On the multiplicity of isolated roots of sparse polynomial systems*

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Abstract

We give formulas for the multiplicity of any affine isolated zero of a generic polynomial system of n equations in n unknowns with prescribed sets of monomials. First, we consider sets of supports such that the origin is an isolated root of the corresponding generic system and prove formulas for its multiplicity. Then, we apply these formulas to solve the problem in the general case, by showing that the multiplicity of an arbitrary affine isolated zero of a generic system with given supports equals the multiplicity of the origin as a common zero of a generic system with an associated family of supports.

The formulas obtained are in the spirit of the classical Bernstein's theorem, in the sense that they depend on the combinatorial structure of the system, namely, geometric numerical invariants associated to the supports, such as mixed volumes of convex sets and, alternatively, mixed integrals of convex functions.

Keywords: Sparse polynomial systems, Multiplicity of zeros, Newton polytopes, Mixed volumes and mixed integrals

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1 Introduction

The connections between the set of solutions of a polynomial system and the geometry of the supports of the polynomials involved have been studied in the literature, starting with the foundational work of Bernstein [1], Kushnirenko [15] and Khovanskii [13]. They proved that the number of isolated solutions in $(\mathbb{C}^*)^n$ (where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$) of a system with n polynomial equations in n unknowns is bounded from above by the mixed volume of their support sets. Afterwards, combinatorial invariants of the same type also allowed to obtain bounds for the number of isolated solutions of the system in the affine space $\mathbb{C}^n$ (see, for

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example, [22], [23], [16], [11] and [7]). In [21], another refinement of Bernstein's bound was given by introducing mixed integrals of concave functions to estimate the number of isolated solutions in $\mathbb{C} \times (\mathbb{C}^*)^{n-1}$. Concerning algorithmic complexity, counting the number of isolated roots is known to be #**P**-complete already for binomial systems [3]. The complexity of counting irreducible components of algebraic varieties is studied in [2].

Even though the common zeroes of sparse polynomial systems in $(\mathbb{C}^*)^n$ are generically simple, the isolated roots on coordinate hyperplanes may generically have high multiplicity. The aim of this paper is to prove formulas for the multiplicity of the isolated affine zeroes of generic sparse polynomial systems in terms of the geometry of their supports. This dependence is already present in the seminal work of Kushnirenko [14], where the Milnor number of the singularity at the origin of a hypersurface is studied.

Several authors have used geometric tools, including convex sets, volumes and covolumes, to solve related problems. Geometric invariants of this type are considered in [25] to determine multiplicities of monomial ideals in local rings. In [8, Chapter 5], the multiplicity of a singular point on a toric variety is given as a normalized volume. A particular case of this result is recovered in [5], where the multiplicity of the origin as an isolated zero of a generic unmixed polynomial system is computed under the assumption that each polynomial contains a pure power of each variable. A generalization of this result to the mixed case under the same assumption can be found in [12], where the multiplicity of the origin is expressed in terms of mixed covolumes. Recently, in [18] a formula for the intersection multiplicity at the origin of the hypersurfaces defined in $\mathbb{C}^n$ by n generic polynomials with fixed Newton diagrams is proved.

In this paper, we obtain formulas for the multiplicities of all the affine isolated zeros of a generic polynomial system of n polynomials in n variables with given supports in terms of mixed volumes and, alternatively, in terms of mixed integrals of convex functions associated to the supports of the polynomials involved (see Theorem 20 in Section 4.1 below).

First, we consider the case of the origin as an isolated zero of a generic system where each polynomial contains a pure power of each variable (see Theorem 10 in Section 3.1). A formula for the multiplicity of the origin under this particular hypothesis has already been obtained in [12] in terms of different invariants. Then, we analyze the case of generic systems with arbitrary supports such that the origin is an isolated zero (see Proposition 12 and Corollary 13 in Section 3.2).

Finally, in order to deal with arbitrary affine isolated zeros, the result in [10, Proposition 6] enables us to determine all sets $I \subset \{1, \dots, n\}$ such that a generic system with the given supports has isolated zeros whose vanishing coordinates are indexed by I. For such an isolated zero, we prove that its multiplicity equals the multiplicity of the origin as an isolated zero of an associated generic sparse system of $\#I$ polynomials in $\#I$ variables whose supports can be explicitly defined from the input supports and the set I (see Theorem 16 in Section 4.1). Thus, a formula for the multiplicity of an arbitrary affine zero of the system follows from our previous result concerning the multiplicity of the origin.

Our formulas for the multiplicity of the origin can be seen as a generalization of those in [12], in the sense that the only hypotheses on the supports we make are the necessary ones, proved in [10, Proposition 6], so that the origin is an isolated zero of a generic system with the given supports. An earlier approach from [18] to compute the multiplicity of the origin under no further assumptions on the supports leads to a formula which, unlike ours, is not symmetric in the input polynomials, as already stated by the author. Furthermore, in this paper we give formulas for the multiplicity of arbitrary isolated affine zeroes of a generic sparse system.

The paper is organized as follows: Section 2 recalls the definitions and basic properties of mixed volumes and mixed integrals, and describes the algorithmic approach to compute multiplicities of isolated zeros of polynomial systems by means of basic linear algebra given in [6], which we use as a tool. In Section 3, formulas for the multiplicity of the origin are obtained, first for systems where each polynomial contains a pure power of each variable and then, in the general case. Finally, Section 4 is devoted to computing the multiplicity of an arbitrary affine isolated zero of a generic system.

2 Preliminaries

2.1 Mixed volume and stable mixed volume

Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be finite subsets of $(\mathbb{Z}_{\geq 0})^n$. A *sparse polynomial system supported on* $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ is given by polynomials

$$f_j = \sum_{a \in \mathcal{A}_j} c_{j,a} x^a$$

in the variables $x = (x_1, \dots, x_n)$, with $c_{j,a} \in \mathbb{C} \setminus \{0\}$ for each $a \in \mathcal{A}_j$ and $1 \leq j \leq n$.

We denote by $MV_n(\mathcal{A}) = MV_n(\mathcal{A}_1, \dots, \mathcal{A}_n)$ the *mixed volume* of the convex hulls of $\mathcal{A}_1, \dots, \mathcal{A}_n$ in $\mathbb{R}^n$, which is defined as

$$MV_n(\mathcal{A}) = \sum_{J \subset \{1,\dots,n\}} (-1)^{n-\#J}\, Vol_n\Big(\sum_{j \in J} \mathrm{conv}(\mathcal{A}_j)\Big)$$

(see, for example, [4, Chapter 7]). The mixed volume of $\mathcal{A}$ is an upper bound for the number of isolated roots in $(\mathbb{C}^*)^n$ of a sparse system supported on $\mathcal{A}$ (see [1]).

The *stable mixed volume* of $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$, denoted by $SM_n(\mathcal{A}) = SM_n(\mathcal{A}_1, \dots, \mathcal{A}_n)$, is introduced in [11] to estimate the number of isolated roots in $\mathbb{C}^n$ of a sparse polynomial system supported on $\mathcal{A}$ and is defined as follows. Let $\mathcal{A}^0 = (\mathcal{A}_1^0, \dots, \mathcal{A}_n^0)$ be the family with $\mathcal{A}_j^0 := \mathcal{A}_j \cup \{0\}$ for every $1 \leq j \leq n$, and let $\omega^0 = (\omega_1^0, \dots, \omega_n^0)$ be the lifting function for $\mathcal{A}^0$ defined by $\omega_j^0(q) = 0$ if $q \in \mathcal{A}_j$ and $\omega_j^0(0) = 1$ if $0 \notin \mathcal{A}_j$. Consider the polytope Q^0 in $\mathbb{R}^{n+1}$ obtained by taking the Minkowski (pointwise) sum of the convex hulls of the graphs of $\omega_1^0, \dots, \omega_n^0$. The projection of the lower facets of Q^0 (that is, the n-dimensional faces with inner normal vector with a positive last coordinate) induces a subdivision of $\mathcal{A}^0$. A cell $C = (C_1, \dots, C_n)$, with $C_j \subset \mathcal{A}_j^0$ for every $1 \leq j \leq n$, of this subdivision is said to be *stable* if it corresponds to a facet of Q^0 having an inner normal vector with all non-negative coordinates. The stable mixed volume $SM_n(\mathcal{A}_1, \dots, \mathcal{A}_n)$ is the sum of the mixed volumes of all the stable cells in the subdivision of $\mathcal{A}^0$.

