CERTAIN FRACTIONAL TYPE OPERATORS WITH HÖRMANDER CONDITIONS

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Abstract. In this paper we study fractional type operators with more than one kernel, defined by

$$T_{\alpha,m}f(x) = \int_{\mathbb{R}^n} k_1(x - A_1y)k_2(x - A_2y)\dots k_m(x - A_my)f(y)dy,$$

where, for $1 \leq i \leq m$, each k_i satisfies a fractional size condition and generalized fractional Hörmander condition, and A_i are invertibles matrices. We obtain weighted Coifman type estimates, strong and weak type inequalities and BMO estimates for this operator. We also present some examples different from those in the literature.

1. Introduction

The classical integral operators, for example the Calderón-Zygmund operator or the fractional integral I_α, have kernels with only one possible singularity. For the study of integral operators with more that one singularity in the kernel, we write the kernel as product of functions where each function has only one possible singularity.

In [18], Ricci and Sjögren obtain the $L^p(\mathbb{R}, dx)$ boundedness, $p > 1$, for a family of maximal operators on the three dimensional Heisenberg group. Some of these operators arise in the study of the boundary behavior of Poisson integrals on the symmetric space $SL\mathbb{R}^3/SO(3)$. To get the principal result, they study the boundedness on $L^2(\mathbb{R})$ of the operator

$$T_\alpha f(x) = \int_{\mathbb{R}} |x - y|^{-\alpha}|x + y|^{\alpha-1}f(y)dy, \tag{1.1}$$

for $0 < \alpha < 1$. Later, in [12], Godoy and Urciuolo study a generalization of (1.1) for $\mathbb{R}^n$.

More recently, in [19] the second author and Urciuolo analyze the following generalization of these operators. Let $0 \leq \alpha < n$ and $m \in \mathbb{N}$. For $1 \leq i \leq m$, let A_i be matrices such that

$$A_i \text{ is invertible and } A_i - A_j \text{ is invertible for } i \neq j, 1 \leq i, j \leq m. \tag{H}$$

For any $f \in L^\infty_{\text{loc}}(\mathbb{R}^n)$, they define

$$T_{\alpha,m}f(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy, \tag{1.2}$$

where

$$K(x, y) = k_1(x - A_1y)k_2(x - A_2y)\dots k_m(x - A_my), \tag{1.3}$$

and k_i is a fractional rough kernel defined is the following way, let $1 < q_i < \infty$ such that $\frac{n}{q_1} + \dots + \frac{n}{q_m} = n - \alpha$. Let Σ the unit sphere in $\mathbb{R}^n$, $\Omega_i \in L^1(\Sigma)$ homogeneous of degree 0.

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Then they consider

$$k_i(x) = \frac{\Omega_i(x/|x|)}{|x|^{n/q_i}}, \tag{1.4}$$

and proved the weighted Coifman type estimates, strong and weak type inequalities and BMO estimates for this operator.

During the last years, several authors studied operators of the form (1.2) in different contexts: weighted Lebesgue and Hardy spaces with constant and variable exponent, also the endpoint estimates and boundedness in BMO and weighted BMO. See for example [9, 11, 13, 14, 20, 21, 22, 23, 24, 25, 26, 27].

These operators generalized classical operators as I_α, the fractional integral operator, and rough fractional and singular operators. In the case of $\alpha = 0$, $T_{0,m}$ behaves like a singular integral operator in sense of L^p boundedness. For $\alpha > 0$, if $1 < p < n/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ then $T_{\alpha,m}$ is bounded from L^p into L^q. It is well known that if $0 < p < 1$ the operator I_α is bounded from H^p into H^q, for some q. In several cases the operators consider in this paper are not bounded from H^p into H^q, but instead are bounded from H^p into L^q, $0 < p < 1$ and some q (see [23, 24]).

In this paper, we consider the operator $T_{\alpha,m}$ defined by (1.2) and (1.3) with the matrices A_i satisfying the condition (H). Let $0 \le \alpha_i < n$, $1 \le i \le m$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$, and assume that k_i satisfies a fractional size condition and a generalized fractional Hörmander condition. The definition of spaces and objets involved in this paper are described in section 2.

Our first result is a pointwise estimate that relates the sharp delta maximal function of $T_{\alpha,m}f$, $M^\sharp_\delta(T_{\alpha,m}f)$, $0 < \delta \le 1$, with a generalized fractional maximal function of f. This estimate is a fundamental key to obtain weighted inequalities for the operator $T_{\alpha,m}$. These inequalities are developed in section 3. These weighted inequalities are the Coifman type estimates, the endpoint estimates and strong type estimates with $A_{p,q}$ weights and bump conditions.

In the section 4, we present new examples of this type of operators different than the ones described above. In section 5 we present the weak type (1,1) estimate with respect to the Lebesgue measure for $T_{0,m}$. In section 6 we give the proofs of the results.

2. Preliminaries

In this section we present some notions about Young function, Luxemburg average and weights that will be fundamental throughout all this work.

Young Function and Luxemburg average. For more details of this topic see [16] or [17]. A function $\Psi : [0,\infty) \to [0,\infty)$ is said to be a Young function if Ψ is continuous, convex, no decreasing and satisfies $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$.

The average of the Luxemburg norm of a function f induced by a Young function Ψ in the ball B is defined by

$$\|f\|_{\Psi,B} := \inf\left\{\lambda > 0 : \frac{1}{|B|}\int_B \Psi\left(\frac{|f|}{\lambda}\right) \le 1\right\}.$$

Observe that if $\Psi(t) = t^r$, $r \ge 1$, $\|f\|_{\Psi,B} = \|f\|_{r,B} = \left(\frac{1}{|B|}\int_B |f|^r\right)^{1/r}$.

Each Young function Ψ has an associated complementary Young function $\overline{\Psi}$ satisfying the generalized Hölder inequality

$$\frac{1}{|B|}\int_B |fg| \leq 2\|f\|_{\Psi,B}\|g\|_{\overline{\Psi},B}.$$

Remark 2.1. Observe that if you see the proof of this last inequality in [16], the ball B can be replaced by any measurable set E such that $|E|<\infty$.

If $\Psi_1,\dots,\Psi_m,\phi$ are Young functions satisfying that for some $t_0>0$, $\Psi_1^{-1}(t)\cdots\Psi_m^{-1}(t)\phi^{-1}(t)\leq ct$, for all $t\geq t_0$, then

$$\|f_1\cdots f_m g\|_{1,B} \leq c\|f_1\|_{\Psi_1,B}\cdots\|f_m\|_{\Psi_m,B}\|g\|_{\phi,B}. \tag{2.1}$$

The function ϕ is called the complementary of the functions $\Psi_1,\dots,\Psi_m$.

Given $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and $0\leq\alpha<n$, the fractional maximal operator associated to the Young function Ψ is defined as

$$M_{\alpha,\Psi}f(x) := \sup_{B\ni x}|B|^{\alpha/n}\|f\|_{\Psi,B}.$$

Now we compile some examples of maximal operators related to certain Young functions.

- If $\Psi(t)=t$ then $M_{\alpha,\Psi}=M_\alpha$, the classical fractional maximal operator.
- $\Psi(t)=t^r$ with $1<r<\infty$. In this case $M_{\alpha,\Psi}=M_{\alpha,r}$, where $M_{0,r}f=M(f^r)^{1/r}$.
- $\Psi(t)=\exp(t)-1$. Then, $M_{\alpha,\Psi}=M_{\alpha,\exp}$.
- If $\beta\geq 0$ and $1\leq r<\infty$, $\Psi(t)=t^r\log(e+t)^\beta$ is a Young function then $M_{\alpha,\Psi}=M_{\alpha,L^r(\log L)^\beta}$.
- If $\alpha=0$ and $k\in\mathbb{N}$, $\Psi(t)=t\log(e+t)^k$ it can be proved that $M_\Psi\approx M^{k+1}$, where M^{k+1} is M iterated $k+1$ times.

