Doctopic: Lie algebras ARTICLE IN PRESS CRASS1:6161

C. R. Acad. Sci. Paris, Ser. I ••• (••••) •••–•••

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Lie algebras

On the SO$(n+3)$ to SO(n) branching multiplicity space ☆

Sur l'espace de multiplicité de branchement de SO$(n+3)$ *vers* SO(n)

Emilio A. Lauret, Fiorela Rossi Bertone

CIEM–FaMAF (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, Argentina

A R T I C L E I N F O

Article history:
Received 25 August 2018
Accepted after revision 11 September 2018
Available online xxxx

Presented by Michèle Vergne

A B S T R A C T

We study the decomposition as an SO(3)-module of the multiplicity space corresponding to the branching from SO$(n+3)$ to SO(n). Here, SO(n) (resp. SO(3)) is considered embedded in SO$(n+3)$ in the upper left-hand block (resp. lower right-hand block). We show that when the highest weight of the irreducible representation of SO(n) interlaces the highest weight of the irreducible representation of SO$(n+3)$, then the multiplicity space decomposes as a tensor product of $\lfloor (n+2)/2 \rfloor$ reducible representations of SO(3).

R É S U M É

Nous étudions la décomposition de l'espace de multiplicité, comme SO(3)-module, correspondant au branchement de SO$(n+3)$ vers SO(n). Ici, SO(n) (resp. SO(3)) est considéré comme plongé dans SO$(n+3)$ dans le bloc en haut à gauche (resp. le bloc en bas à droite). Nous montrons que, lorsque le plus grand poids de la représentation irréductible de SO(n) s'entrelace avec le plus grand poids de la représentation irréductible de SO$(n+3)$, alors l'espace de multiplicité se décompose en un produit tensoriel de $\lfloor (n+2)/2 \rfloor$ représentations réductibles de SO(3).

1. Introduction

The branching law from a compact Lie group G to a closed subgroup K describes how an irreducible representation π of G decomposes when it is restricted to K (see [11, Ch. IX] and [5, Ch. 8] for comprehensive texts, and [10, §1] for a detailed historical review). Since

$$\pi \simeq \bigoplus_{\sigma\in\widehat{K}} \sigma \otimes \mathrm{Hom}_K(\sigma,\pi) \tag{1.1}$$

☆ This research was partially supported by grants from CONICET and Agencia Nacional de Promoción Científica y Tecnológica (PICT-2015-0274 and PICT-2014-2706). The first named author was supported by the Alexander von Humboldt Foundation.

E-mail addresses: elauret@famaf.unc.edu.ar (E.A. Lauret), rossib@famaf.unc.edu.ar (F. Rossi Bertone).

https://doi.org/10.1016/j.crma.2018.09.004
1631-073X/© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Please cite this article in press as: E.A. Lauret, F. Rossi Bertone, On the SO$(n+3)$ to SO(n) branching multiplicity space, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.09.004

as K-modules ($\widehat{K}$ denotes the unitary dual of K, and K acts on the right-hand side at the left in each term), the branching law is determined by the dimension of the *branching multiplicity space* (or just *multiplicity space*) $\mathrm{Hom}_K(\sigma,\pi)$, for each $\sigma\in\widehat{K}$. In other words, $\dim\mathrm{Hom}_K(\sigma,\pi)$ is the number of times that σ occurs in $\pi|_K$.

For d,d' positive integers, let G, K and H be given by a row in the table

G	K	H	type
$\mathrm{SO}(d+d')$	$\mathrm{SO}(d)$	$\mathrm{SO}(d')$	orthogonal
$\mathrm{U}(d+d')$	$\mathrm{U}(d)$	$\mathrm{U}(d')$	unitary
$\mathrm{Sp}(d+d')$	$\mathrm{Sp}(d)$	$\mathrm{Sp}(d')$	symplectic

(1.2)

In the sequel, we assume K (resp. H) embedded in G in the upper left-hand block (resp. lower right-hand block). The quotient $G/(K\times H)$ is a compact symmetric space called (real, complex or quaternionic) Grassmannian space.

We now examine some consequences from the fact that the subgroups K and H commute to each other. The subgroup of G generated by K and H is isomorphic to $K\times H$; we denote it by $K\times H$. Thus, any irreducible representation of $K\times H$ is given by the outer tensor product $\sigma\otimes\tau$ for some $\sigma\in\widehat{K}$ and $\tau\in\widehat{H}$. Furthermore, the branching multiplicity space $\mathrm{Hom}_K(\sigma,\pi)$ carries the structure of an H-module. The action is given by $(h\cdot\varphi)(v)=\pi(h)\cdot\varphi(v)$, for $h\in H$, $\varphi\in\mathrm{Hom}_K(\sigma,\pi)$ and v in the underlying vector space V_σ of σ. We conclude that the multiplicity of $\sigma\otimes\tau$ in $\pi|_K$ is equal to

$$\dim\mathrm{Hom}_{K\times H}(\sigma\otimes\tau,\pi)=\dim\mathrm{Hom}_H(\tau,\mathrm{Hom}_K(\sigma,\pi)). \tag{1.3}$$

Therefore, an explicit decomposition as an H-module of the multiplicity space $\mathrm{Hom}_K(\sigma,\pi)$ may be seen as a more precise branching law from G to $K\times H$.

The branching law from G to $K\times H$ is known only for specific choices of d and d'. We first review the case when $d'=1$. In the orthogonal case, $H\simeq\mathrm{SO}(1)=\{1\}$, thus the problem reduces to the classical branching from $\mathrm{SO}(d+1)$ to $\mathrm{SO}(d)$. A similar situation takes place in the unitary case where $H\simeq\mathrm{U}(1)$. Both cases can be described in terms of Gelfand–Tsetlin patterns [4] (see also [13]). In conclusion, in these cases, under standard choices for a Cartan subalgebra and positiveness in the associated root systems (e.g., as in [11] or [5]), if $\lambda=(\lambda_1,\lambda_2,\dots)$ and $\mu=(\mu_1,\mu_2,\dots)$ are the highest weights of $\pi\in\widehat{G}$ and $\sigma\in\widehat{K}$, respectively, then σ occurs in $\pi|_K$ if and only if μ *simply interlaces* λ, which roughly speaking means

$$\lambda_1\geq\mu_1\geq\lambda_2\geq\mu_2\geq\lambda_3\geq\cdots. \tag{1.4}$$

The symplectic case still assuming $d'=1$, which was established by Lepowsky [12], presents more difficulties than the previous cases. For example, it is not multiplicity-free. Moreover, the necessary condition $\mathrm{Hom}_K(\sigma,\pi)\neq0$ becomes in a *doubly interlacing* $\lambda_i\geq\mu_i\geq\lambda_{i+2}$ for all $1\leq i\leq d$ (with $\lambda_{d+2}=\lambda_{d+3}=0$) between the coefficients $(\lambda_1,\dots,\lambda_{d+1})$ and $(\mu_1,\dots,\mu_d)$ of the highest weights of π and σ respectively. Wallach and Yacobi [16] gave a clean decomposition for the multiplicity space. They proved that

$$\mathrm{Hom}_K(\sigma,\pi)\simeq\tau^{(1)}\otimes\cdots\otimes\tau^{(d+1)} \tag{1.5}$$

as H-modules, where $\tau^{(i)}$ denotes the irreducible representation of $H=\mathrm{Sp}(1)\simeq\mathrm{SU}(2)$ of dimension s_i-t_i+1, and $\{s_1\geq t_1\geq\cdots\geq s_{d+1}\geq t_{d+1}\}$ is the decreasing rearrangement of $\{\lambda_1,\dots,\lambda_{d+1},\mu_1,\dots,\mu_d,0\}$. In their proof, they extended the Clebsch–Gordan formula to an arbitrary tensor product of irreducible representations of $\mathrm{SU}(2)$ (see Theorem 2.1).

We now consider $d'=2$. In the orthogonal case, $H=\mathrm{SO}(2)$ is abelian, so its representations are one-dimensional. In conclusion, a decomposition as an H-module of the multiplicity space does not provide more information than its dimension. An implicit branching law from $G=\mathrm{SO}(d+2)$ to $K\times H=\mathrm{SO}(d)\times\mathrm{SO}(2)$ was given by Tsukamoto [14]. The implicit term refers to the fact that the number of times that an irreducible representation $\sigma\otimes\tau_k$ of $K\times H$ appears in $\pi|_{K\times H}$ is given by the k-th coefficient of certain power series, where $\tau_k(h)$ acts on $\mathbb{C}$ by multiplication by h^k for any $h\in S^1\simeq H$.

In his Ph.D. thesis, Yacobi studied the so-called *branching algebra* associated with the pair $(\mathrm{Sp}(n),\mathrm{Sp}(n-1))$ obtaining an alternative proof of (1.5) (see [17, Prop. 3.3]). Moreover, he also gave a decomposition of the branching multiplicity space from $\mathrm{U}(n)$ to $\mathrm{U}(n-2)$ as a $\mathrm{SU}(2)$-module similar to (1.5) (see [17, Prop. 3.2]). In order to describe such decomposition, let $\pi\in\widehat{G}$ and $\sigma\in\widehat{K}$ with highest weights $\lambda=(\lambda_1,\dots,\lambda_{d+2})$ and $\mu=(\mu_1,\dots,\mu_d)$, respectively, and write $\{s_1\geq t_1\geq\cdots\geq s_{d+1}\geq t_{d+1}\}$ for the decreasing rearrangement of $\{\lambda_1,\dots,\lambda_{d+2},\mu_1,\dots,\mu_d\}$. The decomposition is

$$\mathrm{Hom}_K(\sigma,\pi)\simeq\mathbb{C}\otimes\tau^{(1)}\otimes\cdots\otimes\tau^{(d+1)}, \tag{1.6}$$

where $\mathbb{C}$ is the one-dimensional representation given by $\det(h)^{\mu_1+\cdots+\mu_d}$, and $\tau^{(i)}$ denotes the (s_i-t_i+1)-dimensional representation $\mathbb{C}\otimes\mathrm{Sym}^{t_i-s_i}(\mathbb{C}^2)$ of H, with $h\in H$ acting on $\mathbb{C}$ by $\det(h)^{s_i}$ and $\mathbb{C}^2$ denotes the standard representation of H. It is important to note that (1.6) is a slightly more general decomposition due to Kim (see [8, Thm. 3.5]), since the branching multiplicity space is decomposed as an $\mathrm{U}(2)$-module. A deeper study on the branching algebra of $(\mathrm{Sp}(n),\mathrm{Sp}(n-1))$ was done in [9].

The next challenge is the case $d'=3$. The orthogonal case seems to be the simplest one since $\mathrm{SO}(3)\simeq\mathrm{SU}(2)/\{\pm1\}$ is three-dimensional, while $\mathrm{U}(3)$ and $\mathrm{Sp}(3)$ have dimensions 9 and 21, respectively.

The aim of this paper is to study, for $G = \mathrm{SO}(d+3)$, $K = \mathrm{SO}(d)$, $H = \mathrm{SO}(3)$, whether there is a clean decomposition of the G to K branching multiplicity space as an H-module as in (1.5) and (1.6). The conclusion is that this decomposition is pretty dirty.