Note that $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ is a stable cell in the defined subdivision of $\mathcal{A}^0$, namely, the cell with associated inner normal vector $(0, \dots, 0, 1)$; therefore, we have that

$$MV_n(\mathcal{A}_1, \dots, \mathcal{A}_n) \leq SM_n(\mathcal{A}_1, \dots, \mathcal{A}_n) \leq MV_n(\mathcal{A}_1 \cup \{0\}, \dots, \mathcal{A}_n \cup \{0\}).$$

2.2 Mixed integrals for concave and convex functions

Let $P_1, \dots, P_n$ be polytopes in $\mathbb{R}^{n-1}$, and, for $1 \leq j \leq n$, let $\sigma_j : P_j \to \mathbb{R}$ be a concave function and $\rho_j : P_j \to \mathbb{R}$ a convex function. Following [20], we can define concave (respectively convex) functions as:

$$\begin{aligned}&\sigma_i \boxplus \sigma_j : P_i + P_j \to \mathbb{R},\\ &\sigma_i \boxplus \sigma_j(x) = \max\{\sigma_i(y) + \sigma_j(z) : y \in P_i,\, z \in P_j,\, y + z = x\}\end{aligned}$$

and

$$\begin{aligned}&\rho_i \boxplus' \rho_j : P_i + P_j \to \mathbb{R},\\ &\rho_i \boxplus' \rho_j(x) = \min\{\rho_i(y) + \rho_j(z) : y \in P_i,\, z \in P_j,\, y + z = x\}.\end{aligned}$$

Note that $\rho_i \boxplus' \rho_j = -(-\rho_i) \boxplus (-\rho_j)$.

In the same way, for every non-empty subset $J \subset \{1, \dots, n\}$, we can define

$$\boxplus_{j\in J}\sigma_j : \sum_{j\in J} P_j \to \mathbb{R} \quad \text{and} \quad \boxplus'_{j\in J}\,\rho_j : \sum_{j\in J} P_j \to \mathbb{R}.$$

The mixed integrals of $\sigma_1, \dots, \sigma_n$ (respectively, $\rho_1, \dots, \rho_n$) are defined as:

$$MI_n(\sigma_1, \dots, \sigma_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{J\subset\{1,\dots,n\}\\ \#J=k}} \int_{\sum_{j\in J} P_j} \boxplus_{j\in J}\sigma_j(x)dx,$$

$$MI'_n(\rho_1, \dots, \rho_n) = \sum_{k=1}^{n} (-1)^{n-k} \sum_{\substack{J\subset\{1,\dots,n\}\\ \#J=k}} \int_{\sum_{j\in J} P_j} \boxplus'_{j\in J}\rho_j(x)dx.$$

For a polytope $P \subset \mathbb{R}^{n-1}$, a convex function $\rho : P \to \mathbb{R}$ and a concave function $\sigma : P \to \mathbb{R}$ such that $\rho(x) \leq \sigma(x)$ for every $x \in P$, we denote

$$P_{\rho,\sigma} = \operatorname{conv}(\{(x, \rho(x)) : x \in P\} \cup \{(x, \sigma(x)) : x \in P\}).$$

Given a polytope $Q \subset \mathbb{R}^n$, if $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the projection to the first $n - 1$ coordinates, we may define a concave function $\sigma_Q : \pi(Q) \to \mathbb{R}$ and a convex function $\rho_Q : \pi(Q) \to \mathbb{R}$ as:

$$\sigma_Q(x) = \max\{x_n \in \mathbb{R} : (x, x_n) \in Q\} \quad \text{and} \quad \rho_Q(x) = \min\{x_n \in \mathbb{R} : (x, x_n) \in Q\}.$$

Remark 1 *The functions σ_Q and ρ_Q defined above parameterize the lower and upper envelopes of Q respectively. Moreover, $\pi(Q)_{\rho_Q,\sigma_Q} = Q$.*

Let $Q_1, \dots, Q_n$ be polytopes in $\mathbb{R}^n$. For $1 \leq j \leq n$, let $\sigma_j = \sigma_{Q_j}$ and $\rho_j = \rho_{Q_j}$. Let $J \subset \{1, \dots, n\}$, $J \neq \emptyset$. Then, $\boxplus_{j\in J}\sigma_j : \sum_{j\in J} \pi(Q_j) \to \mathbb{R}$ and $\boxplus'_{j\in J}\rho_j : \sum_{j\in J} \pi(Q_j) \to \mathbb{R}$ parameterize the upper and lower envelopes of $\sum_{j\in J} Q_j$ respectively.

2.3 Multiplicity matrices

In order to compute multiplicities of isolated zeros of polynomial systems, we will follow the algorithmic approach from [6] based on duality theory, which we briefly recall in this section.

Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a system of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$. Denote $\mathcal{I}$ the ideal of $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_n$.

For an isolated zero $\zeta \in \mathbb{C}^n$ of the system $\mathbf{f}$, we denote $\text{mult}_\zeta(\mathbf{f})$ its multiplicity, defined as the dimension (as a $\mathbb{C}$-vector space) of the local ring $\mathbb{C}[x]_{\mathfrak{m}_\zeta}/\mathcal{I}\mathbb{C}[x]_{\mathfrak{m}_\zeta}$, where $\mathfrak{m}_\zeta = (x_1 - \zeta_1, \ldots, x_n - \zeta_n)$ is the maximal ideal associated with ζ (see, for instance, [4, Chapter 4, Definition (2.1)]).

Let $\mathcal{D}_\zeta(\mathcal{I})$ the dual space of the ideal $\mathcal{I}$ at ζ; namely, the vector space

$$\mathcal{D}_\zeta(\mathcal{I}) = \Big\{ c = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n} c_\alpha \, \partial_\alpha[\zeta] \mid c(f) = 0 \text{ for all } f \in \mathcal{I} \Big\},$$

where, for every $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}_{\geq 0})^n$, $c_\alpha \in \mathbb{C}$,

$$\partial_\alpha = \frac{1}{\alpha_1! \ldots \alpha_n!} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots , x_n^{\alpha_n}}, \tag{1}$$

and

$$\partial_\alpha[\zeta] : \mathbb{C}[x] \to \mathbb{C}, \quad \partial_\alpha[\zeta](f) = (\partial_\alpha f)(\zeta).$$

The dimension of $\mathcal{D}_\zeta(\mathcal{I})$ equals the multiplicity of ζ as a zero of $\mathcal{I}$ (see [17], [24]).

For every $k \geq 0$, consider the subspace

$$\mathcal{D}^k_\zeta(\mathcal{I}) = \Big\{ c = \sum_{\alpha \in (\mathbb{Z}_{\geq 0})^n, \ |\alpha| \leq k} c_\alpha \, \partial_\alpha[\zeta] \mid c(f) = 0 \text{ for all } f \in \mathcal{I} \Big\}$$

of all functionals in $\mathcal{D}_\zeta(\mathcal{I})$ with differential order bounded by k. Since ζ is an isolated common zero of $\mathcal{I}$, there exists $k_0 \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{D}_\zeta(\mathcal{I}) = \mathcal{D}^{k_0}_\zeta(\mathcal{I}) = \mathcal{D}^k_\zeta(\mathcal{I})$ for all $k \geq k_0$ and $\dim(\mathcal{D}^k_\zeta(\mathcal{I})) < \dim(\mathcal{D}^{k+1}_\zeta(\mathcal{I}))$ for every $0 \leq k < k_0$ (see [6, Lemma 1]).

Following [6, Section 4], the dimension of the vector spaces $\mathcal{D}^k_\zeta(\mathcal{I})$ can be computed by means of the *multiplicity matrices*, defined as follows. For $k = 0$, set $S_0(\mathbf{f}, \zeta) = [f_1(\xi) \cdots f_n(\xi)]^t = 0 \in \mathbb{C}^{n \times 1}$. Take $\prec$ a graded monomial ordering. For $k \geq 1$, consider the sets $\mathbb{I}_k = \{\alpha \in (\mathbb{Z}_{\geq 0})^n \mid |\alpha| \leq k\}$ ordered by $\prec$, and $\mathbb{I}_{k-1} \times \{1, \ldots, n\}$ with the ordering $(\beta, j) \prec (\beta', j')$ if $\beta \prec \beta'$ or $\beta = \beta'$ and $j < j'$. Let $S_k(\mathbf{f}, \zeta)$ be the $\binom{k-1+n}{k-1} n \times \binom{k+n}{k}$ matrix whose columns are indexed by $\mathbb{I}_k$ (corresponding to the differential functionals ∂_α for $\alpha \in \mathbb{I}_k$) and whose rows are indexed by $(\beta, j) \in \mathbb{I}_{k-1} \times \{1, \ldots, n\}$ (corresponding to the polynomials $(x - \zeta)^\beta f_j$) such that the entry at the intersection of the row indexed by (β, j) and the column indexed by α is

$$(S_k(\mathbf{f}, \zeta))_{(\beta,j),\alpha} = \partial_\alpha((x - \zeta)^\beta f_j)(\zeta).$$

(Here, $(x - \zeta)^\beta = (x_1 - \zeta_1)^{\beta_1} \cdots (x_n - \zeta_n)^{\beta_n}$.) Then, the dimension of $\mathcal{D}^k_\zeta(\mathcal{I})$ equals the dimension of the nullspace of $S_k(\mathbf{f}, \zeta)$ (see [6, Theorems 1 and 2]). As a consequence:

Proposition 2 *With the previous assumptions and notation, if*

$$k_0 = \min\{k \in \mathbb{Z}_{\geq 0} \mid \dim(\ker(S_k(\mathbf{f}, \zeta))) = \dim(\ker(S_{k+1}(\mathbf{f}, \zeta)))\},$$

the multiplicity of ζ *as an isolated zero of* $\mathbf{f}$ *is* $\mathrm{mult}_\zeta(\mathbf{f}) = \dim(\ker(S_k(\mathbf{f}, \zeta)))$ *for any* $k \geq k_0$.