Remark 2.2. Observe that if $\Psi(t)=t^r$ then a simple computation show that

$$M_{\alpha,r}f=\left(M_{\alpha r}|f|^r\right)^{1/r}.$$

Fractional size and fractional Hörmander conditions. Now we present the fractional size condition and a generalized fractional Hörmander condition. For more details of these objects see [2] or [10].

Let Ψ be a Young function and let $0\leq\alpha<n$. Let us introduce some notation: $|x|\sim s$ means $s<|x|\leq 2s$ we write $\|f\|_{\Psi,|x|\sim s}=\|f\chi_{|x|\sim s}\|_{\Psi,B(0,2s)}$.

The function K_α is said to satisfies the fractional size condition and we set $K_\alpha\in S_{\alpha,\Psi}$, if there exists a constant $C>0$ such that

$$\|K_\alpha\|_{\Psi,|x|\sim s}\leq Cs^{\alpha-n}.$$

When $\Psi(t)=t$ we write $S_{\alpha,\Psi}=S_\alpha$. Observe that if $K_\alpha\in S_\alpha$, then there exists a constant $c>0$ such that

$$\int_{|x|\sim s}|K_\alpha(x)|dx\leq cs^\alpha.$$

The function K_α satisfies the $L^{\alpha,\Psi}$-Hörmander condition and we set $K\in H_{\alpha,\Psi}$, if there exists $c_\Psi>1$ and $C_\Psi>0$ such that for all x and $R>c_\Psi|x|$,

$$\sum_{m=1}^\infty (2^mR)^{n-\alpha}\|K_\alpha(\cdot-x)-K_\alpha(\cdot)\|_{\Psi,|y|\sim 2^mR}\leq C_\Psi.$$

We say that $K_\alpha \in H_{\alpha,\infty}$ if K_α satisfies the previous condition with $\|\cdot\|_{L^\infty,|x|\sim 2^mR}$ in place of $\|\cdot\|_{\Psi,|x|\sim 2^mR}$.

When $\Phi(t) = t^r$, $1 \le r < \infty$, we recover the fractional L^r-Hörmander condition and simply write $H_{\alpha,r}$ instead of $H_{\alpha,\Psi}$.

Weights. We say that a function w is a weight if w is a non negative function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let $0 \le \alpha < n$, $1 \le p, q \le \infty$, we say that a weight w belong to the class $A_{p,q}$ if

$$[w]_{A_{p,q}} = \sup_B \|w\|_{q,B}\|w^{-1}\|_{p',B} < \infty.$$

If $1 \le p < \infty$, A_p denotes the classical Muckenhoupt class of weights. Note that $w \in A_{p,p}$ is equivalent to $w^p \in A_p$. We recall that $A_\infty = \cup_{p\ge 1} A_p$, and the statement $w \in A_{\infty,\infty}$ is equivalent to $w^{-1} \in A_1$.

The fractional B_p condition, which is denote by B_p^α was introduced by Cruz-Uribe and Moen in [5]: Let $0 \le \alpha < n$, $1 < p < n/\alpha$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and ϕ be a Young function, we say $\phi \in B_p^\alpha$ if

$$\int_1^\infty \frac{\phi(t)^{q/p}}{t^q}\frac{dt}{t} < \infty.$$

They proved, in Theorem 3.3 in [5], that if $\phi \in B_p^\alpha$ then $M_{\alpha,\phi} : L^p(dx) \to L^q(dx)$ and

$$\|M_{\alpha,\phi}\|_{L^p\to L^q} \le c\left(\int_1^\infty \frac{\phi(t)^{q/p}}{t^q}\frac{dt}{t}\right)^{1/q}.$$

We will consider the following bump conditions: let $1 < q < \infty$ and Ψ be a Young function, then a weight $w \in A_{q,\Psi}$ if

$$[w]_{A_{q,\Psi}} = \sup_Q \|w\|_{q,Q}\|w^{-1}\|_{\Psi,Q} < \infty.$$

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, the sharp maximal function is defined by

$$M^\# f(x) = \sup_{B\ni x}\frac{1}{|B|}\int_B \left| f - \frac{1}{|B|}\int_B f\right|.$$

A locally integrable function f has bounded mean oscillation ($f \in BMO$) if $M^\# f \in L^\infty$ and the norm $\|f\|_{BMO} = \|M^\# f\|_\infty$

Observe that the BMO norm is equivalent to

$$\|f\|_{BMO} = \|M^\# f\|_\infty \sim \sup_B \inf_{a\in\mathbb{C}}\frac{1}{|B|}\int_B |f(x) - a|dx.$$

There is also a weighted version of BMO, this is denoted by $BMO(w)$, and it is described by the seminorm

$$|||f|||_w = \sup_B \|w\chi_B\|_\infty \left(\int_B \left| f(x) - \frac{1}{|B|}\int_B f\right| dx\right).$$

It is easy to check that

$$|||f|||_w \simeq \|wM^\# f\|_\infty.$$

3. Main results

In this section, we present the main results of this paper. We start with the pointwise estimates of the sharp delta maximal function.

Theorem 3.1. *Let $0 \le \alpha < n$, $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.2). For $1 \le i \le m$, let Ψ_i be a Young function and let $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Let $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and let the matrices A_i satisfy the hypothesis (H).*
If $\alpha = 0$, suppose $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.
If ϕ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then there exists $C > 0$ such that, for $0 < \delta \le 1$ and $f \in L^\infty_c(\mathbb{R}^n)$ (f a bounded function with compact support)

$$M^\sharp_\delta |T_{\alpha,m} f|(x) := M^\sharp \left(|T_{\alpha,m} f|^\delta \right)(x)^{1/\delta} \le C \sum_{i=1}^m M_{\alpha,\phi} f(A_i^{-1} x). \tag{3.1}$$

Remark 3.2. Observe that in Theorem 3.1 if $\alpha = 0$ then $m > 1$. Indeed $\alpha = 0$ and $m = 1$ imply $\alpha_1 = n$. So $T_{0,1}$ is a singular integral operator and the size condition has no sense. Nevertheless the result of the Theorem is still true, see [15].

For the weighted estimates we need an extra condition for the weights. There exists $c > 0$ such that

$$w(A_i x) \le c w(x), \tag{3.2}$$

$a.e. x \in \mathbb{R}^n$ and for all $1 \le i \le m$.

Theorem 3.3. *Let $0 \le \alpha < n$ and $m \in \mathbb{N}$ and let $T_{\alpha,m}$ be the integral operator defined by (1.2). For $1 \le i \le m$, let Ψ_i be Young functions, $0 \le \alpha_i < n$ such that $\alpha_1 + \cdots + \alpha_m = n - \alpha$. Also suppose $k_i \in S_{n-\alpha_i,\Psi_i} \cap H_{n-\alpha_i,\Psi_i}$ and that matrices A_i satisfy the hypothesis (H).*
If $\alpha = 0$, suppose $T_{0,m}$ be of strong type (p_0, p_0) for some $1 < p_0 < \infty$.
Let $0 < p < \infty$. If ϕ is the complementary of the functions $\Psi_1, \ldots, \Psi_m$, then there exists $C > 0$ such that, for $f \in L^\infty_c(\mathbb{R}^n)$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |T_{\alpha,m} f(x)|^p w(x) dx \le C \sum_{i=1}^m \int_{\mathbb{R}^n} |M_{\alpha,\phi} f(x)|^p w(A_i x) dx, \tag{3.3}$$

whenever the left-hand side is finite.
Futhermore, if w satisfies (3.2), then

$$\int_{\mathbb{R}^n} |T_{\alpha,m} f(x)|^p w(x) dx \le C \int_{\mathbb{R}^n} |M_{\alpha,\phi} f(x)|^p w(x) dx.$$

By (3.3), the Coifman type estimate, we can obtain weighted inequalities for $T_{\alpha,m}$. To obtain these inequalities we need a relationship between M_Φ and M_r. Caldarelli, Lerner and Ombrosy in [3], and Di Plinio and Lerner in [7], proved the following

Lemma 3.4. [3, 7] *Let Φ be a Young function. For all $x \in \mathbb{R}^n$ and $r > 1$,*

$$M_\Phi f(x) \le \left(2 \sup_{t \ge \Phi^{-1}(1/2)} \frac{\Phi(t)}{t^r} \right)^{1/r} M_r f(x) =: \kappa_r M_r f(x).$$

It follows, in analogous way, that,

$$M_{\alpha,\Phi} f(x) \le c \kappa_r M_{\alpha,r} f(x). \tag{3.4}$$

First, we get a weighted BMO estimate for weights in the class $A(\frac{n}{\alpha r}, \infty)$.

