KILLING-YANO 2-FORMS ON HOMOGENEOUS SPACES

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Abstract. Riemannian manifolds carrying 2-forms satisfying the Killing-Yano equation are natural generalizations of nearly Kähler manifolds. In this article we exhibit new solutions of these equations on flag manifolds.

1. Introduction

We consider Riemannian manifolds (M^{2n}, g, H) where H is a $(1,1)$ tensor satisfying $g(HX,Y) = -g(X,HY)$ for every X,Y vector fields on M. When $H^2 = -I$ we have an almost Hermitian manifold. Notice that when H is invertible, associated to (M^{2n}, g, H) one has an almost Hermitian manifold (M^{2n}, g, J) by considering a polar decomposition $H = PJ$, P a positive definite $(1,1)$ tensor.

Let ∇ denote the Levi Civita connection associated to g. A result of Peter Petersen [17] asserts that when H is invertible and parallel, that is, $(\nabla_X H)Y = 0$ for any X,Y vector fields on M and M is irreducible then $H = cJ$, with $c \in \mathbb{R}$ and (M^{2n}, g, J) Kähler. This happens in the case of a Riemannian homogeneous space $M = G/K$ of maximal rank, for example full flag manifolds, since they are irreducible by a result of [10].

If one requires $(\nabla_X H)X = 0$ for all X, the analog of the nearly Kähler condition when H is an almost complex structure, then the associated 2-form $\omega(X,Y) = g(HX,Y)$ satisfies the Killing Yano equation.

Killing Yano forms were first introduced by K. Yano [23], who showed that they give rise to quadratic first integrals of the geodesic equation. This was first used by R. Penrose and M. Walker ([16]) to integrate the equation of motion.

In 1952 K. Yano considered the generalization of Killing 1-forms (duals of Killing vector fields) defining Killing tensors of order p. They are p-forms η on M such that

$$\nabla_X \eta(Z, Y_1, Y_2, \dots, Y_{p-1}) + \nabla_Z \eta(X, Y_1, Y_2, \dots, Y_{p-1}) = 0 \tag{1}$$

is satisfied for all vector fields $X, Z, Y_1, Y_2, \dots, Y_{p-1}$. In many references these forms are known as Killing-Yano p-forms. They satisfy the Killing-Yano equation:

$$\nabla_X \eta = \frac{1}{p+1}\iota(X)d\eta. \tag{2}$$

Observe that the covariant derivative is totally skew symmetric.

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In the case of Killing 2-forms ω on a Riemannian manifold (M, g) one can consider the associated skew-symmetric tensor $H : TM \to TM$ defined by $\omega(X, Y) = g(HX, Y)$. In this case the 2-form ω is KY if and only if

$$(\nabla_X H)Y + (\nabla_Y H)X = 0,$$

or equivalently,

$$(\nabla_X H)X = 0, \quad \text{for all } X \in \mathfrak{X}(M).$$

As a particular case, if (M, g, J) is an almost Hermitian manifold with Kähler form $\omega(X, Y) = g(JX, Y)$ then ω is a Killing-Yano 2-form if and only if $(\nabla_X J)X = 0$, equivalently (M, g, J) is nearly Kähler.

In [4] the authors showed that a compact simply connected symmetric space carries a non-parallel Killing p-form if and only if it is isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$.

In this note we will consider non degenerate homogeneous solutions to equation (5). We analyse G-invariant Killing-Yano tensors on reductive homogeneous spaces G/K. As an application we study the Killing-Yano equation on generalized flag manifolds and we give examples of invariant Killing-Yano tensors on full flag manifolds of dimension six, eight and twelve. In all cases, we look at the behaviour of the associated almost complex structures and we study their inherited properties.

In section 3 we will prove that the full flag $\mathbb{F}_n = SU(n)/S(U(1)^n), n \geq 3$ carries an invariant non degenerate solution to the Killing Yano equation. In the case of $\mathbb{F}_3 = SU(3)/S(U(1)^3)$ we provide families of solutions. As an 8-dimensional example we show that $SO(5)/T$, T a maximal torus also carries solutions to equation 5. The last example toghether with $F_n n \geq 4$ do not carry invariant nearly Kähler structures (see [], []). In the case of $\mathbb{F}_3$ non degenerate and degenerate solutions are given. The above examples show that on flag manifolds, which may be thought of as the opposite extreme to a symmetric space, there are plenty of invariant solutions to the Killing-Yano equation.

2. Preliminaries

M a differentiable manifold, $f : M \to M$ a diffeomorphism, X a vector field on M. With X^f we will denote the vector field $X^f f = df X$. If ∇ is an affine connection on M, a diffeomorphism $f : M \to M$ is affine if $(\nabla_X Y)^f = \nabla_{X^f} Y^f$ for all X, Y vector fields on M.

If (M, g) is a riemannian manifold , ∇ the Levi Civita connection and X, Y, Z are Killing vector fields on M then

$$g(\nabla_X Y, Z) = 1/2(g([X, Y], Z) + g([Y, Z], X) + g([X, Z], Y)). \tag{3}$$

If $f : M \to M$ is an isometry and X is a Killing vector field on M then X^f is also a Killing vector field on M.

Let (M, g) be a Riemannian manifold, and $H : TM \to TM$ a skew-symmetric endomorphism of the tangent bundle TM of M with its associated 2-form ω given by $\omega(X, Y) = g(HX, Y)$ for all X, Y vector fields on M.

Let N_H be the Nijenhuis tensor of H It is defined by

$$N_H(X, Y) := [HX, HY] - H([X, HY] + [HX, Y]) + H^2[X, Y]. \tag{4}$$

The endomorphism H is called integrable when $N_H \equiv 0$, and the tensor field H is called parallel with respect to ∇ when $\nabla H = 0$, that is, $(\nabla_X H)Y = 0$ for all X, Y vector fields on M.