3 Multiplicity of the origin

Consider a family $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of finite sets in $(\mathbb{Z}_{\geq 0})^n$ such that $0 \notin \mathcal{A}_j$ for all $1 \leq j \leq n$. Under this assumption, $0 \in \mathbb{C}^n$ is a common zero of any sparse system of polynomials $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ supported on $\mathcal{A}$.

We are interested in the case when 0 is an *isolated* common zero of the system. By [10, Proposition 6], for a *generic* family of polynomials $\mathbf{f} = f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ supported on $\mathcal{A}$, we have that 0 is an isolated point of $V(\mathbf{f})$ if and only if $\#I + \#J_I \geq n$ for all $I \subset \{1, \ldots, n\}$, where J_I is the set of subindexes of all polynomials that do not vanish when we evaluate $x_i = 0$ for all $i \in I$.

Every $\mathbf{c} = (\mathbf{c}_1, \ldots, \mathbf{c}_n) \in \mathbb{C}^{\#\mathcal{A}_1} \times \cdots \times \mathbb{C}^{\#\mathcal{A}_n}$ defines a system $\mathbf{f_c}$ of polynomials with coefficients $\mathbf{c}$ supported on a family of subsets of $\mathcal{A}_1, \ldots, \mathcal{A}_n$. If 0 is an isolated zero of $\mathbf{f_c}$, we define $\mathrm{mult}_{\mathcal{A}}(\mathbf{c}) := \mathrm{mult}_0(\mathbf{f_c}) \in \mathbb{Z}_{>0}$.

Lemma 3 *Under the previous assumptions and notation, let* $\mu_{\mathcal{A}}$ *be the minimum of the function* $\mathrm{mult}_{\mathcal{A}}$. *Then,* $\{\mathbf{c} \in \mathbb{C}^{\#\mathcal{A}_1} \times \cdots \times \mathbb{C}^{\#\mathcal{A}_n} \mid \mathrm{mult}_{\mathcal{A}}(\mathbf{c}) = \mu_{\mathcal{A}}\}$ *contains a non-empty Zariski open set of* $\mathbb{C}^{\#\mathcal{A}_1} \times \cdots \times \mathbb{C}^{\#\mathcal{A}_n}$.

Proof: It is straightforward, for example, from the computation of multiplicities by using multiplicity matrices (see Section 2.3). □

In this sense, we may speak of $\mu_{\mathcal{A}}$ as the multiplicity of 0 as an isolated root of a *generic* sparse system supported on $\mathcal{A}$. Explicit conditions on the coefficients satisfying $\mathrm{mult}_{\mathcal{A}}(\mathbf{c}) = \mu_{\mathcal{A}}$ are given in [18, Theorem 4.12].

Therefore, in this section, we will focus on the computation of the multiplicity of the origin as a common zero of a generic polynomial system supported on $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$, under the following assumptions:

(H1) $0 \notin \mathcal{A}_j \subset (\mathbb{Z}_{\geq 0})^n$ for every $1 \leq j \leq n$;

(H2) for all $I \subset \{1, \ldots, n\}$, if $J_I := \{j \in \{1, \ldots, n\} \mid \exists a \in \mathcal{A}_j : a_i = 0 \, \forall i \in I\}$, then $\#I + \#J_I \geq n$.

Moreover, in [10, Proposition 5], these conditions are proved to be equivalent to the fact that, for a generic system $\mathbf{f}$ supported on $\mathcal{A}$ and vanishing at $0 \in \mathbb{C}^n$, the variety $V(\mathbf{f})$ consists only of isolated points in $\mathbb{C}^n$.

Under these assumptions, by [11, Theorem 2], the number of common zeros of $\mathbf{f}$ in $\mathbb{C}^n$ counted with multiplicities is the stable mixed volume $SM_n(\mathcal{A})$. In particular, since the number of common zeros of the system in $(\mathbb{C}^*)^n$ is the mixed volume $MV_n(\mathcal{A})$ (see [1]), we have that

$$\mathrm{mult}_0(\mathbf{f}) \leq SM_n(\mathcal{A}) - MV_n(\mathcal{A}) \leq MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}), \tag{2}$$

where $\mathcal{A}^0 = (\mathcal{A}_1 \cup \{0\}, \ldots, \mathcal{A}_n \cup \{0\})$.

3.1 A particular case

The first case we are going to consider is when the following stronger assumption on $\mathcal{A}$ holds:

(H3) For every $1 \leq i, j \leq n$, there exists $\mu_{ij} \in \mathbb{N}$ such that $\mu_{ij} e_i \in \mathcal{A}_j$, where e_i is the ith vector of the canonical basis of $\mathbb{Q}^n$.

Note that assumption (H3) implies that assumption (H2) holds.

Under condition (H3), in [12, Theorem 7.6] the multiplicity of the origin as an isolated common zero of a generic polynomial system supported on $\mathcal{A}$ is computed in terms of covolumes of coconvex bodies associated to $\mathcal{A}$. Here, we will first re-obtain this result by proving a formula using mixed volumes of convex polytopes and then, we will reformulate this formula in terms of mixed integrals of convex functions.

We start by comparing stable mixed volumes with mixed volumes in our particular setting.

Lemma 4 *With the previous notation, if assumptions (H1) and (H3) hold, we have that*

$$SM_n(\mathcal{A}_1, \dots, \mathcal{A}_n) = MV_n(\mathcal{A}_1^0, \dots, \mathcal{A}_n^0).$$

Proof: It suffices to prove that every cell in the subdivision of $\mathcal{A}^0 = (\mathcal{A}_1^0, \dots, \mathcal{A}_n^0)$ induced by the lifting function introduced in Section 2.1 is stable.

Consider a cell $C = (C_1, \dots, C_n)$ of the stated subdivision different from $(\mathcal{A}_1, \dots, \mathcal{A}_n)$ (for which the result is trivial), and let $\eta = (\eta_1, \dots, \eta_n, 1)$ be its associated inner normal vector. We have to show that $\eta_i \geq 0$ for every $1 \leq i \leq n$.

For every $1 \leq j \leq n$, there exists $a_{C_j} \in \mathbb{R}$ such that $a_{C_j} = \eta.(q, \omega_j^0(q))$ for all $q \in C_j$ and $a_{C_j} \leq \eta.(q, \omega_j^0(q))$ for all $q \in \mathcal{A}_j^0$. As the cell C is not $(\mathcal{A}_1, \dots, \mathcal{A}_n)$, there exists j_0 such that $0 \in C_{j_0}$ and $0 \notin \mathcal{A}_{j_0}$; then, $a_{C_{j_0}} = \eta.(0,1) = 1$. Since, by assumption (H3), for all $1 \leq i \leq n$, there exists $\mu_{ij_0} \in \mathbb{N}$ such that $\mu_{ij_0} e_i \in \mathcal{A}_{j_0}^0$, then, $1 = a_{C_{j_0}} \leq \eta \,.\, \mu_{ij_0}(e_i, 0) = \eta_i \mu_{ij_0}$. The result follows from the fact that $\mu_{ij_0} > 0$ for all $1 \leq i \leq n$. □

Now, we can state our first formula for the multiplicity of the origin.