Theorem 3.5. *Let $T_{\alpha,m}$ be as in Theorem 3.3. Suppose there exists $r>1$ such that $\kappa_r<\infty$. If $w^r\in A(\frac{n}{\alpha r},\infty)$ and satisfies (3.2), then there exists $C>0$ such that for $f\in L^\infty_c(\mathbb{R}^n)$,*

$$\||T_{\alpha,m}f|\|_w \le C\|fw\|_{L^{n/\alpha}}.$$

In [19] it is proved a result analogous to the weighted BMO estimate for $T_{\alpha,m}$, so we omit the proof.

Theorem 3.6. *Let $T_{\alpha,m}$ be as in Theorem 3.3. Let $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Suppose there exists $1<r<p$ such that $\kappa_r<\infty$. If $w^r\in A(1,\frac{n}{n-\alpha r})$ and satisfies (3.2) then there exists $C>0$ such that for $f\in L^\infty_c(\mathbb{R}^n)$,*

$$\sup_{\lambda>0}\lambda(w^{\frac{rn}{n-\alpha r}}\{x\in\mathbb{R}^n:|T_{\alpha,m}f(x)|>\lambda\})^{\frac{n-\alpha r}{rn}}\le C\left(\int|f(x)|^r w^r(x)dx\right)^{1/r}.$$

The strong type inequality follows from the boundedness of $M_{\alpha,\phi}$, Theorem 2.6 in [1].

Theorem 3.7. *Let $T_{\alpha,m}$ be as in Theorem 3.3. Let $1\le r<p<n/\alpha$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let η and φ be Young functions such that $\eta^{-1}(t)t^{\frac{\alpha}{n}}\lesssim\varphi^{-1}(t)$ for every $t>0$. If $\varphi^{1+\frac{sn}{n-\alpha}}\in B_{\frac{sn}{n-\alpha}}$ for every $s>r(n-\alpha)/(n-\alpha r)$ and $w^r\in A(\frac{p}{r},\frac{q}{r})$,*

$$\|T_{\alpha,m}f\|_{L^q(w^q)}\le C\|f\|_{L^p(w^p)}.$$

Observe that Theorems 3.5 and 3.6 depend on an auxiliary exponent r. These exponents r give rise to a class of weights that is sufficient to prove a boundedness condition.

Taking a class of weights satisfying bump condition that does not depend on the exponent r, we are able to prove another weighted strong inequality. Indeed, we first recall Theorem 5.37 in [6]:

Theorem 3.8. [6] *Let $0\le\alpha<n$, $1<p<n/\alpha$, let $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let ϕ,B and C be Young functions such that $B^{-1}(t)C^{-1}(t)\le c\phi^{-1}(t)$, $t\ge t_0>0$. If $C\in B^\alpha_p$ and $w\in A_{q,B}$, then there exits $c>0$ such that for every $f\in L^p(w^p)$,*

$$\int(M_{\alpha,\phi}f)^q w^q\le c\int|f|^p w^p.$$

Now, from Theorem 3.8 we obtain

Theorem 3.9. *Let $T_{\alpha,m}$ be as in Theorem 3.3. Let $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let ϕ, B and C be Young functions such that $B^{-1}(t)C^{-1}(t)\le c\phi^{-1}(t)$, $t\ge t_0>0$. If $C\in B^\alpha_p$ and $w\in A_{q,B}$, then there exists $c>0$ such that for every $f\in L^p(w^p)$,*

$$\|T_{\alpha,m}f\|_{L^q(w^q)}\le c\|f\|_{L^p(w^p)}.$$

4. Examples

Now we present some examples of this type of operator. For $1\le r<\infty$, let r' be the conjugate exponent of r.
Let $\Psi_1(t)=t^r$,$\Psi_2(t)=\exp(t)-1$ and $\phi(t)=t^r\log(e+t)^{r'}$. Observe that

$$\Psi_1^{-1}(t)\Psi_2^{-1}(t)\phi^{-1}(t)\simeq t^{1/r}\log(e+t)\frac{t^{1/r'}}{\log(e+t)}=t$$

then ϕ is the complementary function of Ψ_1,Ψ_2.
For $\beta_i>0$, $i=1,2$, we define

$$\tilde{k}_i(t+4)=\Psi_i^{-1}\left(\frac{1}{t(1-\log(t))^{1+\beta_i}}\right)\chi_{(0,1)}(t).$$

By Theorem 5 in [15], we have $\tilde{k}_i \in H_{\Psi_i}$. For the size condition, observe that

$$\int_{\mathbb{R}} \Psi_i(\tilde{k}_i(t))dt = \int_0^1 \frac{1}{t(1-\log(t))^{1+\beta_i}}dt = \frac{1}{\beta_i} < \infty.$$

If $s > 1$, then $\tilde{k}_i \chi_{s<|x|\leq 2s} \equiv 0$. If $s < 1$,

$$\|\tilde{k}_i\|_{\Psi_i,|x|\sim s} = \|\tilde{k}_i \chi_{s<|x|\leq 2s}\|_{\Psi_i,B(0,2s)} \leq 1 + \frac{1}{4s}\int_s^{2s} \Psi_i(\tilde{k}_i(t))dt \leq 1 + \frac{1}{4s}\left(\frac{1}{\beta_i}\right) \leq \frac{1}{s}\left(1+\frac{1}{4\beta_i}\right).$$

Then, we get $\tilde{k}_i \in S_{\Psi_i}$.

Let $0 < \alpha, \alpha_1, \alpha_2 < 1$ such that $\alpha_1 + \alpha_2 = 1 - \alpha$. By Proposition 4.1 in [2], we know that if $k_i(t) = t^{1-\alpha_i}\tilde{k}_i(t)$ then $k_i \in H_{1-\alpha_i,\Psi_i} \cap S_{1-\alpha_i,\Psi_i}$. We define the operator,

$$Tf(x) = \int k_1(x - A_1y)k_2(x - A_2y)f(y)dy, \tag{4.1}$$

where k_i are defined as above and A_1, A_2 are invertibles matrices such that $A_1 - A_2$ is invertible. This operator satisfies the hypothesis of the Theorem 3.3 and we have the following

Theorem 4.1. *Let $0 < \alpha < 1$. Let T be the operator defined by (4.1). Then,*
(a) For all $1 < q < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} |Tf(x)|^q w(x)dx \leq C \sum_{i=1}^{2} \int_{\mathbb{R}^n} |M_{\alpha,L^{r'}\log L^{r'}} f(x)|^q w(A_i x)dx.$$

(b) Let $1 < p < 1/\alpha$ and $\frac{1}{q} = \frac{1}{p} - \alpha$. If w satisfies (3.2) and $w^{r'} \in A_{\frac{p}{r'},\frac{q}{r'}}$ then

$$\int_{\mathbb{R}^n} |Tf(x)|^q w^q(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w^p(x)dx.$$

(c) $\kappa_{r'+1} < \infty$ and if $w^{r'+1} \in A(\frac{1}{\alpha(r'+1)}, \infty)$ and satisfies (3.2) then

$$\||Tf|\|_w \leq C\|fw\|_{L^{1/\alpha}(dx)}.$$

(d) Let $s = \frac{r'+1}{1-\alpha(r'+1)}$. If $w^{r'+1} \in A(1, \frac{s}{r'+1})$ and satisfies (3.2) then

$$\sup_{\lambda>0} \lambda(w^s\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\})^{\frac{1}{s}} \leq C\left(\int |f(x)|^{r'+1} w^{r'+1}(x)dx\right)^{1/(r'+1)}.$$

Remark 4.2. Observe that to prove (b), we can use Theorem 2.3 in [1]. This result asserts that $M_{\alpha,L^r\log L^\gamma}$ is bounded from $L^p(w^p)$ into $L^q(w^q)$ if and only if $w^r \in A_{\frac{p}{r},\frac{q}{r}}$.