The endomorphism H is called Killing-Yano (or KY) if the associated 2-form ω, $\omega(X,Y) = g(HX,Y)$ satisfies the *Killing-Yano equation*,

$$(\nabla_X \omega)(Y,Z) + (\nabla_Y \omega)(X,Z) = 0, \tag{5}$$

where ∇ is the Levi-Civita connection and X, Y, Z are arbitrary vector fields on M (see [23]). Equivalently

$$g(\nabla_X H)Y, Z) + g(\nabla_Y H)X, Z) = 0, \tag{6}$$

for all X, Y, Z vector fields on M. When the endomorphism H is parallel, it is clearly KY. If the endomorphism H is KY and not parallel we will say it is Killing Yano strict.

If H is Killing-Yano and also integrable then it is parallel with respect to ∇ ([2] for a proof).

Note that if (J,g) is an almost Hermitian structure, then the fundamental 2 form ω given by $\omega(X,Y) = (JX,Y)$ is Killing-Yano if and only if (J,g) is nearly Kähler. Also, an integrable nearly Kähler structere is Kähler, that is parallel.

2.1. **Homogeneous Riemannian manifolds.** Let $M = G/K$ be a homogeneous space where G is a connected Lie group and K is a closed subgroup of G. If $x \in G$, left invariant translations L_x on G, induce diffeomorphisms $\tau(x)$ on G/K in such a way that $\tau(x)\pi = \pi L_x$ where π stands for the projection $G \to G/K$. If g^M is a G -invariant metric on $M = G/K$, that is $\tau(x)$ are isometries, for all $x \in G$ we will say that M is a riemannian homogeneous space. We may assume the action of G on G/K given by τ is effective, that is, $\tau(x) = I$, I the identity map implies $x = e$, e the identity of the group G.

If ∇^M stands for the associated Levi Civita connection then

$$d\tau(x^{-1})_{\pi(x)}(\nabla^M_X Y)_{\pi(x)} = ((\nabla^M_X Y)^{\tau(x^{-1})})_{\tau(x^{-1})\pi(x)} = (\nabla^M_{X^{\tau(x^{-1})}} Y^{\tau(x^{-1})})_{\pi(e)},$$

thus the Levi Civita connection ∇^M is determined by its value at $\pi(e)$.

Given a G-invariant $(1,1)$-tensor H on M, that is, $Hd\tau(x) = d\tau(x)H$ then $H_{\pi(e)}$ is an endomorphism of $T_{\pi(e)}$ commuting with $d\tau(k)_{\pi(e)}$ for all $k \in K$ and conversely. Also, if X, Y are vector fields on M then

$$\begin{aligned} d\tau(x^{-1})_{\pi(x)}(\nabla^M_X H)Y_{\pi(x)} &= d\tau(x^{-1})_{\pi(x)}(\nabla^M_X HY)_{\pi(x)} - d\tau(x^{-1})_{\pi(x)}(H\nabla^M_X Y)_{\pi(x)} \\ &= (\nabla^M_{X^{\tau(x^{-1})}}(HY)^{\tau(x^{-1})})_{\pi(e)} - H(\nabla^M_{X^{\tau(x^{-1})}} Y^{\tau(x^{-1})}_{\pi(e)} \\ &= ((\nabla^M_{X^{\tau(x^{-1})}} H)Y^{\tau(x^{-1})})_{\pi(e)}, \end{aligned}$$

thus, $\nabla^M_X H$ is also determined by its value at $\pi(e)$.

Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G and $\mathfrak{X}(M)$ the space of all vector fields on M. Each $x \in \mathfrak{g}$ gives a vector field $x^* \in \mathfrak{X}(M)$ defined by $x^*_{\pi(g)} = d/dt_{t=0} exptx.\pi(g)$. Notice that $[x^*, y^*] = -[x,y]^*$ and $x^*_{\pi(g)} = d\pi_g x^r_g$ where $x^r_g = d/dt_{t=0} exptx.g$ is the right invariant vector field defined by $x \in \mathfrak{g}$.

The vector fields x^* are Killing vector fields and they give, in a neighborhood U of $\pi(e)$ a basis of all vector fields $\mathfrak{X}(U)$. Therefore, from 3 the Levi Civita connection is given by

$$g^M(\nabla^M_{x^*}y^*, z^*)_{\pi(e)} = 1/2(g^M([x^*, y^*], z^*)_{\pi(e)} + g^M([y^*, z^*], x^*)_{\pi(e)} - g^M([z^*, x^*], y^*)_{\pi(e)} \tag{7}$$

$$= -1/2(g^M([x, y]^*, z^*)_{\pi(e)} - g^M([y, z]^*, x^*)_{\pi(e)} + g^M([z, x]^*, y^*)_{\pi(e)}. \tag{8}$$

Homogeneous spaces $M = G/K$ carrying a G-invariant metric are reductive, that is, the Lie algebra $\mathfrak{g}$ of G decomposes $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant complement. Choosing one such a complement $\mathfrak{m}$ and identifying with $T_{\pi(e)}M$ via $d(\pi)_e$, the isotropy representation of K on $T_{\pi(e)}M$ is identified with the adjoint representation of K on $\mathfrak{m}$.

The G- invariant metrics on reductive homogeneous M are in one to one correspondence with $\mathrm{Ad}(K)$-invariant inner products on $\mathfrak{m}$. More generally, there is a one to one correspondence between G- invariant (r, s) tensor fields on M and $\mathrm{Ad}(K)$-invariant tensors on the vector space $\mathfrak{m}$.

Given an $\mathrm{Ad}(K)$-invariant inner product $(,)$ on $\mathfrak{m}$ corresponding to the G-invariant metric g^M on M and $x, y, z \in \mathfrak{m}$, 7 becomes

$$g^M(\nabla^M_{x^*}y^*, z^*)_{\pi(e)} = 1/2(-([x, y]_{\mathfrak{m}}, z) + ([z, y]_{\mathfrak{m}}, x) + ([z, x]_{\mathfrak{m}}, y)).$$

(Compare [7] Proposition 7.28).