Nevertheless, when μ simply interlaces λ, we decompose the multiplicity space $\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda)$ as a tensor product of $\lfloor \frac{d+2}{2} \rfloor$ (reducible) representations of H (Theorems 3.1 and 4.1). Generically, each factor of the tensor product is non-trivial. We recall that $\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda) \neq 0$ if and only if μ triply interlaces λ (cf. last paragraph in [11, §IX.3]).

Furthermore, Theorems 3.5 and 4.5 establish a partial decomposition of $\mathrm{Hom}_K(\sigma, \pi)$ as an H-module under certain coincidence among the coefficients of the highest weights of π and σ. Moreover, the decomposition is reduced to a simple branching law from U(3) to $H = \mathrm{SO}(3)$ by using a duality by Knapp [10].

Tsukamoto [15] showed an implicit branching law from G to $K \times H$. Similarly, as mentioned above, his result gives the multiplicity of an irreducible representation $\sigma \otimes \tau$ of $K \times H$ in $\pi|_{K\times H}$ for $\pi \in \widehat{G}$ as the coefficient of certain power series. El Chami [1][2] applied the same method to the branching laws from $\mathrm{SO}(d+d')$ to $\mathrm{SO}(d) \times \mathrm{SO}(d')$ and from $\mathrm{Sp}(d+d')$ to $\mathrm{Sp}(d) \times \mathrm{Sp}(d')$. In all cases, their main goal was to describe the spectra of the corresponding symmetric spaces.

The tools used in the proofs include Kostant's branching formula, Tsukamoto's implicit branching law from $\mathrm{SO}(d+3)$ to $\mathrm{SO}(d) \times \mathrm{SO}(3)$, and a duality of Knapp [10] between the SO(3)-representation $V_\pi^{\mathrm{SO}(d)}$ and certain canonical associated representation of U(3).

The article is organized as follows. Section 2 reviews standard facts used in the sequel. The case when $G = \mathrm{SO}(2n+3)$, $K = \mathrm{SO}(2n)$ and $H = \mathrm{SO}(3)$, called the type-B case, is considered in Section 3. Similarly, Section 4 deals with the type-D case, that is, when $G = \mathrm{SO}(2n+4)$, $K = \mathrm{SO}(2n+1)$ and $H = \mathrm{SO}(3)$.

2. Preliminaries

In this section, we introduce several tools used in the sequel, divided in subsections. Such tools are Kostant's branching formula, Wallach and Yacobi's extension of the Clebsch–Gordan rule, and standard facts on characters of compact groups and representations of U(3).

In what follows, we denote compact Lie groups by capital letters (e.g., G), their Lie algebras by the corresponding Gothic letter with the subscript 0 (e.g., $\mathfrak{g}_0$), and their complexified Lie algebras by the corresponding Gothic letter (e.g., $\mathfrak{g}$).

Furthermore, each time that a maximal torus T is fixed in a compact Lie group G, therefore a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is picked, we will use the following notation without any mention: $\Phi(\mathfrak{g}, \mathfrak{t})$ denotes the associated root system, $W_\mathfrak{g}$ the Weyl group, and $P(G)$ the weight lattice of G. Moreover, a positive system $\Phi^+(\mathfrak{g}, \mathfrak{t})$ will be assumed, unless an order on $\mathfrak{t}^*$ is explicitly chosen. In any of these cases, we denote by $P^{++}(G)$ the set of G-integral dominant weights in $\mathfrak{t}^*$ and by $\rho_\mathfrak{g}$ half of the sum of the positive roots.

2.1. Characters

We first review standard facts for characters. We refer to [11, Ch. IV–V] for further details. Let G be a compact connected semisimple Lie group, let T be a maximal torus in G, and let π be a finite-dimensional representation of G, that is, a continuous homomorphism $\pi : G \to \mathrm{GL}(V_\pi)$, where V_π is the complex finite-dimensional underlying vector space of π.

We denote by $\chi_\pi : G \to \mathbb{C}$ the character of π, that is, $\chi_\pi(g) = \mathrm{tr}(\pi(g))$. It is well known that χ_π determines π, that is, $\chi_\pi = \chi_{\pi'}$ if and only if π and π' are equivalent. It will be useful to consider χ_π as a formal power series $\sum_{\eta \in P(G)} m_\pi(\eta)\, \mathrm{e}^{\eta}$, with $m_\pi(\eta) \in \mathbb{N}_0$ the multiplicity of η in π. The identification satisfies $\chi_\pi(\exp(X)) = \sum_{\eta \in P(G)} m_\pi(\eta)\, \mathrm{e}^{\eta(X)}$ for all $X \in \mathfrak{t}_0$. Of course, $m_\pi(\eta) = 0$ for all but finitely many $\eta \in P(G)$. For $\eta \in P(G)$, we set

$$\xi_G(\eta) = \sum_{\omega \in W_\mathfrak{g}} \mathrm{sgn}(\omega)\, \mathrm{e}^{\omega(\eta)}. \tag{2.1}$$

For $\lambda \in P^{++}(G)$, let π_λ denote the irreducible representation of G with highest weight λ. The Weyl character formula ensures that

$$\chi_{\pi_\lambda} = \frac{\xi_G(\lambda + \rho_\mathfrak{g})}{\xi_G(\rho_\mathfrak{g})}. \tag{2.2}$$

Furthermore,

$$\xi_G(\rho_\mathfrak{g}) = \prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} (\mathrm{e}^{\alpha/2} - \mathrm{e}^{-\alpha/2}). \tag{2.3}$$

Let K be a closed subgroup of G. Suppose that a maximal torus S of K is contained in T. For $\beta \in \mathfrak{t}^*$, we denote by $\bar\beta$ its restriction to $\mathfrak{s}^*$. We extend this operator to the formal power series discussed above by setting $\overline{\mathrm{e}^{\eta}} = \mathrm{e}^{\bar\eta}$ for any $\eta \in P(G)$. It turns out that, for a representation π of G, the character of its restriction $\pi|_K$ to K satisfies

$$\chi_{\pi|_K} = \overline{\chi_\pi}. \tag{2.4}$$

2.2. Kostant's branching formula

We will follow [11, §IX.4] (see also [5, §8.2]). This formula is valid for a big amount of homogeneous spaces including all symmetric spaces.

Let G be a connected compact Lie group, and let K be a connected closed subgroup. We assume that the centralizer T in G of a maximal torus S of K is abelian. Thus, T is a maximal torus in G. Equivalently, there is a regular element of K that is regular in G. This allows us to introduce compatible positive systems $\Phi^+(\mathfrak{g},\mathfrak{t})$ and $\Phi^+(\mathfrak{k},\mathfrak{s})$ by defining positivity relative to a regular element in $\mathrm{i}\mathfrak{s}_0$. Set

$$\Sigma = \overline{\Phi^+(\mathfrak{g},\mathfrak{t})} \smallsetminus \Phi^+(\mathfrak{k},\mathfrak{s}). \tag{2.5}$$

More precisely, Σ is the multiset given by the elements $\overline{\alpha}$ for $\alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})$, repeated according to their multiplicity, but deleting the elements in $\Phi^+(\mathfrak{k},\mathfrak{s})$, each with multiplicity one. The Kostant partition function $\mathcal{P}_\Sigma$ is defined as follows: $\mathcal{P}_\Sigma(\nu)$ is the number of ways that a member ν of $\mathfrak{s}^*$ can be written as a sum of members of Σ, with the multiple versions of a member of Σ being regarded as distinct.

Under the notation above, for $\lambda\in P^{++}(G)$ and $\mu\in P^{++}(K)$, Kostant's branching formula tells us that the multiplicity of the irreducible representation σ_μ of K with highest weight μ in the restriction of the irreducible representation π_λ of G with highest weight λ is given by

$$\dim \operatorname{Hom}_K(\sigma_\mu,\pi_\lambda) = \sum_{w\in W_{\mathfrak{g}}} \operatorname{sgn}(\omega)\, \mathcal{P}_\Sigma(\overline{\omega(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}-\mu). \tag{2.6}$$

2.3. Generalized Clebsch–Gordan formula

We now recall the extension of the Clebsch–Gordan formula given by Wallach and Yacobi [16] for an arbitrary tensor product of irreducible representations of SU(2). It is well known that the representations of SU(2) are parametrized by non-negative integer numbers. We denote them by $\tau_{k/2}$ for $k\in\mathbb{Z}_{\geq 0}$, whose underlying vector space $V_{\tau_{k/2}}$ has dimension $k+1$.

Let $\{\varepsilon_1,\dots,\varepsilon_{n+1}\}$ denote the canonical basis of $\mathbb{C}^{n+1}$. We set

$$\Sigma' = \{\varepsilon_i \pm \varepsilon_{n+1} : 1\leq i\leq n\}. \tag{2.7}$$

For $\nu\in\mathbb{C}^{n+1}$, denote by $\mathcal{P}_{\Sigma'}(\nu)$ the number of ways of writing ν as a sum of elements in Σ',

$$\mathcal{P}_{\Sigma'}(\nu) = \#\left\{ \{a_\alpha\}_{\alpha\in\Sigma'} : a_\alpha\in\mathbb{N}_0,\ \sum_{\alpha\in\Sigma'} a_\alpha\alpha = \nu \right\}. \tag{2.8}$$

Theorem 2.1. *[16, Thm. 2.3] For non-negative integers $r_1,\dots,r_{n+1}$, the number of times that $\tau_{k/2}$ appears in $\tau_{r_1/2}\otimes\cdots\otimes\tau_{r_{n+1}/2}$ is equal to*

$$\dim \operatorname{Hom}_{\mathrm{SU}(2)}(\tau_{k/2}, \otimes_{i=1}^{n+1}\tau_{r_i/2}) = \mathcal{P}_{\Sigma'}\left(\sum_{i=1}^{n+1} r_i\varepsilon_i - k\varepsilon_{n+1}\right) - \mathcal{P}_{\Sigma'}\left(\sum_{i=1}^{n+1} r_i\varepsilon_i + (k+2)\varepsilon_{n+1}\right).$$

Write $[\![1,n]\!] = \{m\in\mathbb{Z} : 1\leq m\leq n\}$, and for any $I\subset[\![1,n]\!]$, set $\beta_I=\sum_{i\in I}\varepsilon_i$. The following elementary lemma will be very useful in the sequel.

Lemma 2.2. *We have that $\sum_{I\subset[\![1,n]\!]} \mathcal{P}_{\Sigma'}(\nu-\beta_I) = \mathcal{P}_{\Sigma'}(2\nu)$ for every $\nu\in\mathbb{C}^{n+1}$.*

Proof. Write $u_i^\pm = \varepsilon_i\pm\varepsilon_{n+1}$ for $i=1,\dots,n$. Let

$$\mathcal{A} = \left\{ (a_1,b_1,\dots,a_n,b_n)\in\mathbb{N}_0^{2n} : \sum_{i=1}^n a_iu_i^+ + b_iu_i^- = \nu-\beta_I \text{ for some } I\subset[\![1,n]\!] \right\}; \tag{2.9}$$

$$\mathcal{B} = \left\{ (c_1,d_1,\dots,c_n,d_n)\in\mathbb{N}_0^{2n} : \sum_{i=1}^n c_iu_i^+ + d_iu_i^- = 2\nu \right\}. \tag{2.10}$$

We will prove that there exists a bijective correspondence between $\mathcal{A}$ and $\mathcal{B}$.