5. Auxiliaries results

In this section, we obtain an auxiliary lemma and the weak type $(1,1)$ estimate for the case $\alpha = 0$ with respect to the Lebesgue measure. These results are used in the proof of the main results.

Lemma 5.1. *Let $T_{\alpha,m}$ be as in Theorem 3.3. Let $\frac{n-\alpha}{n} < q < \infty$ and $\nu \in A_s$ for some $s > 1$. If $f \in L^\infty_c(\mathbb{R}^n)$ then $T_{\alpha,m}f \in L^q(\nu)$.*

Remark 5.2. Let $1 < p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $w^r \in A(\frac{p}{r}, \frac{q}{r})$ for some $1 < r < \infty$, then $w^q \in A_s$ with $s = \frac{q}{n}(n - \alpha)$.

Let Ψ be a Young function and $w \in A_{p,\Psi}$. If $t^{q'} \leq c\Psi(t)$ then $w^q \in A_q$. On the other hand, if $t^{p'} \leq c\Psi(t)$ then $w \in A_{p,q}$.

Theorem 5.3. *Under the hypothesis of Theorem 3.1 for $\alpha = 0$, $T_{0,m}$ is weak type $(1,1)$ respect to the Lebesgue measure, in other words*

$$|\{x \in \mathbb{R}^n : |T_{0,m}f(x)| > \lambda\}| \leq \frac{c}{\lambda}\int_{\mathbb{R}^n} |f|,$$

for all $f \in L^1(\mathbb{R}^n)$.

6. Proofs of the results

6.1. **Proofs of main results.** In the proof of Theorem 3.1, we follow the idea of Theorem 2.2 in [19].

Proof of Theorem 3.1. Les us consider the case $m = 2$. The general follows in an analogous way.

Let $f \in L_c^\infty(\mathbb{R}^n)$ and $0 < \delta \leq 1$. Let $x \in \mathbb{R}^n$ and let $B = B(c_B, R)$ be a ball that contains x, centered at c_B with radius R. We write $\tilde{B} = B(c_B, 2R)$ and for $1 \leq i \leq 2$, set $\tilde{B}_i = A_i^{-1}\tilde{B}$. Let $f_1 = f\chi_{\cup_{i=1}^2 \tilde{B}_i}$ and $f_2 = f - f_1$.
Suppose that $a := T_\alpha(f_2)(c_B) < \infty$. Then,

$$\begin{aligned}
\left(\frac{1}{|B|}\int_B |T_\alpha f(y) - a|^\delta dy\right)^{1/\delta} &\leq \left(\frac{1}{|B|}\int_B |T_\alpha f(y) - a|^\delta dy\right)^{1/\delta} \\
&\leq C\left(\frac{1}{|B|}\int_B |T_\alpha(f_1)(y)|^\delta dy\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|B|}\int_B |T_\alpha(f_2)(y) - T_\alpha(f_2)(c_B)|^\delta dy\right)^{1/\delta} \\
&\quad = C(I + II). \qquad (6.1)
\end{aligned}$$

First, we consider the case $0 < \alpha < n$. For I, using Jensen inequality we have,

$$\begin{aligned}
I &\leq \frac{1}{|B|}\int_B |T_\alpha(f_1)(y)|dy \\
&\leq \frac{1}{|B|}\int_B \int_{\tilde{B}_1 \cup \tilde{B}_2} |K(y,z)||f_1(z)|dzdy \\
&\leq \sum_{i=1}^2 \frac{1}{|B|}\int_{\tilde{B}_i} |f_1(z)| \int_B |K(y,z)|dydz. \qquad (6.2)
\end{aligned}$$

Let us estimate the first summand, i.e. $z \in \tilde{B_1}$. The case $z \in \tilde{B_2}$ is analogous.
Now,

$$\int_B |K(y,z)|dy \leq \int_{\{y \in B: |y - A_1z| \leq |y - A_2z|\}} |K(y,z)|dy + \int_{\{y \in B: |y - A_2z| \leq |y - A_1z|\}} |K(y,z)|dy. \qquad (6.3)$$

For $j \in \mathbb{N}$, let consider the set

$$C_j^1 := \{y \in B : |y - A_1z| \leq |y - A_2z|, |y - A_1z| \sim 2^{-j-1}R\}.$$

Observe that if $y \in B$ and $z \in \tilde{B_1}$ then $|y - A_1z| \leq 3R < 4R$ and so $B \subset B(A_1z, 4R)$.
Then, by Hölder's inequality

$$\int_{\{y \in B: |y - A_1z| \leq |y - A_2z|\}} |K(y,z)|dy \leq \sum_{j=-2}^\infty \int_{C_j^1} |K(y,z)|dy$$

$$\leq \sum_{j=-2}^{\infty} \frac{|B(A_1z, 2^{-j}R)|}{|B(A_1z, 2^{-j}R)|} \int_{B(A_1z,2^{-j}R)} |K(y,z)|\chi_{C_j^1} dy$$

$$\leq C \sum_{j=-2}^{\infty} |B(A_1z, 2^{-j}R)| \|k_1(\cdot - A_1z)\|_{\Psi_1,|y-A_1z|\sim 2^{-j-1}R} \|k_2(\cdot - A_2z)\|_{\Psi_2,|y-A_1z|\sim 2^{-j-1}R}.$$

Observe that of $y \in C_j^1$ then $|y - A_2z| \geq |y - A_1z| > 2^{j-1}R$. Then, since $k_2 \in S_{n-\alpha_2,\Psi_2}$

$$\begin{aligned} \|k_2(\cdot - A_2z)\|_{\Psi_2,|y-A_1z|\sim 2^{-j-1}R} &\leq \sum_{k\geq 0} \|k_2(\cdot - A_2z)\|_{\Psi_2,|y-A_2z|\sim 2^{-j+k-1}R} \\ &\leq \sum_{k\geq 0} \|k_2(\cdot)\|_{\Psi_2,|y|\sim 2^{-j+k-1}R} \\ &\leq \sum_{k\geq 0} (2^{-j+k}R)^{-\alpha_2} = c(2^{-j}R)^{-\alpha_2}. \end{aligned} \tag{6.4}$$

Inequality (6.4) and the fact that $k_1 \in S_{n-\alpha_1,\Psi_1}$, gives

$$\int_{\{y\in B:|y-A_1z|\leq|y-A_2z|\}} |K(y,z)|dy \leq C \sum_{j=-2}^{\infty} (2^{-j}R)^{n-\alpha_1-\alpha_2} = CR^{\alpha}.$$

In analogous way, we get

$$\int_{\{y\in B:|y-A_2z|\leq|y-A_1z|\}} |K(y,z)|dy \leq CR^{\alpha}. \tag{6.5}$$

Then, by (6.2) and (6.5), we have

$$\begin{aligned} I &\leq CR^{\alpha} \sum_{i=1}^{2} \frac{1}{|B|} \int_{\tilde{B}_i} |f(z)|dz \leq CR^{\alpha} \sum_{i=1}^{2} \frac{1}{|\tilde{B}_i|} \int_{\tilde{B}_i} |f(z)|dz \\ &\leq C \sum_{i=1}^{2} M_{\alpha} f(A_i^{-1}x) \leq c \sum_{i=1}^{2} M_{\alpha,\varphi} f(A_i^{-1}x). \end{aligned}$$

