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ORIGINAL PAPER

The spectrum on p-forms of a lens space

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Abstract We give an explicit description of the spectrum of the Hodge–Laplace operator on p-forms of an arbitrary lens space for any p. We write the two generating functions encoding the p-spectrum as rational functions. As a consequence, we prove a geometric characterization of lens spaces that are p-isospectral for every p in an interval of the form $[0, p_0]$.

Keywords Spectrum · Lens space · Isospectrality · One-norm

Mathematics Subject Classification (2010) 58J50 · 58J53

1 Introduction

Each compact Riemannian manifold (M, g) has an associated *Hodge–Laplace operator* Δ_p acting on smooth p-forms given by $dd^* + d^*d$, where d denotes the exterior derivative and d^* denotes the formal adjoint. Here Δ_0 is the *Laplace–Beltrami operator*. The spectrum of Δ_p is discrete and is usually called the *p-spectrum of* (M, g), denoted by $\mathrm{Spec}_p(M, g)$ or just $\mathrm{Spec}_p(M)$. Two compact Riemannian manifolds will be said to be *p-isospectral* if their p-spectra coincide and in the literature, one abbreviates 0-isospectral to *isospectral*. It is well known that $\mathrm{Spec}_p(M, g)$ does not determine the geometry of (M, g), as shown by many

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examples of non-isometric p-isospectral manifolds, via the so called *generalized Sunada method* (see for instance [4,19,24,25]) working for all p and also with methods working for individual values of p (see for instance [5–7,9,18]).

Lens spaces are compact manifolds with positive constant curvature and cyclic fundamental group. This class of spaces has provided many isospectral examples of different kinds. Ikeda [8] gave families of lens spaces mutually 0-isospectral (see also [22]). Later in [9], using the same family, he found pairs of lens spaces that are p-isospectral for all $0 \leq p < p_0$ but not p_0-isospectral, for any $p_0 > 0$. More recently, Miatello, Rossetti and the author [15] found families of pairs, in any odd dimension $n \geq 5$, of lens spaces that are p-isospectral for all p, but are not *strongly isospectral*. In particular such pair cannot be constructed by the generalized Sunada method due to DeTurck and Gordon [4], which uses representation equivalent discrete subgroups.

Furthermore, [15] also gives an explicit formula for the multiplicity of each eigenvalue in the 0-spectrum of a lens space and a geometric characterization of 0-isospectral lens spaces. The *one-norm* $\|\cdot\|_1$ of an element in $\mathbb{Z}^n$ is given by the sum of the absolute values of its entries, and two subsets $\mathcal{L}$ and $\mathcal{L}'$ of $\mathbb{Z}^n$ are called $\|\cdot\|_1$-isospectral if for each positive integer k, there are the same number of elements in $\mathcal{L}$ and in $\mathcal{L}'$ with one norm equal to k. Each $(2n-1)$-dimensional lens space L has associated a *congruence lattice* $\mathcal{L} \subset \mathbb{Z}^n$ [see (2.5)]. Hence, the above mentioned characterization for 0-isospectral lens spaces can be stated as follows: L and L' are 0-isospectral if and only if $\mathcal{L}$ and $\mathcal{L}'$ are $\|\cdot\|_1$-*isospectral*. (See [1,12,16,20,21] for related results.)

The aim of this paper, as a continuation of the study begun in [15], is to give an explicit description of the p-spectrum of a lens space for any value of p (Theorem 3.3). As a consequence, for any $p_0 \geq 0$, we obtain a geometric characterization of lens spaces p-isospectral for all $0 \leq p \leq p_0$ (Corollary 2.3). The main tool is a closed explicit formula from [17] for the multiplicity of weights in the irreducible representations of $\mathrm{SO}(2n)$ occurring in the decomposition of $\mathrm{Sym}^k(\mathbb{C}^{2n}) \otimes \bigwedge^p(\mathbb{C}^{2n})$ for any $k \geq 0$ and $0 \leq p \leq n-1$.

Some results of this article have been recently used in [13] to prove the non-existence of p-isospectral lens spaces (and lens orbifolds) for p in certain subsets of $\{0, 1, \ldots, n-1\}$.

The paper is organized as follows. Section 2 introduces the notation and states the results. The proofs are included in Sect. 3.

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2 Results

We set $G = \mathrm{SO}(2n)$ and $K = \mathrm{SO}(2n-1)$, thus the homogeneous space G/K is diffeomorphic to the $(2n-1)$-dimensional sphere S^{2n-1}. We consider the round metric on S^{2n-1}.

If Γ is a discrete subgroup of G (thus Γ is finite), then $\Gamma\backslash S^{2n-1}$ has a structure of a good orbifold, which is a manifold if Γ acts freely on S^{2n-1}. In this case, $\Gamma\backslash S^{2n-1}$ is usually called a *spherical space form* and Γ is isomorphic to the fundamental group of $\Gamma\backslash S^{2n-1}$.

For $\Gamma \subset G$ finite, let us denote by $\Delta_{\Gamma,p}$ the Hodge-Laplace operator on p-forms of $\Gamma\backslash S^{2n-1}$, which is given by the restriction of Δ_p on S^{2n-1} to Γ-invariant smooth p-forms on S^{2n-1}. The space $\Gamma\backslash S^{2n-1}$ is orientable, thus $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1}) = \mathrm{Spec}_{2n-1-p}(\Gamma\backslash S^{2n-1})$, hence p-isospectrality for every $0 \leq p \leq n-1$ is actually equivalent to p-isospectrality for every p.

In order to exhibit a description of $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1})$, we introduce some notation related to the root system associated to $\mathfrak{g}_{\mathbb{C}} := \mathfrak{so}(2n, \mathbb{C})$. We use standard choices for the Cartan subalgebra $\mathfrak{h}$ and the root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ (see Notation 3.1). In particular, the set of positive roots is $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) := \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ and the lattice of G-integral weights is $P(G) := \bigoplus_{j=1}^n \mathbb{Z}\varepsilon_j \simeq \mathbb{Z}^n$.

Given a dominant G-integral weight Λ, we will write π_Λ for the irreducible representation of G with highest weight Λ. Set $\Lambda_p = \varepsilon_1 + \cdots + \varepsilon_p$ for any $1 \leq p \leq n$, $\Lambda_0 = 0$, and $\bar{\Lambda}_n = \Lambda_n - 2\varepsilon_n$. For $k \geq 0$, define

$$\pi_{k,p} = \begin{cases} 0 & \text{if } p = 0, \\ \pi_{k\varepsilon_1+\Lambda_p} & \text{if } 1 \leq p < n, \\ \pi_{k\varepsilon_1+\Lambda_n} \oplus \pi_{k\varepsilon_1+\bar{\Lambda}_n} & \text{if } p = n. \end{cases} \tag{2.1}$$

It is well known that $V_{\pi_{0,p}} \simeq \bigwedge^p(\mathbb{C}^{2n})$ and $\mathrm{Sym}^{k+1}(\mathbb{C}^{2n}) \simeq V_{\pi_{k,1}} \oplus \mathrm{Sym}^{k-1}(\mathbb{C}^{2n})$ as G-modules. For $\Gamma \subset G$, write V_π^Γ for the subset of Γ-invariant elements in V_π. For $k \geq 1$, we set

$$\lambda_{k,p} = \begin{cases} 0 & \text{if } p = -1, \\ (k+p)(k+2n-2-p) & \text{if } 0 \leq p \leq n-1. \end{cases} \tag{2.2}$$

The next theorem is well known (see for instance [10, Thm. 4.2], [9, Prop. 2.1], [14, Thm. 1.1], [15, Prop. 2.2]). It describes the p-spectrum of $\Gamma\backslash S^{2n-1}$ in algebraic terms.