For $(a_1,b_1,\dots,a_n,b_n)\in\mathcal{A}$, we have $a_i+b_i=\nu_i-1$ for all $i\in I$, $a_i+b_i=\nu_i$ for all $i\notin I$, and $\sum_{i=1}^n a_i-b_i=\nu_{n+1}$, for some $I\subset[\![1,n]\!]$. Define

$$c_i = \begin{cases} 2a_i + 1 & \text{if } i \in I, \\ 2a_i & \text{if } i \notin I; \end{cases} \qquad \text{and} \qquad d_i = \begin{cases} 2b_i + 1 & \text{if } i \in I, \\ 2b_i & \text{if } i \notin I. \end{cases} \tag{2.11}$$

Hence, $(c_1, d_1, \dots, c_n, d_n) \in \mathcal{B}$, since $2c_i + 2d_i = 2\nu_i$ for all i and $\sum_{i=1}^n c_i - d_i = 2\nu_{n+1}$.

Moreover, if $(c_1, d_1, \dots, c_n, d_n) \in \mathcal{B}$, then for each $i = 1, \dots, n$ we have $c_i \equiv d_i \pmod 2$. This implies that the correspondence is bijective and the lemma follows. □

2.4. Representations of U(3)

We now fix the notation to parametrize the irreducible representations of $\mathrm{U}(3) = \{g \in \mathrm{GL}(3, \mathbb{C}) : g^* g = I_3\}$. Let $T' = \{\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \mathrm{e}^{\mathrm{i}\theta_2}, \mathrm{e}^{\mathrm{i}\theta_3}) : \theta_j \in \mathbb{R}\ \forall j\}$ with associated Cartan subalgebra $\mathfrak{h}'$ of $\mathfrak{u}(3)$ given by $\mathfrak{h}' = \{\mathrm{diag}(\theta_1, \theta_2, \theta_3) : \theta_j \in \mathbb{C}\ \forall j\}$. Let $\varepsilon'_j \in (\mathfrak{h}')^*$ for $1 \le j \le 3$ given by $\varepsilon'_j(\mathrm{diag}(\theta_1, \theta_2, \theta_3)) = \theta_j$.

We consider the standard order given by the lexicographic order with respect to the ordered basis $\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$. Thus, the irreducible representations of U(3) are in correspondence with $P^{++}(\mathrm{U}(3)) = \{\sum_{j=1}^3 \lambda'_j \varepsilon'_j : a_j \in \mathbb{Z}\ \forall j,\ a_1 \ge a_2 \ge a_3\}$. For $\lambda' \in P^{++}(\mathrm{U}(3))$, let $\pi'_{\lambda'}$ denote the irreducible representation of U(3) with highest weight λ'.

3. Type-B case

Throughout this section, we set

$$G = \mathrm{SO}(2n+3), \qquad K = \mathrm{SO}(2n), \qquad H = \mathrm{SO}(3),$$

for any $n \ge 2$. We have that $\mathfrak{g} = \mathfrak{so}(2n+3, \mathbb{C})$ is a classical Lie algebra of type B_{n+1}.

3.1. Root system notation for type-B case

We first fix compatible notation for the corresponding root systems associated with G, K, H, and $K \times H$. We pick the maximal torus of G given by

$$T := \{\mathrm{diag}(R(\theta_1), \dots, R(\theta_{n+1}), 1) : \theta_j \in \mathbb{R}\ \forall j\}, \tag{3.1}$$

where $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, whose the associated Cartan subalgebra is given by

$$\mathfrak{t} := \left\{ \mathrm{diag}\left(\begin{pmatrix} 0 & \mathrm{i}\theta_1 \\ -\mathrm{i}\theta_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \mathrm{i}\theta_{n+1} \\ \mathrm{i}\theta_{n+1} & 0 \end{pmatrix}, 0 \right) : \theta_j \in \mathbb{C}\ \forall j \right\}. \tag{3.2}$$

For any $1 \le j \le n+1$, let $\varepsilon_j \in \mathfrak{t}^*$ given by $\varepsilon_j(X) = \theta_j$ for X in $\mathfrak{t}$ as above. Then, the set of roots is $\Phi(\mathfrak{g}, \mathfrak{t}) = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n+1\} \cup \{\pm\varepsilon_j : 1 \le j \le n+1\}$.

The maximal torus $T \cap K$ of K satisfies $(\mathfrak{k} \cap \mathfrak{t})^* = \mathrm{span}_{\mathbb{C}}\{\varepsilon_1, \dots, \varepsilon_n\}$, thus $\Phi(\mathfrak{k}, \mathfrak{t}) = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\}$. Similarly, $T \cap H$ is a maximal torus in H satisfying that $(\mathfrak{h} \cap \mathfrak{t})^* = \mathrm{span}_{\mathbb{C}}\{\varepsilon_{n+1}\}$, thus $\Phi(\mathfrak{h}, \mathfrak{t}) = \{\pm\varepsilon_{n+1}\}$.

We fix the order on $\mathfrak{t}^*$ given by the lexicographic order with respect to the ordered basis $\{\varepsilon_1, \dots, \varepsilon_{n+1}\}$. We have compatible order on $(\mathfrak{k} \cap \mathfrak{t})^*$ and $(\mathfrak{h} \cap \mathfrak{t})^*$. Then

$$\Phi^+(\mathfrak{g}, \mathfrak{t}) = \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n+1\} \cup \{\varepsilon_j : 1 \le j \le n+1\}, \tag{3.3}$$

$$\Phi^+(\mathfrak{k}, \mathfrak{t}) = \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\}, \qquad \Phi^+(\mathfrak{h}, \mathfrak{t}) = \{\varepsilon_{n+1}\}, \tag{3.4}$$

$$P(G) = \oplus_{j=1}^{n+1} \mathbb{Z}\varepsilon_j, \qquad P^{++}(G) = \{\textstyle\sum_{j=1}^{n+1} \lambda_j \varepsilon_j \in P(G) : \lambda_1 \ge \dots \ge \lambda_{n+1} \ge 0\}, \tag{3.5}$$

$$P(K) = \oplus_{j=1}^{n} \mathbb{Z}\varepsilon_j, \qquad P^{++}(K) = \{\textstyle\sum_{j=1}^{n} \lambda_j \varepsilon_j \in P(K) : \lambda_1 \ge \dots \ge \lambda_{n-1} \ge |\lambda_n|\}, \tag{3.6}$$

$$P(H) = \mathbb{Z}\varepsilon_{n+1}, \qquad P^{++}(H) = \{k\varepsilon_{n+1} \in P(H) : k \ge 0\}, \tag{3.7}$$

$$\rho_{\mathfrak{g}} := \sum_{i=1}^{n+1} (n + \tfrac{3}{2} - \mathrm{i})\varepsilon_i, \qquad \rho_{\mathfrak{k}} := \sum_{i=1}^{n} (n - \mathrm{i})\varepsilon_i, \qquad \rho_{\mathfrak{h}} := \tfrac{1}{2}\varepsilon_{n+2}. \tag{3.8}$$

We will denote by $\pi_\lambda, \sigma_\mu, \tau_{k\varepsilon_{n+1}}$ the irreducible representations of G, K and H with highest weights $\lambda \in P^{++}(G)$, $\mu \in P^{++}(K)$, and $k\varepsilon_{n+1} \in P^{++}(H)$, respectively. We will abbreviate $\tau_k = \tau_{k\varepsilon_{n+1}}$.

We now describe the Weyl group $W_{\mathfrak{g}}$. Any element $\omega \in W_{\mathfrak{g}}$ is of the form $\omega = sp$, with p a permutation of the $n+1$ coordinates and s a multiplication by -1 on a subset of coordinates. For $1 \le i \le n+1$, write $s_i : \mathfrak{t}^* \to \mathfrak{t}^*$ the reflexion with respect to the axis i, that is, $s_i(\varepsilon_i) = -\varepsilon_i$ and $s_i(\varepsilon_j) = \varepsilon_j$ for all $j \ne i$.

We consider the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{g}$ given by $\langle X, Y \rangle = \frac{1}{2}\,\mathrm{tr}(XY)$. With respect to its extension to $\mathfrak{t}^*$, $\{\varepsilon_1, \dots, \varepsilon_{n+1}\}$ is an orthonormal basis.

3.2. Main theorem for the type-B case

The main result in this section is the following.

Theorem 3.1. *Let $n \geq 2$, $G = \mathrm{SO}(2n+3)$, $K = \mathrm{SO}(2n)$, $H = \mathrm{SO}(3)$, $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \in P^{++}(G)$, $\mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \in P^{++}(K)$. If μ simply interlaces λ, i.e. $\lambda_i \geq |\mu_i| \geq \lambda_{i+1}$ for $1 \leq i \leq n$, then*

$$\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda) \simeq \tau_{\lambda_{n+1}} \otimes \bigotimes_{j=1}^{n} \left(\bigoplus_{m=0}^{\lfloor (\lambda_j - |\mu_j|)/2 \rfloor} \tau_{\lambda_j - |\mu_j| - 2m} \right) \tag{3.9}$$

as H-modules.

The absolute value on every μ_i simplifies the notation, though $|\mu_i| = \mu_i$ for all $i < n$.

Kostant's Branching Formula 2.6 will be the main tool to prove Theorem 3.1. We next give the first steps to apply it to the symmetric space $G/(K \times H)$. The maximal torus T in G (see (3.1)) is also a maximal torus in $K \times H$. In particular, the restriction denoted by a bar is the identity operator on $\mathfrak{t}^*$. Furthermore, $\Phi^+(\mathfrak{k} \times \mathfrak{h}, \mathfrak{t}) = \Phi^+(\mathfrak{k}, \mathfrak{t} \cap \mathfrak{k}) \cup \Phi^+(\mathfrak{h}, \mathfrak{t} \cap \mathfrak{h})$. From (3.3) and (3.4), it follows that

$$\Sigma = \{\varepsilon_i \pm \varepsilon_{n+1} : 1 \leq i \leq n\} \cup \{\varepsilon_i : 1 \leq i \leq n\}, \tag{3.10}$$

each element with multiplicity one. Clearly, for $\nu \in \mathfrak{t}^*$,

$$\mathcal{P}_\Sigma(\nu) > 0 \quad \Longrightarrow \quad \nu_i := \langle \varepsilon_i, \nu \rangle \in \mathbb{Z} \quad \forall i, \ \nu_i \geq 0 \quad \forall 1 \leq i \leq n, \ |\nu_{n+1}| \leq \sum_{i=1}^{n} \nu_i. \tag{3.11}$$

Let $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \in P^{++}(G)$, $\mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \in P^{++}(K)$, and $k\varepsilon_{n+1} \in P^{++}(H)$. We obtain from (2.6) that the number of times that $\sigma_\mu \otimes \tau_k \in \widehat{K \times H}$ occurs in $\pi_\lambda|_{K \times H}$ is given by

$$\dim \mathrm{Hom}_{K \times H}(\sigma_\mu \otimes \tau_k, \pi_\lambda) = \sum_{\omega \in W_\mathfrak{g}} \mathrm{sgn}(\omega)\, \mathcal{P}_\Sigma\big(\overline{\omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g}} - \mu - k\varepsilon_{n+1}\big). \tag{3.12}$$

The next lemmas will indicate the set of elements $\omega \in W_\mathfrak{g}$ such that the ω-th term in the above formula does not vanish.