Another basis of vector fields in a neighborhood of $\pi(e)$ is given by the analogues of left invariant vector fields as considered in [15]. Notice that left invariant vector fields on G do not descend to vector fields on M but restricting to a convenient subset of G do so. Indeed, there exists a neighborhood V of e in G homeomorphic to a product $N \times K_1$ such that $\pi_{|N}$ is an homeomorphism onto a neighborhood N_* of $\pi(e)$. Every $x \in \mathfrak{m}$ induces a vector field on N_* given by $(x_*)_{\pi(c)} = d\tau(c)_{\pi(e)} dpi_e)x$ where $c \in N$. Since $\tau(c)$ are isometries and $[x_*, y_*] = ([x, y]_{\mathfrak{m}})_*$ it follows that,for $x, y, z \in \mathfrak{m}$ one has

$$2g^M(\nabla^M_{x_*}y_*, z_*)_{\pi(e)} = g^M(([x, y]_{\mathfrak{m}})_*, z_*)_{\pi(e)} - g^M(([y, z]_{\mathfrak{m}})_*, x_*)_{\pi(e)} + g^M(([z, x]_{\mathfrak{m}})_*, y_*)_{\pi(e)}.$$

Therefore, for a given $\mathrm{Ad}(K)$-invariant inner product $(,)$ on $\mathfrak{m}$, the Levi Civita connection on M for the corresponding G-invariant metric is given by the mapping $\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$:

$$\Lambda(x)y = 1/2[x, y] + U(x, y),$$

where

$$(2U(x, y), z) = (([z, y]_{\mathfrak{m}}, x) + ([z, x]_{\mathfrak{m}}, y)).$$

The correspondence is given by $d\pi_e\Lambda(x)y = (\nabla^M_{x_*}y_*)_{\pi(e)}$. Moreover, the mapping $\Lambda : \mathfrak{m} \to \mathfrak{so}(\mathfrak{m})$ is $\mathrm{Ad}(K)$-invariant and satisfies

$$\Lambda(x)y - \Lambda(y)x = [x, y]_{\mathfrak{m}}, \quad (\Lambda(x)y, z) + (\Lambda(y)x, z) = 0.$$

G-invariant endomorphisms on G/K are determined by $\mathrm{Ad}(K)$-invariant endomorphisms of $\mathfrak{m}$. Such endomorphisms commute with ad_x for all $x \in \mathfrak{k}$. The adjoint representation $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{m})$ gives an orthogonal decomposition of $\mathfrak{m}$ into mutually orthogonal irreducible subspaces, such that each subspace is ad_x invariant for all $x \in \mathfrak{k}$.

3. The Killing-Yano equation on homogenous manifolds

Let M be a homogeneous Riemannian space $(M = G/K, g^M)$ where G/K is a homogeneous reductive space with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, G is a connected Lie group, K is a closed subgroup of G, π stands for the projection $G \to G/K$ and g^M is a G-invariant metric. The corresponding $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{m}$ will be denoted by $(,)$.

There is a one to one correspondence between homogeneous $(1, 1)$ tensors H on M which are skew symmetric with respect to g^M and linear endomorphisms H on $\mathfrak{m}$ which commute with Ad (K) and are skew symmetric with respect $(,)$ on $\mathfrak{m}$. Examples of this kind of structures are given by almost Hermitian homogeneous spaces and by metric Lie groups carrying skew symmetric endomorphism at $\mathfrak{g}$.

The following proposition gives an algebraic condition that ensure the fulfillment of the Killing-Yano equation (5) on homogeneous reductive spaces. It generalizes to arbitrary isotropy the one obtained in Proposition 1.4 of [3].

Proposition 3.1. *The G-invariant skew-adjoint endomorphism H of TM, where $M = G/K$ is a homogeneous riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{m}$ satisfies $(\nabla^M_X H)Y = 0$ for all vector fields X, Y on M if and only if $\beta_H(x, y, z) = 0$ for all $x, y, z \in \mathfrak{m}$ where*

$$\begin{aligned}\beta_H(x, y, z) = ([x, Hy]_{\mathfrak{m}}, z) + ([z, x]_{\mathfrak{m}}, Hy) + ([z, Hy]_{\mathfrak{m}}, x) + ([x, y]_{\mathfrak{m}}, Hz)+\\ + ([Hz, x]_{\mathfrak{m}}, y) + ([Hz, y]_{\mathfrak{m}}, x).\end{aligned} \tag{9}$$

It satisfies the Killing Yano equation $(\nabla^M_X H)X = 0$ for all vector fields X on M if and only if $\alpha_H(x, y, z) = 0$ for all $x, y, z \in \mathfrak{m}$ where

$$\begin{aligned}\alpha_H(x, y, z) := \beta(x, y, z) + \beta(y, x, z) =\\ ([Hx, y]_{\mathfrak{m}} - [x, Hy]_{\mathfrak{m}}, z) + (-H[y, z]_{\mathfrak{m}} + [Hy, z]_{\mathfrak{m}} + 2[y, Hz]_{\mathfrak{m}}, x)\\ + (-H[x, z]_{\mathfrak{m}} + [Hx, z]_{\mathfrak{m}} + 2[x, Hz]_{\mathfrak{m}}, y).\end{aligned} \tag{10}$$

It is integrable, $N_H(X, Y) = 0$, for all vector fields X, Y on M if and only if $\gamma_H(x, y) = 0$ for all $x, y \in \mathfrak{m}$ where

$$\gamma_H(x, y) := [Hx, Hy]_{\mathfrak{m}} - H([x, Hy]_{\mathfrak{m}} + [Hx, y]_{\mathfrak{m}}) + H^2[x, y]_{\mathfrak{m}}. \tag{11}$$

Moreover, the following identities hold:

1. $\beta_H(x, y, z) = -\beta_H(x, z, y)$,
2. $\alpha_H(x, y, z) = \alpha_H(y, x, z)$
3. $\alpha_H(x, y, x) = \beta_H(x, y, x)$
4. $\beta_H(x, y, z) = 0$ *if* $Hy = az$ *and* $Hz = -ay$.
5. $\alpha_H(x, y, z) + \alpha_H(y, z, x) + \alpha_H(z, x, y) = 0$.

