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MULTIFRACTAL SPECTRUM OF QUOTIENTS OF BIRKHOFF AVERAGES FOR A FAMILY OF QUADRATIC MAPS

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Abstract. In a recent article, Chung and Takahashi (Erg. Th & Dynan. Sys. 34, 1116 (2014)) effected a multifractal description of the Birkhoff spectrum for a set of quadratic one dimensional functions known as the *Benedicks-Carleson maps*. They obtained a variational formula for the dimension spectrum of Birkhoff averages. In this article we try to complete the analysis studying the spectrum of quotients of Birkhoff averages.

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1. Introduction

Let (X, f) , be a topological dynamical system, with X a compact metric space and f a continuous map. The multifractal decomposition of the phase space X in level sets K_α is done in the form

$$K_\alpha = \{x \in X : \ F(x) = \alpha\}, \tag{1}$$

where F is a real valuated function defined on X. The Multifractal Analysis treats with the problem of describing these level sets by mean of a function defined on sets. In the area of Dimension Theory of Dynamical Systems the functions used are characteristic dimensions like Hausdorff dimension or topological entropy.

The *irregular part, or historic set,* of the multifractal spectrum is the set

$$\widehat{K} = \{x \in X : \ F(x) \quad \text{is not defined}\}. \tag{2}$$

The *Birkhoff spectrum* corresponds to the following decomposition: if $\varphi : X \to \mathbf{R}$ then consider the statistical sums

$$S_n(\varphi)(x) = \sum_{k=0}^{n-1} \varphi\left(f^k(x)\right) \tag{3}$$

and set

$$K_{\alpha,\varphi} = \left\{x \in X : \lim_{n\to\infty} \frac{1}{n} S_n(\varphi)(x) \ = \alpha\right\}. \tag{4}$$

These equations motive to introduce the terminology "sets of points with historic behavior" or, simply, "historic set", for the irregular part. The name has to do with the fact that the divergence points of the Birkhoff averages describes the history of the system and may be interpreted as the changes in the "epochs" of the system.

It is very interesting to study regularity properties of the multifractal maps. So, in some cases, the map $\alpha \mapsto \dim_H K_{\alpha,\varphi}$ is analytic. This happens when the dynamics are hyperbolic and the potentials φ are Hölder continuous. In some other cases, the multifractal map is continuous. The notable thing of these cases is that the decomposition of the phase space in multifractal level sets, even having a complex structure, can be described by functions with a good behavior. Another observed phenomena is that even the irregular part of the spectrum have zero measure.

Multifractal formalisms to study the dimension spectrum were developed for systems with uniform hyperbolicity or without critical points[10],[12],[4],[5],[9]. In reference [8], Moo- Chung and Takahashi obtained a variational description of the dimension spectrum, where the dynamics are maps with critical points. More specifically, in this theory a family of quadratic maps, called the Benedicks and Carleson maps (BC maps), is considered:

(5) $$f_a : X \to X,\ X = [-1,1],\ \text{defined by } f_a(x) = 1 - ax^2,\ 0 < a \leq 2.$$

The following facts were proved by Benedicks and Carleson[1] and Benedicks and Young[3].

For $a_0 < 2$, there is set $\Delta \subset (a_0, 2)$, with full Lebesgue measure such that for any $a \in \Delta$:

$(C1)$ $f = f_a$.

$(C2)$ $|Df^n(0)| \geq \exp(\lambda n)$, $\lambda = \dfrac{9}{10}\log 2$, $n \geq 0$.

$(C3)$ There is abundance of parameters near to 2 such that

$|f^n(0)| \geq \exp(\dfrac{1}{100}\sqrt{n})$, $n \geq 1$.

$(C4)$ f is topologically mixing in $\left[f^2(0), f(0)\right]$.

Also was established that there exists a set of values $\Delta \subset (0,2)$, with Lebesgue measure, such that for every $a \in \Delta$ the map f_a has not attractive cycles. Due to the presence of the critical point $x_0 = 0$ it must be designed a particular technique to obtain a symbolic representation of the maps. This is done by the method of towers or induced schemes. The problem of the construction of towers is known as *liftability problem*. It is well known that for hyperbolic systems Markov partitions can be constructed. The fundamental idea in [8] is the construction of induced systems with a special property of recurrence. In the seminal work of Benedicks and Carleson is studied the types of return of the orbits $f_a^n(0)$ to an interval $I^* = (-\delta, \delta)$ with $\delta = \exp(-\sqrt{a})$, the interval I^* is divided in subintervals I_γ, and are considered three types of orbits called free, bound and inessential. The ideas of Benedicks and Carleson in references [1] and [2] are followed and conveniently adapted by Moo-Chung and Takahashi to analyze the growing of the derivative of the maps f_a outside a small neighborhood of the critical point $x_0 = 0$. The binding argument presented in [1] and [2] is also modified in [8] to estimates the return times of the orbits and how close are to the critical point. In reference [8] their authors proved that the BC maps are uniformly expanding outside a neighborhood of $x_0 = 0$. To estimate the return times the pieces I_γ in which is subdivided the neighborhood $I^* = (-\delta, \delta)$, are treated independently and in each subinterval are constructed dynamical partitions which lead to the desired towers. We shall recall a bit more explicitly the techniques of [8] in the next section. For more details see that article or reference [7].