Lemma 3.2. *Assume $\mu_n \geq 0$. If $\omega \in W_\mathfrak{g}$ satisfies that the ω-th term in (3.12) is non-zero, then $\omega = p$ or $\omega = s_{n+1}p$, for some permutation p.*

Proof. Write $\omega = sp$ as above. Suppose that $s(\varepsilon_j) = -\varepsilon_j$ for some $1 \leq j \leq n$. Since $\mathcal{P}_\Sigma(\omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g} - \mu - k\varepsilon_{n+1}) > 0$ by assumption, (3.11) forces $0 \leq \langle \varepsilon_j, \omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g} - \mu - k\varepsilon_{n+1} \rangle$, so $0 < \langle \varepsilon_j, \rho_\mathfrak{g} + \mu + k\varepsilon_{n+1} \rangle \leq \langle \varepsilon_j, \omega(\lambda + \rho_\mathfrak{g}) \rangle = \langle \varepsilon_j, sp(\lambda + \rho_\mathfrak{g}) \rangle = -\langle \varepsilon_j, p(\lambda + \rho_\mathfrak{g}) \rangle \leq 0$, which is a contradiction. Hence, $s(\varepsilon_j) = \varepsilon_j$ for all $1 \leq j \leq n$ and the claim follows. □

Lemma 3.3. *Assume $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n$. If $w \in W_\mathfrak{g}$ satisfies that the ω-th term in (3.12) is non-zero, then $\omega = 1$ or $\omega = s_{n+1}$.*

Proof. By Lemma 3.2, we may assume that $w = sp$ with p a permutation and $s = s_{n+1}$ or $s = 1$. Let l be an index distinct to $n+1$ and let r be the index such that $p(\varepsilon_r) = \varepsilon_l$. Since $\mathcal{P}_\Sigma(\omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g} - \mu - k\varepsilon_{n+1}) > 0$ by assumption, (3.11) gives $0 \leq \langle \varepsilon_l, \omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g} - \mu - k\varepsilon_{n+1} \rangle = \langle \varepsilon_r, \lambda + \rho_\mathfrak{g} \rangle - \langle \varepsilon_l, \rho_\mathfrak{g} + \mu + k\varepsilon_{n+1} \rangle = \lambda_r - \mu_l + (l - r)$. We conclude that $\lambda_r \geq \mu_l + r - l$.

If $l = 1$, then $r = 1$. Indeed, if $r \geq 2$, then $\lambda_2 \geq \lambda_r \geq \mu_1 + r - 1 > \lambda_2$ by assumption, which is a contradiction. Suppose $l \leq n-1$ and $\omega(\varepsilon_i) = \varepsilon_i$ for all $1 \leq i < l$, and let r be an index such that $\omega(\varepsilon_r) = \varepsilon_l$. Clearly, $r \geq l$. If $r \geq l+1$, then $\lambda_{l+1} \geq \lambda_r \geq \mu_l + r - l > \mu_l \geq \lambda_{l+1}$ by assumption, which is again a contradiction. We conclude that $r = l$. We have shown that $\omega(\varepsilon_i) = \varepsilon_i$ for all $1 \leq i \leq n-1$. In other words, $\omega = s$ or $\omega = sp_{n,n+1}$ where $p_{n,n+1}$ preserves ε_i for all $1 \leq i \leq n-1$ and switches ε_n and ε_{n+1}. If $\omega = sp_{n,n+1}$, then $0 \leq \langle \varepsilon_n, \omega(\lambda + \rho_\mathfrak{g}) - \rho_\mathfrak{g} - \mu - k\varepsilon_{n+1} \rangle = \lambda_{n+1} - 1 - \mu_n$ which is impossible when $\mu_n \geq 0$ because $|\mu_n| \geq \lambda_{n+1}$. This completes the proof. □

We now show that it is sufficient to prove Theorem 3.1 for $\mu_n \geq 0$. We set $\widetilde{\mu} = \sum_{i=1}^{n-1} \mu_i \varepsilon_i - \mu_n \varepsilon_n$ for any $\mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \in P^{++}(K)$. Note $\widetilde{\mu} \in P^{++}(K)$.

Lemma 3.4. *For any $\lambda \in P^{++}(G)$, $\mu \in P^{++}(K)$, $k \geq 0$, $\sigma_\mu \otimes \tau_k$ and $\sigma_{\widetilde{\mu}} \otimes \tau_k$ occur in $\pi_\lambda|_{K \times H}$ the same number of times.*

Proof. We set $g_0 \in \mathrm{diag}(1,\dots,1,-1,1,1,1) \in \mathrm{O}(2n+3)$. Although g_0 is not in G, the map $\varphi : x \mapsto g_0 x g_0$ is an automorphism of G. It turns out that $\pi_\lambda \circ \varphi \simeq \pi_\lambda$, $\sigma_\mu \circ \varphi|_K \simeq \sigma_{\widetilde{\mu}}$, and $\tau_k \circ \varphi|_H = \tau_k$. It follows immediately that $\mathrm{Hom}_{K\times H}(\sigma_\mu \otimes \tau_k, \pi_\lambda) \simeq \mathrm{Hom}_{K\times H}(\sigma_{\widetilde{\mu}} \otimes \tau_k, \pi_\lambda)$ as complex vector spaces, as asserted. □

We are now in position to prove the main theorem of this section.

Proof of Theorem 3.1. To establish the isomorphism of H-modules in (3.9), we will show that each irreducible representation τ_k of H occurs in both sides with the same multiplicity. For the right-hand side, we will use Theorem 2.1. For the left-hand side, we use that

$$\dim \mathrm{Hom}_H(\tau_k, \mathrm{Hom}_K(\sigma_\mu, \pi_\lambda)) = \dim \mathrm{Hom}_{K\times H}(\sigma_\mu \otimes \tau_k, \pi_\lambda), \tag{3.13}$$

and then we apply Kostant's branching formula to $G/(K \times H)$. Note that (3.13) and Lemma 3.4 allow us to assume $\mu_n \geq 0$ for the rest of the proof.

Lemmas 3.2 and 3.3 tell us that the only non-zero terms in (3.12) are given by $\omega = 1$ and $\omega = s_{n+1}$. The term corresponding to the last choice is equal to

$$\begin{aligned}\mathcal{P}_\Sigma\big(\overline{s_{n+1}(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}-\mu-k\varepsilon_{n+1}\big) &= \mathcal{P}_\Sigma\Big(\sum_{i=1}^n \lambda_i\varepsilon_i - \mu - (\lambda_{n+1}+k+1)\varepsilon_{n+1}\Big)\\ &= \mathcal{P}_\Sigma\Big(s_{n+1}\big(\sum_{i=1}^n \lambda_i\varepsilon_i - \mu + (\lambda_{n+1}+k+1)\varepsilon_{n+1}\big)\Big) = \mathcal{P}_\Sigma\big(\lambda-\mu+(k+1)\varepsilon_{n+1}\big).\end{aligned} \tag{3.14}$$

The last identity follows by $\mathcal{P}_\Sigma(s_{n+1}\nu) = \mathcal{P}_\Sigma(\nu)$ for all ν. We have established so far that

$$\dim \mathrm{Hom}_H(\tau_k, \mathrm{Hom}_K(\sigma_\mu, \pi_\lambda)) = \mathcal{P}_\Sigma(\lambda-\mu-k\varepsilon_{n+1}) - \mathcal{P}_\Sigma(\lambda-\mu+(k+1)\varepsilon_{n+1}). \tag{3.15}$$

Since $\Sigma' = \{\varepsilon_i \pm \varepsilon_{n+1} : 1 \leq i \leq n\} \subset \Sigma$, it follows that the previous expression becomes

$$\begin{aligned}&\sum_{\nu_1=0}^{\lambda_1-\mu_1}\cdots\sum_{\nu_n=0}^{\lambda_n-\mu_n}\Big(\mathcal{P}_{\Sigma'}(\lambda-\mu-k\varepsilon_{n+1}-\nu) - \mathcal{P}_{\Sigma'}(\lambda-\mu+(k+1)\varepsilon_{n+1}-\nu)\Big)\\ &\quad= \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor}\cdots\sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor}\Bigg(\sum_{I\subset[\![1,n]\!]}\mathcal{P}_{\Sigma'}(\lambda-\mu-k\varepsilon_{n+1}-2\gamma-\beta_I)\\ &\qquad\qquad - \sum_{I\subset[\![1,n]\!]}\mathcal{P}_{\Sigma'}(\lambda-\mu+(k+1)\varepsilon_{n+1}-2\gamma-\beta_I)\Bigg).\end{aligned} \tag{3.16}$$

Here and subsequently, we write $\nu = \sum_{i=1}^n \nu_i\varepsilon_i$ and $\gamma = \sum_{i=1}^n \gamma_i\varepsilon_i$, whenever ν_i and γ_i for $1 \leq i \leq n$ are determined. The notation for β_I was introduced before Lemma 2.2. Such lemma implies

$$\begin{aligned}&\dim \mathrm{Hom}_H(\tau_k, \mathrm{Hom}_K(\sigma_\mu, \pi_\lambda))\\ &\quad= \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor}\cdots\sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor}\Big(\mathcal{P}_{\Sigma'}\big(2(\lambda-\mu-k\varepsilon_{n+1}-2\gamma)\big) - \mathcal{P}_{\Sigma'}\big(2(\lambda-\mu+(k+1)\varepsilon_{n+1}-2\gamma)\big)\Big).\end{aligned} \tag{3.17}$$

On the other hand, the right-hand side of (3.9) is

$$\bigoplus_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor}\cdots\bigoplus_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor}\tau_{\lambda_1-\mu_1-2\gamma_1}\otimes\cdots\otimes\tau_{\lambda_n-\mu_n-2\gamma_n}\otimes\tau_{\lambda_{n+1}}. \tag{3.18}$$

One has that $H = \mathrm{SO}(3) \simeq \mathrm{SU}(2)/\{\pm 1\}$. The irreducible representation $\tau_{k/2}$ of $\mathrm{SU}(2)$ with k odd does not descend to a representation of $\mathrm{SO}(3)$. Furthermore, the $(2k+1)$-dimensional representation τ_k of $\mathrm{SU}(2)$ descends to $\mathrm{SO}(3)$. Therefore, Theorem 2.1 ensures that the number of times that τ_k occurs in the factor $\tau_{\lambda_1-\mu_1-2\gamma_1}\otimes\cdots\otimes\tau_{\lambda_n-\mu_n-2\gamma_n}\otimes\tau_{\lambda_{n+1}}$ is equal to

$$\mathcal{P}_{\Sigma'}\left(2(\lambda-\mu-2\gamma)-2k\varepsilon_{n+1}\right) - \mathcal{P}_{\Sigma'}\left(2(\lambda-\mu-2\gamma)+(2k+2)\varepsilon_{n+1}\right). \tag{3.19}$$

It follows that the number of times that τ_k occurs in (3.18) coincides with (3.17), as asserted. □

3.3. Prescribed highest weight end for the type-B case

The aim in this subsection is to prove Theorem 3.5. Roughly speaking, it is a decomposition of the multiplicity space $\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda)$ as H-module when the ending coefficients of λ coincide with a part of the coefficients of μ.