For II, by Jensen inequality

$$\begin{aligned} II &\leq \frac{1}{|B|} \int_B |T_{\alpha}(f_2)(y) - T_{\alpha}(f_2)(c_B)|dy \\ &\leq \frac{1}{|B|} \int_B \int_{(\tilde{B}_1\cup\tilde{B}_2)^c} |K(y,z) - K(c_B,z)||f_2(z)|dzdy \\ &\leq \frac{1}{|B|} \int_B \sum_{l=1}^{2} \int_{Z^l} |K(y,z) - K(c_B,z)||f_2(z)|dzdy, \end{aligned}$$

where

$$Z^l = (\tilde{B}_1 \cup \tilde{B}_2)^c \cap \{z : |c_B - A_lz| \leq |c_B - A_rz|, r \neq l, 1 \leq r \leq 2\}.$$

For $y \in B$ and $z \in Z^l$, let estimate

$$\begin{aligned} |K(y,z) - K(c_B,z)| \leq &|k_1(y - A_1z) - k_1(c_B - A_1z)||k_2(y - A_2z)| \\ &+ |k_1(c_B - A_1z)||k_2(y - A_2z) - k_2(c_B - A_2z)|. \end{aligned} \tag{6.6}$$

For simplicity we control the first summand of (6.6), the other summand follows in analogous way. For $j \in \mathbb{N}$, let

$$D_j^l = \{z \in Z^l : |c_B - A_l z| \sim 2^{j+1}R\}.$$

Observe that $D_j^l \subset \{z : |c_B - A_l z| \sim 2^{j+1}R\} \subset A_l^{-1}B(c_B, 2^{j+2}R) =: \tilde{B}_{l,j}$ and $Z^l = \bigcup_{j\in\mathbb{N}} D_j^l$. Using generalized Hölder inequality we get

$$\begin{aligned}
&\int_{Z^l} |k_1(y - A_1z) - k_1(c_B - A_1z)||k_2(y - A_2z)||f(z)|dz \\
&\quad\leq \sum_{j=1}^{\infty} \int_{D_j^l} |k_1(y - A_1z) - k_1(c_B - A_1z)||k_2(y - A_2z)|f(z)|dz \\
&\quad\leq \sum_{j=1}^{\infty} \frac{|\tilde{B}_{l,j}|}{|\tilde{B}_{l,j}|} \int_{\tilde{B}_{l,j}} \left[\chi_{D_j^l}|k_1(y - A_1z) - k_1(c_B - A_1z)||k_2(y - A_2z)||f(z)|\right] dz \\
&\quad\leq \sum_{j=1}^{\infty} |\tilde{B}_{l,j}|\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^l}\|_{\Psi_1,\tilde{B}_{l,j}}\|k_2(y - A_2\cdot)\chi_{D_j^l}\|_{\Psi_2,\tilde{B}_{l,j}}\|f_2\|_{\varphi,\tilde{B}_{l,j}} \\
&\quad\leq c\sum_{j=1}^{\infty} |\tilde{B}_{l,j}|\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^l}\|_{\Psi_1,\tilde{B}_{l,j}}\|k_2(y - A_2\cdot)\chi_{D_j^l}\|_{\Psi_2,\tilde{B}_{l,j}}\|f_2\|_{\varphi,\tilde{B}_{l,j}}.
\end{aligned}$$

If $y \in B$ and $z \in Z^l$ then $|c_B - A_lz|/2 \leq |y - A_lz| < 2|c_B - A_lz|$ and if $z \in D_j^l$ then $2^jR \leq |y - A_lz| \leq 2^{j+2}R$.
For the case $l = 1$, observe that if $z \in D_j^1$ then $|c_B - A_2z| \geq |c_B - A_1z| \geq 2^{j+1}R$. So we decompose $D_j^1 = \bigcup_{k\geq j}(D_j^1)_{k,2}$ where

$$(D_j^1)_{k,2} = \{z \in D_j^1 : |c_B - A_2z| \sim 2^{j+1}R\}.$$

Note that $(D_j^1)_{k,2} \subset \{z : |c_B - A_2z| \sim 2^{k+1}R\}$. As $k_2 \in S_{n-\alpha_2,\Psi_2}$, then

$$\begin{aligned}
\|k_2(y - A_2\cdot)\chi_{D_j^1}\|_{\Psi_2,\tilde{B}_{1,j}} &\leq \sum_{k\geq j} \|k_2(y - A_2\cdot)\chi_{(D_j^1)_{k,2}}\|_{\Psi_2,\tilde{B}_{2,k}} \\
&\leq \sum_{k\geq j} \|k_2(\cdot)\|_{\Psi_2,|x|\sim 2^kR} + \|k_2(\cdot)\|_{\Psi_2,|x|\sim 2^{k+1}R} \\
&\leq c\sum_{k\geq j}(2^kR)^{-\alpha_2} = c(2^jR)^{-\alpha_2}.
\end{aligned}$$

Finally using $k_1 \in H_{n-\alpha_1,\Psi_1}$ and since $A_1^{-1}x \in \tilde{B}_{1,j}$ we get

$$\begin{aligned}
&\int_{Z^1} |k_1(y - A_1z) - k_1(c_B - A_1z)||k_2(y - A_2z)||f_2(z)|dz \\
&\quad\leq c\sum_{j=1}^{\infty}(2^jR)^{n-\alpha_2}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^1}\|_{\Psi_1,\tilde{B}_{1,j}}\|f\|_{\varphi,\tilde{B}_{1,j}} \\
&\quad\leq cM_{\alpha,\varphi}f(A_1^{-1}x)\sum_{j=1}^{\infty}(2^jR)^{n-\alpha_2-\alpha}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^1}\|_{\Psi_1,\tilde{B}_{1,j}} \\
&\quad\leq cM_{\alpha,\varphi}f(A_1^{-1}x).
\end{aligned}$$

The case $l = 2$ follows the same argument with minimals changes. As $k_2 \in S_{n-\alpha_2,\Psi_2}$, we get

$$\|k_2(y - A_2\cdot)\chi_{D_j^2}\|_{\Psi_2,\tilde{B}_{2,j}} \leq c(2^j R)^{-\alpha_2}.$$

Then, as above

$$\begin{aligned}
&\int_{Z^l} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)||f(z)|dz \\
&\quad \leq c\sum_{j=1}^{\infty} |\tilde{B}_{2,j}|\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^2}\|_{\Psi_1,\tilde{B}_{2,j}}\|f_2\|_{\varphi,\tilde{B}_{2,j}} \\
&\quad \leq cM_{\alpha,\varphi}f(A_2^{-1}x)\sum_{j=1}^{\infty}(2^j R)^{n-\alpha_2-\alpha}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^2}\|_{\Psi_1,\tilde{B}_2^j} \\
&\quad \leq cM_{\alpha,\varphi}f(A_2^{-1}x)\sum_{j=1}^{\infty}(2^j R)^{n-\alpha_2-\alpha}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{D_j^2}\|_{\Psi_1,\tilde{B}_2^j} \\
&\quad \leq cM_{\alpha,\varphi}f(A_2^{-1}x)\sum_{k=1}^{\infty}\left(\sum_{j=1}^{k}(2^{-\alpha_1})^{k-j}\right)(2^k R)^{\alpha_1}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{(D_j^l)_{k,1}}\|_{\Psi_1,\tilde{B}_1^k} \\
&\quad \leq cM_{\alpha,\varphi}f(A_2^{-1}x)\sum_{k=1}^{\infty}(2^k R)^{\alpha_1}\|(k_1(y - A_1\cdot) - k_1(c_B - A_1\cdot))\chi_{(D_j^l)_{k,1}}\|_{\Psi_1,\tilde{B}_1^k} \\
&\quad \leq cM_{\alpha,\varphi}f(A_2^{-1}x),
\end{aligned}$$

where the last inequality holds since $k_1 \in H_{n-\alpha_1,\Psi_1}$.