Our objective in this paper is to describe the spectrum of quotients of Birkhoff averages for BC maps, i.e. given potentials $\varphi, \psi : X \to \mathbf{R}$, obtain a variational formula for the multifractal decomposition

$$K_{\alpha,\varphi,\psi} = \left\{ x : \lim_{n\to\infty} \frac{S_n(\varphi(x))}{S_n(\psi(x))} = \alpha \right\}. \tag{6}$$

In ref.[6], Iommi and Jordan have analyzed the problem of the multifractal description of spectrum of quotients of Birkhoff averages for countable Markov maps (EMR maps). These kind of maps are defined on a countable union of intervals in $[0,1]$ and are topologically conjugate to a shift with a countable alphabet. The Gauss map falls in this class: The potentials for the Birkhoff averages are functions uniformly bounded by below and with summable variation. The partition used to conjugate the EMR maps with a shift is a partition by cylinders. Thus the symbolic representation is more direct.

Here we consider the potentials within a class $\mathcal{C}$ of maps, which we will do explicit later. For $\varphi, \psi \in \mathcal{C}$, denote

$$\overline{\alpha}_{\varphi,\psi} := \sup \left\{ \lim_{n\to\infty} \frac{S_n(\varphi(x))}{S_n(\psi(x))} \right\} \tag{7a}$$

$$\underline{\alpha}_{\varphi,\psi} := \inf \left\{ \lim_{n\to\infty} \frac{S_n(\varphi(x))}{S_n(\psi(x))} \right\}. \tag{7b}$$

Within this context, the main result to be proved in this work reads:

Theorem 1: *Let us consider a set of parameters a such that $f = f_a$, where f_a is a family of Benedicks-Carleson maps. Let $\varphi, \psi \in \mathcal{C}$, and with ψ such that $S_n(\psi(x)) > \eta n$, for some $\eta > 0$, then for any $\alpha \in (\underline{\alpha}_{\varphi,\psi}, \overline{\alpha}_{\varphi,\psi})$ holds*

$$\dim_H K_{\alpha,\varphi,\psi} = \lim_{\varepsilon\to 0} \sup \left\{ \frac{h_\mu(f)}{\lambda(\mu,f)} : \frac{\int \varphi d\mu}{\int \psi d\mu} \in (\alpha-\varepsilon, \alpha+\varepsilon) \right\}, \tag{8}$$

where $h_\mu(f)$ and $\lambda(\mu,f)$ are respectively the entropy and the Lyapunov exponent of f with respect to the measure μ and the supremum is taking over the all the f-invariant measures μ with $\int \psi d\mu < \infty$ (this class will be denoted $\mathcal{M}^0_{inv}(X,f)$.

The analysis of the continuity of the spectrum can be done in a similar way that in [6], once the variational formula for the dimension spectrum be estalished. Thus like in [6], consider sets

$$\overline{A}_{\varphi,\psi} := \sup\left\{\alpha : \exists\ (x_n)\,,\ \text{with } x_n \to 0 \text{ and } \limsup_{n\to\infty} \frac{\varphi(x_n)}{\psi(x_n)} = \alpha\right\} \tag{9a}$$

$$\underline{A}_{\varphi,\psi} := \inf\left\{\alpha :\ \exists\ (x_n)\,,\ \text{with } x_n \to 0 \text{ and } \liminf_{n\to\infty} \frac{\varphi(x_n)}{\psi(x_n)} = \alpha\right\}. \tag{9b}$$

Set $I = \left(\underline{\alpha}_{\varphi,\psi}.\overline{\alpha}_{\varphi,\psi}\right) - \left[\underline{A}_{\varphi,\psi}.\overline{A}_{\varphi,\psi}\right]$, and prove that the map $\alpha \in I \mapsto D(\alpha) := \dim_H K_{\varphi,\psi}$ is continuous. For completeness we outline the proof.

Let $\{\alpha_n\}$ be a sequence in I converging to α_0, firstly we prove that

$$\inf\left\{\lim_{k\to\infty} D\left(\alpha_{n_{(k)}}\right)\right\} \geq D(\alpha_0)\,,$$

with the infimum taken over all the subsequences $\left\{\alpha_{n_{(k)}}\right\}$ of $\{\alpha_n\}$. Let $\{\mu_n\}$ be a sequence of measures in $\mathcal{M}_{inv}(X,f)$ with $\int \psi d\mu_n < \infty$ such that $\dfrac{h_{\mu_n}(f)}{\lambda(\mu_n, f)} > D(\alpha_0) - \varepsilon$ and $\lim\limits_{n\to\infty} \dfrac{\int \varphi d\mu_n}{\int \psi d\mu_n} = \alpha_0$. By [6] there are sequence of numbers (r_n), (C_n) such that if (μ_n) in a sequence of measures in $\mathcal{M}_{inv}(X,f)$ with $\int \psi d\mu_n < \infty$ such that if $\dfrac{\int \varphi d\mu_n}{\int \psi d\mu_n} \in B_{r_n}(\alpha_n)$ then $\int \psi d\mu_n \leq C_n$. Let us choose subsequences $\left\{\nu^1_{n(k)}\right\}, \left\{\nu^2_{n(k)}\right\}$ of the sequences $\left\{\nu^1_n\right\}, \left\{\nu^2_n\right\}$ such that

$$\frac{\int \varphi d\nu^1_{n(k)}}{\int \psi d\nu^1_{n(k)}} \leq\ \alpha_{n(k)}\ \leq \frac{\int \varphi d\nu^2_{n(k)}}{\int \psi d\nu^1_{n(k)}}. \tag{10}$$