Proposition 5 *Let $\mathbf{f} = (f_1, \dots, f_n)$ be a generic polynomial system in $\mathbb{C}[x_1, \dots, x_n]$ supported on a family $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ of finite sets of $(\mathbb{Z}_{\geq 0})^n$ satisfying assumptions (H1) and (H3). Then, the origin is an isolated common zero of $\mathbf{f}$ and*

$$\operatorname{mult}_0(\mathbf{f}) = MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}).$$

Proof: Assumption (H1) implies that the origin is a common zero of the polynomials $\mathbf{f}$. In addition, by assumption (H3), the only common zero of $\mathbf{f}$ not in $(\mathbb{C}^*)^n$ is the origin. Then, all the common zeros of $\mathbf{f}$ in $\mathbb{C}^n$ are isolated and so, the number of these common zeros is $SM_n(\mathcal{A})$ (see [11]). Finally, since the number of common zeros of $\mathbf{f}$ in $(\mathbb{C}^*)^n$ is $MV_n(\mathcal{A})$ (see [1]) and all these zeros have multiplicity 1 (see [19]), we deduce that $MV_n(\mathcal{A}) + \operatorname{mult}_0(\mathbf{f}) = SM_n(\mathcal{A})$. Thus, the result follows from Lemma 4. □

Example 1 *Consider the generic polynomial system* $\mathbf{f} = (f_1, f_2, f_3)$ *with*

$$\begin{array}{rcl} f_1 & = & c_{11}x_1 + c_{12}x_2 + c_{13}x_2^2 + c_{14}x_1^2x_2x_3 + c_{15}x_3^7 \\ f_2 & = & c_{21}x_1^2 + c_{22}x_1^3 + c_{23}x_1^2x_2 + c_{24}x_3^3 + c_{25}x_2^7 \\ f_3 & = & c_{31}x_1 + c_{32}x_1x_2 + c_{33}x_3^2 + c_{34}x_2x_3^3 + c_{35}x_2^7 \end{array}$$

with support family $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$*, where*

$$\begin{array}{rcl} \mathcal{A}_1 & = & \{(1,0,0),(0,1,0),(0,2,0),(2,1,1),(0,0,7)\} \\ \mathcal{A}_2 & = & \{(2,0,0),(3,0,0),(2,1,0),(0,0,3),(0,7,0)\} \\ \mathcal{A}_3 & = & \{(1,0,0),(1,1,0),(0,0,2),(0,1,3),(0,7,0)\} \end{array}$$

satisfying assumptions (H1) and (H3). Then, Proposition 5 states that 0 *is an isolated common root of* $\mathbf{f}$ *with multiplicity*

$$\mathrm{mult}_0(\mathbf{f}) = MV_3(\mathcal{A}^0) - MV_3(\mathcal{A}) = 147 - 144 = 3.$$

In order to restate the formula in the previous proposition by means of a mixed integral of suitable convex functions, we first introduce further notation and prove some auxiliary results.

For $1 \leq j \leq n$, let $Q_j = \mathrm{conv}(\mathcal{A}_j)$ and $\Delta_j = \mathrm{conv}\{0, \lambda_{1j}e_1, \ldots, \lambda_{nj}e_n\}$, where

$$\lambda_{ij} = \min\{\mu \in \mathbb{N} \mid \mu e_i \in Q_j\} \quad \text{for } 1 \leq i \leq n. \tag{3}$$

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection to the first $n-1$ coordinates. As in Section 2.2, let $\sigma_j : \pi(Q_j) \to \mathbb{R}$ denote the concave function that parameterizes the upper envelope of Q_j and $\rho_j : \pi(Q_j) \to \mathbb{R}$ the convex function that parameterizes its lower envelope. Since $\pi(\Delta_j) \subset \pi(Q_j)$, we may consider

$$\overline{\sigma}_j = \sigma_j|_{\pi(\Delta_j)} \quad \text{and} \quad \overline{\rho}_j = \rho_j|_{\pi(\Delta_j)}, \tag{4}$$

the restrictions of these functions to $\pi(\Delta_j)$.

For a non-empty set $J \subset \{1, \ldots, n\}$, we denote

$$\Delta_J := \sum_{j \in J} \Delta_j, \qquad Q_J := \sum_{j \in J} Q_j.$$

Lemma 6 *Let* $J \subset \{1, \ldots, n\}$ *be a non-empty set. Then, every facet of* Δ_J *that is not contained in a hyperplane* $\{x_i = 0\}$*, for* $1 \leq i \leq n$*, has an inner normal vector with all negative coordinates. We will call these facets the* non-trivial *facets of* Δ_J*.*

Proof: If $J = \{j\}$ for some $1 \leq j \leq n$, the result is straightforward because the only facet satisfying the required conditions is $F = \mathrm{conv}\{\lambda_{1j}e_1, \ldots, \lambda_{nj}e_n\}$, and $\lambda_{ij} \in \mathbb{N}$ for every $1 \leq i \leq n$.

Let F be a non-trivial facet of Δ_J and $\eta = (\eta_1, \ldots, \eta_n)$ an inner normal vector of F. Then, $F = \sum_{j \in J} F_j$, where F_j is a face of Δ_j with inner normal vector η. For every $1 \leq i \leq n$, since F is not contained in the hyperplane $\{x_i = 0\}$, there exists $j_i \in J$ such that $\lambda_{ij_i}e_i \in F_{j_i}$; then,

$$0 = \eta . 0 \geq \eta . \lambda_{ij_i}e_i = \eta_i \lambda_{ij_i} \tag{5}$$

and, so $\eta_i \leq 0$.

If $0 \in F_j$ for some $j \in J$, then $\eta. q \geq \eta \cdot 0 = 0$ for every $q \in \Delta_j$; in particular, $\eta_k \lambda_{kj} = \eta. \lambda_{kj} e_k \geq \eta. 0 = 0$ for every $1 \leq k \leq n$. This implies that $\eta = 0$, a contradiction. Then, $0 \notin F_j$ for every $j \in J$, the inequalities in (5) are strict and, therefore, $\eta_i < 0$ for every $1 \leq i \leq n$. □

Lemma 7 *Let $J \subset \{1, \dots, n\}$ be a non-empty set. Then, for every point x in a non-trivial facet of $\pi(\Delta_J)$ we have that $(\boxplus'_{j\in J}\rho_j)(x) = 0$.*

Proof: If $J = \{j\}$, we have $x \in \operatorname{conv}\{\lambda_{1j}\pi(e_1), \dots, \lambda_{n-1,j}\pi(e_{n-1})\}$, that is, $x = \sum_{i=1}^{n-1} t_i \lambda_{ij}\pi(e_i)$ for $t_i \geq 0$ with $\sum_{i=1}^{n-1} t_i = 1$. Then, since ρ_j is convex, $0 \leq \rho_j(x) \leq \sum_{i=1}^{n-1} t_i \rho_j(\lambda_{ij}\pi(e_i)) = 0$.

If $\#J > 1$, let x be in a nontrivial facet F of $\pi(\Delta_J)$. We have that $F = \sum_{j\in J} F_j$, with F_j a face of $\pi(\Delta_j)$ such that $0 \notin F_j$; then, $x = \sum_{j\in J} p_j$ with $p_j \in F_j$. Hence, $\rho_j(p_j) = 0$ and, by the definition of $\boxplus'_{j\in J}\rho_j$, it follows that $0 \leq (\boxplus'_{j\in J}\rho_j)(x) \leq \sum_{j\in J} \rho_j(p_j) = 0$. □

Lemma 8 *For every non-empty subset J of $\{1, \dots, n\}$, the convex function $\boxplus'_{j\in J}\overline{\rho}_j$ defined over $\pi(\Delta_J)$ parameterizes the lower envelope of Q_J over the points of $\pi(\Delta_J)$.*

Proof: For every $J \subset \{1, \dots, n\}$, we denote

$$P_J := \sum_{j\in J} \pi(Q_j), \quad D_J := \sum_{j\in J} \pi(\Delta_j)$$

and $\overline{\boxplus'_{j\in J}\rho_j}$ to the restriction of $\boxplus'_{j\in J}\rho_j : P_J \to \mathbb{R}$ to $D_J \subset P_J$. With this notation, we have to prove that

$$\boxplus'_{j\in J}\, \overline{\rho}_j = \overline{\boxplus'_{j\in J}\rho_j}. \tag{6}$$

Before proceeding, we will state three basic results that will be applied throughout the proof. We use the notation

$$\rho_J := \boxplus'_{j\in J}\rho_j.$$

Claim I. If p_1 lies in a non-trivial facet of D_J and $p_2 \in P_J$, then for every x lying on the line segment p_1p_2, we have $\rho_J(x) \leq \rho_J(p_2)$: as $x = (1-t)p_1 + tp_2$ for $0 \leq t \leq 1$, ρ_J is convex and $\rho_J \equiv 0$ on the non-trivial facets of D_J, $\rho_J(x) \leq (1-t)\rho_J(p_1) + t\,\rho_J(p_2) = t\,\rho_J(p_2)$.

Claim II. If $p_1 \in D_J$ and $p_2 \notin D_J$, then for every $x \neq p_2$ lying on the line segment p_1p_2, since D_J is a convex set, $d(p_1, D_J) < d(p_2, D_J)$, where $d(\cdot, D_J)$ is the distance to D_J.