Theorem 2.1 *Fix $0 \leq p \leq n-1$. Each eigenvalue in $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1})$ is of the form $\lambda_{k,p-1}$ or $\lambda_{k,p}$ for some $k \geq 1$, with multiplicity*

$$\mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k,p-1}) = \dim V^\Gamma_{\pi_{k-1,p}} \quad \textit{and} \quad \mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k,p}) = \dim V^\Gamma_{\pi_{k-1,p+1}}.$$

A *lens space* is a spherical space form $\Gamma\backslash S^{2n-1}$ with Γ cyclic, which can be assumed included in the standard maximal torus

$$T := \{\mathrm{diag}(R(\theta_1), \ldots R(\theta_n)) : \theta_1, \ldots, \theta_n \in \mathbb{R}\} \qquad \left(R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) \tag{2.3}$$

of G. These spaces are parametrized as follows: for each $q \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{Z}$ satisfying $\gcd(q, s_1, \ldots, s_n) = 1$, set $\gamma = \mathrm{diag}(R(2\pi s_1/q), \ldots, R(2\pi s_1/q)) \in T$ and let Γ be the cyclic group of order q generated by γ; the space $L(q; s_1, \ldots, s_n) := \Gamma\backslash S^{2n-1}$ is called an *orbifold lens space*. The group Γ acts freely on S^{2n-1} if and only if $\gcd(q, s_j) = 1$ for all j, and in this case $L(q; s_1, \ldots, s_n)$ is a *lens space* (see [2] for details). We will actually consider spaces more general than orbifold lens spaces. Namely, good orbifolds of the form $\Gamma\backslash S^{2n-1}$ with Γ any discrete subgroup of T. Note that Γ must be finite and abelian.

For Γ a finite subgroup of T, we want to determine $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1})$ by using Theorem 2.1. If $\Gamma \subset T$ is finite, we have (see [15, Lem. 3.1] or [12, Prop. 2.6]) that

$$\dim V^\Gamma_{\pi_{k,p}} = \sum_{\mu \in \mathcal{L}_\Gamma} m_{\pi_{k,p}}(\mu), \tag{2.4}$$

where $\mathcal{L}_\Gamma = \{\mu \in P(G) : \gamma^\mu = 1 \quad \forall \gamma \in \Gamma\}$, $\gamma^\mu = e^{\mu(H)}$ if $\gamma = \exp(H)$, and $m_{\pi_{k,p}}(\mu)$ denotes the multiplicity of the weight μ in the representation $\pi_{k,p}$. In the case of a lens space $L(q; s_1, \ldots, s_n) = \Gamma\backslash S^{2n-1}$, it turns out that

$$\mathcal{L}_\Gamma = \left\{a_1\varepsilon_1 + \cdots + a_n\varepsilon_n \in P(G) \simeq \mathbb{Z}^n : a_1 s_1 + \cdots + a_n s_n \equiv 0 \pmod{q}\right\}. \tag{2.5}$$

The main tool will be a closed explicit expression for $m_{\pi_{k,p}}(\mu)$ obtained in [17] (see Lemma 3.2 below). The expression for $\mu = \sum_{j=1}^{n} a_j \varepsilon_j \in P(G)$ depends only on

$$\|\mu\|_1 := \sum_{j=1}^{n} |a_j| \quad \text{and} \quad Z(\mu) := \#\{1 \le j \le n : a_j = 0\}. \tag{2.6}$$

Combining this expression, (2.4) and Theorem 2.1, we get an explicit formula for $\mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k,p})$ for Γ a finite subgroup of T (Theorem 3.3). The formula is written in terms of the following numbers: for any subset $\mathcal{L}$ of $P(G)$, set

$$\begin{aligned} N_{\mathcal{L}}(k,\ell) &= \#\{\mu \in \mathcal{L} : \|\mu\|_1 = k,\ Z(\mu) = \ell\}, \\ N_{\mathcal{L}}(k) &= \#\{\mu \in \mathcal{L} : \|\mu\|_1 = k\}. \end{aligned} \tag{2.7}$$

Our next goal is to give an explicit description of $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1})$ for $\Gamma \subset T$ finite, alternative to Theorem 3.3. This description uses generating functions, giving more elegant results. Generating functions were first used for spectral problems concerning lens spaces by Ikeda and Yamamoto [11].

Ikeda in [9] associated to an arbitrary finite subgroup Γ of $G = \mathrm{SO}(2n)$, the n generating functions given by

$$F_{\Gamma}^{p}(z) = \sum_{k\ge 0} \mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k+1,p}) z^k = \sum_{k\ge 0} \dim V_{\pi_{k,p+1}}^{\Gamma} z^k, \tag{2.8}$$

for each $0 \le p \le n-1$. From Theorem 2.1, $\mathrm{Spec}_p(\Gamma\backslash S^{2n-1})$ is encoded by $F_{\Gamma}^{p-1}(z)$ and $F_{\Gamma}^{p}(z)$. In particular ([9, Prop. 2.1]),

$$\Gamma\backslash S^{2n-1} \text{ and } \Gamma'\backslash S^{2n-1} \text{ are } p\text{-}isospectral \text{ if and only if } \begin{cases} F_{\Gamma}^{p-1}(z) = F_{\Gamma'}^{p-1}(z), \\ F_{\Gamma}^{p}(z) = F_{\Gamma'}^{p}(z). \end{cases} \tag{2.9}$$

Ikeda gave expressions for the functions $F_{\Gamma}^{p}(z)$ for $0 \le p \le n-1$ (see [9, (2.13)]). These expressions were used in the computational study [7] of the p-spectrum of lens spaces (equation (3) in [7] works only for q prime). As a consequence of Lemma 3.2, we will give alternative expressions (only when $\Gamma \subset T$) in terms of the numbers in (2.7).

We next assume again that Γ is any finite subgroup of T. A formula for $m_{\pi_{k,1}}(\mu) = m_{\pi_{(k+1)\varepsilon_1}}(\mu)$ is previously known (see [15, Lem. 3.2]). This fact and (2.4) applied to (2.8) when $p = 0$ give (see [12, Thm. 3.6])

$$F_{\Gamma}^{0}(z) = \frac{1}{z}\left(\frac{\vartheta_{\mathcal{L}_\Gamma}(z)}{(1-z^2)^{n-1}} - 1\right), \tag{2.10}$$

where $\vartheta_{\mathcal{L}_\Gamma}(z)$ is the *one-norm generating function* associated to $\mathcal{L}_\Gamma$ given by (see also [23])

$$\vartheta_{\mathcal{L}_\Gamma}(z) = \sum_{k\ge 0} N_{\mathcal{L}_\Gamma}(k) z^k. \tag{2.11}$$

Equation (2.10) gives a neat way to describe the 0-spectrum of $\Gamma\backslash S^{2n-1}$ for any $\Gamma \subset T$.