We can take convex combinations of $\dfrac{\int \varphi d\mu_n}{\int \psi d\mu_n}$ and $\dfrac{\int \varphi d\nu^1_{n(k)}}{\int \psi d\nu^1_{n(k)}}$ or $\dfrac{\int \varphi d\nu^2_{n(k)}}{\int \psi d\nu^1_{n(k)}}$, thus we have adequate values $0 \leq p_n \leq 1$, such that

$$\alpha_{n(k)} = p_{n(k)}\ \frac{\int \varphi d\mu_{n(k)}}{\int \psi d\mu_{n(k)}} + (1-p_n)\frac{\int \varphi d\nu^1_{n(k)}}{\int \psi d\nu^1_{n(k)}}$$

or

$$\alpha_{n(k)} = p_{n(k)}\ \frac{\int \varphi d\mu_{n(k)}}{\int \psi d\mu_{n(k)}} + (1-p_n)\frac{\int \varphi d\nu^2_{n(k)}}{\int \psi d\nu^2_{n(k)}},$$

according to the position of α_n. From this convex combinations of measures we can define

$$\nu_{_{n(k)}} = p_{n(k)}\mu_{n(k)} + (1-p_n)\,\nu^1_{n(k)} \text{ and } \nu_{_{n(k)}} = p_{n(k)}\mu_{n(k)} + (1-p_n)\,\nu^2_{n(k)}.$$

Therefore

$$D\left(\alpha_{n(k)}\right)=D\left(p_{n(k)}\,\frac{\int\varphi d\mu_{n(k)}}{\int\psi d\mu_{n(k)}}+\left(1-p_n\right)\frac{\int\varphi d\nu^1_{n(k)}}{\int\psi d\nu^1_{n(k)}}\right)$$

or

$$D\left(\alpha_{n(k)}\right)=D\left(p_{n(k)}\,\frac{\int\varphi d\mu_{n(k)}}{\int\psi d\mu_{n(k)}}+\left(1-p_n\right)\frac{\int\varphi d\nu^2_{n(k)}}{\int\psi d\nu^2_{n(k)}}\right),$$

so that $D\left(\alpha_{n(k)}\right)\geq\dfrac{h_{\mu_{n(k)}}(f)}{\lambda\left(\mu_{n(k)},f\right)}>D\left(\alpha_0\right)-\varepsilon$, for n enough large and $p_n\to 1$ (since $\alpha_n\to\alpha_0$).

To prove that $\sup\left\{\lim\limits_{k\to\infty}D\left(\alpha_{n_{(k)}}\right)\right\}\leq D\left(\alpha_0\right)$, the argument is similar. We consider measures $\{\mu_n\}$ such that $\lim\limits_{n\to\infty}\dfrac{h_{\mu_n}(f)}{\lambda\left(\mu_n,f\right)}>D\left(\alpha_0\right)$ and convex combinations of these measures with subsequences of $\left\{\nu^1_n\right\},\left\{\nu^2_n\right\}$.

After proving that the spectrum is continuous in I we have

$$\dim_H K_{\alpha,\varphi,\psi}=\sup\left\{\frac{h_\mu(f)}{\lambda(\mu,f)}:\frac{\int\varphi d\mu}{\int\psi d\mu}=\alpha\right\},\tag{11}$$

for any $\alpha\in I$.

2. Preliminaries

Let us recall the definition of *Hausdorff dimension*, let (X,d) be a metric space and $Z\subset X$, let $\mathcal{B}_\varepsilon$ be a cover of Z with *diam* $\mathcal{B}_\varepsilon<\varepsilon$. For any $s>0$.the $s-$Hausdorff measure of Z is defined by

$$M_H(Z,s)=\lim_{\varepsilon\to 0}\inf_{\mathcal{B}_\varepsilon}\left\{\sum_{U\in\mathcal{B}_\varepsilon}\operatorname{diam}(U)^s\right\}.\tag{12}$$

The Hausdorff dimension of Z is defined as

$$\dim_H Z=\sup\left\{s:\ M_H(Z,s)=+\infty\right\}=\inf s:\ M_H(Z,s)=0\tag{13}$$

The *point-wise dimension* of a measure μ, denoted $D_\mu(x)$, is defined as:

$$D_\mu(x)=\lim_{r\to 0}\frac{\mu\left(B_r(x)\right)}{\log r},\tag{14}$$

where $B_r(x)$ is the ball of centre x and radius r.

By $\mathcal{M}(X)$ we denote the space of probability measures on X, and by $\mathcal{M}_{inv}(X,f)$ the space of $f-$invariant measures on X. By $\mathcal{M}_{inv}^{E}(X,f)$ is denoted the set of ergodic measures of $\mathcal{M}_{inv}(X,f)$.

The space $\mathcal{M}(X)$ is endowed the weak $*-$ topology, and if X is compact then $\mathcal{M}(X)$ is compact in the weak $*-$ topology.

Volume lemma[11]: Let $Z\subset X$, $\alpha\in\mathbf{R}$, and $\mu\in\mathcal{M}(X)$, if for any $x\in Z$ holds $D_{\mu}(x)\geq\alpha$ then $\dim_H Z\geq\alpha$.