Claim III. If $p_1 \in D_J$ and $p_2 \in (\mathbb{R}_{\geq 0})^{n-1} \setminus D_J$ there exists $t \in (0, 1]$ such that $tp_1 + (1-t)p_2$ lies in a non-trivial facet of D_J.

The proof will be done recursively. For a fixed non-empty set $J \subset \{1, \dots, n\}$, let J_1, J_2 be disjoints sets such that $J = J_1 \cup J_2$ and assume that identity (6) holds for each of them. We will prove that if $\overline{\rho}_{J_k} := \boxplus'_{j\in J_k}\overline{\rho}_j$, for $k = 1, 2$, then $\overline{\rho}_{J_1} \boxplus' \overline{\rho}_{J_2} = \overline{\rho_{J_1} \boxplus' \rho_{J_2}}$.

Let $x \in D_J$. Then, there exist $y_0 \in D_{J_1}$ and $z_0 \in D_{J_2}$ such that $x = y_0 + z_0$. Let $y' \in P_{J_1}$ and $z' \in P_{J_2}$ be such that $x = y' + z'$ and $\overline{\rho_{J_1} \boxplus' \rho_{J_2}}(x) = \rho_{J_1}(y') + \rho_{J_2}(z')$. If $y' \in D_{J_1}$ and $z' \in D_{J_2}$ the result follows.

We first show that there exist y' and z' as before satisfying that $y' \in D_{J_1}$ or $z' \in D_{J_2}$. For every $0 \leq t \leq 1$, if $y_t = (1-t)y_0 + ty'$ and $z_t = (1-t)z_0 + tz'$, then $x = y_t + z_t$. If

$y' \notin D_{J_1}$ and $z' \notin D_{J_2}$, there exist $0 < t_1, t_2 \leq 1$ such that y_{t_1} and z_{t_2} lie in non-trivial facets of D_{J_1} and D_{J_2} respectively. Consider $t_0 = \min\{t_1, t_2\}$; then $x = y_{t_0} + z_{t_0}$ and, by Claim I, $\overline{\rho_{J_1} \boxplus \rho_{J_2}}(x) = \rho_{J_1}(y_{t_0}) + \rho_{J_2}(z_{t_0})$.

Now, without loss of generality, assume that $z' \in D_{J_2}$. Consider the compact set

$$C_x = \{y \in P_{J_1} \mid x - y \in D_{J_2} \text{ and } \overline{\rho_{J_1} \boxplus' \rho_{J_2}}(x) = \rho_{J_1}(y) + \rho_{J_2}(x-y)\}.$$

We will prove that $C_x \cap D_{J_1} \neq \emptyset$. If not, let $y \in C_x$ be such that $d(C_x, D_{J_1}) = d(y, D_{J_1}) > 0$.

First, assume that $z := x - y$ does not lie in a non-trivial facet of D_{J_2}. This implies that $z + w \in D_{J_2}$ for every w with sufficiently small non-negative coordinates. Let $0 < \epsilon < 1$ such that $(1-\epsilon)y \notin D_{J_1}$ and that $z + \epsilon y \in D_{J_2}$. Claims III and I imply that $\rho_{J_1}((1-\epsilon)y) \leq \rho_{J_1}(y)$ and that $\rho_{J_2}(z + \epsilon y) \leq \rho_{J_2}(z)$ and, therefore, $\rho_{J_1} \boxplus' \rho_{J_2}(x) = \rho_{J_1}((1-\epsilon)y) + \rho_{J_2}(z + \epsilon y)$. As, by Claim II, $d((1-\epsilon)y, D_{J_1}) < d(y, D_{J_1})$ we have a contradiction.

Assume now that $z := x - y$ lies in non-trivial facets of D_{J_2}.

Recall that $x = y_0 + z_0$ with $y_0 \in D_{J_1}$, $z_0 \in D_{J_2}$. If z and z_0 lie in the same non-trivial facet of D_{J_2}, then the line segment zz_0 is contained in this facet. On the other hand, there exists $0 \leq t \leq 1$ such that $(1-t)y_0 + ty$ lies in a non-trivial facet of D_{J_1}. Therefore, $x = ((1-t)y_0 + ty) + ((1-t)z_0 + tz)$, $\rho_{J_1} \boxplus' \rho_{J_2}(x) = \rho_{J_1}((1-t)y_0 + ty) + \rho_{J_2}((1-t)z_0 + tz) = 0$ and so, $(1-t)y_0 + ty \in C_x \cap D_{J_1}$, which is a contradiction.

If z_0 does not lie in any of the non-trivial facets of D_{J_2} containing z, let $\eta^1, \dots, \eta^k$ be inner normal vectors to these facets and consider the hyperplanes parallel to them and containing y, which are defined by the equations $\eta^\ell.(Y - y) = 0$ for $1 \leq \ell \leq k$. As $\eta^\ell.y + \eta^\ell.z = \eta^\ell.y_0 + \eta^\ell.z_0$ and $\eta^\ell.z < \eta^\ell.z_0$, then $\eta^\ell.y_0 < \eta^\ell.y$. In addition, since all the coordinates of η^ℓ are negative (see Lemma 6) and $y \in (\mathbb{R}_{\geq 0})^{n-1}$, then $\eta^\ell.y < 0$. Therefore, the hyperplane $\eta^\ell \cdot (Y - y) = 0$ intersects the line segment $0y_0$ in a point $\lambda_\ell y_0$ with $0 \leq \lambda_\ell \leq 1$. If $\lambda = \max\{\lambda_\ell \ / \ 1 \leq \ell \leq k\}$, consider $y_t = (1-t)y + t\lambda y_0$ and $z_t = x - y_t$ for $0 \leq t \leq 1$. For t sufficiently small, we will show that $z_t \in D_{J_2}$, that $\rho_{J_1} \boxplus' \rho_{J_2}(x) = \rho_{J_1}(y_t) + \rho_{J_2}(z_t)$ and that $d(y_t, D_{J_1}) < d(y, D_{J_1})$, which leads to a contradiction.

For $1 \leq \ell \leq k$, as $\lambda \geq \lambda_\ell$, $\eta^\ell.(y - \lambda y_0) \geq 0$; then $\eta^\ell \cdot (y - y_t) \geq 0$ and so $\eta^\ell.z_t = \eta^\ell.z + \eta^\ell.(y - y_t) \geq \eta^\ell.z$. If z lies in a trivial facet of D_{J_2}, that is, $z_i = 0$ for some $1 \leq i \leq n$, then $y_i = x_i$; as $(y_0)_i \leq x_i$, we have that $(z_t)_i = t(y_i - \lambda(y_0)_i) \geq 0$. Taking t sufficiently small, z_t satisfies all the remaining inequalities defining D_{J_2} and so, $z_t \in D_{J_2}$. Moreover, since $y \notin D_{J_1}$, for t sufficiently small, $y_t \notin D_{J_1}$. Then, by Claim I, $\rho_{J_1}(y_t) \leq \rho_{J_1}(y)$. On the other hand, z_t lies in the same non-trivial facet of D_{J_2} as z, namely, the facet defined by $\eta^{\ell_0} \cdot (Z - z) = 0$ for ℓ_0 such that $\lambda = \lambda_{\ell_0}$ and, therefore, $\rho_{J_2}(z_t) = 0$. We conclude that $\rho_{J_1}(y_t) + \rho_{J_2}(z_t) = \rho_{J_1} \boxplus' \rho_{J_2}(x)$. Finally, the inequality $d(y_t, D_{J_1}) < d(y, D_{J_1})$ holds by Claim II. □

For every $1 \leq j \leq n$, let $Q_j^0 = \mathrm{conv}(\mathcal{A}_j \cup \{0\})$ and σ_j^0, ρ_j^0 the functions that parameterize its upper and lower envelopes respectively. Assumption (H3) ensures that $\pi(Q_j^0) = \pi(Q_j)$.

Lemma 9 *For every $1 \leq j \leq n$,* $\rho_j^0(x) = \begin{cases} 0 & \text{if } x \in \pi(\Delta_j) \\ \rho_j(x) & \text{if } x \notin \pi(\Delta_j) \end{cases}$ *and* $\sigma_j^0 = \sigma_j$.

Proof: Since $Q_j \subset Q_j^0$, then $\rho_j^0(x) \leq \rho_j(x)$ and $\sigma_j(x) \leq \sigma_j^0(x)$ for every $x \in \pi(Q_j^0)$.

If $x \in \pi(\Delta_j)$, there exists $x_n \geq 0$ such that $(x, x_n) \in \Delta_j$. Then, $(x, x_n) = \sum_{i=1}^n t_i \lambda_{ij} e_i$, where $\sum_{i=1}^n t_i = 1$ and $t_i \geq 0$ for every $1 \leq i \leq n$. Taking $y = \sum_{i=1}^{n-1} t_i \lambda_{ij} e_i$, we have that $\pi(y) = x$, $(y)_n = 0$ and $y \in Q_j^0$. Hence, $\rho_j^0(x) = 0$.

