then the $\mathfrak{sl}_n(\mathbb{C})$-module F coincides with the exceptional module F_0. Due to Theorem 6, we have to take the quotient of $M(F_0)$ by the submodule generated by all its nontrivial singular vectors to make the module irreducible.

Finally, if $\lambda = (0, 0, \ldots, 1)$, the Freudenthal's formula gives that the multiplicities for $\mu = -2(\alpha_{n-2} + \alpha_{n-1})$ are equal to one. This implies that $E_{n,n-2} E_{n,n-2} v_\lambda \neq 0$, therefore Eq. (43) is nonzero and the induced representation $M(F)$ is not a representation of the n-Lie algebra S^n, for $n \geq 3$. Conversely, it is straightforward to check that if $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) = (0, \ldots, 0)$, the corresponding irreducible quotient of the induced module is an S^n module, finishing our proof. □

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