Let $f_a:[-1,1]\to[-1,1]$ a family of Benedicks and Carleson maps with parameter values sufficiently close to 2, accordingly to condition $(C1)$ we can consider $f=f_a$..

If $\mu\in\mathcal{M}_{inv}([-1,1],f)$ then the Lyapunov exponent of μ is

$$\lambda(\mu,f)=\int|Df|\,d\mu. \tag{16}$$

The phase space X will be the interval $[-1,1]$.

Now we display a sketch of the constructions of [1], [2],[8],[7] which allows to produce a horseshoe and symbolic dynamics. Let $p>0$, recall that the techniques of Benedicks-Carleson essentially consist in subdividing a neighborhood $(-\delta,\delta)$ of the critical point and treating any segment independently, let $0<\varepsilon<<1$, $N>>1$, set

$$\delta_p=\sqrt{\frac{\exp(-\varepsilon p)}{10}\left(\sum_{i=0}^{p-1}\frac{Df^i(0)}{f^{i+1}(0)}\right)^{-1}}, \tag{17}$$

thus if $\delta_p<|x|<\delta_{p-1}$, $x\in X$, then $|Df^p(x)|\geq\exp((\lambda/3)\,p)$, with $\lambda=\lambda(\mu,f)$ and $p>N$[8]. Let $\widehat{x}$ be a fixed point of f in $[-1,1]$ and set $\widehat{X}=[-\widehat{x},\widehat{x}]$, a sequence $\widehat{\mathcal{P}}=\left(\widehat{\mathcal{P}_n}\right)_n$ of partitions of $\widehat{X}$ in intervals is constructed in such a way that holds a bounded distortion property for the map on members of the partition whose iterations are "free" (see [8] for definition of free and bound states). The construction is inductive with the process beginning with $\widehat{\mathcal{P}_0}=([-\widehat{x},-\delta],[\delta,\widehat{x}]\cup I_{p,j})_{j=1,2,..,[\exp(3\epsilon p]}$, where $I_{p,j}$ are the subintervals in which any $I_{p,j}$ is subdivided. It holds that for any pair p,j there exist a $x\in I_{p,j}$ such that $|f^n(x)|\geq\delta_N\exp(-\varepsilon n)$ for $n>1/\varepsilon$. So these points return slowly to the critical point 0, these points are used for the partition. Let $\Lambda^+=I_{N,1}$ and $\Lambda^-=-\Lambda^+$ and set $\Lambda=\Lambda^+\cup\Lambda^-$, from the sequence of partitions $\widehat{\mathcal{P}}=\left(\widehat{\mathcal{P}_n}\right)_n$ is defined inductively a new partition by intervals $\mathcal{Q}$ and a time return map $R:\mathcal{Q}\to\mathbf{N}$. The function is defined in the following way: for $\mathcal{A}\in\widehat{\mathcal{P}_n}\mid\Lambda:=\{A\cap\Lambda : A\in\mathcal{A}\}$, such that $f^n(\mathcal{A})$ free (see [8]) and $3\Lambda^+\subset f^n(\mathcal{A})$ (or $3\Lambda^-\subset f^n(\mathcal{A})$) set $\mathcal{A}\cap f^{-n}(\Lambda^+)\subset\mathcal{Q}$ or $\mathcal{A}\cap f^{-n}(\Lambda^-)\subset\mathcal{Q}$ and in this case define $R(\mathcal{A})=n$. The parts $f^n(\mathcal{A})-\Lambda^+$ and $f^n(\mathcal{A})-\Lambda^-$ are iterated and the map is defined following the same procedure. The iteration $f^{R(\mathcal{A})}$ is difeomorphism between $\mathcal{A}$ and Λ^+ or Λ^-. Then is defined the tail set

$$\{R>n\}:=\bigcup_{\substack{\mathcal{A}\in\mathcal{Q}\\ R(\mathcal{A})>n}}\mathcal{A} \tag{18}$$

From $\left(\widehat{\mathcal{P}_n}\mid\{R>n\}\right)_n$ another new partition $\mathcal{P}=(\mathcal{P}_n)_n$ is defined inductively by gluing elements of $\widehat{\mathcal{P}_n}$. In this case the inductive process begins with $\mathcal{P}_0=(\Lambda^-,\Lambda^+)$. The iteration of the members of the partition returns quickly to Λ. For the construction of towers proceeds in the following way: let $R:\mathcal{Q}\to\mathbf{N}$ the time-return map and let $\Delta=\{(x,\ell):x\in\Lambda,\ \ell=0,1,...,R(x)-1\}$ and consider the induced function on Δ

$$\widetilde{f}(x,\ell)=\begin{cases}(x,\ell+1) & \text{if } \ell+1<R(x)\\ \left(f^{R(x)},0\right) & \text{if } \ell+1=R(x)\end{cases}. \tag{19}$$

This map is interpreted like a climbing in the first case and a falling down in the second of any pair (x,ℓ).