Proof. If g^M is a G-invariant metric on $M = G/K$ and ∇^M stands for the associated Levi Civita connection then H is parallel if and only if $g^M((\nabla^M_X H)Y, Z)_{\pi(e)} = 0$. As considered in [15], there exists a neighborhood V of e in G homeomorphic to a product $N \times K_1$ such that $\pi_{|N}$ is an homeomorphism onto a neighborhood N_* of $\pi(e)$. Every $x \in \mathfrak{m}$ induces a vector field on N_* given by $(x_*)_{\pi(c)} = d\tau(c)_{\pi(e)} d\pi_e x$ where $c \in N$. Notice that $(x_*)_{\pi(c)} = d\pi_c x|_N$

and $[x_*, y_*] = ([x,y]_{\mathfrak{m}})_*$. Thus, H is parallel if and only if $g^M((\nabla^M_{x_*}H)y_*, z_*)_{\pi(e)} = 0$, for $x, y, z \in \mathfrak{m}$.

Since $g^M(\nabla^M_{x_*}y_*, z_*)_{\pi(e)} = (\Lambda(x)y, z)$ and $Hy_* = (Hy)_*$ it follows that H is parallel if and only if $(\Lambda(x)Hy, z) - (H\Lambda(x)y, z) = 0$ and this is equivalent to $\beta_H(x,y,z) = 0$ for all $x, y, z \in \mathfrak{m}$.

H satisfies the Killing-Yano equation if and only if $g^M((\nabla^M_{x_*}H)y_*, z_*)_{\pi(e)} + g^M((\nabla^M_{y_*}H)x_*, z_*)_{\pi(e)} = 0$ hence, if and only if $\beta_H(x,y,z) + \beta_H(y,x,z) = \alpha_H(x,y,z) = 0$ for all $x, y, z \in \mathfrak{m}$.

Finally, the Nijenhuis tensor is also G- invariant thus, H is integrable if and only if $N_H(x_*, y_*)_{\pi(e)} = 0$ for all $x, y \in \mathfrak{m}$. Since

$$\begin{aligned} N_H(x_*, y_*)_\pi(e) =& [Hx_*, Hy_*]_\pi(e) - H([x_*, Hy_*]_\pi(e) + [Hx_*, y_*]_\pi(e)) + H^2[x_*, y_*]_\pi(e) \\ =& (d\pi)_e \gamma_H(x,y) \end{aligned}$$

and $\gamma_H(x,y) \in \mathfrak{m}$ the last assertion follows.

The last identities can be easily verified.

□

Remark 1. It follows from $\beta_H(x,y,z) = -\beta_H(x,z,y)$ that β_H vanishes on $\mathrm{span}\{x,y,z\}$ if and only if

$$\beta_H(x,y,z) = \beta_H(y,z,x) = \beta_H(z,x,y) = 0.$$

Furthermore, from $\alpha_H(x,y,z) = \alpha_H(y,x,z)$ and $\alpha_H(x,y,z) + \alpha_H(y,z,x) + \alpha_H(z,x,y) = 0$ it follows that α_H vanishes on $\mathrm{span}\{x,y,z\}$ if and only if

$$\alpha_H(x,y,z) = \alpha_H(z,x,y) = 0$$

Definition 3.2. If a G-invariant skew-adjoint endomorphism H of TM, where $M = G/K$ is a homogeneous riemannian manifold with reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ satisfies $(\nabla^M_X H)X = 0$ for all vector fields X in M, equivalently, $\alpha_H(x,y,z) = 0$ for all $x, y, z \in \mathfrak{m}$ where in $\mathfrak{m}$ the $Ad(K)$-invariant inner product corresponding to the $G-$invariant metric on M is considered then we will say it is Killing Yano (KY for short). When $(\nabla^M_X H)Y = 0$ for all vector fields X, Y in M, equivalently, $\beta_H(x,y,z) = 0$ for all $x, y, z \in \mathfrak{m}$ we will say it is a parallel skew adjoint endomorphism.

In the case of a symmetric space G/K with canonical decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ every skew symmetric isomorphism H which is $Ad(K)$ invariant is integrable, $\gamma_H(x,y,z) = 0$ for all $x, y, z \in \mathfrak{p}$, because $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. If moreover H satisfies the Killing-Yano equation then it is parallel. Hence symmetric spaces do not carry invariant solutions (non parallel) to the KY equation. But, according to [**?**], spheres do so (of course not invariant).

All the examples known to us of homogeneous skew symmetric endomorphisms H of TM satisfying the Killing-Yano equation occur on Lie groups with a left invariant metric. In the following section we will present homogeneous solutions to equation (5) on various flag manifolds.

When the isotropy is trivial we obtained in [3] an algebraic characterization of non-degenerate Killing-Yano 2-forms on 2-step nilpotent Lie groups. They cannot be parallel, according to Theorem 5.1 in [2]. Furthermore, in [2] a method is shown to build Killing-Yano 2-forms on Lie groups with left invariant metrics, starting with a Lie group equipped with such a tensor and a suitable representation of its Lie algebra. Choosing appropriate representations of the nilpotent Lie algebras obtained in [3] and applying the referred method one may produce many non parallel KY endomorphisms on Lie algebras (not necessarily nilpotent).

If M has dimension 4 there are no non parallel solutions to equation (5)

Theorem 3.3. *Let $M = G/K$ be a 4-dimensional homogeneous reductive space with a G-invariant metric and let H be a G-invariant skewsymmetric endomorphism of TM. If H satisfies the Killing Yano equation then it is parallel.*

Proof. We have to prove that when $\mathfrak{m}$ is 4-dimensional $\alpha_H = 0$ implies $\beta_H = 0$. It is immediate that $\beta_H(x, y, z) = -\beta_H(x, z, y)$. From $\alpha_H(x, y, z) = 0$ it follows that $\beta_H(x, y, z) = -\beta_H(y, x, z)$. Thus β_H is totally skew symmetric. Using $\beta(x, y, z) = 0$ if $Hy = az$ and $Hz = -ay$ and that $\mathfrak{m}$ is $4-$ dimensional the assertion follows.