In the next result we exhibit an expression of $F_{\Gamma}^{p}(z)$, analogous to (2.10), valid for any p. For each $0 \le \ell \le n$, set

$$\vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \sum_{k\ge 0} N_{\mathcal{L}_\Gamma}(k,\ell) z^k. \tag{2.12}$$

Theorem 2.2 *Let Γ be a finite subgroup of T. For each $1 \leq p \leq n$, there exist Laurent polynomials $A_p^{(\ell)}(z)$ for $0 \leq \ell \leq n$, with $-p \leq \deg(A_p^{(\ell)}(z)) \leq n - \ell - 1 - 2p$, satisfying that*

$$F_\Gamma^{p-1}(z) = \frac{1}{(1-z^2)^{n-1}} \sum_{\ell=0}^{n} \vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z)\, A_p^{(\ell)}(z) + \frac{(-1)^p}{z^p} \tag{2.13}$$

for every $1 \leq p \leq n$. Explicit expressions for $A_p^{(\ell)}(z)$ are given in (3.2).

We next derive spectral consequences from Theorem 2.2. For Γ and Γ' finite subgroups of T, one obtains immediately from (2.10) the following geometric characterization proved in [15, Thm. 3.6(i)],

$$\begin{aligned} &\Gamma\backslash S^{2n-1} \text{ and } \Gamma'\backslash S^{2n-1} \text{ are 0-isospectral if and only if } \mathcal{L}_\Gamma \text{ and } \mathcal{L}_{\Gamma'} \\ &\quad \text{are } \|\cdot\|_1\text{-}isospectral \text{ (i.e. } \vartheta_{\mathcal{L}_\Gamma}(z) = \vartheta_{\mathcal{L}_{\Gamma'}}(z)). \end{aligned} \tag{2.14}$$

Generalizing this result, we obtain the following geometric characterization.

Corollary 2.3 *Let $0 \leq p_0 \leq n-1$ and let Γ and Γ' be finite subgroups of T. Then, $\Gamma\backslash S^{2n-1}$ and $\Gamma'\backslash S^{2n-1}$ are p-isospectral for all $0 \leq p \leq p_0$ if and only if*

$$\sum_{\ell=0}^{n} \ell^h\, \vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \sum_{\ell=0}^{n} \ell^h\, \vartheta_{\mathcal{L}_{\Gamma'}}^{(\ell)}(z) \qquad \textit{for all } 0 \leq h \leq p_0, \tag{2.15}$$

or equivalently,

$$\sum_{\ell=0}^{n} \ell^h\, N_{\mathcal{L}_\Gamma}(k,\ell) = \sum_{\ell=0}^{n} \ell^h\, N_{\mathcal{L}_{\Gamma'}}(k,\ell) \quad \textit{for all } k \geq 0, \qquad \textit{for all } 0 \leq h \leq p_0. \tag{2.16}$$

It turns out that the geometric characterization for lens spaces p-isospectral for all p, proved in [15, Thm. 3.6(ii)], coincides with the previous result when $p_0 = n-1$ (see Remark 3.5). Namely, for Γ and Γ' finite subgroups of T,

$$\begin{aligned} &\Gamma\backslash S^{2n-1} \text{ and } \Gamma'\backslash S^{2n-1} \text{ are } p\text{-isospectral for all } p \text{ if and only if } \mathcal{L}_\Gamma \\ &\quad \text{and } \mathcal{L}_{\Gamma'} \text{ are } \|\cdot\|_1^*\text{-}isospectral \text{ (i.e. } \vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \vartheta_{\mathcal{L}_{\Gamma'}}^{(\ell)}(z) \text{ for all } \ell). \end{aligned} \tag{2.17}$$

Here, $\|\cdot\|_1^*$-isospectrality means $N_{\mathcal{L}_\Gamma}(k,\ell) = N_{\mathcal{L}_{\Gamma'}}(k,\ell)$ for all $k \geq 0$ and all $0 \leq \ell \leq n$. Examples of lens spaces p-isospectral for all p can be found in [15,16] and [3].

Ikeda's expression [9, (2.13)] for $F_\Gamma^p(z)$ ensures that $F_\Gamma^p(z)$ is a rational function. As a last main result we obtain explicit expressions for those polynomials in terms of the associated lattice $\mathcal{L}_\Gamma$. This expression is very useful for explicit computational purposes. Moreover, it is required only finitely many calculations to determine the rational expression. Consequently, the p-spectrum of $\Gamma\backslash S^{2n-1}$ is determined by a finite part of it.

With the above goal in mind, we will show that $\vartheta_{\mathcal{L}}^{(\ell)}(z)$ has a rational expression for any ℓ, obtaining the required expression for $F_\Gamma^p(z)$ from Theorem 2.2. We first introduce some more notation. For q a positive integer and $\mathcal{L} \subset P(G)$, we define

$$\begin{aligned} C(q) &= \{\mu = \textstyle\sum_i a_i \varepsilon_i \in P(G) : |a_i| < q \quad \forall i\}, \\ N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) &= \#\{\mu \in C(q) \cap \mathcal{L} : \|\mu\|_1 = k,\ Z(\mu) = \ell\}, \\ \Phi_{\mathcal{L}}^{(\ell)}(z) &= \textstyle\sum_{k \geq 0} N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) z^k. \end{aligned} \tag{2.18}$$

It is important to note that $\Phi_{\mathcal{L}}^{(\ell)}(z)$ is actually a polynomial of degree at most $(n-\ell)(q-1)$. Indeed, if $\|\mu\|_1 > (n-\ell)(q-1)$, then $\mu \notin C(q)$. Hence, $N_{\mathcal{L}}^{\mathrm{red}}(k,\ell) = 0$ for all $k > (n-\ell)(q-1)$. We set $q(\Gamma) = \min\{m \in \mathbb{N} : \gamma^m = 1 \quad \forall \gamma \in \Gamma\}$.

Theorem 2.4 *Let Γ be a finite subgroup of T with $q(\Gamma) = q$. Then, for each $0 \le \ell \le n$,*

$$\vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \frac{1}{(1-z^q)^{n-\ell}} \sum_{s=0}^{n-\ell} 2^s \binom{\ell+s}{s} z^{sq}\, \Phi_{\mathcal{L}_\Gamma}^{(\ell+s)}(z).$$

It was already shown in [12, Thm. 3.9], by using Ehrhart's theory for counting integral points in rational polytopes, that $\vartheta_{\mathcal{L}_\Gamma}(z)$ has a rational expression. Actually, the author has recently found the article [23], where a more general result was proved with the identical method. The next corollary refines these results by giving an explicit expression for the polynomial at the numerator in terms of the polynomials $\Phi_{\mathcal{L}_\Gamma}^{(\ell)}(z)$.

Corollary 2.5 *Let Γ be a finite subgroup of T with $q(\Gamma) = q$. Then,*

$$\vartheta_{\mathcal{L}_\Gamma}(z) = \frac{1}{(1-z^q)^n} \sum_{t=0}^{n} z^{tq} \sum_{\ell=t}^{n} \binom{\ell}{t} \Phi_{\mathcal{L}_\Gamma}^{(\ell)}(z).$$

3 Spectra on forms of lens spaces

In this section we prove the statements in Sect. 2. We first describe in Theorem 3.3 the spectrum of the Hodge-Laplace operator $\Delta_{\Gamma,p}$ acting on p-forms of $\Gamma\backslash S^{2n-1}$ with $\Gamma \subset T$. Then, Theorem 2.2, Corollary 2.3, Theorem 2.4 and Corollary 2.5 are proved. We first set up notation.