Let $\Delta_\ell=\{(x,\ell)\in\Delta:R(x)>\ell\}$, the partition $\mathcal{P}_\ell$ of $\{R>\ell\}$ can be carried to a partition of Δ_ℓ via the natural identification of any point x with $R(x)>\ell$ with the element $(x,\ell)\in\Delta_\ell$. This partition is also denoted by $\mathcal{P}_\ell$. If $\mathcal{D}=\bigcup_{\ell\geq0}\mathcal{P}_\ell$ then $\mathcal{D}$ is a Markov partition. For a partition $\mathcal{A}=\{A_1,A_2,...,A_k\}$ of a space X, the name of a point $x\in X$ of length n, with respect to $\mathcal{A}$ and a ,map $f:X\to X$, is the string $(a_0,a_1.,,,a_{n-1})$ accordingly $f^{a_i}\in A_{a_i}$. The partition of X by sets with the same name of length n is a refinement of $\mathcal{A}$ which is denoted by $\mathcal{A}^n$. Let us consider, for each n, the partitions $\mathcal{D}^n$, and denote their members by D^n, i.e. each D^n is formed by points with the same name of length n with respect to $\mathcal{D},\widetilde{f}$. Now consider

$$\mathcal{D}_0^n=\left\{D^n\subset\Delta_0:\ \left|\frac{S_n(\varphi(x))}{S_n(\psi(x))}-\alpha\right|<\varepsilon \text{ for some } x\in D^n\right\}. \tag{20}$$

In [8] is proved that there exists an integer r such that from this partition is generated a horseshoe for $F = f^r$. This means that there exists a collection of intervals $\mathcal{J} = \{I_1, I_2, ..., I_M\}$, contained in Λ, such that

$F|_{I_j}: I_j \to \widehat{X} = [-\widehat{x}, \widehat{x}]$ is a diffeomorphism for any $j = 1, 2, ..., M$. The horseshoe is constructed as

$$H(\mathcal{J}) = \bigcap_{j=1}^{\infty} F^{-j}\left(\bigcup_{i=1}^{M} I_i\right). \tag{21}$$

The intervals are extracted from the partition $\mathcal{D}_0^n$. From the above horseshoe can be obtained a code map which conjugates Λ with a full subshift Σ_M with alphabet $\{1, 2, ..., M\}$ and such that $F = f^r$ is uniformly expanding on each interval $I_i \in \mathcal{J}$. The code map is given by

$$\begin{gathered} \pi : \Sigma_M \to \Lambda \\ \pi(x = \ell_0 \ell_1 ...) = \bigcap_{i=1}^{\infty} F^{-i}(I_{\ell_i}). \end{gathered} \tag{22}$$

We are now in condition to specify the class of potentials $\mathcal{C}$ considered for the theorem. A potential $\varphi : X \to \mathbf{R}$ belongs to the class $\mathcal{C}$ if there is a number $K > 0$ such that for $n > N$ for any $x, y \in \mathcal{A} \in \mathcal{P}_n$ holds

$$|S_n(\varphi(x)) - S_n(\varphi(y))| < K. \tag{23}$$

3. Proof of the theorem

To obtain the upper bound are considered, like in [8], the following decomposition sets

$$K_{k,\alpha,\varepsilon} = \left\{x \in \Lambda : \left|\frac{S_n(\varphi(x))}{S_n(\psi(x))} - \alpha\right| < \varepsilon, \text{ for any } n > k\right\}. \tag{24}$$

We have that $K_{\varphi,\psi} \cap \Lambda \subset \bigcup_{k \geq n} K_{k,\alpha,\varepsilon}$, for any $n \geq 0$. Any set $K_{k,\alpha,\varepsilon}$ is covered by the partitions $\mathcal{D}_0^n$, with $n > k$. If $s_n = \inf\left\{s : \sum_{A \in \mathcal{D}_0^n} |A|^s \leq 1\right\}$ then, by definition of

Hausdorff dimension, we have $\dim_H K_{k,\alpha,\varepsilon} \leq \lim_{n\to\infty} s_n$. Thus to establish the upper bound this limit must be related with measures supported on the horseshoe $H(\mathcal{J})$. The following lemma is useful for this purpose.

Lemma: Let $\varphi, \psi \in \mathcal{C}$, and with ψ such that $S_n(\psi(x)) > \eta n$, for some $\eta > 0$, then for $x, y \in \mathcal{A} \in \mathcal{P}_n$

$$\left| \frac{S_n(\varphi(x))}{S_n(\psi(x))} - \frac{S_n(\varphi(y))}{S_n(\psi(y))} \right| < \varepsilon, \tag{25}$$

for $n > N$.

Proof: Let $K_1 = K_1(\varphi)$ and $K_2 = K_2(\psi)$ be the constants for φ and ψ in the class $\mathcal{C}$. If $x, y \in \mathcal{A} \in \mathcal{P}_n$ then

$$\frac{S_n(\varphi(y)) - K_1}{S_n(\psi(y)) + K_2} \leq \frac{S_n(\varphi(x))}{S_n(\psi(x))} \leq \frac{S_n(\varphi(y)) + K_1}{S_n(\psi(y)) - K_2}.$$