□

Remark 2. More generally, it was proved in [1] that on a 4-dimensional Riemannian manifold, any Killing-Yano 2-form of constant length is parallel.

As a consequence of results in [10] and [17] one has

Theorem 3.4. *Let M be a homogeneous riemannian manifold $M = G/T$ with a G-invariant metric g, where G is a compact, connected and simple Lie group and T is a maximal torus. If H is an invertible endomorphism of TM satisfying $g(HX, Y) = -g(X, HY)$ for every X, Y vector fields on M then $H = cJ$ with J a Kähler structure with respect to g and $c \in \mathbb{R}, c \neq 0$.*

Proof. If G is a compact connected Lie group and T is a maximal torus of G then $\chi(G/T) > 0$ (see for example [**?**] for a proof). Moreover, homogeneous Riemannian manifolds with G compact and simple and Euler characteristic $\neq 0$ are irreducible as Riemannian manifold ([10]). Hence, if H is an invertible parallel endomorphism which is skew-symmetric with respect to the G-invariant metric g then, according to Theorem 10.3.2 in [17], $H = cJ$ with J a Kähler structure with respect to g and $c \in \mathbb{R}, c \neq 0$.

□

It follows from the Theorem above that in maximal flags parallel invertible $(1, 1)$-tensors are multiples of a Kähler structure.

3.1. **Invariant Killing Yano endomorphisms on flag manifolds.** In this section we obtain explicit invariant solutions to the Killing Yano equation on full flag manifolds.

We write next the Killing-Yano equation on a flag manifold and we present three examples carrying non parallel KY structureon $SU(n)/S(U(1)^n), n \geq 3$ and on $SO(5)/T$.

There exist many equivalent definitions of flag manifolds. We will use the Lie theoretic approach.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and G a connected complex Lie group such that $\mathrm{Lie}(G) = \mathfrak{g}$. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, denote by $\mathfrak{g}_\alpha$ the following subspace of $\mathfrak{g}$:

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \ \text{ for all } H \in \mathfrak{h}\}.$$

$\alpha \in \mathfrak{h}^*$ is called a root if $\alpha \neq 0$ and the subspace $\mathfrak{g}_\alpha$ is nonzero. If R is the set of all roots, $\mathfrak{g}$ decomposes as follows:

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha .$$

We choose a system Σ of simple roots and denote by $R^+ \subset R$ the set of positive roots. Then

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in R^+} \mathfrak{g}_\alpha$$

is a Borel subalgebra of $\mathfrak{g}$. Let G be the simply connected complex Lie group with Lie algebra $\mathfrak{g}$. We denote by B the connected Lie subgroup of G with Lie algebra $\mathfrak{b}$, which is the normalizer of $\mathfrak{b}$ in G. We call the homogeneous space $\mathbb{F} := G/B$ a full (or maximal) flag manifold.

We denote by $\langle \cdot, \cdot \rangle$ the Killing form of $\mathfrak{g}$. Its restriction to $\mathfrak{h}$ is non degenerate, hence, for each $\alpha \in R$, there exists a unique $h_\alpha \in \mathfrak{h}$ such that $\alpha(\cdot) = \langle h_\alpha, \cdot \rangle$. We will work with a Cartan-Weyl basis $X_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in R$, satisfying:

- $\langle X_\alpha, X_{-\alpha} \rangle = 1$,
- $[h, X_\alpha] = \alpha(h) X_\alpha, \forall h \in \mathfrak{h}$,
- $[X_\alpha, X_{-\alpha}] = h_\alpha$,
- $[X_\alpha, X_\beta] = m_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha + \beta \in R$, and zero otherwise,
- $m_{\alpha,\beta} \in \mathbb{Q}$ and $m_{\alpha,\beta} = -m_{-\alpha,-\beta}$.

For each $\alpha \in R$ we set: $A_\alpha := X_\alpha - X_{-\alpha}$, $S_\alpha := X_\alpha + X_{-\alpha}$. It follows that

$$\mathfrak{u} := \mathrm{Span}\{A_\alpha, iS_\alpha, i\mathfrak{h}_{\mathbb{R}}\}_{\alpha \in R^+}$$

is a compact real form of $\mathfrak{g}$, unique up to inner automorphisms, where $\mathfrak{h}_{\mathbb{R}} = \mathrm{span}_{\mathbb{R}}\{h_\alpha : \alpha \in R\}$.

Let $U = \exp(\mathfrak{u})$ be the connected Lie subgroup of G with Lie algebra $\mathfrak{u}$. U is a compact real form of G. The intersection $T = U \cap B$ is a maximal torus of U. U acts transitively on $\mathbb{F}$, hence $\mathbb{F} = G/B = U/T$. Denote by $\pi : G \to G/B$ (resp $\pi : U \to U/T$) and by $\pi(e)$ the origin of the homogeneous spaces

Let $\mathfrak{m}$ be the orthogonal complement (with respect to the Killing form) of $\mathfrak{t}$. $\mathfrak{m}$ is multiplicity free as a $\mathfrak{t}$-module under the adjoint representation and identifies naturally with

$$\mathfrak{m} = \sum_{\alpha \in R} \mathrm{Span}\{A_\alpha, iS_\alpha\} = \sum_{\alpha \in R} \mathfrak{m}_\alpha ,$$

where $\mathfrak{m}_\alpha$, $\alpha \in R$ are the irreducible components.