Consider now $x \in \pi(Q_j) \backslash \pi(\Delta_j)$. Take $(x, \rho_j^0(x)) \in Q_j^0 = \mathrm{conv}(Q_j \cup \{0\})$. Then, $(x, \rho_j^0(x)) = tq$, with $q \in Q_j$ and $0 < t \leq 1$. Since $(x, \rho_j^0(x)) \notin \Delta_j$, there exists $0 < t' < 1$ such that $t'(x, \rho_j^0(x))$ lies in the nontrivial facet of Δ_j and so, $q' := t'(x, \rho_j^0(x)) \in Q_j$. Then, the line segment qq' is contained in Q_j; in particular, $(x, \rho_j^0(x)) \in Q_j$. It follows that $\rho_j(x) \leq \rho_j^0(x)$.

If $x \in \pi(Q_j) = \pi(Q_j^0) \subset \mathbb{R}^{n-1}$, consider $(x, \sigma_j^0(x)) \in Q_j^0$. Then, $(x, \sigma_j^0(x)) = tq$ with $q \in Q_j$ and $0 \leq t \leq 1$. If $y = tq + (1-t)\lambda_{nj} e_n \in Q_j$, then $\pi(y) = x$ and so, $\sigma_j(x) \geq (y)_n = \sigma_j^0(x) + (1-t)\lambda_{nj} \geq \sigma_j^0(x)$. □

Now, we can restate the formula for the multiplicity of the origin in Proposition 5 as a mixed integral of convex functions:

Theorem 10 *Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^n$ satisfying assumptions (H1) and (H3). Let $\mathbf{f} = (f_1, \dots, f_n)$ be a generic system of sparse polynomials in $\mathbb{C}[x_1, \dots, x_n]$ supported on $\mathcal{A}$. For every $1 \leq j \leq n$, let $\bar{\rho}_j$ be the convex function defined in (4). Then, the origin is an isolated common zero of $\mathbf{f}$ and*

$$\mathrm{mult}_0(\mathbf{f}) = MI'_n(\overline{\rho}_1, \dots, \overline{\rho}_n).$$

Proof: By Proposition 5, it suffices to show that

$$MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}) = MI'_n(\overline{\rho}_1, \dots, \overline{\rho}_n).$$

For every $1 \leq j \leq n$, consider $\nu_j \in \mathbb{R}$ such that $\nu_j \geq \max(\rho_j) \geq \max(\overline{\rho}_j)$.

For $J \subset \{1, \dots, n\}$, let $D_J = \sum_{j \in J} \pi(\Delta_j)$ and $\nu_J = \sum_{j \in J} \nu_j$. Since $\nu_J \geq \max(\boxplus'_{j \in J} \overline{\rho}_j)$, we have that

$$\int_{D_J} \boxplus'_{j \in J} \overline{\rho}_j \; dx_1 \dots dx_n = \nu_J Vol_{n-1}(D_J) - Vol_n\left((D_J)_{\boxplus'_{j \in J} \overline{\rho}_j, \nu_J} \right).$$

Then, by Lemma 8,

$$\int_{D_J} \boxplus'_{j \in J} \overline{\rho}_j \, dx_1 \dots dx_n = Vol_n\left((D_J)_{0, \nu_J}\right) - Vol_n\left((D_J)_{(\boxplus'_{j \in J} \rho_j)|_{D_J}, \nu_J} \right).$$

Now, if $P_J = \sum_{j \in J} \pi(Q_j^0) = \sum_{j \in J} \pi(Q_j)$, Lemma 9 implies that

$$\begin{aligned} Vol_n\left((D_J)_{0, \nu_J}\right) - Vol_n\left((D_J)_{(\boxplus'_{j \in J} \rho_j)|_{D_J}, \nu_J} \right) &= \\ = Vol_n((P_J)_{\boxplus'_{j \in J} \rho_j^0, \nu_J}) - Vol_n((P_J)_{\boxplus'_{j \in J} \rho_j, \nu_J}) &= \\ = Vol_n\left((P_J)_{\boxplus'_{j \in J} \rho_j^0, \boxplus_{j \in J} \sigma_j^0} \right) - Vol_n\left((P_J)_{\boxplus'_{j \in J} \rho_j, \boxplus_{j \in J} \sigma_j} \right). \end{aligned}$$

Finally, by Remark 1,

$$Vol_n\left((P_J)_{\boxplus'_{j \in J} \rho_j^0, \boxplus_{j \in J} \sigma_j^0} \right) = Vol_n\Big(\sum_{j \in J} Q_j^0 \Big)$$

and

$$Vol_n\left((P_J)_{\boxplus'_{j\in J}\rho_j,\boxplus_{j\in J}\sigma_j}\right) = Vol_n\Big(\sum_{j\in J} Q_j\Big),$$

and so,

$$\int_{D_J} \boxplus'_{j\in J}\overline{\rho}_j \, dx_1 \dots dx_n = Vol_n\Big(\sum_{j\in J} Q_j^0\Big) - Vol_n\Big(\sum_{j\in J} Q_j\Big).$$

The theorem follows from the definitions of the mixed integral and the mixed volume.
□

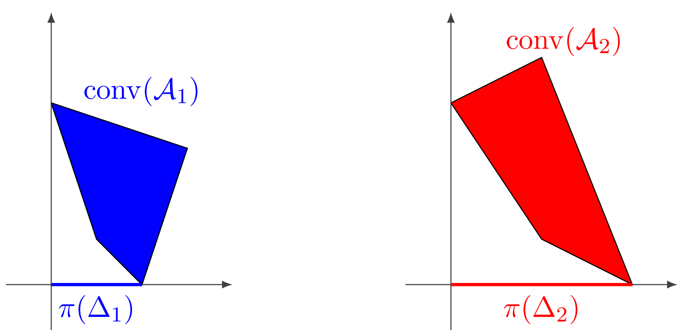
Example 2 *Consider the generic sparse polynomial system*

$$\begin{array}{l} f_1 = c_{1,20}x_1^2 + c_{1,11}x_1x_2 + c_{1,04}x_2^4 + c_{1,13}x_1x_2^3 + c_{1,33}x_1^3x_2^3 \\ f_2 = c_{2,40}x_1^4 + c_{2,21}x_1^2x_2 + c_{2,04}x_2^4 + c_{2,25}x_1^2x_2^5 + c_{2,13}x_1x_2^3 \end{array}$$

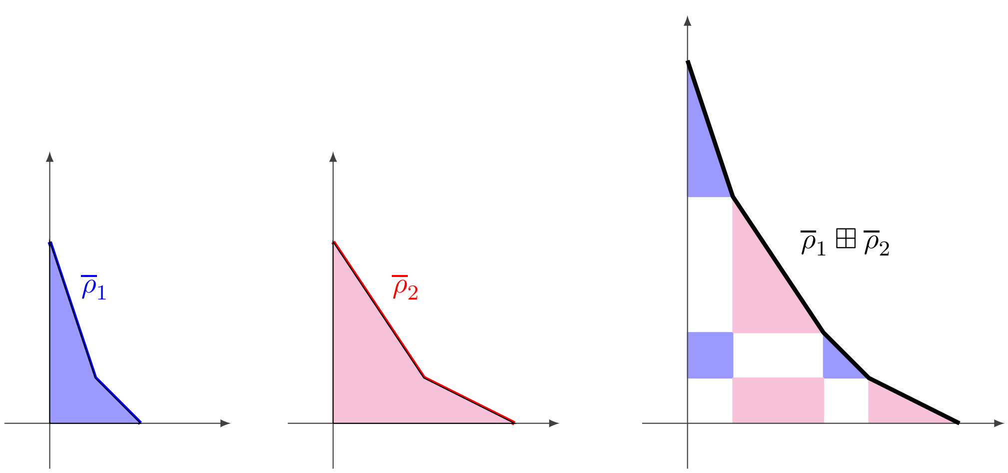
with supports

$$\mathcal{A}_1 = \{(2,0),(1,1),(0,4),(1,3),(3,3)\}, \qquad \mathcal{A}_2 = \{(4,0),(2,1),(0,4),(2,5),(1,3)\}.$$

Here, we have $\Delta_1 = \text{conv}\{(0,0),(2,0),(0,4)\}$ *and* $\Delta_2 = \text{conv}\{(0,0),(4,0),(0,4)\}$.