Thus since $S_n(\psi(x)) > \eta n$, for some $\eta > 0$ the result follows. □

According to ref.[8] we define a sequence of measures $\{\nu_k\}$ supported on the horseshoe as follows: if $I_{\ell_0\ell_1\ldots\ell_k} = I_{.\ell_k} \cap F^{-1}(I_{\ell_1}) \cap \ldots \cap F^{-k}(I_{\ell k})$ then set

$$\nu_n = \nu_{n,s} = \frac{1}{Z_{k,s}} \sum_{\ell_0\ell_1\ldots\ell_n} |I_{\ell_0\ell_1\ldots\ell_n}|^s \sum_{x\in P_{n+1}(F)} \delta_x,$$

where $P_{n+1}(F)$ are the $n+1$- periodic points of F on $I_{\ell_0\ell_1\ldots\ell_n}$ and $Z_{n,s}$ is the normalization factor $Z_{n,s} = \sum_{\ell_0\ell_1\ldots\ell n} |I_{\ell_0\ell_1\ldots\ell n}|^s$. Let ν be an $*$-weak accumulation point of the sequence$\{\nu_{n,s}\}$

By [8] is valid that for any $\gamma > 0$ there exists a n_0 such that for $n \geq n_0$ holds

$$\sum_{i=1}^{M} |I_i|^s \geq \exp(-3\sqrt{\gamma} n) \sum_{A\in\mathcal{D}_0^n} |A|^{\overline{s}}, \tag{26}$$

and the integer r such that $H(\mathcal{J})$ generates a horseshoe for f^r satisfies $r \geq (1-\gamma)n$. Let $\overline{s}$ be a number such that $\sum_{A\in\mathcal{D}_0^n} |A|^{\overline{s}} > 1$, and $\varepsilon > 0$,we shall see that for any $\gamma > 0$ there exists a measure m such that $\dfrac{\int \varphi dm}{\int \psi dm} \in (\alpha - 2\varepsilon, \alpha + 2\varepsilon)$ and such that

$\dfrac{h_m(f)}{\lambda(m,f)}-\gamma>\overline{s}$. Let

$$m_n=\frac{1}{r}\sum_{i=1}^{r}\left(f^i\right)_*(\nu_n),$$

and let m be an $*-$weak accumulation point of the sequence $\{m_n\}$, i.e. $m=\frac{1}{r}\sum_{i=1}^{r}\left(f^i\right)_*(\nu)$. We claim that this measure satisfies the above conditions. Let us assume that at $\dfrac{h_m(f)}{\lambda(m,f)}<\overline{s}+\gamma$ then, by [8],[7], it holds

$$\limsup_{k\to\infty}\frac{1}{k}\log\sum_{\ell_0\ell_1\ldots\ell_k}\left|I_{\ell_0\ell_1\ldots\ell_k}\right|^{\overline{s}}\leq h_\nu(F)-\overline{s}\lambda(\nu,F)\,(F=f^r),$$

and

$$\limsup_{k\to\infty}\frac{1}{k}\log\sum_{\ell_0\ell_1\ldots\ell_k}\left|I_{\ell_0\ell_1\ldots\ell_k}\right|^{\overline{s}}\geq\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}-\overline{s}\log M.$$

Thus

$$r\left(h_m(f)-\overline{s}\lambda(m,f)\right)\geq\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}-\overline{s}\log M\geq\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}-4\sqrt{\gamma}n\overline{s}-\overline{s}\log M.$$

Since we are assuming that $r\geq(1-\gamma)\,n$ and $\dfrac{h_m(f)}{\lambda(m,f)}<\overline{s}+\gamma$ we obtain

$$-r\gamma\lambda(m,f)\geq\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}-4\sqrt{\gamma}\overline{s}n+\overline{s}\log M$$

and

$$[-(1-\gamma)\gamma\lambda(m,f)+\ 4\sqrt{\gamma}\overline{s}]\,n+\overline{s}\log M\geq\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}.$$

So, for n sufficiently large, $\log\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}\leq-\gamma\lambda(m,f)<0$, or $\sum_{A\in\mathcal{D}_0^n}|A|^{\overline{s}}<1$, contradicting the election of $\overline{s}$. The fact that $\dfrac{\int\varphi dm}{\int\psi dm}\in(\alpha-2\varepsilon,\alpha+2\varepsilon)$ follows from the lemma, since for n enough large is

$$\int\frac{S_n\left(\varphi(x)\right)\ d\nu_n}{S_n\left(\psi(x)\ d\nu_n\right.}\in(\alpha-2\varepsilon,\alpha+2\varepsilon)\,.$$

Therefore

$$\overline{s}\leq\sup\left\{\frac{h_\mu(f)}{\lambda(\mu,f)}:\mu\in\mathcal{M}_{inv}(X,f),\ \frac{\int\varphi d\mu}{\int\psi d\mu}\in(\alpha-2\varepsilon,\alpha+2\varepsilon)\right\}.$$

Thus, $\limsup_{n\to\infty}\left|s_n - \dfrac{h_{m_m}(f)}{\lambda(m_n,f)}\right| = 0$ and $\int \dfrac{S_n(\varphi(x))\ dm_n}{S_n(\psi(x)\ dm_n} \in (\alpha-2\varepsilon,\alpha+2\varepsilon)$. From this we get

$$\dim_H K_{\alpha,\varphi,\psi} \le \lim_{\varepsilon\to 0}\sup\left\{\frac{h_\mu(f)}{\lambda(\mu,f)} : \frac{\int\varphi d\mu}{\int\psi d\mu}\in(\alpha-\varepsilon,\alpha+\varepsilon)\right\}. \tag{27}$$