So,

$$\sum_{l=1}^{2}\int_{Z^l} |k_1(y - A_1 z) - k_1(c_B - A_1 z)||k_2(y - A_2 z)||f(z)|dz \leq c\sum_{l=1}^{2} M_{\alpha,\varphi}f(A_l^{-1}x),$$

and

$$II \leq c\sum_{l=1}^{2} M_{\alpha,\varphi}f(A_l^{-1}x).$$

For the case $\alpha = 0$, proceed as in (6.1). The estimate for I follows, since $T_{0,2}$ is of weak-type $(1,1)$ with respect to the Lebesgue measure (see Lemma 5.3). Using Kolmogorov's inequality (see Lemma 5.16 in [8]), we get

$$I \leq \frac{C}{|B|}\int_{\mathbb{R}^n} |f_1(y)|dy = \sum_{i=1}^{2}\frac{C}{|B|}\int_{\tilde{B}_i} |f(y)|dy \leq C\sum_{i=1}^{2} Mf(A_i^{-1}f(x)).$$

The term II is analogous to the case $0 < \alpha < n$, and so the theorem follows in this case. □

Proof of Theorem 3.3. By the extrapolation result Theorem 1.1 in [4], estimate (3.3) holds for all $0 < p < \infty$ and all $w \in A_\infty$ if, and only if, it holds for some $0 < p_0 < \infty$ and all $w \in A_\infty$. Therefore, we will show that (3.3) is true for p_0, which is taken such that $\frac{n-\alpha}{n} < p_0 < \infty$.

Let $w \in A_\infty$, then there exists $r > 1$ such that $w \in A_r$. Let $0 < \delta < 1$ such that $1 < r < p_0/\delta$, thus $w \in A_{p_0/\delta}$. Then, by Lemma (5.1), we have $\|T_{\alpha,m}f\|_{L^{p_0}(w)} < \infty$, and $\|(T_{\alpha,m}f)^\delta\|_{L^{p_0/\delta}(w)} < \infty$.

Applying Fefferman-Stein inequality (see Lemma 7.10 in [8], p. 144) and Theorem 3.1 we get

$$\begin{aligned}\int_{\mathbb{R}^n}|T_{\alpha,m}f(x)|^{p_0}w(x)dx &\leq \int_{\mathbb{R}^n}|M(T_{\alpha,m}f)^{\delta}(x)|^{p_0/\delta}w(x)dx\\ &\leq \int_{\mathbb{R}^n}(M_{\delta}^{\sharp}(T_{\alpha,m}f)(x))^{p_0}w(x)dx\\ &\leq C\sum_{i=1}^{m}\int_{\mathbb{R}^n}(M_{\alpha,\phi}f(A_i^{-1}x))^{p_0}w(x)dx.\end{aligned}$$

Hence, for all $w\in A_\infty$, (3.3) holds for p_0, that is

$$\int_{\mathbb{R}^n}|T_{\alpha,m}f(x)|^{p_0}w(x)dx \leq C\sum_{i=1}^{m}\int_{\mathbb{R}^n}(M_{\alpha,\phi}f(A_i^{-1}x))^{p_0}w(x)dx. \tag{6.7}$$

Thus, as mentioned, using the extrapolation results obtained in [4], (3.3) holds for all $0<p<\infty$ and $w\in A_\infty$.

If w satisfies (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^n}|T_{\alpha,m}f(x)|^{p}w(x)dx &\leq C\sum_{i=1}^{m}\int_{\mathbb{R}^n}(M_{\alpha,\phi}f(A_i^{-1}x))^{p}w(x)dx = C\sum_{i=1}^{m}\int_{\mathbb{R}^n}(M_{\alpha,\phi}f(x))^{p}w(A_ix)dx\\ &\leq C\sum_{i=1}^{m}\int_{\mathbb{R}^n}(M_{\alpha,\phi}f(x))^{p}w(x)dx.\end{aligned}$$

□

6.2. **Proof of weighted inequalities.**

Proof of Theorem 3.6. Let $t>1$ such that $\frac{1}{t}=\frac{1}{r}-\frac{\alpha}{n}=\frac{n-\alpha r}{rn}$, by Theorem 3.3 and inequality (3.4) we have

$$\begin{aligned}(w^t\{x\in\mathbb{R}^n:|T_{\alpha,m}f(x)|>\lambda\})^{\frac{1}{t}} &\leq C(w^t\{x\in\mathbb{R}^n:\sum_{i=1}^{m}M_{\alpha,\phi}f(A_i^{-1}x)>c\gamma\lambda\})^{\frac{1}{t}}\\ &\leq C(w^t\{x\in\mathbb{R}^n:\sum_{i=1}^{m}M_{\alpha,r}f(A_i^{-1}x)>c\gamma\lambda\})^{\frac{1}{t}}\\ &\leq C(w^t\{x\in\mathbb{R}^n:\sum_{i=1}^{m}M_{\alpha r}|f|^r(A_i^{-1}x)>\lambda^r\})^{\frac{1}{t}},\end{aligned}$$

where the last inequality holds by Remark 2.2.

Since w satisfies (3.2), we have

$$\begin{aligned}\sup_{\lambda>0}\lambda(w^t\{x\in\mathbb{R}^n:|T_{\alpha,m}f(x)|>\lambda\})^{\frac{1}{t}} &\leq C\sup_{\lambda>0}\lambda(w^t\{x\in\mathbb{R}^n:M_{\alpha r}|f|^r(x)>\lambda^r\})^{\frac{1}{t}}\\ &\leq C\left(\int|f|^r(x)w^r(x)dx\right)^{1/r},\end{aligned}$$

where the last inequality follows since $w^r\in A_{1,\frac{n}{n-\alpha r}}$ and $M_{\alpha r}$ is of weak type $(1,\frac{n}{n-\alpha r})$ in other words of weak type $(1,t/r)$. □

Proof of Theorem 3.7. Since $\kappa_r < \infty$ and $w^r \in A_{\frac{p}{r},\frac{q}{r}}$, by Lemma 5.1 we have that if $f \in L^\infty_c(\mathbb{R}^n)$ then $T_{\alpha,m}f \in L^q(w^q)$. Now, from Theorem 3.3 and Theorem 2.6 in [1], we obtain

$$\left(\int_{\mathbb{R}^n} |T_{\alpha,m}f(x)|^q w^q(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |M_{\alpha,\phi}f(x)|^q w^q(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w^p(x)dx\right)^{1/p}.$$

□

6.3. **Proof of the Auxiliaries results.**

Proof of Lemma 5.1. Let $M = \max_{1\leq j\leq 2}\|A_j\|_\infty$. Suppose $\mathrm{supp}f \subset B(0,R)$. If $|x| > 2MR$ and $y \in \mathrm{supp}f$, then for $1 \leq i \leq 2$, $|A_iy| \leq MR < \frac{|x|}{2}$ and

$$\frac{|x|}{2} \leq |x| - RM \leq |x - A_iy| \leq |x| + |A_iy| < \frac{3}{2}|x|.$$

Analogous to the proof of Theorem 3.1,

$$\begin{aligned}|Tf(x)| &= |\int_{B(0,R)} k_1(x-A_1y)k_2(x-A_2y)f(y)dy| \\ &\leq |\int_{y\in B(0,R):|x-A_2y|\leq|x-A_1y|} k_1(x-A_1y)k_2(x-A_2y)f(y)dy| \\ &\quad + |\int_{y\in B(0,R):|x-A_1y|\leq|x-A_2y|} k_1(x-A_1y)k_2(x-A_2y)f(y)dy|.\end{aligned}$$