Theorem 3.5. *Let $G = \mathrm{SO}(2n+3)$, $K = \mathrm{SO}(2n)$, and $H = \mathrm{SO}(3)$ for any $n \geq 2$. For $\mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \in P^{++}(K)$ and $\lambda = \sum_{i=1}^{n+1} \lambda_i \varepsilon_i \in P^{++}(G)$ with $\lambda_{i+3} = \mu_i$ for all $1 \leq i \leq n-2$ and $\mu_{n-1} \leq \lambda_{n+1}$, we have that*

$$\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda) \simeq \pi'_{\lambda'}|_H \otimes \left(\bigoplus_{j=|\mu_n|}^{\mu_{n-1}} \tau_j \right) \tag{3.20}$$

as H-modules, where $\pi'_{\lambda'}$ denotes the irreducible representation of $\mathrm{U}(3)$ with highest weight $\lambda' := \lambda_1 \varepsilon'_1 + \lambda_2 \varepsilon'_2 + \lambda_3 \varepsilon'_3$ (see Subsection 2.4).

We note that the condition $\mu_{n-1} \leq \lambda_{n+1}$ is redundant unless $n = 2$.

It turns out that Kostant's branching formula is not adequate to prove this result, since the number of non-zero terms in the formula (3.12) is too high. We will use the branching law from G to $K \times H$ given by Tsukamoto [15]. We first recall such a result.

Fix $\lambda \in P^{++}(G)$. It will be convenient to set $\lambda_{n+2} = \lambda_{n+3} = 0$ and $a_0 = \mu_0 = \lambda_1$. Furthermore, for $\mu \in P^{++}(K)$, let $\widetilde{\sigma}_\mu$ denote the representation of K given by $\widetilde{\sigma}_\mu = \sigma_\mu$ if $\mu_n = 0$ and $\widetilde{\sigma}_\mu = \sigma_\mu \oplus \sigma_{\bar{\mu}}$ otherwise.

Tsukamoto established in the proof of Theorem 1 in [15] that

$$\chi_{\pi_\lambda|_{K\times H}} = \frac{1}{(\mathrm{e}^{\frac{1}{2}\varepsilon_{n+1}} - \mathrm{e}^{-\frac{1}{2}\varepsilon_{n+1}})} \sum_{\mu} \chi_{\widetilde{\sigma}_\mu} \sum_{(a_1,\dots,a_n)} \frac{\prod_{i=1}^{n+1} (\mathrm{e}^{l_i \varepsilon_{n+1}} - \mathrm{e}^{-l_i \varepsilon_{n+1}})}{(\mathrm{e}^{\varepsilon_{n+1}} - \mathrm{e}^{-\varepsilon_{n+1}})^n}, \tag{3.21}$$

where the first sum is over every $\mu \in P^{++}(K)$ *triply interlacing* λ, that is, $\lambda_i \geq \mu_i \geq \lambda_{i+3}$ for all $1 \leq i \leq n$, the second sum is over the n-tuples $(a_1, \dots, a_n) \in \mathbb{Z}^n$ satisfying that $a_1 \geq \dots \geq a_n \geq 0$ and $\max(\mu_i, \lambda_{i+2}) \leq a_i \leq \min(\mu_{i-1}, \lambda_i)$ for all $1 \leq i \leq n$, and the parameters $l_1, \dots, l_{n+1}$ are given by

$$\begin{cases} l_i = \min(\lambda_i, a_{i-1}) - \max(\lambda_{i+1}, a_i) + 1 & \text{for all } 1 \leq i \leq n, \\ l_{n+1} = \min(\lambda_{n+1}, a_n) + 1/2. \end{cases} \tag{3.22}$$

Since $\chi_{\tau_k} = \xi_H(k\varepsilon_{n+1} + \rho_{\mathfrak{h}})/\xi_H(\rho_{\mathfrak{h}})$ by (2.2), with $\xi_H(k\varepsilon_{n+1} + \rho_{\mathfrak{h}}) = \xi_H((k + \frac{1}{2})\varepsilon_{n+1}) = \mathrm{e}^{(k+\frac{1}{2})\varepsilon_{n+1}} - \mathrm{e}^{-(k+\frac{1}{2})\varepsilon_{n+1}}$ and $\xi_H(\rho_{\mathfrak{h}}) = \mathrm{e}^{\frac{1}{2}\varepsilon_{n+1}} - \mathrm{e}^{-\frac{1}{2}\varepsilon_{n+1}}$, Tsukamoto thus obtained the following implicit branching rule from $G = \mathrm{SO}(2n+3)$ to $K \times H = \mathrm{SO}(2n) \times \mathrm{SO}(3)$.

Theorem 3.6. *[15, Theorem 1] Let $\lambda \in P^{++}(G)$, $\mu \in P^{++}(K)$, and $k\varepsilon_{n+1} \in P^{++}(H)$. If $\lambda_i \geq \mu_i \geq \lambda_{i+3}$ for all $1 \leq i \leq n-1$ and $\lambda_n \geq |\mu_n|$, then the number of times that $\sigma_\mu \otimes \tau_k$ occurs in $\pi_\lambda|_{K\times H}$ is given by m_k, where the coefficients m_p for $p \geq 0$ are defined by*

$$\sum_{(a_1,\dots,a_n)} \frac{\prod_{i=1}^{n+1} (\mathrm{e}^{l_i \varepsilon_{n+1}} - \mathrm{e}^{-l_i \varepsilon_{n+1}})}{(\mathrm{e}^{\varepsilon_{n+1}} - \mathrm{e}^{-\varepsilon_{n+1}})^n} = \sum_{p \geq 0} m_p \left(\mathrm{e}^{(p+\frac{1}{2})\varepsilon_{n+1}} - \mathrm{e}^{-(p+\frac{1}{2})\varepsilon_{n+1}} \right), \tag{3.23}$$

where the sum at the left is over the n-tuples $(a_1, \dots, a_n) \in \mathbb{Z}^n$ satisfying that $a_1 \geq \dots \geq a_n \geq 0$, $\max(\mu_i, \lambda_{i+2}) \leq a_i \leq \min(\mu_{i-1}, \lambda_i)$ for all $1 \leq i \leq n-1$, $\max(|\mu_n|, \lambda_{n+2}) \leq a_n \leq \min(\mu_{n-1}, \lambda_n)$, and $l_1, \dots, l_{n+1}$ are given by (3.22). Otherwise, $\sigma_\mu \otimes \tau_k$ does not occur in $\pi_\lambda|_{K\times H}$.

Proof of Theorem 3.5. We will assume throughout the proof that $\mu_n \geq 0$, which is possible by Lemma 3.4. We will first show that

$$\mathrm{Hom}_K(\sigma_\mu, \pi_\lambda) \simeq \mathrm{Hom}_K(\sigma_0, \pi_{\lambda'}) \otimes \left(\bigoplus_{j=\mu_n}^{\mu_{n-1}} \tau_j \right) \tag{3.24}$$

as H-modules, where $\lambda' = \lambda_1 \varepsilon_1 + \lambda_2 \varepsilon_2 + \lambda_3 \varepsilon_3$. To do that, we will check that the term accompanying χ_{σ_μ} in χ_{π_λ} coincides with the term accompanying χ_{σ_0} in $\chi_{\pi_{\lambda'}}$ times $\sum_{j=\mu_n}^{\mu_{n-1}} \chi_{\tau_j}$.

From (3.21), the term accompanying χ_{σ_μ} in χ_{π_λ} equals

$$\frac{1}{\xi_H(\rho_{\mathfrak h})}\sum_{(a_1,\dots,a_n)}\frac{\prod_{i=1}^{n+1}(e^{l_i\varepsilon_{n+1}}-e^{-l_i\varepsilon_{n+1}})}{(e^{\varepsilon_{n+1}}-e^{-\varepsilon_{n+1}})^n},\tag{3.25}$$

where the sum is over $(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfying that $a_1\geq\cdots\geq a_n\geq 0$ and $\max(\mu_i,\lambda_{i+2})\leq a_i\leq\min(\mu_{i-1},\lambda_i)$ for all $1\leq i\leq n$, and the parameters $l_1,\dots,l_{n+1}$ are given by (3.22). By assumption, $\lambda_{i+3}=\mu_i$ for all $1\leq i\leq n-2$. This implies that the sum reduces to $(a_1,\dots,a_n)\in\mathbb{Z}^n$ satisfying that $\lambda_3\leq a_1\leq\lambda_1$, $\mu_n\leq a_n\leq\mu_{n-1}$, and $a_i=\lambda_{i+2}$ for all $1\leq i\leq n-1$, thus $l_1=\lambda_1+1-\max(\lambda_2,a_1)$, $l_2=\min(\lambda_2,a_1)-\lambda_3+1$, $l_i=1$ for all $3\leq i\leq n$, $l_{n+1}=a_n+1/2$. Hence, the previous expression becomes

$$\sum_{a_1=\lambda_3}^{\lambda_1}\frac{(e^{l_1\varepsilon_{n+1}}-e^{-l_1\varepsilon_{n+1}})(e^{l_2\varepsilon_{n+1}}-e^{-l_2\varepsilon_{n+1}})}{(e^{\varepsilon_{n+1}}-e^{-\varepsilon_{n+1}})^2}\sum_{a_n=\mu_n}^{\mu_{n-1}}\frac{(e^{l_{n+1}\varepsilon_{n+1}}-e^{-l_{n+1}\varepsilon_{n+1}})}{\xi_H(\rho_{\mathfrak h})}$$
$$=\left(\sum_{a_1=\lambda_3}^{\lambda_1}\frac{(e^{l_1\varepsilon_{n+1}}-e^{-l_1\varepsilon_{n+1}})(e^{l_2\varepsilon_{n+1}}-e^{-l_2\varepsilon_{n+1}})}{(e^{\varepsilon_{n+1}}-e^{-\varepsilon_{n+1}})^2}\right)\left(\sum_{a_n=\mu_n}^{\mu_{n-1}}\chi_{\tau_{a_n}}\right),\tag{3.26}$$

where $l_1=\lambda_1+1-\max(\lambda_2,a_1)$ and $l_2=\min(\lambda_2,a_1)-\lambda_3+1$. We conclude that the proof of (3.24) follows by checking that the second term in the right-hand side of the last expression coincides with the term accompanying χ_{σ_0} in $\chi_{\pi_{\lambda'}}$. This can be easily checked by (3.21) since $\lambda'_{i+3}=\mu_i=0$ for all $i\geq 1$. Indeed, one obtains $\lambda_3\leq a_1\leq\lambda_1$, $a_i=0$ for all $2\leq i\leq n$, $l_1=\lambda_1+1-\max(\lambda_2,a_1)$, $l_2=\min(\lambda_2,a_1)-\lambda_3+1$, $l_i=1$ for all $3\leq i\leq n$, and $l_{n+1}=1/2$.