For the lower bound we use the constructions of [8] to obtain a fractal set F and also consider a specific sequence of measures $\{\mu_n\}\subset\mathcal{M}(X)$ such that

$$\lim_{n\to\infty}\frac{h_{\mu_n}(f)}{\lambda(\mu_n,f)} = t, \tag{28}$$

where

$$t := \lim_{\varepsilon\to 0}\sup\left\{\frac{h_\mu(f)}{\lambda(\mu,f)} : \mu\in\mathcal{M}^0_{inv}(X,f),\ \frac{\int\varphi d\mu}{\int\psi d\mu}\in(\alpha-\varepsilon,\alpha+\varepsilon)\right\}. \tag{29}$$

To do this we must find a sequence of measures $\{\mu_n\}$ such that $\lim_{n\to\infty}\dfrac{\int\varphi d\mu_n}{\int\psi d\mu_n}=\alpha$, and such that for any measure μ with $\dfrac{\int\varphi d\mu}{\int\psi d\mu}=\alpha$, $\int\psi d\mu<\infty$ and $h_\mu(f)<\infty$ holds $\lim_{n\to\infty}\dfrac{h_{\mu_n}(f)}{\lambda(\mu_n,f)}=\dfrac{h_\mu(f)}{\lambda(\mu,f)}$. Let $\mathcal{A}$ be generating partition and $\mathcal{A}^n$ be the induced partition by names of length n, let us consider the measures $\varsigma_n(\mathcal{A}^n)=\mu(\mathcal{A}^n)$ then like in[6] we introduce the (ergodic) family of measures $\mu_n := \dfrac{1}{n}\sum_{i=0}^{n-1} f_*(\varsigma_n)$. As generating partition may be used the "cylinders" $I_{\ell_0\ell_1\ldots\ell_k}$. Now we have $\lim_{n\to\infty}\dfrac{\int\varphi d\mu_n}{\int\psi d\mu_n}=\alpha$, $\lim_{n\to\infty}\dfrac{h_{\mu_n}(f)}{\lambda(\mu_n,f)}=\dfrac{h_\mu(f)}{\lambda(\mu,f)}$ and then $\lim_{n\to\infty}\dfrac{h_{\mu_n}(f)}{\lambda(\mu_n,f)}=t$.

The next step is to find a measure ν such that $D_\nu(x)\ge t$.

Any ergodic measure with positive entropy can be approximated for a horseshoe. This means that for any n there is a sequence of numbers (j_n), closed intervals (L_n) and disjoint intervals $\mathcal{I}_n$ contained in the interior of L_n such that

1. For any $I\in\mathcal{I}_n$ holds $f^{j_n}(I)=L_n$.
2. For any $x\in\bigcup_{I\in\mathcal{I}_n} I$ and $\alpha_n := \dfrac{\int\varphi d\mu_n}{\int\psi d\mu_n}$ there exists a sequence (j_n) such that $\left|\dfrac{S_j(\varphi(x))}{S_j(\psi(x))}-\alpha_n\right|<\dfrac{1}{n}$, for $j>j_n$.
3. $\dfrac{1}{j_n}\log card\,\mathcal{I}_n \ge h_{\mu_n}(f)-\dfrac{1}{n}$ (Katok formula)

Let $k \geq 1$ and numbers $n = n(k)$, $r = r(k)$ and γ_i such that

$$k = \sum_{i=1}^{n} k_i + r, \text{ with } 0 \leq r < k_{n+1},$$

where $\{k_n\}$ is a sequence inductively defined in [8]. If $(I_1, I_2, ..., I_k)$ denotes an element belonging to $\mathcal{I}^{(k)} := \mathcal{I}_1^{k_1} \times \mathcal{I}_2^{k_2} \times ... \times \mathcal{I}_n^{k_n} \times \mathcal{I}_{n+1}^{r}$ then by [8], can be associated an interval $\mathcal{J}(I_1, I_2, ..., I_k)$ such that for $I \in \mathcal{I}_1$, set $\mathcal{J}(I) = I$. and, inductively, if $\mathcal{J}(I_1, ..., I_k)$ is supposed constructed, the $\mathcal{J}(I_1, ..., I_k, I_{k+1})$ is defined as

(30)

$$\mathcal{J}(I_1, ..., I_k, I_{k+1}) := \begin{cases} \left(f^t \mid \mathcal{J}(I_1, ..., I_k)\right)^{-1}(I_{k+1}) & \text{if } r < k_{n+1} - 1 \\ \left(f^t \mid \mathcal{J}(I_1, ..., I_k)\right)^{-1}\left(f^{s_j} \mid I_{k+1}\right) (I_{n+1}) & \text{if } r = k_{n+1} - 1 \end{cases}$$

for some $t = t_{n(k),r} = \sum_{i=1}^{n} (j_i k_i + \gamma_i) + j_{n+1} r$.

Let

$$F^{(k)} = \left\{\mathcal{J}(I_1, ..., I_k) : (I_1, ..., I_k) \in \mathcal{I}^{(k)}\right\}, \tag{31}$$

we have that $\left\{ \bigcup_{I \in F^{(k)}} I \right\}$ is a nested sequence of closed intervals, so it can be defined the non-empty set

$$F := \bigcap_{k=1}^{\infty} \bigcup_{I \in F^{(k)}} I. \tag{32}$$

For any I we associate a point $x_I \in F \cap I$. Let $M_k = \left\{x = x_I : I \in F^{(k)}\right\}$, it can be defined a sequence of measures equidistributed on F by setting

$$\nu_k = \frac{1}{card\ M_k} \sum_{x \in M_k} \delta_x. \tag{33}$$

If ν is the weak $*-$ limit of the sequence $\{\nu_k\}$ then $\nu(F) = 1$.