The complex tangent space $\mathfrak{m}^C$ identifies with $\sum_{\alpha \in R} \mathfrak{g}_\alpha$. A U-invariant metric on $\mathbb{F}$ is determined by $\langle \cdot, \cdot \rangle_Q = -\langle Q \cdot, \cdot \rangle$, where $Q \in GL(\mathfrak{m})$ is symmetric, positive definite and commutes with $\mathrm{Ad}(k)$ $\forall k \in T$. The inner product $\langle \cdot, \cdot \rangle_Q$ admits a natural extension to a symmetric bilinear form on $\mathfrak{m}^C$ that we denote again Q. Since the inner product on $\mathfrak{m}$ is $\mathrm{Ad}(T)$ invariant, the elements of the standard basis $\{A_\alpha, iS_\alpha,\ \alpha \in R\}$ are eignvectors of Q with the same eigenvalue. In the complex tangent space one has $QX_\alpha = \lambda_\alpha X_\alpha, \alpha \in R$, with $\lambda_\alpha > 0$ and $\lambda_{-\alpha} = \lambda_\alpha$. A U-invariant endomorphism H of the tangent bundle $T(U/T)$ skew symmetric with respect to a U-invariant metric correponds to $H : \mathfrak{m} \to \mathfrak{m}$ an $\mathrm{Ad}(T)$-invariant skew-symmetric endomorphism with respect to the inner product $\langle \cdot, \cdot \rangle_Q$, where $\langle \cdot, \cdot \rangle$ is the Killing form. On each component $\mathfrak{m}_\alpha$ we have

$$[H_0]_{\mathfrak{m}_\alpha} = \begin{bmatrix} 0 & -v_\alpha \\ v_\alpha & 0 \end{bmatrix} \tag{12}$$

In the complex tangent space one has $HX_\alpha = iv_\alpha X_\alpha, \alpha \in R$, with $\alpha \in R$ and $v_{-\alpha} = -v_\alpha$.

We will call *triple of roots* to the roots α, β, $\gamma \in R$ such that $\alpha + \beta + \gamma = 0$. We denote $\{\alpha, \beta, \gamma\}$.

Proposition 3.5. *Let $H \in GL(\mathfrak{m})$ be $\mathrm{Ad}(T)$-invariant and skew-symmetric with respect to $\langle \cdot, \cdot \rangle_Q$. Then H is KY if and only if the following equations are satisfied $\forall$ α, β, $\gamma \in R$ such that $\alpha + \beta + \gamma = 0$:*

$$(v_\beta - v_\alpha)\lambda_\gamma + (\lambda_\beta - \lambda_\alpha)(v_\alpha + v_\beta + 2v_\gamma) = 0, \tag{13}$$

$$(v_\gamma - v_\alpha)\lambda_\beta + (\lambda_\gamma - \lambda_\alpha)(v_\alpha + v_\gamma + 2v_\beta) = 0; \tag{14}$$

H is parallel if and only if the following equations are satisfied $\forall$ α, β, $\gamma \in R$ such that $\alpha + \beta + \gamma = 0$:

$$(v_\beta + v_\gamma)(\lambda_\gamma + \lambda_\beta - \lambda_\alpha) = 0 \tag{15}$$

$$(v_\alpha + v_\gamma)(\lambda_\gamma - \lambda_\beta + \lambda_\alpha) = 0 \tag{16}$$

$$(v_\beta + v_\alpha)(-\lambda_\gamma + \lambda_\beta + \lambda_\alpha) = 0. \tag{17}$$

Proof. According to Proposition 3.1, H satisfies the Killing-Yano equation if and only if $\alpha_H = 0$ on $\mathfrak{m}$ if and only if $\alpha_H = 0$ on $\mathfrak{m}^{\mathbb{C}}$. Observe that $\alpha_H = 0$ on the span of x, y, z if and only if $\alpha_H(x, y, z) = 0$ and $\alpha_H(z, x, y) = 0$ (see Remark 1). In our situation $\mathfrak{m}^{\mathbb{C}} = \sum \mathfrak{g}_\alpha$ with $\alpha \in R$ so we analyze the vanishing of α_H in the span of $X_\alpha, X_\beta, X_\gamma$. We compute

$$\begin{aligned} \alpha_H(X_\alpha, X_\beta, X_\gamma) :=& \langle [HX_\alpha, X_\beta] - [X_\alpha, HX_\beta], QX_\gamma \rangle \\ &+ \langle -H[X_\beta, X_\gamma] + [HX_\beta, X_\gamma] + 2[X_\beta, HX_\gamma], QX_\alpha \rangle \\ &+ \langle -H[X_\alpha, X_\gamma] + [HX_\alpha, X_\gamma] + 2[X_\alpha, HX_\gamma], QX_\beta \rangle \\ &= i\, m_{\alpha,\beta} \langle X_{\alpha+\beta}, X_\gamma \rangle (\lambda_\gamma(v_\alpha - v_\beta) + (\lambda_\alpha - \lambda_\beta)(v_\alpha + v_\beta + 2v_\gamma)). \end{aligned} \tag{18}$$

Similarly

$$\alpha_H(X_\gamma, X_\alpha, X_\beta) := i\, m_{\gamma,\alpha} \langle X_{\gamma+\alpha}, X_\beta \rangle (\lambda_\beta(v_\gamma - v_\alpha) + (\lambda_\gamma - \lambda_\alpha)(v_\gamma + v_\alpha + 2v_\beta)). \tag{19}$$

□

Remark 3. The analogue of the above proposition for generalized flag manifolds has been obtained in [11].

3.2. **A strict KY solution for $F_n = SU(n)/S(U(1)^n)$.** In the following theorem we will show that a particular solution on F_3 can be extended to give a KY solution on F_n. According to [], see also[], $F_n, n \geq 4$ do not carry invariant Nearly Kähler structures.

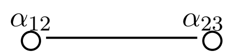
Theorem 3.6. *$SU(n)/S(U(1)\times ... \times U(1)), n\geq 3$ admits a non parallel KY solution. The pair (H,Q) is given by:*

$$HX_{\alpha_{jk}} = (k-j)viX_{\alpha_{jk}},\ \forall k>j, v\in\mathbb{R} \tag{20}$$

$$\lambda_{\alpha_{jk}} = (k-j)^2\lambda, \lambda\in\mathbb{R}^+ \tag{21}$$

In other words v_{jk} is equal to a multiple of the height of the root, and λ_{jk} is a multiple of the square of the height of the root.

Proof. We will prove it by induction. We start with $n=3$.