To compute the multiplicity of the origin following Theorem 10, consider the convex functions $\overline{\rho}_1 : \pi(\Delta_1) \to \mathbb{R}$ *and* $\overline{\rho}_2 : \pi(\Delta_2) \to \mathbb{R}$*:*



Therefore,

$$\mathrm{mult}_0(\mathbf{f}) = MI_2'(\overline{\rho}_1, \overline{\rho}_2) = \int_0^6 \overline{\rho}_1 \boxplus \overline{\rho}_2(x)\, dx - \int_0^2 \overline{\rho}_1(x)\, dx - \int_0^4 \overline{\rho}_2(x)\, dx = 7.$$

Remark 11 *The computation of the multiplicity of the origin by means of mixed integrals following Theorem 10 may involve smaller polytopes than its computation using mixed volumes according to Proposition 5, since it depends only on the points of the lower envelopes of the polytopes $Q_j = \mathrm{conv}(\mathcal{A}_j)$ that lie above the simplices $\pi(\Delta_j)$ for $j = 1, \ldots, n$.*

Following [11], this computation can also be done by locating the stable mixed cells with positive inner normals in a subdivision of $\mathcal{A}^0$ induced by a suitable lifting, and computing and adding the mixed volumes of those cells, which may also involve smaller polytopes. Moreover, the proof of [11, Theorem 2] implies that the mixed integral in Theorem 10 also counts the number of Puiseux series expansions around the origin of the solution set of the system under generic perturbation of the constant terms of the polynomials.

3.2 General case

Consider now a family $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of finite sets in $(\mathbb{Z}_{\geq 0})^n$ satisfying conditions (H1) and (H2). Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a system of generic sparse polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ supported on $\mathcal{A}$.

For $M \in \mathbb{Z}_{>0}$, let $\Delta_M := \{Me_i\}_{i=1}^n$ and, for all $1 \leq j \leq n$, let $\mathcal{A}_j^{\Delta_M} := \mathcal{A}_j \cup \Delta_M$ and $\mathcal{A}_j^{\Delta_M,0} := \mathcal{A}_j^{\Delta_M} \cup \{0\}$. Set $\mathcal{A}^{\Delta_M} := (\mathcal{A}_1^{\Delta_M}, \ldots, \mathcal{A}_n^{\Delta_M})$ and $\mathcal{A}^{\Delta_M,0} := (\mathcal{A}_1^{\Delta_M,0}, \ldots, \mathcal{A}_n^{\Delta_M,0})$.

Proposition 12 *With the previous assumptions and notation, we have that 0 is an isolated common zero of $\mathbf{f}$ and, for every $M \gg 0$, its multiplicity is*

$$\mathrm{mult}_0(\mathbf{f}) = MV_n(\mathcal{A}^{\Delta_M,0}) - MV_n(\mathcal{A}^{\Delta_M}).$$

Moreover, the identity holds for every $M > \mathrm{mult}_0(\mathbf{f})$. In particular, it suffices to take $M = MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}) + 1$.

Proof: Conditions (H1) and (H2) imply that 0 is an isolated common zero of the generic system $\mathbf{f}$ supported on $\mathcal{A}$.

Take $M > \mathrm{mult}_0(\mathbf{f})$ and consider polynomials

$$g_j = f_j + \sum_{i=1}^n c_{j,Me_i}\, x_i^M$$

with support sets $\mathcal{A}_j^{\Delta_M} = \mathcal{A}_j \cup \Delta_M$ and generic coefficients for all $1 \leq j \leq n$.

Since $\mathcal{A}^{\Delta_M} = (\mathcal{A}_1^{\Delta_M}, \ldots, \mathcal{A}_n^{\Delta_M})$ fulfills the conditions (H1) and (H3) stated in Section 3.1, by Proposition 5 the multiplicity of the origin as a common isolated root of $\mathbf{g} := (g_1, \ldots, g_n)$ is $\mathrm{mult}_0(\mathbf{g}) = MV_n(\mathcal{A}^{\Delta_M,0}) - MV_n(\mathcal{A}^{\Delta_M})$.

Let us prove that $\mathrm{mult}_0(\mathbf{f}) = \mathrm{mult}_0(\mathbf{g})$. To do so, we consider the matrices $S_k(\mathbf{f}, 0)$ and $S_k(\mathbf{g}, 0)$, for $k \geq 0$, introduced in Section 2.3. Note that, since $M > \mathrm{mult}_0(\mathbf{f})$, in order to compute $\mathrm{mult}_0(\mathbf{f})$, it suffices to compare the dimensions of the nullspaces of the matrices

$S_k(\mathbf{f},0)$ for $0 \leq k \leq M-1$. Now, for every $k \leq M-1$, $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n$, with $|\alpha| \leq k$ and $|\beta| \leq k-1$, and every $1 \leq j \leq n$, we have that

$$(S_k(\mathbf{f},0))_{(\beta,j),\alpha} = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}(x^\beta f_j)(0) = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}(x^\beta g_j)(0) = (S_k(\mathbf{g},0))_{(\beta,j),\alpha}.$$

Since the dimensions of the nullspaces of $S_k(\mathbf{f},0) = S_k(\mathbf{g},0)$ stabilize for $k < M$, then, $\mathrm{mult}_0(\mathbf{f}) = \mathrm{mult}_0(\mathbf{g})$.

The fact that we can take $M = MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}) + 1$ follows from inequality (2). □

From the previous result and Theorem 10 we can express the multiplicity of the origin as an isolated zero of a generic sparse system via mixed integrals:

Corollary 13 *Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^n$ satisfying assumptions (H1) and (H2). Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a generic family of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ supported on $\mathcal{A}$. Let $M := MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}) + 1$ and, for $1 \leq j \leq n$, let $\rho_j^{\Delta_M}$ be the convex function that parameterizes the lower envelope of the polytope $\mathrm{conv}(\mathcal{A}_j^{\Delta_M})$ and $\overline{\rho_j}^{\Delta_M}$ its restriction defined as in (4). Then,*

$$\mathrm{mult}_0(\mathbf{f}) = MI_n'(\overline{\rho_1}^{\Delta_M}, \ldots, \overline{\rho_n}^{\Delta_M}).$$

The following property enables us to deal with smaller support sets when computing multiplicities.

Proposition 14 *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a generic system of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ supported on a family $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ of finite subsets of $(\mathbb{Z}_{\geq 0})^n$. Assume that 0 is an isolated common zero of $\mathbf{f}$. Let $f_1 = \sum_{a \in \mathcal{A}_1} c_{1,a} x^a$. If $\alpha, \alpha+\beta \in \mathcal{A}_1$ with $\beta \in (\mathbb{Z}_{\geq 0})^n \setminus \{0\}$, then*

$$\mathrm{mult}_0(\mathbf{f}) = \mathrm{mult}_0(f_1 - c_{1,\alpha+\beta}x^{\alpha+\beta}, f_2, \ldots, f_n).$$

Proof: Let $h_1, \ldots, h_n$ be polynomials of the form $h_j = f_j + \sum_{i=1}^n c_{j,Me_i}\, x_i^M$ with $c_{j,Me_i} \in \mathbb{C}$ generic coefficients and $M \in \mathbb{N}$ sufficiently big such that

$$\mathrm{mult}_0(f_1, \ldots, f_n) = \mathrm{mult}_0(h_1, \ldots, h_n),$$

$$\mathrm{mult}_0(f_1 - c_{1,\alpha+\beta}x^{\alpha+\beta}, f_2, \ldots, f_n) = \mathrm{mult}_0(h_1 - c_{1,\alpha+\beta}x^{\alpha+\beta}, h_2, \ldots, h_n),$$

$\alpha + \beta \neq Me_i$ for all $1 \leq i \leq n$ and $\mathcal{A}_1 \subset \mathrm{conv}(\{0, Me_1, \ldots, Me_n\})$. The existence of M is ensured by Proposition 12 and its proof.

To prove that $\mathrm{mult}_0(h_1, \ldots, h_n) = \mathrm{mult}_0(h_1 - c_{1,\alpha+\beta}x^{\alpha+\beta}, h_2, \ldots, h_n)$, by Proposition 5, it suffices to show that $\mathrm{conv}(\mathcal{A}_1 \cup \{Me_i\}_{i=1}^n \setminus \{\alpha+\beta\}) = \mathrm{conv}(\mathcal{A}_1 \cup \{Me_i\}_{i=1}^n)$. This follows from the fact that $\alpha + \beta \in \mathrm{conv}(\{\alpha, Me_1, \ldots, Me_n\})$, since

$$\alpha + \beta = \Big(1 - \frac{|\beta|}{M - |\alpha|}\Big)\alpha + \sum_{i=1}^n \Big(\frac{\beta_i}{M} + \frac{|\beta|\alpha_i}{(M-|\alpha|)M}\Big) Me_i$$

is a convex linear combination of $\alpha, Me_1, \ldots, Me_n$. □

As a consequence of Proposition 14 we are able to obtain a refined formula for the multiplicity of the origin for generic polynomials supported on a family $\mathcal{A}$ satisfying conditions (H1) and (H2), with no need of adding extra points to the supports whenever they intersect the coordinate axes.