Recall that the point-wise dimension of a measure ν is given by $D_\nu(x) = \lim_{\eta \to 0} \frac{\nu(B_\eta(x))}{\log \eta}$, where $B_\eta(x)$ is the ball with centre in x and radius η (Eq.14). We must to estimate $\nu(B_\eta(x))$, for $x \in F$ and apply the volume lemma for the point-wise dimension of a measure. Let $t_{\ell,r} = \sum_{i=1}^{\ell} (j_i k_i + \gamma_i) + j_{n+1} r$, $\ell = 2, ..., n$, $r = 0, 1, , ..., k_{\ell-1}$. Hence for any $x \in F$, $I \in \mathcal{I}^{(k)}$ such that $x \in I$ and $f^{t_{\ell,r}}(x) \in I$, we have by 2. and since $\alpha_n \to \alpha$, as $j \to \infty$, that

$\lim_{j \to \infty} \frac{S_j(\varphi(x))}{S_j(\psi(x))} = \alpha$, for any x, $\nu - a.e.$

Let $J \in F^{(k)}$, recall that

$$\nu_k(J) = \frac{card\ \{M_k \cap J\}}{card\ M_k}, \tag{34}$$

for any $t \geq k$, this can be rewritten as

$$\nu_t(J) = \frac{card\ \left\{ I \in F^{(t)} : I \subset J \right\}}{card\ F^{(t)}}. \tag{35}$$

Therefore

$$\lim_{t \to \infty} \nu_t(I) = \frac{1}{card\ F^{(k)}}, \text{for any } I \in F^{(t)}. \tag{38}$$

To estimate $\nu(B_\eta(x))$, for $x \in F$ and for a small η, $card\ \{\ J \cap B_\eta(x)\}$ must be bounded for any $J \in F^{(k)}$. This is done by the computations in [8]. It holds

$$card\ F^{(k)} \geq (card\ \mathcal{I}_n)^{k_n} (card\ \mathcal{I}_{n+1})^r, \text{ for } n = n(k),\ r = r(k). \tag{37}$$

Like in [8], can be chosen k such that the radius η satisfies

$$\eta > \exp\left[j_n k_n \left(\lambda(\mu_n) + \frac{2}{n} \right) - j_{n+1} \left(\lambda(\mu_n + 1) + \frac{2}{n+1} \right) \right] \tag{38}$$

for $n = n(k)$, $r = r(k)$ and

$$\eta < \exp\left[j_n k_n \left(\lambda(\mu_n) + \frac{2}{n} \right) - j_{n+1} \left(\lambda(\mu_n + 1) + \frac{1}{n+1} \right) \right] \tag{39}$$

for $n = n(k-1)$, $r = r(k-1)$.

From [8] and the Katok formula to approximate the entropy we have

$$card\ \left(B_\eta(x) \cap F^{(k)} \right) \leq 2 \exp\left[j_{n+1} r \left(\frac{2}{n} - \frac{1}{n+1} \right) \right] \text{ for any } x \in F$$

and

$$\begin{aligned} (card\ \mathcal{I}_n)^{k_n} (card\ \mathcal{I}_{n+1})^r &\geq \\ \exp\left[j_n k_n \left(h_{\mu_n}(f) - \frac{1}{n} \right) \right] &\exp\left[j_{n+1} \left(h_{\mu_{n+1}}(f) \right) - \frac{1}{n+1} \right]. \end{aligned}$$

These equations, together with Eqs. (27), (34) and (35), get

$$\nu(B_\eta(x)) \leq \frac{2}{card\ F^{(k)}} \exp\left[-j_n k_n \left(h_{\mu_n}(f) - \frac{1}{n} \right) - j_{n+1} r \left(h_{\mu_{n+1}}(f) - \frac{2}{n} \right) \right].$$

Hence

$$\frac{\log \nu\left(B_{\eta}(x)\right)}{\log \eta} \geq$$
$$\frac{\log\left(\frac{2}{card\ F^{(k)}} \exp\left[-j_n k_n\left(h_{\mu_n}(f)-\frac{1}{n}\right)-j_{n+1} r\left(h_{\mu_{n+1}}(f)-\frac{2}{n}\right)\right]\right)}{\left[j_n k_n\left(\lambda\left(\mu_n\right)+\frac{2}{n}\right)-j_{n+1}\left(\lambda\left(\mu_n+1\right)+\frac{1}{n+1}\right)\right]}.$$

Then taking $\lim\limits_{\eta \to 0}$, so $n \to \infty$ we get

$$D_{\nu}(x) \geq \lim_{n\to\infty} \frac{h_{\mu_n}(f)}{\lambda\left(\mu_n, f\right)} = t. \tag{40}$$

Therefore by the volume lemma

(41)
$$\dim_H K_{\varphi,\psi} \geq t = \lim_{\varepsilon\to 0} \sup\left\{\frac{h_{\mu}(f)}{\lambda(\mu, f)} : \mu \in \mathcal{M}^0_{inv}(X,f),\ \frac{\int \varphi d\mu}{\int \psi d\mu} \in (\alpha-\varepsilon, \alpha+\varepsilon)\right\}$$

and the lower bound is obtained □

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