Notation 3.1 We recall that $G = \mathrm{SO}(2n)$, thus its Lie algebra and complexified Lie algebra are $\mathfrak{g} = \mathfrak{so}(2n)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$. We pick the Cartan subalgebra

$$\mathfrak{h} := \left\{ \mathrm{diag}\left(\begin{bmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -\theta_n \\ \theta_n & 0 \end{bmatrix} \right) : \theta_j \in \mathbb{C}\ \forall j \right\}$$

of $\mathfrak{g}_{\mathbb{C}}$. For any $1 \le j \le n$, let $\varepsilon_j \in \mathfrak{h}^*$ given by $\varepsilon_j \left(\mathrm{diag}\left(\begin{bmatrix} 0 & -\theta_1 \\ \theta_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & -\theta_n \\ \theta_n & 0 \end{bmatrix} \right) \right) = i\theta_j$. Therefore, the root system associated to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ (i.e. the root system of type D_n) is given by $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}) := \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\}$, the lattice of G-integral weights is $P(G) := \bigoplus_{j=1}^n \mathbb{Z}\varepsilon_j$ and $\mu = \sum_{j=1}^n a_j\varepsilon_j \in P(G)$ is dominant if and only if $a_1 \ge \cdots \ge a_{n-1} \ge |a_n|$.

The following lemma, proved in [17, Thm. IV.1], gives a closed explicit expression for the multiplicity $m_{\pi_{k,p}}(\mu)$ of the weight μ in the representation $\pi_{k,p}$. The formula depends only on the one-norm $\|\mu\|_1$ of μ and the number of zero entries of μ, denoted by $Z(\mu)$ [see (2.6)]. This fact was already shown in [15, Lem. 3.3]. We shall use the convention $\binom{b}{a} = 0$ if $b < a$ or $a < 0$.

Lemma 3.2 *Let $k \ge 0$, $1 \le p \le n$ and let $\mu \in P(G)$. Write $r(\mu) = (k + p - \|\mu\|_1)/2$. If $r(\mu)$ is a non-negative integer, then*

$$m_{\pi_{k,p}}(\mu) = \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} \sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-Z(\mu)}{\beta} \binom{Z(\mu)}{p-j-2t-\beta}$$
$$\sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \sum_{i=0}^{j-1} \binom{r(\mu)-i-p+\alpha+t+j+n-2}{n-2},$$

and $m_{\pi_{k,p}}(\mu) = 0$ *otherwise.*

For $\Gamma \subset T$ finite, $0 \le \ell \le n$ and $k \ge 1$, we set

$$M_\Gamma(k,p) = \sum_{\ell=0}^{n} \sum_{r=0}^{\lfloor \frac{k-1+p}{2} \rfloor} N_{\mathcal{L}_\Gamma}(k-1+p-2r,\ell) \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t}$$
$$\sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-2t-\beta}$$
$$\sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \sum_{i=0}^{j-1} \binom{r-i-p+j+\alpha+t+n-2}{n-2}. \tag{3.1}$$

We recall from (2.2) that $\lambda_{k,p} = (k+p)(k+2n-2-p)$ if $0 \le p \le n-1$, and $\lambda_{k,-1} = 0$.

Theorem 3.3 *Let* Γ *be a finite subgroup of* T *and* $0 \le p \le n-1$. *Each eigenvalue in* $\mathrm{Spec}_p(\Gamma \backslash S^{2n-1})$ *is of the form* $\lambda_{k,p-1}$ *or* $\lambda_{k,p}$ *for some* $k \ge 1$, *with multiplicities given by*

$$\mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k,p-1}) = M_\Gamma(k,p-1) \quad \textit{and} \quad \mathrm{mult}_{\Delta_{\Gamma,p}}(\lambda_{k,p}) = M_\Gamma(k,p).$$

In particular, the 0*-spectrum of* $\Gamma \backslash S^{2n-1}$ *has eigenvalues* $k(k+2n-2)$ *for any* $k \ge 0$, *with*

$$\mathrm{mult}_{\Delta_{\Gamma,0}}\big(k(k+2n-2)\big) = \sum_{r=0}^{\lfloor k/2 \rfloor} N_{\mathcal{L}_\Gamma}(k-2r) \binom{r+n-2}{n-2}.$$

Proof From Theorem 2.1, it suffices to calculate $\dim V^\Gamma_{\pi_{k,p}}$ for $\Gamma \subset T$ finite and furthermore, (2.4) implies that $\dim V^\Gamma_{\pi_{k,p}} = \sum_{\mu \in \mathcal{L}_\Gamma} m_{\pi_{k,p}}(\mu)$. Lemma 3.2 ensures that $m_{\pi_{k,p}}(\mu) = 0$ if $k + p - \|\mu\|_1 \notin 2\mathbb{Z}_{\ge 0}$, and moreover $m_{\pi_{k,p}}(\mu) = m_{\pi_{k,p}}(\mu')$ for μ and μ' satisfying $\|\mu\|_1 = \|\mu'\|_1$ and $Z(\mu) = Z(\mu')$. Thus,

$$\dim V^\Gamma_{\pi_{k,p}} = \sum_{\ell=0}^{n} \sum_{r=0}^{\lfloor \frac{k+p}{2} \rfloor} \sum_{\substack{\mu \in \mathcal{L}_\Gamma : Z(\mu)=\ell \\ \|\mu\|_1 = k+p-2r}} m_{\pi_{k,p}}(\mu) = \sum_{\ell=0}^{n} \sum_{r=0}^{\lfloor \frac{k+p}{2} \rfloor} N_{\mathcal{L}}(k+p-2r,\ell)\, m_{\pi_{k,p}}(\mu_0),$$

where μ_0 is any weight satisfying $\|\mu_0\|_1 = k+p-2r$ and $Z(\mu_0) = \ell$. The rest follows by the formula for $m_{\pi_{k,p}}(\mu_0)$ in Lemma 3.2. □

For $1 \le p \le n-1$ and $0 \le \ell \le n$, we set

$$A_p^{(\ell)}(z) = \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t}$$
$$\sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-2t-\beta} \sum_{\alpha=0}^{\beta} \binom{\beta}{\alpha} \sum_{i=0}^{j-1} z^{p-2(j+t+\alpha-i)}. \tag{3.2}$$

Remark 3.4 One can check that $A_p^{(\ell)}(z)$ is a Laurent polynomial of degree in between $-p$ and $p-2$. Furthermore, $A_p^{(\ell)}(z)$ is a polynomial on the variable ℓ, with coefficients in the space of complex Laurent polynomials on z, of degree at most p.