We only estimate the first summand the other is analogous. Let

$$Z = \{y \in B(0,R) : |x - A_1y| \leq 4|x|\} \subset B(0,R).$$

By Hölder's inequality

$$\begin{aligned}&\left|\int_{y\in B(0,R):|x-A_2y|\leq|x-A_1y|} k_1(x-A_1y)k_2(x-A_2y)f(y)dy\right| \\ &\leq \frac{|Z|}{|Z|}\|f\|_{L^\infty}\int_{y\in B(0,R):|x-A_2y|\leq|x-A_1y|} |k_1(x-A_1y)k_2(x-A_2y)|dy \\ &\leq \|f\|_{L^\infty}|Z|\|k_1(x-A_1\cdot)\chi_{\{y:\frac{|x|}{2}\leq|x-A_1y|<\frac{3}{2}|x|\}}\|_{\Psi_1,Z}\|k_2(x-A_2\cdot)\chi_{\{y:\frac{|x|}{2}\leq|x-A_2y|<\frac{3}{2}|x|\}}\|_{\Psi_2,Z} \\ &\leq c\|f\|_{L^\infty}|Z||x|^{-\alpha_1-\alpha_2} \\ &\leq c\|f\|_{L^\infty}|B(0,R)||x|^{\alpha-n} \\ &\leq c|x|^{\alpha-n}.\end{aligned}$$

Hence, if $|x| > 2MR$, then $|Tf(x)| \leq c|x|^{\alpha-n}$.
On the other hand, if $|x| < 2MR$, $|x - A_iy| \leq |x| + |A_iy| < 3MR$. Then, we proceed just as above to get $|Tf(x)| \leq cR^{\alpha-n}$ and for $1 \leq s < \infty$,

$$\int_{B(0,2MR)} |Tf(x)|^s dx < C.$$

The rest of the proof follows the same steps as the proof of Lemma 3.2 in [19]: if $\nu \in A_s$ for some $s > 1$, we get

$$\int |Tf(x)|^q \nu(x)dx \leq C.$$

□

Proof of Theorem 5.3. We consider $T = T_{0,2}$.
Let f be a function in the Schwartz space and $\lambda > 0$. By the Calderón-Zygmund decomposition for f at the height λ, we get $\Omega_\lambda = \cup_j Q_j$, where Q_j are disjoint dyadic cubes in $\mathbb{R}^n$. Then there exist g and $h = \sum_j h_j$ functions such that $f = g + h$, $\|g\|_{p_0} \le c_n \lambda^{1/p_0'} \|f\|_1^{1/p_0}$, $\operatorname{supp}(h_j) \subset Q_j$ and $\int h_j = 0$. Thus,

$$\begin{aligned}|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| &\le |\{x \in \mathbb{R}^n : |Tg(x)| > \lambda/2\}| + |\{x \in \mathbb{R}^n : |Th(x)| > \lambda/2\}|\\ &= I + II.\end{aligned}$$

For I, using that T is of weak type (p_0, p_0), we obtain

$$I = |\{x \in \mathbb{R}^n : |Tg(x)| > \lambda/2\}| \le c\frac{2^{p_0}}{\lambda^{p_0}} \|g\|_{p_0}^{p_0} \le c\frac{2^{p_0}}{\lambda^{p_0}} \|f\|_1 \lambda^{p_0-1} = \frac{c}{\lambda}\int_{\mathbb{R}^n} |f|.$$

For II, let $\tilde{Q}_{j,i}$ the cube with center $A_i c_j$ and $l(\tilde{Q}_{j,i}) = 4Ml(Q_j)$, where $M = \max_{1\le i\le 2} \|A_i\|_\infty$,

$$\begin{aligned}II &= |\{x \in \mathbb{R}^n : |Th(x)| > \lambda/2\}|\\ &\le |\{x \in \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) : |Th(x)| > \lambda/2\} + |\{x \notin \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) : |Th(x)| > \lambda/2\}\\ &\le |\bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})| + |\{x \notin \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) : |Th(x)| > \lambda/2\}.\end{aligned}$$

For the first term, we have

$$\begin{aligned}|\bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2})| &\le \sum_j |\tilde{Q}_{j,1}| + |\tilde{Q}_{j,2}| = 2\sum_j (4Ml(Q_j))^n\\ &= 2(4M)^n \sum_j l(Q_j)^n = 2(4M)^n |\bigcup_j Q_j|\\ &\le \frac{c}{\lambda}\int_{\mathbb{R}^n} |f|.\end{aligned}$$

For the second term

$$\begin{aligned}|\{x \notin \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) : |Th(x)| > \lambda/2\}| &\le \frac{2c}{\lambda}\int_{(\bigcup_j (\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}))^c} |Th(x)|dx\\ &\le \frac{2c}{\lambda}\sum_j \int_{(\bigcup_j (\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2}))^c} \int_{Q_j} |K(x,y) - K(x,c_j)||h_j(y)|\,dy\,dx\\ &= \frac{2c}{\lambda}\sum_j \int_{Q_j} |h_j(y)| \int_{(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})^c} |K(x,y) - K(x,c_j)|\,dx\,dy.\end{aligned}$$

If we have

$$\int_{(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})^c} |K(x,y) - K(x,c_j)|dx \le C, \tag{6.8}$$

then

$$|\{x \notin \bigcup_j (\tilde{Q}_{j,1} \cup \tilde{Q}_{j,2}) : |Th(x)| > \lambda/2\}| \le \frac{C}{\lambda}\sum_j \int_{Q_j} |h_j(y)|dy \le \frac{C}{\lambda}\|f\|_1.$$

Hence, T is of weak-type $(1,1)$.

Now, let us prove (6.8). Observe that $B_{j,i} = B(A_i c_j, 2Ml(Q_j)) \subset \tilde{Q}_{j,i}$, then

$$\int_{(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})^c} |K(x,y) - K(x,c_j))|dx \leq \sum_{l=1}^{2} \int_{Z^l} |K(x,y) - K(x,c_j)|dx,$$

where

$$Z^l = (B_{j,1} \cup B_{j,2})^c \cap \{x : |x - A_l y| \leq |x - A_r y|, r \neq l, 1 \leq r \leq 2\}.$$

Let estimate

$$\begin{aligned} |K(x,y) - K(x,c_j)| \leq& |k_1(x - A_1 y) - k_1(x - A_1 c_j)||k_2(x - A_2 y)| \\ &+ |k_1(x - A_1 c_j)||k_2(x - A_2 y) - k_2(x - A_2 c_j)|. \end{aligned} \tag{6.9}$$

We only study the first summand, the second one follows in analogous way. For $t \in \mathbb{N}$,

$$D_t^l = \{x \in Z^l : |x - A_l c_j| \sim 2^t l(Q_j)\}.$$

Observe that $D_t^l \subset \{x : |x - A_l c_j| \sim 2^t l(Q_j)\} \subset B(A_l c_j, 2^{t+1} l(Q_j)) =: \tilde{B}_t^l$. Using generalized Hölder inequality we get

$$\begin{aligned} &\int_{(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})^c} |k_1(x - A_1 y) - k_1(x - A_1 c_j)||k_2(x - A_2 y)|dx \\ &\leq \sum_{l=1}^{2}\sum_{t=1}^{\infty} \int_{D_t^l} |k_1(x - A_1 y) - k_1(x - A_1 c_j)||k_2(x - A_2 y)|dx \\ &\leq \sum_{l=1}^{2}\sum_{t=1}^{\infty} \frac{|\tilde{B}_t^l|}{|\tilde{B}_t^l|} \int_{\tilde{B}_t^l} \chi_{D_t^l} |k_1(x - A_1 y) - k_1(x - A_1 c_j)||k_2(x - A_2 y)|dx \\ &\leq C \sum_{l=1}^{2}\sum_{t=1}^{\infty} |\tilde{B}_t^l| \|k_1(\cdot - A_1 y) - k_1(\cdot - A_1 c_j))\chi_{D_t^l}\|_{\Psi_1,\tilde{B}_t^l} \|k_2(\cdot - A_2 y)\chi_{D_t^l}\|_{\Psi_2,\tilde{B}_t^l}. \end{aligned} \tag{6.10}$$

For $l = 1$, since $k_2 \in S_{n-\alpha_2,\Psi_2}$ and using inequality (6.4), we have

$$\|k_2(\cdot - A_2 y)\chi_{D_t^1}\|_{\Psi_2,\tilde{B}_t^1} \leq c(2^t Ml(Q_j))^{-\alpha_2}.$$