In this case we have the following triple

$$\alpha_{12}+\alpha_{23}-\alpha_{13}=0$$

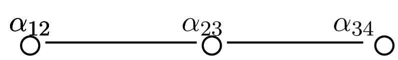
According to the KY condition, it must be satisfied the following equations:

$$(v_{23}-v_{12})\lambda_{13}+(v_{12}+v_{23}-2v_{13})(\lambda_{23}-\lambda_{12})=0$$
$$(-v_{13}-v_{12})\lambda_{23}+(v_{12}-v_{13}+2v_{23})(\lambda_{13}-\lambda_{12})=0$$

Putting $v_{12}=v_{23}=v$, $\lambda_{12}=\lambda_{23}=\lambda$ and $v_{13}=2v$, $\lambda_{13}=4\lambda$, one verifies it is a solution to the previous equations, thus $SU(3)/S(U(1)^3)$ carries an invariant solution to the KY equation. This is one of the solutions considered in....

We proceed next with $SU(4)/S(U(1)^4)$. This will give the idea of how to proceed with the induction in general.

Here we have the following situation



with simple roots : $\alpha_{12},\alpha_{23},\alpha_{34}$, then the positive roots are $\alpha_{12},\alpha_{23},\alpha_{34},\ \alpha_{13},\alpha_{24},\alpha_{14}$ and triples of roots given by

$$\alpha_{12}+\alpha_{23}-\alpha_{13}=0$$
$$\alpha_{23}+\alpha_{34}-\alpha_{24}=0$$
$$\alpha_{13}+\alpha_{34}-\alpha_{14}=0$$
$$\alpha_{12}+\alpha_{24}-\alpha_{14}=0$$

Remember that $\mathfrak{su}(4)=\{A\in\mathfrak{gl}(4,\mathbb{C}) : A+A^*=I \ \wedge \operatorname{Tr}(A)=0\}$, then $\mathfrak{m}=$ $\left\{\begin{bmatrix} 0 & z_1 & z_2 & z_3\\ -\overline{z_1} & 0 & z_4 & z_5\\ -\overline{z_2} & -\overline{z_4} & 0 & z_6\\ -\overline{z_3} & -\overline{z_5} & -\overline{z_6} & 0\end{bmatrix}\right\}$ and $\mathfrak{t}=\left\{\begin{bmatrix} ai & 0 & 0 & 0\\ 0 & bi & 0 & 0\\ 0 & 0 & ci & 0\\ 0 & 0 & 0 & di\end{bmatrix} : a+b+c+d=0\right\}$.

Notice that the positive roots $\alpha_{12},\alpha_{23},\alpha_{13}$ correspond to one $SU(3)$ and $\alpha_{23},\alpha_{34},\alpha_{24}$ to other $SU(3)$ inside $SU(4)$ where a solution was found. Thus one needs to consider only the last two equations involving the longest root α_{14}.

Applying ...., from the last two triples one has four equations on $v's$ and λ's to be satisfied:

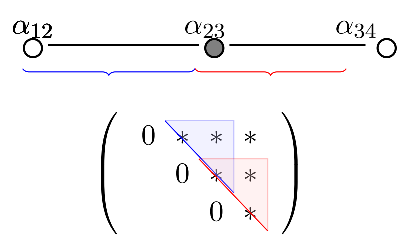
$$
\begin{aligned}
(v_{34}-v_{13})\lambda_{14}+(v_{13}+v_{34}-2v_{14})(\lambda_{34}-\lambda_{13})&=0\\
(-v_{14}-v_{13})\lambda_{34}+(-v_{14}+v_{13}+2v_{34})(\lambda_{14}-\lambda_{13})&=0\\
(v_{24}-v_{12})\lambda_{14}+(v_{24}+v_{12}-2v_{14})(\lambda_{24}-\lambda_{12})&=0\\
(-v_{14}-v_{12})\lambda_{24}+(-v_{14}+v_{12}+2v_{24})(\lambda_{14}-\lambda_{12})&=0
\end{aligned}
$$

Replacing $v_{12}=v_{23}=v$ hence $v_{23}=v_{34}=v$; $\lambda_{12}=\lambda_{23}=\lambda_{34}=\lambda$, $v_{13}=v_{24}=2v$, $\lambda_{13}=\lambda_{24}=4\lambda$ that are the solutions corresponding to both $SU(3)$, the above equations become

$$
\begin{aligned}
(-v)\lambda_{14}+(3v-2v_{14})(-3\lambda)&=0\\
(-v_{14}-2v)\lambda+(-v_{14}+4v)(\lambda_{14}-4\lambda_{)}&=0\\
(v)\lambda_{14}+(3v-2v_{14})(3\lambda)&=0\\
(-v_{14}-v)4\lambda+(-v_{14}+5v)(\lambda_{14}-\lambda)&=0
\end{aligned}
$$

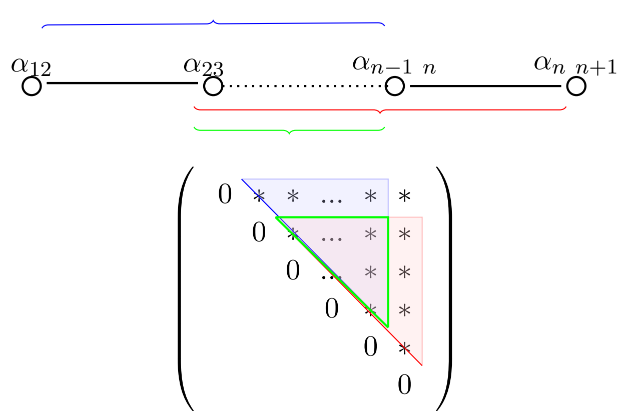
and $v_{14}=3v$, $\lambda_{14}=9\lambda$ gives a solution.