Proof of Theorem 2.2 This proof is quite lengthy, so it will contain several claims in order to facilitate the reading. Write $\mathcal{L} = \mathcal{L}_\Gamma$. Set

$$\Upsilon_m^{(\ell)}(z) = \sum_{h=0}^{m} z^h \sum_{s=0}^{\lfloor \frac{h}{2} \rfloor} N_{\mathcal{L}}(h-2s,\ell)\binom{s+n-2}{n-2}, \tag{3.3}$$

$$B_{\Gamma,p}^{(\ell)}(z) = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta}\binom{n-\ell}{\beta}$$
$$\binom{\ell}{p-j-2t-\beta}\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1} z^{p-2(j+t+\alpha-i)}\ \Upsilon_{2(j+t+\alpha-i)-p-1}^{(\ell)}(z). \tag{3.4}$$

Claim 1 $F_\Gamma^{p-1}(z) = \sum_{\ell=0}^{n}\left(\frac{\vartheta_{\mathcal{L}}^{(\ell)}(z)}{(1-z^2)^{n-1}}\ A_p^{(\ell)}(z) - B_{\Gamma,p}^{(\ell)}(z)\right).$

Proof Theorem 3.3 implies $F_\Gamma^{p-1}(z) = \sum_{k\geq 0} M_\Gamma(k+1,p)z^k$. By (3.1) and reordering the sums, we obtain that

$$F_\Gamma^{p-1}(z) = \sum_{\ell=0}^{n}\sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor}\binom{n-p+j+2t}{t}$$
$$\sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta}\binom{n-\ell}{\beta}\binom{\ell}{p-j-2t-\beta}\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\sum_{i=0}^{j-1} G(z), \tag{3.5}$$

where

$$G(z) = \sum_{k\geq 0} z^k \sum_{r=0}^{\lfloor \frac{k+p}{2} \rfloor} N_{\mathcal{L}}(k+p-2r,\ell)\binom{r-i-p+j+\alpha+t+n-2}{n-2}.$$

Write $\gamma = i+p-j-\alpha-t$. One can check that $\gamma \geq 0$ for all allowed choices for i, j, α and t. We have that

$$\begin{aligned} G(z) &= \sum_{k\geq 0} z^k \sum_{r=\gamma}^{\lfloor \frac{k+p}{2} \rfloor} N_{\mathcal{L}}(k+p-2r,\ell)\binom{r-\gamma+n-2}{n-2} \\ &= \sum_{k\geq 0} z^k \sum_{s=0}^{\lfloor \frac{k+p-2\gamma}{2} \rfloor} N_{\mathcal{L}}(k+p-2\gamma-2s,\ell)\binom{s+n-2}{n-2} \\ &= z^{2\gamma-p}\sum_{h\geq p-2\gamma} z^h \sum_{s=0}^{\lfloor \frac{h}{2} \rfloor} N_{\mathcal{L}}(h-2s,\ell)\binom{s+n-2}{n-2}. \end{aligned}$$

Here we made the changes of variables, $r = s + \gamma$ and $h = k + p - 2\gamma$. Hence

$$G(z) = z^{2\gamma - p}\left(\sum_{h\geq 0} z^h \sum_{s=0}^{\lfloor \frac{h}{2}\rfloor} N_{\mathcal{L}}(h-2s,\ell)\binom{s+n-2}{n-2} - \Upsilon^{(\ell)}_{p-2\gamma-1}(z)\right)$$
$$= z^{2\gamma-p}\left(\frac{\vartheta^{(\ell)}_{\mathcal{L}}(z)}{(1-z^2)^{n-1}} - \Upsilon^{(\ell)}_{p-2\gamma-1}(z)\right).$$

This formula and (3.5) imply the claim. □

By Claim 1, it remains to show that $\sum_{\ell=0}^{n} B^{(\ell)}_{\Gamma,p}(z) = (-1)^{p+1}z^{-p}$. Set

$$C^{(\ell)}_{p,g} = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor \frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t}2^{p-j-2t-\beta}\binom{n-\ell}{\beta}\binom{\ell}{p-j-2t-\beta}$$
$$\sum_{i=0}^{j-1}\binom{\beta}{p+i-j-t-g}. \tag{3.6}$$

Claim 2 $B^{(\ell)}_{\Gamma,p}(z) = \sum_{g=0}^{p-1} z^{2g-p}\ \Upsilon^{(\ell)}_{p-1-2g}(z)\ C^{(\ell)}_{p,g}$.

Proof We note that the sum over α in (3.4) can be extended to $0 \leq \alpha \leq p$ since $\binom{\beta}{\alpha}$ will vanish for $\alpha > \beta$. We make the change of variables $g = p+i-j-t-\alpha$, thus $0 \leq g \leq p-1$ and $\alpha = p+i-j-t-g$. Hence

$$B^{(\ell)}_{\Gamma,p}(z) = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor \frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\beta=0}^{p-j-2t}2^{p-j-2t-\beta}\binom{n-\ell}{\beta}$$
$$\binom{\ell}{p-j-2t-\beta}\sum_{g=0}^{p-1}\sum_{i=0}^{j-1}\binom{\beta}{p+i-j-t-g}z^{2g-p}\ \Upsilon^{(\ell)}_{p-1-2g}(z).$$

The claim follows by reordering the sums. □

By definition, $\Upsilon^{(\ell)}_m(z) = 0$ for $m < 0$. Moreover, $\Upsilon^{(\ell)}_m(z) = 0$ if $n - \ell > m$, since $N_{\mathcal{L}}(k,\ell) = 0$ for all $0 \leq k < n - \ell$. Hence,

$$\sum_{\ell=0}^{n} B^{(\ell)}_{\Gamma,p}(z) = \sum_{g=0}^{\lfloor \frac{p-1}{2}\rfloor} z^{2g-p}\sum_{\ell=n+1+2g-p}^{n}\Upsilon^{(\ell)}_{p-1-2g}(z)\ C^{(\ell)}_{p,g}. \tag{3.7}$$

Claim 3 For any $0 \leq g \leq \frac{p-1}{2}$, $C^{(\ell)}_{p,g} = 0$ if $n+1+2g-p \leq \ell < n$, and $C^{(n)}_{p,g} = (-1)^{p-1+g}\binom{n-1}{g}$.

Proof We first consider the case $\ell = n$. One can easily see from (3.6) that the sum over β vanishes for $\beta > 0$ since $n - \ell = 0$. Thus

$$C_{p,g}^{(n)} = \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} 2^{p-j-2t} \binom{n}{p-j-2t} \sum_{i=0}^{j-1} \binom{0}{p+i-j-t-g}$$
$$= \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=p-g-j}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} 2^{p-j-2t} \binom{n}{p-j-2t}$$

since $\sum_{i=0}^{j-1} \binom{0}{p+i-j-t-g}$ is equal to 1 if $p - g - j - t \leq 0 \leq p - 1 - g - t$ and 0 otherwise. Note that the sum over j in the last row is restricted to $p - g - j \leq \lfloor (p-j)/2 \rfloor$, which is equivalent to $j \geq p - 2g$. By making the change of variables $k = p - j$, we obtain that

$$C_{p,g}^{(n)} = \sum_{k=0}^{2g} (-1)^{p-k-1} \sum_{t=k-g}^{\lfloor k/2 \rfloor} \binom{n-k+2t}{t} 2^{k-2t} \binom{n}{k-2t}. \tag{3.8}$$

We now prove the assertion for $\ell = n$ by induction on g. One can easily check that $C_{p,0}^{(n)} = (-1)^{p-1}$ from (3.8). In the inductive step, we have that

$$(-1)^{p-1} C_{p,g+1}^{(n)} = \sum_{k=0}^{2(g+1)} (-1)^k \sum_{t=k-(g+1)}^{\lfloor \frac{k}{2} \rfloor} \binom{n-k+2t}{t} 2^{k-2t} \binom{n}{k-2t} = (-1)^{p-1} C_{p,g}^{(n)}$$
$$-2n \binom{n-1}{g} + \binom{n}{g+1} + \sum_{k=0}^{2g} (-1)^k \binom{n-2g-2+k}{k-g-1} 2^{2g+2-k} \binom{n}{2g+2-k}.$$