He has established so far the identity (3.24). Now, by using Knapp's duality [10], the multiplicity space $\mathrm{Hom}_K(\sigma_0,\pi_{\lambda'})$ is isomorphic as an H-module to the restriction of the representation $\pi'_{\lambda'}$ of U(3) to $H=\mathrm{SO}(3)$. This completes the proof. □

Remark 3.7. The branching law from U(3) to SO(3) has been thoroughly studied. It states that (see for instance [3, (1.18)]) the number of times that τ_k occurs in $\pi'_{\lambda'}|_{\mathrm{SO}(3)}$ with $\lambda'=\sum_{j=1}^{3}\lambda'_j\varepsilon'_j$, is given by

$$\begin{cases}0 & \text{if } 0\leq p\leq k-1,\\ \lceil\frac{p-k+1}{2}\rceil-\lceil\frac{p-k-q}{2}\rceil & \text{if } k\leq p\leq 2k,\quad 0\leq q\leq p-k,\\ \lceil\frac{p-k+1}{2}\rceil & \text{if } k\leq p\leq 2k,\quad p-k\leq q\leq k,\\ \lceil\frac{p-k+1}{2}\rceil-\lceil\frac{q-k}{2}\rceil & \text{if } k\leq p\leq 2k,\quad k\leq q,\\ \lceil\frac{p-k+1}{2}\rceil-\lceil\frac{p-k-q}{2}\rceil & \text{if } 2k\leq p,\quad 0\leq q\leq k,\\ \lceil\frac{p-k+1}{2}\rceil-\lceil\frac{p-k-q}{2}\rceil-\lceil\frac{q-k}{2}\rceil & \text{if } 2k\leq p,\quad k\leq q\leq p-k,\\ \lceil\frac{p-k+1}{2}\rceil-\lceil\frac{q-k}{2}\rceil & \text{if } 2k\leq p,\quad p-k\leq q,\end{cases}\tag{3.27}$$

where $p=\lambda'_1-\lambda'_3$, $q=\lambda'_2-\lambda'_3$, and $\lceil\cdot\rceil$ stands for the ceiling function (i.e. $\lceil x\rceil$ is the least integer greater than or equal to x). Now, Theorem 3.5 and (3.27) give an explicit expression for the number of times that $\sigma_\mu\otimes\tau_k$ occurs in $\pi_\lambda|_{K\times H}$ for λ and μ as in Theorem 3.5.

Remark 3.8. Knapp's duality [10] was already present in [6] (cf. second-to-last paragraph in §2 of [7]).

4. Type-D case

We consider in this section the case $d=2n+1$, thus for any $n\geq 1$, we set

$$G=\mathrm{SO}(2n+4),\qquad K=\mathrm{SO}(2n+1),\qquad H=\mathrm{SO}(3).$$

The Lie algebra $\mathfrak g=\mathfrak{so}(2n+4,\mathbb{C})$ is a classical Lie algebra of type D_{n+2}. There are several similarities with the previous case considered in Section 3, so we will omit many details.

4.1. Root system notation for the type-D case

We pick the maximal torus

$$T:=\{\mathrm{diag}(R(\theta_1),\dots,R(\theta_{n+2})):\theta_j\in\mathbb{R}\ \forall j\},\tag{4.1}$$
$$\mathfrak t:=\left\{\mathrm{diag}\left(\begin{pmatrix}0 & \mathrm{i}\theta_1\\ -\mathrm{i}\theta_1 & 0\end{pmatrix},\dots,\begin{pmatrix}0 & \mathrm{i}\theta_{n+2}\\ \mathrm{i}\theta_{n+2} & 0\end{pmatrix}\right):\theta_j\in\mathbb{C}\ \forall j\right\}.\tag{4.2}$$

We define $\varepsilon_j\in\mathfrak t^*$ by $\varepsilon_j(X)=\theta_j$ for X in $\mathfrak t$ as above, for $1\leq j\leq n+2$. Then, $\Phi(\mathfrak g,\mathfrak t)=\{\pm\varepsilon_i\pm\varepsilon_j:1\leq i<j\leq n+2\}$.

Please cite this article in press as: E.A. Lauret, F. Rossi Bertone, On the SO(n + 3) to SO(n) branching multiplicity space, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.09.004

The maximal torus $T\cap K$ of K satisfy $(\mathfrak{k}\cap\mathfrak{t})^*=\operatorname{span}_{\mathbb{C}}\{\varepsilon_1,\dots,\varepsilon_n\}$, thus $\Phi(\mathfrak{k},\mathfrak{t})=\{\pm\varepsilon_i\pm\varepsilon_j:1\le i<j\le n\}\cup\{\pm\varepsilon_i:1\le i\le n\}$. Similarly, the maximal torus $T\cap H$ in H satisfies that $(\mathfrak{h}\cap\mathfrak{t})^*=\operatorname{span}_{\mathbb{C}}\{\varepsilon_{n+2}\}$ and $\Phi(\mathfrak{h},\mathfrak{t})=\{\pm\varepsilon_{n+2}\}$.

We pick compatible orders in $\mathfrak{t}^*$, $(\mathfrak{k}\cap\mathfrak{t})^*$ and $(\mathfrak{h}\cap\mathfrak{t})^*$, determined by the lexicographic order with respect to the ordered basis $\{\varepsilon_1,\dots,\varepsilon_{n+2}\}$. Thus

$$\Phi^+(\mathfrak{g},\mathfrak{t})=\{\varepsilon_i\pm\varepsilon_j:1\le i<j\le n+2\},\qquad \Phi^+(\mathfrak{h},\mathfrak{t})=\{\varepsilon_{n+2}\},\tag{4.3}$$

$$\Phi^+(\mathfrak{k},\mathfrak{t})=\{\varepsilon_i\pm\varepsilon_j:1\le i<j\le n\}\cup\{\varepsilon_j:1\le j\le n\},\tag{4.4}$$

$$P(G)=\oplus_{j=1}^{n+2}\mathbb{Z}\varepsilon_j,\qquad P^{++}(G)=\{\textstyle\sum_{j=1}^{n+2}\lambda_j\varepsilon_j\in P(G):\lambda_1\ge\dots\ge\lambda_{n+1}\ge|\lambda_{n+2}|\},\tag{4.5}$$

$$P(K)=\oplus_{j=1}^{n}\mathbb{Z}\varepsilon_j,\qquad P^{++}(K)=\{\textstyle\sum_{j=1}^{n}\lambda_j\varepsilon_j\in P(K):\lambda_1\ge\dots\ge\lambda_{n}\ge 0\},\tag{4.6}$$

$$P(H)=\mathbb{Z}\varepsilon_{n+1},\qquad P^{++}(H)=\{k\varepsilon_{n+2}\in P(H):k\ge 0\},\tag{4.7}$$

$$\rho_{\mathfrak{g}}:=\sum_{i=1}^{n+2}(n+2-\mathrm{i})\varepsilon_i,\qquad \rho_{\mathfrak{k}}:=\sum_{i=1}^{n}(n+\tfrac12-\mathrm{i})\varepsilon_i,\qquad \rho_{\mathfrak{h}}:=\tfrac12\varepsilon_{n+2}.\tag{4.8}$$

For $\lambda\in P^{++}(G)$, $\mu\in P^{++}(K)$, and $k\varepsilon_{n+2}\in P^{++}(H)$, we denote by $\pi_\lambda,\sigma_\mu,\tau_k$ the corresponding irreducible representations of G, K and H, respectively.

The Weyl group $W_{\mathfrak{g}}$ consists in elements $\omega=sp$, with p a permutation of the $n+2$ coordinates and s a multiplication by -1 on a subset of coordinates with even cardinality. We still denote by $s_i:\mathfrak{t}^*\to\mathfrak{t}^*$ the reflexion with respect to the axis i like in Subsection 3.1. Furthermore, we define $p_{i,j}$ to be the transposition of the coordinates i and j.

We consider the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ given by $\langle X,Y\rangle=\frac12\operatorname{tr}(XY)$. We extend it to $\mathfrak{t}^*$. It turns out that $\{\varepsilon_1,\dots,\varepsilon_{n+2}\}$ is an orthonormal basis of $\mathfrak{t}^*$.

4.2. Main theorem for the type-D case

The main result in this section is the following.

Theorem 4.1. *Let $n\ge1$, $G=\mathrm{SO}(2n+4)$, $K=\mathrm{SO}(2n+1)$, $H=\mathrm{SO}(3)$, $\lambda=\sum_{i=1}^{n+2}\lambda_i\varepsilon_i\in P^{++}(G)$, $\mu=\sum_{i=1}^{n}\mu_i\varepsilon_i\in P^{++}(K)$. If μ simply interlaces λ, i.e. $\lambda_i\ge\mu_i\ge\lambda_{i+1}$ for $1\le i\le n$, then*

$$\operatorname{Hom}_K(\sigma_\mu,\pi_\lambda)\simeq\left(\bigoplus_{k=|\lambda_{n+2}|}^{\lambda_{n+1}}\tau_k\right)\otimes\bigotimes_{m=1}^{n}\left(\bigoplus_{j=0}^{\lfloor(\lambda_m-\mu_m)/2\rfloor}\tau_{\lambda_m-\mu_m-2j}\right)\tag{4.9}$$

as H-modules.

Similarly as in the previous section, we first make the first steps of applying Kostant's branching formula (2.6) to $(G,K\times H)$. The maximal torus $S:=T\cap(K\times H)$ in $K\times H$ misses the $(n+1)$-th 2×2-block in (4.1). Thus, $\mathfrak{s}^*=\operatorname{span}_{\mathbb{C}}\{\varepsilon_1,\dots,\varepsilon_n,\varepsilon_{n+2}\}$ and the restriction from $\mathfrak{t}^*$ to $\mathfrak{s}^*$ denoted by a bar removes the $(n+1)$-th coordinate, that is,

$$\overline{\sum_{i=1}^{n+2}\beta_i\varepsilon_i}=\sum_{i=1}^{n}\beta_i\varepsilon_i+\beta_{n+2}\varepsilon_{n+2}.\tag{4.10}$$

Furthermore, $\Phi^+(\mathfrak{k}\times\mathfrak{h},\mathfrak{s})=\Phi^+(\mathfrak{k},\mathfrak{t}\cap\mathfrak{k})\cup\Phi^+(\mathfrak{h},\mathfrak{t}\cap\mathfrak{h})$, thus

$$\Sigma=\{\varepsilon_i\pm\varepsilon_{n+2}:1\le i\le n\}\cup\{\varepsilon_i:1\le i\le n\}\cup\{-\varepsilon_{n+2}\},\tag{4.11}$$

each element with multiplicity one. It follows that, for $\nu\in\mathfrak{s}^*$,

$$\mathcal{P}_\Sigma(\nu)>0\quad\Longrightarrow\quad \nu_i:=\langle\varepsilon_i,\nu\rangle\in\mathbb{Z}\quad\forall i,\ \nu_i\ge0\quad\forall 1\le i\le n,\ \nu_{n+2}\le\sum_{i=1}^{n}\nu_i.\tag{4.12}$$

From (2.6), we obtain that the number of times that $\sigma_\mu\otimes\tau_k$ appears in $\pi_\lambda|_{K\times H}$ is given by

$$\dim\operatorname{Hom}_{K\times H}(\sigma_\mu\otimes\tau_k,\pi_\lambda)=\sum_{\omega\in W_{\mathfrak{g}}}\operatorname{sgn}(\omega)\,\mathcal{P}_\Sigma\big(\overline{\omega(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}-\mu-k\varepsilon_{n+2}\big).\tag{4.13}$$

The next lemmas indicate the non-zero terms in the above sum. The proof of the first one is completely analogous to the proof of Lemma 3.2.