We are considering the following situation that is easy to see in the Dynkin diagram. We note that if we separate in two dynkin diagram we have the dynkin diagram with simples roots $\{\alpha_{12},\alpha_{23},\alpha_{34}\}$



Assume the theorem holds on $SU(n)/S(U(1)^n)$ and we will prove it holds on $SU(n+1)/S(U(1)^{n+1})$.

diagrama A_{n+1}



In the $n+1$ case the simple roots are given by $\Sigma = \{\alpha_{12}, ..., \alpha_{n-1\ n}, \alpha_{n\ n+1}\}$. Split $\Sigma = \{\alpha_{12}, ..., \alpha_{n-1\ n}\} \cup \{\alpha_{23}, ..., \alpha_{n\ n+1}\} = \Sigma_1 \cup \Sigma_2$. The positive roots coming from Σ_1 and from Σ_2 give two $SU(n)$ inside $SU(n+1)$ where the theorem holds. The intersection of the two system is the case $n-1$, where the theorem also holds.

Thus the only triples we need to consider are the ones which the root $\alpha_{1\ n+1}$ is involved:

$$\alpha_{12} + \alpha_{2n+1} - \alpha_{1n+1} = 0$$
$$\alpha_{13} + \alpha_{3n+1} - \alpha_{1n+1} = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$\alpha_{1j} + \alpha_{jn+1} - \alpha_{1n+1} = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$\alpha_{1n} + \alpha_{nn+1} - \alpha_{1n+1} = 0$$

Each triple give rise to two equations (**??**

$$(v_{j(n+1)} - v_{1j})\lambda_{1(n+1)} + (v_{1j} + v_{j(n+1)} - 2v_{1(n+1)})(\lambda_{j(n+1)} - \lambda_{1j}) = 0$$
$$(-v_{1(n+1)} - v_{1j})\lambda_{j(n+1)} + (v_{1j} - v_{1(n+1)} + 2v_{j(n+1)})(\lambda_{1n+1} - \lambda_{1j}) = 0$$

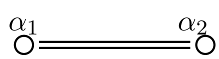
By the inductive hypothesis, if $j > 1$, one has $v_{1j} = (j-1)v$, $v_{j(n+1)} = (n+1-j)v$ and $\lambda_{1j} = (j-1)^2\lambda$, $\lambda_{j(n+1)} = (n+1-j)^2\lambda$. Replacing these values in the above equation

□

3.3. **An additional example:** $SO(5)/T$. The complex simple Lie group $SO(5)$ is defined as $SO(5) = \left\{A \in GL(5,\mathbb{C})/AA^T = I \text{ y } \det(A) = 1\right\}$, its Lie algebra $\mathfrak{so}(5)$ is the set of complex skew symmetric matrix of order 5 with zero trace.

The system of roots of $B_2 = \mathfrak{so}(5)$ is given by $R = \{\pm e_1, \pm e_2, \pm(e_1 - e_2), \pm(e_1 + e_2)\}$. If we take $\Sigma = \{e_1 - e_2, e_2\}$ as the set of simple roots, then the set of positive roots is $\{e_1, e_2, e_1 - e_2, e_1 + e_2\}$. We call $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, $\alpha_3 = e_1$ and $\alpha_4 = e_1 + e_2$.

The Dynkin diagram of this example is:



The following identities define the only two possibles triples for this case:

$$e_1 = (e_1 - e_2) + e_2 = \alpha_1 + \alpha_2$$
$$e_1 + e_2 = ((e_1 - e_2) + e_2) + e_2 = \alpha_1 + \alpha_2 + \alpha_2$$

Hence

$$\alpha_1 + \alpha_2 - \alpha_3 = 0, \quad \alpha_2 + \alpha_3 - \alpha_4 = 0$$

Thus, there are only two possibles triples in this case and a KY solution will be the intersection of the solutions of each triple.

The tangent space $T_{eT}SO(5)/T$ descomposes as $\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_3} \oplus \mathfrak{m}_{\alpha_4}$ and $\dim \mathfrak{m} = 8$.

Given H, a $\mathrm{Ad}(T)$-invariant skew symmetric, invertible and KY endomorphism on $\mathfrak{m} = \mathfrak{m}_{\alpha_1} \oplus \mathfrak{m}_{\alpha_2} \oplus \mathfrak{m}_{\alpha_3} \oplus \mathfrak{m}_{\alpha_4}$ we apply equations **??** and **??** to obtain the following

$$(\lambda_2 - \lambda_1 - \lambda_3)v_1 + (\lambda_3 + \lambda_2 - \lambda_1)v_2 - 2(\lambda_2 - \lambda_1)v_3 = 0$$
$$(\lambda_3 - \lambda_1 - \lambda_2)v_1 + 2v_2(\lambda_3 - \lambda_1) + (\lambda_1 - \lambda_2 - \lambda_3)v_3 = 0$$
$$(-\lambda_4 - \lambda_2 + \lambda_3)v_2 + (\lambda_4 + \lambda_3 - \lambda_2)v_3 + 2(\lambda_2 - \lambda_3)v_4 = 0$$
$$(\lambda_4 - \lambda_3 - \lambda_2)v_2 + 2(\lambda_4 - \lambda_2)v_3 + (-\lambda_3 - \lambda_4 + \lambda_2)v_4 = 0$$

where the first two equation corresponds to the first triple and the last two to the second triple.

The above system has infinitely many solutions. Two curves of solutions are given by

$$Q = (\lambda, \lambda, 3/2\lambda, 3/2\lambda), \quad H = (v, v, 1/3v, 1/3v).$$
$$Q = (\lambda, \lambda, 2/3\lambda, 2/3\lambda), \quad H = (v, v, -3v, 3v).$$

Sakane [19] classify all the Ricci Einstein metrics for this example, they are:

$2\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = 1,\ \lambda_4 = 4$ $\quad \lambda_1 = 4,\ \lambda_2 = 3,\ \lambda_3 = 1,\ \lambda_4 = 2$ $\quad \lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 3,\ \lambda_4 = 4$ $\quad \lambda_1 = 4,\ \lambda_2 = 1,\ \lambda_3 = 3,\ \lambda_4 = 2$ $\quad \lambda_1 = \lambda_4 = \dfrac{24 \pm 4\sqrt{6}}{15},\ \lambda_2 = \dfrac{7 \pm 2\sqrt{6}}{5}$

The first four metrics are Kähler metrics. By direct calculation one can verify that the fifth metric do not admit a solution to the KY equation.

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