By making the change of variable $m = 2g + 2 - k$ and the inductive hypothesis, we have that

$$C_{p,g+1}^{(n)} = C_{p,g}^{(n)} + (-1)^{p-1} \sum_{m=0}^{g+1} (-2)^m \binom{n}{m} \binom{n-m}{g+1-m}$$
$$= (-1)^{p-1} \sum_{u=0}^{g} (-1)^u \binom{n}{u} + (-1)^{p-1} \sum_{m=0}^{g+1} (-2)^m \binom{g+1}{m} \binom{n}{g+1}$$
$$= (-1)^{p-1} \sum_{u=0}^{g} (-1)^u \binom{n}{u} + (-1)^{p-1} (-1)^{g+1} \binom{n}{g+1} = (-1)^{p-1} \sum_{u=0}^{g+1} (-1)^u \binom{n}{u}.$$

The assertion follows by the well known identity

$$\sum_{u=0}^{m} (-1)^u \binom{a}{u} = (-1)^m \binom{a-1}{m}, \qquad a > 0. \tag{3.9}$$

We now consider the other cases. We again use induction on g. When $g = 0$, we assume $n + 1 - p \le \ell < n$ and we have that

$$C_{p,0}^{(\ell)} = \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} \sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-2t-\beta} \sum_{i=0}^{j-1} \binom{\beta}{p+i-j-t}.$$

Clearly, $\binom{\beta}{p+i-j-t} \neq 0$ if and only if $0 \le p + i - j - t \le \beta$, thus $i + t \le 0$ since $\beta \le p - j - 2t$. Thus, $i = t = 0$ and

$$\begin{aligned} C_{p,0}^{(\ell)} &= \sum_{j=1}^{p} (-1)^{j-1} \sum_{\beta=0}^{p-j} 2^{p-j-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-\beta} \binom{\beta}{p-j} \\ &= \sum_{j=1}^{p} (-1)^{j-1} \binom{n-\ell}{p-j} = (-1)^{p-1} \sum_{j=0}^{p-1} (-1)^{j} \binom{n-\ell}{j} = \binom{n-\ell-1}{p-1} = 0 \end{aligned}$$

since $n - \ell - 1 < p - 1$ by assumption. The third equality in the last row follows by (3.9).

We now assume that $C_{p,g}^{(\ell)} = 0$, $g + 1 \le \frac{p-1}{2}$ and $n + 1 + 2(g + 1) - p \le \ell$ ($\iff n - \ell \le p - 2g - 3$). In order to use induction, we first note that

$$\begin{aligned} \sum_{i=0}^{j-1} \binom{\beta}{p+i-j-t-g-1} &= \binom{\beta}{p-j-t-g-1} \\ &\quad + \sum_{i=0}^{j-1} \binom{\beta}{p+i-j-t-g} - \binom{\beta}{p-1-t-g}, \end{aligned}$$

which is the last part in the expression of $C_{p,g+1}^{(\ell)}$ in (3.6). Hence,

$$\begin{aligned} C_{p,g+1}^{(\ell)} &= \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} \sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-2t-\beta} \\ &\quad \binom{\beta}{p-j-t-g-1} + C_{p,g}^{(\ell)} - \sum_{j=1}^{p} (-1)^{j-1} \sum_{t=0}^{\lfloor \frac{p-j}{2} \rfloor} \binom{n-p+j+2t}{t} \\ &\quad \sum_{\beta=0}^{p-j-2t} 2^{p-j-2t-\beta} \binom{n-\ell}{\beta} \binom{\ell}{p-j-2t-\beta} \binom{\beta}{p-1-t-g}. \end{aligned}$$

We have that $C_{p,g}^{(\ell)} = 0$ by induction. Moreover, the last term vanishes since $\binom{n-\ell}{\beta}\binom{\beta}{p-1-t-g} = 0$ in the corresponding cases. Indeed, if this number were nonzero, then $p - 1 - t - g \le \beta \le n - \ell \le p - 2g - 3$, thus $g + 2 \le t$, which is a contradiction since $p - 1 - t - g \le \beta \le p - j - 2t$ yields $t \le g + 1 - j \le g$.

By making the change of variables $\gamma = p - j - 2t - \beta$ to the remaining term, we obtain that

$$C^{(\ell)}_{p,g+1} = \sum_{j=1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\gamma=0}^{p-j-2t}2^{\gamma} \times\binom{\ell}{\gamma}\binom{n-\ell}{p-j-2t-\gamma}\binom{p-j-2t-\gamma}{p-j-t-g-1}.$$

Clearly, $\binom{p-j-2t-\gamma}{p-j-t-g-1} \neq 0$ if and only if $0 \leq p-j-t-g-1 \leq p-j-2t-\gamma$, thus $t+\gamma \leq g+1$. This implies that the sum over t goes from 0 to $\min(g+1, \lfloor\frac{p-j}{2}\rfloor)$. Since $g+1 \leq \lfloor\frac{p-j}{2}\rfloor$ if and only if $j \leq p-2g-2$, we have that

$$\begin{aligned}
C^{(\ell)}_{p,g+1} &= \sum_{j=1}^{p-2g-2}(-1)^{j-1}\sum_{t=0}^{g+1}\binom{n-p+j+2t}{t}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\binom{n-\ell}{p-j-2t-\gamma}\binom{p-j-2t-\gamma}{g+1-t-\gamma}\\
&\quad+\sum_{j=p-2g-1}^{p}(-1)^{j-1}\sum_{t=0}^{\lfloor\frac{p-j}{2}\rfloor}\binom{n-p+j+2t}{t}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\binom{n-\ell}{p-j-2t-\gamma}\binom{p-j-2t-\gamma}{g+1-t-\gamma}\\
&=\sum_{t=0}^{g+1}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\sum_{j=1}^{p-2g-2}(-1)^{j-1}\binom{n-p+j+2t}{t}\binom{n-\ell}{p-j-2t-\gamma}\binom{p-j-2t-\gamma}{g+1-t-\gamma}\\
&\quad+\sum_{t=0}^{g}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\sum_{j=p-2g-1}^{p-2t}(-1)^{j-1}\binom{n-p+j+2t}{t}\binom{n-\ell}{p-j-2t-\gamma}\binom{p-j-2t-\gamma}{g+1-t-\gamma}.
\end{aligned}$$

The last equality follows by reordering the sums in both terms, with the sum extremes handled with particular care in the second row.