Then,

$$\begin{aligned} &\sum_{t=1}^{\infty} |\tilde{B}_t^1| \|k_1(\cdot - A_1 y) - k_1(\cdot - A_1 c_j))\chi_{D_t^1}\|_{\Psi_1,\tilde{B}_t^1} \|k_2(\cdot - A_2 y)\chi_{D_t^1}\|_{\Psi_2,\tilde{B}_t^1} \\ &\qquad \leq c \sum_{t=1}^{\infty} (2^t Ml(Q_j))^{n-\alpha_2} \|k_1(\cdot - A_1 y) - k_1(\cdot - A_1 c_j))\chi_{D_t^1}\|_{\Psi_1,\tilde{B}_t^1} \\ &\qquad \leq C \sum_{t=1}^{\infty} (2^t Ml(Q_j))^{\alpha_1} \|k_1(\cdot - A_1 y) - k_1(\cdot - A_1 c_j))\chi_{D_t^1}\|_{\Psi_1,\tilde{B}_t^1} \\ &\qquad \leq C, \end{aligned}$$

where the last inequality holds by $k_1 \in H_{n-\alpha_1,\Psi_1}$.

If $l = 2$, since $k_2 \in S_{n-\alpha_2,\Psi_2}$, we obtain

$$\|k_2(\cdot - A_2 y)\chi_{D_t^2}\|_{\Psi_2,\tilde{B}_t^2} \leq c(2^t Ml(Q_j))^{-\alpha_2}.$$

Then, proceeding as inequality (6.4), we get

$$\sum_{t=1}^{\infty} |\tilde{B}_t^2| \|k_1(\cdot - A_1 y) - k_1(\cdot - A_1 c_j))\chi_{D_t^2}\|_{\Psi_1,\tilde{B}_t^2} \|k_2(\cdot - A_2 y)\chi_{D_t^2}\|_{\Psi_2,\tilde{B}_t^2}$$

$$\leq C\sum_{t=1}^{\infty}(2^tMl(Q_j))^{\alpha_1}\|k_1(\cdot - A_1y) - k_1(\cdot - A_1c_j))\chi_{D_t^2}\|_{\Psi_1,\tilde{B}_t^2}$$
$$\leq C.$$

Hence,

$$\int_{(\tilde{Q}_{j,1}\cup\tilde{Q}_{j,2})^c}|k_1(x-A_1y)-k_1(x-A_1c_j)||k_2(x-A_2y)|dx$$
$$\leq C\sum_{l=1}^{2}\sum_{t=1}^{\infty}|\tilde{B}_t^l|\|k_1(\cdot - A_1y)-k_1(\cdot - A_1c_j))\chi_{D_t^l}\|_{\Psi_1,\tilde{B}_t^l}\|k_2(\cdot - A_2y)\chi_{D_t^l}\|_{\Psi_2,\tilde{B}_t^l}$$
$$\leq C.$$

Then, we prove (6.8). □

References

[1] Bernardis, A., Dalmasso, E., and Pradolini, G. Generalized maximal functions and related operators on weighted Musielak-Orlicz spaces. *Ann. Acad. Sci. Fenn. Math. 39*, 1 (2014), 23–50.

[2] Bernardis, A. L., Lorente, M., and Riveros, M. S. Weighted inequalities for fractional integral operators with kernel satisfying Hörmander type conditions. *Math. Inequal. Appl 14*, 4 (2011), 881–895.

[3] Caldarelli, M., Lerner, A., and Ombrosi, S. On a counterexample related to weighted weak type estimates for singular integrals. *Proceedings of the American Mathematical Society 145*, 7 (2017), 3005–3012.

[4] Cruz-Uribe, D., Martell, J. M., and Pérez, C. Extrapolation from a_∞ weights and applications. *Journal of Functional Analysis 213*, 2 (2004), 412–439.

[5] Cruz-Uribe, D., and Moen, K. A fractional Muckenhoupt-Wheeden theorem and its consequences. *arXiv preprint arXiv:1303.3424* (2013).

[6] Cruz-Uribe, D. V., Martell, J. M., and Pérez, C. *Weights, extrapolation and the theory of Rubio de Francia*, vol. 215. Springer Science & Business Media, 2011.

[7] Di Plinio, F., and Lerner, A. K. On weighted norm inequalities for the Carleson and Walsh–Carleson operator. *Journal of the London Mathematical Society 90*, 3 (2014), 654–674.

[8] Duoandikoetxea Zuazo, J. *Fourier analysis*, vol. 29. American Mathematical Soc., 2001.

[9] Ferreyra, E. V., and Flores, G. J. Weighted estimates for integral operators on local BMO type spaces. *Mathematische Nachrichten 288*, 8-9 (2015), 905–916.

[10] Gallo, A. L., Firnkorn, G. H. I., and Riveros, M. S. Hörmander's conditions for vector-valued kernels of singular integrals and its commutators. *arXiv preprint arXiv:1706.08357* (2017).

[11] Godoy, T., Saal, L., and Urciuolo, M. About certain singular kernels k (x, y)= k 1 (x–y) k 2 (x+ y). *Mathematica Scandinavica* (1994), 98–110.

[12] Godoy, T., and Urciuolo, M. About the Lp-boundedness of some integral operators. *Revista de la Unión Matemática Argentina 38*, 3 (1993), 192–195.

[13] Godoy, T., and Urciuolo, M. About the L p-boundedness of integral operators with kernels of the form k 1 (x–y) k 2 (x+ y). *Mathematica Scandinavica* (1996), 84–92.

[14] Godoy, T., and Urciuolo, M. On certain integral operators of fractional type. *Acta Mathematica Hungarica 82*, 1-2 (1999), 99–105.

[15] Lorente, M., Riveros, M. S., and de la Torre, A. Weighted estimates for singular integral operators satisfying hörmander's conditions of Young type. *Journal of Fourier analysis and Applications 11*, 5 (2005), 497–509.

[16] O'Neil, R. Fractional integration in Orlicz spaces. I. *Transactions of the American Mathematical Society 115* (1965), 300–328.

[17] Rao, M. M., and Ren, Z. Theory of Orlicz spaces, volume 146 of monographs and textbooks in pure and applied mathematics, 1991.

[18] Ricci, F., and Sjögren, P. Two-parameter maximal functions in the Heisenberg group. *Mathematische Zeitschrift 199*, 4 (1988), 565–575.

[19] Riveros, M., and Urciuolo, M. Weighted inequalities for some integral operators with rough kernels. *Open Mathematics 12*, 4 (2014), 636–647.

[20] Riveros, M. S., and Urciuolo, M. Weighted inequalities for integral operators with some homogeneous kernels. *Czechoslovak Mathematical Journal 55*, 2 (2005), 423–432.

[21] Riveros, M. S., and Urciuolo, M. Weighted inequalities for fractional type operators with some homogeneous kernels. *Acta Mathematica Sinica. English Series 29*, 3 (2013), 449–460.

[22] Rocha, P. A remark on certain integral operators of fractional type. *arXiv preprint arXiv:1703.03287* (2017).

[23] Rocha, P., and Urciuolo, M. On the Hp-Lp boundedness of some integral operators. *Georgian Math. J 18* (2011), 801–808.

[24] Rocha, P., and Urciuolo, M. On the Hp-Lq boundedness of some fractional integral operators. *Czechoslovak Mathematical Journal 62*, 3 (2012), 625–635.

[25] Rocha, P., and Urciuolo, M. About integral operators of fractional type on variable Lp spaces. *Georgian Mathematical Journal 20*, 4 (2013), 805–816.

[26] Urciuolo, M. Weighted inequalities for integral operators with almost homogeneous kernels. *Georgian Mathematical Journal 13*, 1 (2006), 183–191.

[27] Vallejos, L. A. Acotación de ciertos operadores integrales en espacios de Lebesgue variables. B.S. thesis, FAMAF- Universidad Nacional de Córdoba, 2016.

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