Lemma 4.2. *Assume $\lambda_{n+2}\ge0$. If $\omega\in W_{\mathfrak{g}}$ satisfies that the ω-th term in (4.13) is non-zero, then $\omega=p$ or $\omega=s_{n+1}s_{n+2}p$, for some permutation p.*

Lemma 4.3. *Assume $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n$. If $\omega \in W_{\mathfrak{g}}$ satisfies that the ω-th term in (4.13) is non-zero, then ω is in* $\{1,\ s_{n+1}s_{n+2},\ p_{n+1,n+2},\ s_{n+1}s_{n+2}p_{n+1,n+2}\}$.

Proof. We write $\omega = sp$ with $s = 1$ or $s = s_{n+1}s_{n+2}$ by Lemma 4.2. By proceeding as in the proof of Theorem 3.1 we obtain that $\omega(\varepsilon_i) = \varepsilon_i$ for all $1 \leq i \leq n-1$.

Suppose $\omega(\varepsilon_{n+1}) = \varepsilon_n$, then $0 \leq \langle \varepsilon_n, \omega(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}} - \mu - k\varepsilon_{n+2}\rangle = \lambda_{n+1} - \mu_n - 1 < 0$, which is a contradiction. Finally, if $\omega(\varepsilon_{n+2}) = \varepsilon_n$, then $0 \leq \langle \varepsilon_n, \omega(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}} - \mu - k\varepsilon_{n+2}\rangle = \lambda_{n+2} - \mu_n - 2 < 0$, which is again a contradiction. Thus, $\omega(\varepsilon_n) = \varepsilon_n$ and the claim follows. □

We now show that it is sufficient to prove Theorem 4.1 for $\lambda_{n+2} \geq 0$. We set $\widetilde{\lambda} = \sum_{i=1}^{n+1} \lambda_i\varepsilon_i - \lambda_{n+2}\varepsilon_{n+2}$ for any $\lambda = \sum_{i=1}^{n+2} \lambda_i\varepsilon_i \in P^{++}(G)$. Note $\widetilde{\lambda} \in P^{++}(G)$.

Lemma 4.4. *For any $\lambda \in P^{++}(G)$, $\mu \in P^{++}(K)$, $k \geq 0$, we have that $\pi_\lambda|_{K\times H} \simeq \pi_{\widetilde{\lambda}}|_{K\times H}$.*

Proof. We set $g_0 \in \operatorname{diag}(1, \ldots, 1, -1) \in \mathrm{O}(2n+4)$. Although g_0 is not in G, the map $\varphi : x \mapsto g_0xg_0$ is an automorphism of G. It turns out that $\pi_\lambda \circ \varphi \simeq \pi_{\widetilde{\lambda}}$, $\sigma_\mu \circ \varphi|_K = \sigma_\mu$, and $\tau_k \circ \varphi|_H \simeq \tau_k$. The assertions then follow. □

We are now in a position to prove the main theorem of this section.

Proof of Theorem 4.1. The strategy is the same as in the proof of Theorem 3.1. By Lemma 4.4, we may assume $\lambda_{n+2} \geq 0$.

From (4.13) and Lemmas 4.2 and 4.3, we obtain that

$$\begin{aligned}\dim \operatorname{Hom}_{K\times H}(\sigma_\mu \otimes \tau_k, \pi_\lambda) &= \mathcal{P}_\Sigma(\bar{\lambda} - \mu - k\varepsilon_{n+2}) + \mathcal{P}_\Sigma(\bar{\lambda} - \mu - (2\lambda_{n+2} + k)\varepsilon_{n+2}) \\ &\quad - \mathcal{P}_\Sigma(\bar{\lambda} - \mu + (\lambda_{n+1} - \lambda_{n+2} + 1 - k)\varepsilon_{n+2}) - \mathcal{P}_\Sigma(\bar{\lambda} - \mu - (\lambda_{n+1} + \lambda_{n+2} + 1 + k)\varepsilon_{n+2}). \end{aligned} \tag{4.14}$$

Set $\Sigma'' = \Sigma' \cup \{-\varepsilon_{n+2}\}$, where $\Sigma' = \{\varepsilon_i \pm \varepsilon_{n+2} : 1 \leq i \leq n\}$. One has that, see for instance [11, (9.56)], $\mathcal{P}_{\Sigma''}(\nu) = \mathcal{P}_{\Sigma''}(\nu + m\varepsilon_{n+2}) + \sum_{r=0}^{m-1} \mathcal{P}_{\Sigma'}(\nu + r\varepsilon_{n+2})$ for all $\nu \in \mathfrak{s}^*$. Then,

$$\begin{aligned}&\mathcal{P}_\Sigma(\bar{\lambda} - \mu - k\varepsilon_{n+2}) - \mathcal{P}_\Sigma(\bar{\lambda} - \mu + (\lambda_{n+1} - \lambda_{n+2} + 1 - k)\varepsilon_{n+2}) \\ &= \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \sum_{I\subset[\![1,n]\!]} \Big(\mathcal{P}_{\Sigma''}(\bar{\lambda} - \mu - k\varepsilon_{n+2} - 2\gamma - \beta_I) \\ &\qquad - \mathcal{P}_{\Sigma''}(\bar{\lambda} - \mu + (\lambda_{n+1} - \lambda_{n+2} + 1 - k)\varepsilon_{n+2} - 2\gamma - \beta_I)\Big) \\ &= \sum_{r=\lambda_{n+2}}^{\lambda_{n+1}} \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \sum_{I\subset[\![1,n]\!]} \mathcal{P}_{\Sigma'}(\bar{\lambda} - \mu + (r - \lambda_{n+2} - k)\varepsilon_{n+2} - 2\gamma - \beta_I) \\ &= \sum_{r=\lambda_{n+2}}^{\lambda_{n+1}} \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \mathcal{P}_{\Sigma'}\big(2(\bar{\lambda} - \mu + (r - \lambda_{n+2} - k)\varepsilon_{n+2} - 2\gamma)\big). \end{aligned} \tag{4.15}$$

The last equality follows by Lemma 2.2. Similarly, one obtains that

$$\begin{aligned}&\mathcal{P}_\Sigma(\bar{\lambda} - \mu - (2\lambda_{n+2} + k)\varepsilon_{n+2}) - \mathcal{P}_\Sigma(\bar{\lambda} - \mu - (\lambda_{n+2} + \lambda_{n+1} + 1 + k)\varepsilon_{n+2}) \\ &= -\sum_{r=\lambda_{n+2}}^{\lambda_{n+1}} \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \mathcal{P}_{\Sigma'}\big(2(\bar{\lambda} - \mu - (r + \lambda_{n+2} + 1 + k)\varepsilon_{n+2} - 2\gamma)\big). \end{aligned} \tag{4.16}$$

Now, substituting (4.15) and (4.16) in (4.14), we get

$$\begin{aligned}&\dim \operatorname{Hom}_H(\tau_k, \operatorname{Hom}_K(\sigma_\mu, \pi_\lambda)) = \dim \operatorname{Hom}_{K\times H}(\sigma_\mu \otimes \tau_k, \pi_\lambda) \\ &= \sum_{r=\lambda_{n+2}}^{\lambda_{n+1}} \sum_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \sum_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \Big(\mathcal{P}_{\Sigma'}\big(2(\bar{\lambda} - \mu + (r - \lambda_{n+2} - k)\varepsilon_{n+2} - 2\gamma)\big) \\ &\qquad - \mathcal{P}_{\Sigma'}\big(2(\bar{\lambda} - \mu - (r + \lambda_{n+2} + 1 + k)\varepsilon_{n+2} - 2\gamma)\big)\Big). \end{aligned} \tag{4.17}$$

Again, the right-hand side of (4.9) is

$$\bigoplus_{r=\lambda_{n+2}}^{\lambda_{n+1}} \bigoplus_{\gamma_1=0}^{\lfloor\frac{\lambda_1-\mu_1}{2}\rfloor} \cdots \bigoplus_{\gamma_n=0}^{\lfloor\frac{\lambda_n-\mu_n}{2}\rfloor} \tau_{\lambda_1-\mu_1-2\gamma_1} \otimes \cdots \otimes \tau_{\lambda_n-\mu_n-2\gamma_n} \otimes \tau_r. \tag{4.18}$$

By Theorem 2.1, τ_k appears in the factor $\tau_{\lambda_1-\mu_1-2\gamma_1} \otimes \cdots \otimes \tau_{\lambda_n-\mu_n-2\gamma_n} \otimes \tau_r$ with coefficient

$$\mathcal{P}_{\Sigma'}\left(2(\bar\lambda-\mu-2\gamma)+2(r-\lambda_{n+2}-k)\varepsilon_{n+2}\right)-\mathcal{P}_{\Sigma'}\left(2(\bar\lambda-\mu-2\gamma)+2(r-\lambda_{n+2}+k+1)\varepsilon_{n+2}\right).$$

Since $\mathcal{P}_{\Sigma'}(\alpha)=\mathcal{P}_{\Sigma'}(s_{n+2}\alpha)$, it follows that the number of times that τ_k occurs in (4.18) coincides with (4.17), as asserted. □

4.3. Prescribed highest weight end for the type-D case

We next show the analogous result to Theorem 3.5 for the type-D case, which gives an explicit decomposition of the multiplicity space as an H-module when the ending coefficients of λ coincide with a part of the coefficients of μ. This result will be also proved by Tsukamoto's branching law from G to $K \times H$.

Theorem 4.5. *Let $G=\mathrm{SO}(2n+4)$, $K=\mathrm{SO}(2n+1)$, and $H=\mathrm{SO}(3)$ for any $n\geq 1$. For $\mu=\sum_{i=1}^n \mu_i\varepsilon_i \in P^{++}(K)$ and $\lambda=\sum_{i=1}^{n+2}\lambda_i\varepsilon_i \in P^{++}(G)$ with $|\lambda_{i+3}|=\mu_i$ for all $1\leq i\leq n-1$ and $\mu_n\leq|\lambda_{n+2}|$, then*

$$\mathrm{Hom}_K(\sigma_\mu,\pi_\lambda)\simeq \pi'_{\lambda'}|_H\otimes\tau_{\mu_n}, \tag{4.19}$$

as H-modules, where $\pi'_{\lambda'}$ denotes the irreducible representation of $\mathrm{U}(3)$ with highest weight $\lambda':=\lambda_1\varepsilon'_1+\lambda_2\varepsilon'_2+|\lambda_3|\varepsilon'_3$ (see Subsection 2.4).