In the first term in the formula above, the sum over j goes actually from $2g+3-2t-\gamma$ since $\binom{n-\ell}{p-j-2t-\gamma} \neq 0$ implies that $p-j-2t-\gamma \leq n-\ell \leq p-2g-3$. By making the change of variables $k = p-j-2t-\gamma$ in both terms we obtain that

$$\begin{aligned}
C^{(\ell)}_{p,g+1} &= \sum_{t=0}^{g+1}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\sum_{k=2g+2-2t-\gamma}^{p-2g-3}(-1)^{p-k-\gamma-1}\binom{n-k-\gamma}{t}\binom{n-\ell}{k}\binom{k}{g+1-t-\gamma}\\
&\quad+\sum_{t=0}^{g}\sum_{\gamma=0}^{g+1-t}2^{\gamma}\binom{\ell}{\gamma}\sum_{k=-\gamma}^{2g+1-2t-\gamma}(-1)^{p-k-\gamma-1}\binom{n-k-\gamma}{t}\binom{n-\ell}{k}\binom{k}{g+1-t-\gamma}.
\end{aligned}$$

Note that in the second term, the sum over k goes from 0 in order to $\binom{n-\ell}{k} \neq 0$, and the sum over t can be extended up to $g+1$ since it vanishes for $t = g+1$. Hence,

$$\begin{aligned}
C^{(\ell)}_{p,g+1} &= (-1)^{p-1}\sum_{t=0}^{g+1}\sum_{\gamma=0}^{g+1-t}(-2)^{\gamma}\binom{\ell}{\gamma}\sum_{k=0}^{p-2g-3}(-1)^{k}\binom{n-k-\gamma}{t}\binom{n-\ell}{k}\binom{k}{g+1-t-\gamma}\\
&= (-1)^{p-1}\sum_{k=0}^{p-2g-3}(-1)^{k}\binom{n-\ell}{k}\sum_{\gamma=0}^{g+1}(-2)^{\gamma}\binom{\ell}{\gamma}\sum_{t=0}^{g+1-\gamma}\binom{n-k-\gamma}{t}\binom{k}{g+1-t-\gamma}.
\end{aligned}$$

The well known identity $\sum_{t=0}^{m}\binom{a}{t}\binom{b}{m-t}=\binom{a+b}{m}$ implies that the sum over t in the formula above equals $\binom{n-k-\gamma+k}{g+1-\gamma}=\binom{n-\gamma}{g+1-\gamma}$, which does not depend on k. Hence

$$\begin{aligned}C^{(\ell)}_{p,g+1}&=(-1)^{p-1}\sum_{\gamma=0}^{g+1}(-2)^{\gamma}\binom{\ell}{\gamma}\binom{n-\gamma}{g+1-\gamma}\sum_{k=0}^{p-2g-3}(-1)^{k}\binom{n-\ell}{k}\\&=(-1)^{p-1}\sum_{\gamma=0}^{g+1}(-2)^{\gamma}\binom{\ell}{\gamma}\binom{n-\gamma}{g+1-\gamma}(-1)^{p-2g-3}\binom{n-\ell-1}{p-2g-3}.\end{aligned}$$

The last row follows by (3.9). We conclude that $C^{(\ell)}_{p,g+1}=0$ since $n-\ell-1<n-\ell\leq p-2g-3$ by assumption. □

Claim 4 $\displaystyle\sum_{\ell=0}^{n}B^{(\ell)}_{\Gamma,p}(z)=\frac{(-1)^{p-1}}{z^{p}}.$

Proof Equation (3.7) and Claim 3 imply that

$$\sum_{\ell=0}^{n}B^{(\ell)}_{\Gamma,p}(z)=\sum_{g=0}^{\lfloor\frac{p-1}{2}\rfloor}z^{2g-p}\,\Upsilon^{(n)}_{p-1-2g}(z)\,C^{(n)}_{p,g}.$$

By Claim 3, since $N_{\mathcal{L}}(k,n)=1$ if $k=0$ and 0 otherwise, we have that

$$\begin{aligned}\sum_{\ell=0}^{n}B^{(\ell)}_{\Gamma,p}(z)&=\frac{(-1)^{p-1}}{z^{p}}\sum_{g=0}^{\lfloor\frac{p-1}{2}\rfloor}z^{2g}(-1)^{g}\binom{n-1}{g}\sum_{h=0}^{\lfloor\frac{p-1}{2}\rfloor-g}z^{2h}\binom{h+n-2}{n-2}\\&=\frac{(-1)^{p-1}}{z^{p}}\sum_{m=0}^{\lfloor\frac{p-1}{2}\rfloor}z^{2m}\sum_{g=0}^{m}(-1)^{g}\binom{n-1}{g}\binom{m-g+n-2}{n-2}.\end{aligned}$$

We note that $\sum_{g=0}^{m}(-1)^{g}\binom{n-1}{g}\binom{m-g+n-2}{n-2}$ is the m-th term of the series

$$\begin{aligned}&\left(\sum_{h\geq 0}\binom{h+n-2}{n-2}z^{2h}\right)\left(\sum_{g=0}^{\lfloor\frac{p-1}{2}\rfloor}(-z^{2})^{g}\binom{n-1}{g}\right)\\&=\frac{1}{(1-z^{2})^{n-1}}\left((1-z^{2})^{n-1}-\sum_{g=\lfloor\frac{p-1}{2}\rfloor+1}^{n-1}(-z^{2})^{g}\binom{n-1}{g}\right)\\&=1-\frac{1}{(1-z^{2})^{n-1}}\sum_{g=\lfloor\frac{p-1}{2}\rfloor+1}^{n-1}(-z^{2})^{g}\binom{n-1}{g},\end{aligned}$$

which is 1 for $m=0$ and 0 for $1\leq m\leq\lfloor\frac{p-1}{2}\rfloor$. Then $\sum_{\ell=0}^{n}B^{(\ell)}_{\Gamma,p}(z)=(-1)^{p-1}/z^{p}$ as claimed. □

This completes the proof of Theorem 2.2. □

Before proving Corollary 2.3, we make an observation.

Remark 3.5 Corollary 2.3 contains the characterizations in [15, Thm. 3.6] in the extreme cases $p_0 = 0$ and $p_0 = n-1$. Indeed, if $p_0 = 0$, then (2.15) tells us that $\vartheta_{\mathcal{L}_\Gamma}(z) = \sum_{\ell=0}^{n} \vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \sum_{\ell=0}^{n} \vartheta_{\mathcal{L}_{\Gamma'}}^{(\ell)}(z) = \vartheta_{\mathcal{L}_{\Gamma'}}(z)$, or equivalently that, $\mathcal{L}_\Gamma$ and $\mathcal{L}_{\Gamma'}$ are $\|\cdot\|_1$-isospectral.

When $p_0 = n-1$, the condition (2.15) consists of n equations in the n variables $\vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) - \vartheta_{\mathcal{L}_{\Gamma'}}^{(\ell)}(z)$, $0 \le \ell \le n-1$ (clearly $\vartheta_{\mathcal{L}_\Gamma}^{(n)}(z) = \vartheta_{\mathcal{L}_{\Gamma'}}^{(n)}(z) = 1$). The matrix associated to this linear equation is the Vandermonde matrix, which is non-singular. Hence, (2.15) is equivalent to have $\vartheta_{\mathcal{L}_\Gamma}^{(\ell)}(z) = \vartheta_{\mathcal{L}_{\Gamma'}}^{(\ell)}(z)$ for all $0 \le \ell \le n$, that is, $\mathcal{L}_\Gamma$ and $\mathcal{L}_{\Gamma'}$ are $\|\cdot\|_1^*$-isospectral.

Proof of Corollary 2.3 Write $\mathcal{L} = \mathcal{L}_\Gamma$ and $\mathcal{L}' = \mathcal{L}_{\Gamma'}$. By Theorem 2.2, we have that

$$F_\Gamma^p(z) - F_{\Gamma'}^p(z) = \frac{1}{(1-z^2)^{n-1}} \sum_{\ell=0}^{n} \left(\vartheta_{\mathcal{L}}^{(\ell)}(z) - \vartheta_{\mathcal{L}'}^{(\ell)}(z) \right) A_{p+1,\ell}(z) \tag{3.10}$$

for all $0 \le p \le n-1$.

On the one hand, we know that $\Gamma \backslash S^{2n-1}$ and $\Gamma' \backslash S^{2n-1}$ are p-isospectral for all $0 \le p \le p_0$ if and only if $F_\Gamma^p(z) = F_{\Gamma'}^p(z)$ for every $0 \le p \le p_0$ by (2.9). On the other hand, (2.15) is equivalent to $\sum_{\ell=0}^{n} \left(\vartheta_{\mathcal{L}}^{(\ell)}(z) - \vartheta_{\mathcal{L}'}^{(\ell)}(z) \right) A_{p+1,\ell}(z) = 0$, since $A_{p+1,\ell}(z)$ is a polynomial on ℓ of degree $\le p_0 + 1$ (see Remark 3.4) for every $0 \le p \le p_0$. These both facts complete the proof. □

Proof of Theorem 2.4 In [15, Thm. 4.2], the following formula was proved, which actually holds for any sublattice $\mathcal{L}$ of $P(G) \simeq \mathbb{Z}^n$ satisfying that $\mu \in \mathcal{L}$ if and only if $\mu + q\nu \in \mathcal{L}$ for any $\nu \in P(G)$: for $a \ge 0$ and $0 \le r < q$,

$$N_{\mathcal{L}}(aq + r, \ell) = \sum_{s=0}^{n-\ell} 2^s \binom{\ell+s}{s} \sum_{t=s}^{a} \binom{t-s+n-\ell-1}{n-\ell-1} N_{\mathcal{L}}^{\mathrm{red}}((a-t)q + r, \ell + s). \tag{3.11}$$

One can check that $\mathcal{L}_\Gamma$ with $\Gamma \subset T$ finite and $q(\Gamma) = q$, satisfies that $\mu \in \mathcal{L}_\Gamma$ if and only if $\mu + q\nu \in \mathcal{L}_\Gamma$ for any $\nu \in P(G)$. Write $\mathcal{L} = \mathcal{L}_\Gamma$. Applying (3.11) to (2.12), we obtain that

$$\begin{aligned}
\vartheta_{\mathcal{L}}^{(\ell)}(z) &= \sum_{r=0}^{q-1} \sum_{a \ge 0} N_{\mathcal{L}_\Gamma}(aq + r, \ell) z^{aq+r} \\
&= \sum_{s=0}^{n-\ell} 2^s \binom{\ell+s}{s} \sum_{r=0}^{q-1} \sum_{a \ge 0} z^{aq+r} \sum_{t=s}^{a} \binom{t-s+n-\ell-1}{n-\ell-1} N_{\mathcal{L}}^{\mathrm{red}}((a-t)q + r, \ell + s) \\
&= \sum_{s=0}^{n-\ell} 2^s \binom{\ell+s}{s} z^{sq} \sum_{r=0}^{q-1} z^r \sum_{b \ge 0} z^{bq} \sum_{u=0}^{b} \binom{u+n-\ell-1}{n-\ell-1} N_{\mathcal{L}}^{\mathrm{red}}((b-u)q + r, \ell + s).
\end{aligned}$$

In the last row we made the change of variables $t = u + s$ and $a = b + s$.

Since $\sum_{k\geq 0}\binom{k+n-\ell-1}{n-\ell-1}z^{kq}=(1-z^q)^{-(n-\ell)}$, we get

$$\vartheta_{\mathcal{L}}^{(\ell)}(z)=\sum_{s=0}^{n-\ell}2^s\binom{\ell+s}{s}z^{sq}\sum_{r=0}^{q-1}z^r\left(\frac{1}{(1-z^q)^{n-\ell}}\sum_{h\geq 0}N_{\mathcal{L}}^{\mathrm{red}}(hq+r,\ell+s)z^{hq}\right)$$
$$=\frac{1}{(1-z^q)^{n-\ell}}\sum_{s=0}^{n-\ell}2^s\binom{\ell+s}{s}z^{sq}\sum_{k\geq 0}N_{\mathcal{L}}^{\mathrm{red}}(k,\ell+s)z^k,$$

which completes the proof. □

Proof of Corollary 2.5 Write $\mathcal{L}=\mathcal{L}_\Gamma$. Theorem 2.4 implies that

$$\vartheta_{\mathcal{L}}(z)=\sum_{\ell=0}^{n}\vartheta_{\mathcal{L}}^{(\ell)}(z)=\frac{1}{(1-z^q)^n}\sum_{\ell=0}^{n}(1-z^q)^\ell\sum_{s=0}^{n-\ell}2^s\binom{\ell+s}{s}z^{sq}\,\Phi_{\mathcal{L}}^{(\ell+s)}(z)$$
$$=\frac{1}{(1-z^q)^n}\sum_{\ell=0}^{n}\left(\sum_{r=0}^{\ell}\binom{\ell}{r}(-1)^r z^{rq}\right)\left(\sum_{s=0}^{n-\ell}2^s\binom{\ell+s}{s}z^{sq}\,\Phi_{\mathcal{L}}^{(\ell+s)}(z)\right).$$

The two sums over the square $\{(r,s):0\leq r\leq\ell,\ 0\leq s\leq n-\ell\}$ can be replaced by the two sums over $\{(t,s):0\leq t\leq n,\ 0\leq s\leq n\}$ by making the change of variables $t=r+s$, since $\binom{\ell}{r}=0$ for any $r>\ell$ and by letting $\Phi_{\mathcal{L}}^{(\ell)}(z)=0$ for any $\ell>n$. Hence

$$\vartheta_{\mathcal{L}}(z)=\frac{1}{(1-z^q)^n}\sum_{\ell=0}^{n}\sum_{t=0}^{n}z^{tq}\sum_{s=0}^{n}\Phi_{\mathcal{L}}^{(\ell+s)}(z)\,2^s\binom{\ell+s}{s}\binom{\ell}{t-s}(-1)^{t-s}$$
$$=\sum_{t=0}^{n}\frac{z^{tq}}{(1-z^q)^n}\sum_{s=0}^{n}2^s(-1)^{t-s}\sum_{\ell=s}^{n}\Phi_{\mathcal{L}}^{(\ell)}(z)\binom{\ell}{s}\binom{\ell-s}{t-s}.$$

We next make a new change of variables. The sums over the quadrilateral $\{(s,\ell):0\leq s\leq n,\ s\leq\ell\leq n\}$ is replaced by the sums over the triangle $\{(s,\ell):0\leq\ell\leq n,\ \ell\leq s\leq n\}$ since $\binom{\ell}{s}=0$ if $s>n\leq\ell$. Then $\vartheta_{\mathcal{L}}(z)=\sum_{t=0}^{n}\frac{z^{tq}}{(1-z^q)^n}\sum_{\ell=0}^{n}\binom{\ell}{t}\Phi_{\mathcal{L}}^{(\ell)}(z)\sum_{s=0}^{n}\binom{t}{s}2^s(-1)^{t-s}$, which concludes the proof since the last sum is equal to $(-1+2)^n=1$ and $\binom{\ell}{t}=0$ for $\ell<t$. □

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