We note that the condition $\mu_n\leq|\lambda_{n+2}|$ is redundant unless $n=1$. Furthermore, $|\lambda_{i+3}|=\lambda_{i+3}$ for all $i\leq n-2$.

Fix $\lambda\in P^{++}(G)$ with $\lambda_{n+2}\geq 0$. It will be convenient to set $\lambda_{n+3}=\mu_{n+1}=0$ and $\mu_0=\mu_{-1}=\lambda_1$. We recall from (2.4), that the character of $\pi|_{K\times H}$ is given by $\overline{\chi_\pi}$, for any finite dimensional representation π of G. Tsukamoto established in the proof of Theorem 4 in [15] that the character of the restriction of π_λ to $K\times H$ is given by

$$\chi_{\pi_\lambda|_{K\times H}}=\overline{\chi_{\pi_\lambda}}=\frac{1}{(e^{\frac12\varepsilon_{n+1}}-e^{-\frac12\varepsilon_{n+1}})}\sum_\mu \chi_{\sigma_\mu}\sum_{(a_1,\dots,a_{n+1})}\frac{\prod_{i=1}^{n+1}(e^{l_i\varepsilon_{n+2}}-e^{-l_i\varepsilon_{n+2}})}{(e^{\varepsilon_{n+2}}-e^{-\varepsilon_{n+2}})^n}, \tag{4.20}$$

where the first sum is over every $\mu\in P^{++}(K)$ *triply interlacing* λ, that is, $\lambda_i\geq\mu_i\geq\lambda_{i+3}$ for all $1\leq i\leq n$, the second sum is over the $(n+1)$-tuples $(a_1,\dots,a_{n+1})\in\mathbb{Z}^{n+1}$ satisfying that $a_1\geq\cdots\geq a_{n+1}\geq 0$ and $\max(\mu_i,\lambda_{i+1})\leq a_i\leq\min(\mu_{i-2},\lambda_i)$ for all $1\leq i\leq n+1$, and the parameters $l_1,\dots,l_{n+1}$ are given by

$$\begin{cases} l_i=\min(\mu_{i-1},a_i)-\max(\mu_i,a_{i+1})+1 & \text{for all } 1\leq i\leq n,\\ l_{n+1}=\min(\mu_n,a_{n+1})+1/2. \end{cases} \tag{4.21}$$

He thus got the next implicit branching law from $G=\mathrm{SO}(2n+4)$ to $K\times H=\mathrm{SO}(2n+1)\times\mathrm{SO}(3)$.

Theorem 4.6. *[15, Theorem 4] Let $\lambda\in P^{++}(G)$, $\mu\in P^{++}(K)$, and $k\varepsilon_{n+1}\in P^{++}(H)$. If $\lambda_i\geq\mu_i\geq\lambda_{i+3}$ for all $1\leq i\leq n-2$, $\lambda_{n-1}\geq\mu_{n-1}\geq|\lambda_{n+2}|$, and $\lambda_n\geq\mu_n$, then the number of times that $\sigma_\mu\otimes\tau_k$ occurs in $\pi_\lambda|_{K\times H}$ is given by m_k, where the coefficients m_p for $p\geq 0$ are defined by*

$$\sum_{(a_1,\dots,a_n)}\frac{\prod_{i=1}^{n+1}(e^{l_i\varepsilon_{n+2}}-e^{-l_i\varepsilon_{n+2}})}{(e^{\varepsilon_{n+2}}-e^{-\varepsilon_{n+2}})^n}=\sum_{p\geq 0}m_p\left(e^{(p+\frac12)\varepsilon_{n+2}}-e^{-(p+\frac12)\varepsilon_{n+2}}\right), \tag{4.22}$$

where the sum at the left is over the $(n+1)$-tuples $(a_1,\dots,a_{n+1})\in\mathbb{Z}^{n+1}$ satisfying that $a_1\geq\cdots\geq a_{n+1}\geq 0$, $\max(\mu_i,\lambda_{i+1})\leq a_i\leq\min(\mu_{i-2},\lambda_i)$ for all $1\leq i\leq n$, $|\lambda_{n+2}|\leq a_{n+1}\leq\min(\mu_{n-1},\lambda_{n+1})$, and $l_1,\dots,l_{n+1}$ are given by (4.21). Otherwise, $\sigma_\mu\otimes\tau_k$ does not occur in $\pi_\lambda|_{K\times H}$.

Proof of Theorem 4.5. From Lemma 4.4, we assume that $\lambda_{n+2}\geq 0$. We will first show that

$$\mathrm{Hom}_K(\sigma_\mu,\pi_\lambda)\simeq\mathrm{Hom}_K(\sigma_0,\pi_{\lambda'})\otimes\tau_{\mu_n} \tag{4.23}$$

as H-modules, where $\lambda' = \lambda_1\varepsilon_1 + \lambda_2\varepsilon_2 + \lambda_3\varepsilon_3$. We need to check that the term accompanying χ_{σ_μ} in $\chi_{\pi_\lambda|_{K\times H}}$ coincides with the term accompanying χ_{σ_0} in $\chi_{\pi_{\lambda'}|_{K\times H}}$ times $\chi_{\tau_{\mu_n}}$.

By assumption, $\lambda_{i+3} = \mu_i$ for all $1 \le i \le n-1$. Then, (4.20) yields that the term accompanying χ_{σ_μ} in $\chi_{\pi_\lambda|_{K\times H}}$ is equal to

$$\chi_{\tau_{\mu_n}} \sum_{a_1=\lambda_2}^{\lambda_1} \sum_{a_2=\lambda_3}^{\lambda_2} \frac{e^{(a_1-a_2+1)\varepsilon_{n+1}} - e^{-(a_1-a_2+1)\varepsilon_{n+1}}}{e^{\varepsilon_{n+1}} - e^{-\varepsilon_{n+1}}}. \tag{4.24}$$

On the other hand, also by (4.20), the term accompanying χ_{σ_0} in $\chi_{\pi_{\lambda'}|_{K\times H}}$ is given by

$$\sum_{a_1=\lambda_2}^{\lambda_1} \sum_{a_2=\lambda_3}^{\lambda_2} \frac{e^{(a_1-a_2+1)\varepsilon_{n+1}} - e^{-(a_1-a_2+1)\varepsilon_{n+1}}}{e^{\varepsilon_{n+1}} - e^{-\varepsilon_{n+1}}}, \tag{4.25}$$

which completes the proof of (4.23).

The proof follows by applying Knapp's duality [10] to the right-hand side of (4.23). □

Remark 4.7. Theorem 4.5 and (3.27), give an explicit expression for the number of times that $\sigma_\mu \otimes \tau_k$ occurs in $\pi_\lambda|_{K\times H}$ for λ and μ as in Theorem 4.5.

References

[1] F. El Chami, Spectra of the Laplace operator on Grassmann manifolds, Int. J. Pure Appl. Math. 12 (4) (2004) 395–418.
[2] F. El Chami, A branching law from Sp(n) to Sp(q) × Sp(n-q) and an application to Laplace operator spectra, Indian J. Pure Appl. Math. 43 (1) (2012) 71–86, https://doi.org/10.1007/s13226-012-0005-4.
[3] H.W. Galbraith, J.D. Louck, Canonical solution of the SU(3) ↓ SO(3) reduction problem from the SU(3) pattern calculus, Acta Appl. Math. 24 (1) (1991) 59–108.
[4] I.M. Gelfand, M.L. Cetlin, Finite-dimensional representations of the group of unimodular matrices, Dokl. Akad. Nauk SSSR (N.S.) 71 (1950) 825–828.
[5] R. Goodman, N. Wallach, Symmetry, Representations, and Invariants, Grad. Texts in Math., vol. 255, Springer-Verlag, New York, 2009.
[6] K. Gross, R. Kunze, Finite-dimensional induction and new results on invariants for classical groups, I, Amer. J. Math. 106 (4) (1984) 893–974, https://doi.org/10.2307/2374328.
[7] R. Howe, E.-C. Tan, J.F. Willenbring, Stable branching rules for classical symmetric pairs, Trans. Amer. Math. Soc. 357 (4) (2005) 1601–1626, https://doi.org/10.1090/S0002-9947-04-03722-5.
[8] S. Kim, Tiling branching multiplicity spaces with GL_2 pattern blocks, J. Aust. Math. Soc. 94 (3) (2013) 362–374, https://doi.org/10.1017/S1446788712000560.
[9] S. Kim, O. Yacobi, A basis for the symplectic group branching algebra, J. Algebraic Comb. 35 (2) (2012) 269–290, https://doi.org/10.1007/s10801-011-0303-7.
[10] A.W. Knapp, Branching theorems for compact symmetric spaces, Represent. Theory 5 (2001) 404–463, https://doi.org/10.1090/S1088-4165-01-00139-X.
[11] A.W. Knapp, Lie Groups Beyond an Introduction, Progr. Math., vol. 140, Birkhäuser Boston Inc., 2002.
[12] J. Lepowsky, Multiplicity formulas for certain semisimple Lie groups, Bull. Amer. Math. Soc. 77 (1971) 601–605, https://doi.org/10.1090/S0002-9904-1971-12767-2.
[13] A.I. Molev, Gelfand–Tsetlin bases for classical Lie algebras, in: Handbook of Algebra, in: Handb. Algebr., vol. 4, Elsevier/North-Holland, Amsterdam, 2006, pp. 109–170.
[14] C. Tsukamoto, Spectra of Laplace–Beltrami operators on $SO(n+2)/SO(2)\times SO(n)$ and $Sp(n+1)/Sp(1)\times Sp(n)$, Osaka J. Math. 18 (2) (1981) 407–426.
[15] C. Tsukamoto, Branching rules for $SO(n+3)/SO(3)\times SO(n)$, Bull. Fac. Tex. Sci., Kyoto Inst. Technol. 30 (2005) 11–20, http://hdl.handle.net/10212/1686.
[16] N. Wallach, O. Yacobi, A multiplicity formula for tensor products of SL_2 modules and an explicit Sp_{2n} to $Sp_{2n-2}\times Sp_2$ branching formula, in: Symmetry in Mathematics and Physics, in: Contemp. Math., vol. 490, Amer. Math. Soc., Providence, RI, USA, 2009, pp. 151–155.
[17] O. Yacobi, An analysis of the multiplicity spaces in branching of symplectic groups, Sel. Math. (N.S.) 16 (4) (2010) 819–855, https://doi.org/10.1007/s00029-010-0033-z.

Please cite this article in press as: E.A. Lauret, F. Rossi Bertone, On the $SO(n+3)$ to $SO(n)$ branching multiplicity space, C. R. Acad. Sci. Paris, Ser. I (2018), https://doi.org/10.1016/j.crma.2018.09.004