Proposition 15 *Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a family of finite subsets of $(\mathbb{Z}_{\geq 0})^n$ satisfying assumptions (H1) and (H2), and let $\mathbf{f} = (f_1, \ldots, f_n)$ be a generic sparse polynomial system supported on $\mathcal{A}$. Let $M \in \mathbb{Z}$, $M \geq MV_n(\mathcal{A}^0) - MV_n(\mathcal{A}) + 1$. Then, 0 is an isolated common zero of $\mathbf{f}$ with multiplicity*

$$\mathrm{mult}_0(\mathbf{f}) = MV_n(\mathcal{A}_1^{M,0}, \ldots, \mathcal{A}_n^{M,0}) - MV_n(\mathcal{A}_1^M, \ldots, \mathcal{A}_n^M),$$

where, for every $1 \leq j \leq n$, $\mathcal{A}_j^M := \mathcal{A}_j \cup \{Me_i : 1 \leq i \leq n,\ \mathcal{A}_j \cap \{\mu e_i \mid \mu \in \mathbb{Z}_{\geq 0}\} = \emptyset\}$ and $\mathcal{A}_j^{M,0} := \mathcal{A}_j^M \cup \{0\}$.

Example 3 *Consider the generic polynomial system $\mathbf{f} = (f_1, f_2, f_3)$ with*

$$\begin{aligned} f_1 &= c_{11}x_1 + c_{12}x_2 + c_{13}x_2^2 + c_{14}x_1^2x_2x_3 \\ f_2 &= c_{21}x_1^2 + c_{22}x_1^3 + c_{23}x_1^2x_2 + c_{24}x_3^3 \\ f_3 &= c_{31}x_1 + c_{32}x_1x_2 + c_{33}x_3^2 + c_{34}x_2x_3^3 \end{aligned}$$

with support family $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$, where

$$\begin{aligned} \mathcal{A}_1 &= \{(1,0,0),(0,1,0),(0,2,0),(2,1,1)\} \\ \mathcal{A}_2 &= \{(2,0,0),(3,0,0),(2,1,0),(0,0,3)\} \\ \mathcal{A}_3 &= \{(1,0,0),(1,1,0),(0,0,2),(0,1,3)\} \end{aligned}$$

which satisfies assumptions (H1) and (H2). Then, 0 is an isolated common root of $\mathbf{f}$. In order to compute its multiplicity according to Proposition 15, let

$$M := MV_3(\mathcal{A}^0) - MV_3(\mathcal{A}) + 1 = 28 - 22 + 1 = 7,$$

and consider the modified support sets

$$\begin{aligned} \mathcal{A}_1^7 &= \{(1,0,0),(0,1,0),(0,2,0),(2,1,1),(0,0,7)\} \\ \mathcal{A}_2^7 &= \{(2,0,0),(3,0,0),(2,1,0),(0,0,3),(0,7,0)\} \\ \mathcal{A}_3^7 &= \{(1,0,0),(1,1,0),(0,0,2),(0,1,3),(0,7,0)\} \end{aligned},$$

which coincide with the supports of the polynomials in Example 1. Therefore,

$$\mathrm{mult}_0(\mathbf{f}) = MV_3(\mathcal{A}_1^{7,0}, \mathcal{A}_2^{7,0}, \mathcal{A}_3^{7,0}) - MV_3(\mathcal{A}_1^7, \mathcal{A}_2^7, \mathcal{A}_3^7) = 3.$$

4 Multiplicity of other roots with zero coordinates

Let $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_n)$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^n$ and $\mathbf{f} = (f_1, \ldots, f_n) \subset \mathbb{C}[x_1, \ldots, x_n]$ a generic family of polynomials with support set $\mathcal{A}$.

For $I \subset \{1, \ldots, n\}$, recall that

$$J_I = \{j \in \{1, \ldots, n\} \mid \exists a \in \mathcal{A}_j : a_i = 0\ \forall i \in I\}$$

is the set of indices of the polynomials in **f** that do not vanish identically under the specialization $x_i = 0$ for every $i \in I$. Also, for every $j \in J_I$, we denote

$$\mathcal{A}_j^I = \{a \in \mathcal{A}_j \mid a_i = 0 \ \forall i \in I\}.$$

Following [10, Section 3.2.1], the system **f** has isolated common zeros lying in $O_I := \{x \in \mathbb{C}^n \mid x_i = 0 \text{ if and only if } i \in I\}$ if and only if

(A1) $\#I + \#J_I = n$,

(A2) for every $\widetilde{I} \subset I$, $\#\widetilde{I} + \#J_{\widetilde{I}} \geq n$,

(A3) for every $J \subset J_I$, $\dim(\sum_{j \in J} \mathcal{A}_j^I) \geq \#J$.

From now on, we will consider a non-empty set $I \subset \{1, \dots, n\}$ satisfying the conditions above and we will study the multiplicity of the isolated common zeros of **f** lying in O_I.

4.1 Multiplicity of affine isolated roots

The aim of this section is to compute multiplicities of the isolated zeros of **f** in O_I in terms of mixed volumes and mixed integrals associated to the system supports. The key result that allows us to do this shows that these multiplicities coincide with the multiplicity of the origin as an isolated root of an associated generic sparse system:

Theorem 16 *Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^n$ and $\mathbf{f} = (f_1, \dots, f_n)$ a generic sparse system of polynomials in $\mathbb{C}[x_1, \dots, x_n]$ supported on $\mathcal{A}$. Assume that $\emptyset \neq I \subset \{1, \dots, n\}$ satisfies conditions (A1), (A2) and (A3). Let $\zeta \in \mathbb{C}^n$ be an isolated zero of* **f** *with $\zeta \in O_I$. Then*

$$\mathrm{mult}_\zeta(\mathbf{f}) = \mathrm{mult}_0(\mathbf{g}),$$

for a system $\mathbf{g} := (g_j)_{j \notin J_I}$ of generic polynomials with supports $\mathcal{B}_j^I := \pi_I(\mathcal{A}_j)$ for every $j \notin J_I$, where $\pi_I : \mathbb{Z}^n \to \mathbb{Z}^{\#I}$ is the projection onto the coordinates indexed by I.

For this statement to make sense, we need the following:

Lemma 17 *Under the previous assumptions and notation, let $\mathcal{B}^I = (\mathcal{B}_j^I)_{j \notin J_I}$. Then, $0 \in \mathbb{C}^{\#I}$ is an isolated zero of a generic polynomial system supported on $\mathcal{B}^I$.*

Proof: It suffices to show that $\mathcal{B}^I$ satisfies conditions (H1) and (H2) stated at the beginning of Section 3 (see [10, Proposition 6]).

By the definition of J_I, it follows that $0 \notin \pi_I(\mathcal{A}_j) = \mathcal{B}_j^I$ for every $j \notin J_I$.

In order to simplify notation, we will index the coordinates of $\mathbb{Z}^{\#I}$ by the corresponding elements of I.

To prove that condition (H2) holds, we must show that $\#\widetilde{I} + \#J_{\widetilde{I}}(\mathcal{B}^I) \geq \#I$ for every $\widetilde{I} \subset I$, where $J_{\widetilde{I}}(\mathcal{B}^I) = \{j \notin J_I \mid \exists b \in \mathcal{B}_j^I : b_i = 0 \ \forall i \in \widetilde{I}\}$. Now, for every $\widetilde{I} \subset I$, we have that

$$J_{\widetilde{I}}(\mathcal{A}) = J_I \cup \{j \notin J_I \mid \exists a \in \mathcal{A}_j : a_i = 0 \ \forall i \in \widetilde{I}\} = J_I \cup J_{\widetilde{I}}(\mathcal{B}^I).$$

Under assumption (A2) on I, the inequality $\#\widetilde{I} + \#J_{\widetilde{I}}(\mathcal{A}) \geq n$ holds; then,

$$\#\widetilde{I} + \#J_{\widetilde{I}}(\mathcal{B}^I) = \#\widetilde{I} + \#J_{\widetilde{I}}(\mathcal{A}) - \#J_I \geq n - \#J_I = \#I,$$

where the last identity follows from assumption (A1). □

In order to prove Theorem 16, we first introduce some notation and prove some auxiliary results.

For a polynomial $g \in \mathbb{C}[x_1, \dots, x_n]$, g_I will denote the polynomial in $\mathbb{C}[(x_i)_{i \notin I}]$ obtained from g by specializing $x_i = 0$ for every $i \in I$, and $\mathbf{f}_I$ the associated family of polynomials

$$\mathbf{f}_I = ((f_j)_I)_{j \in J_I}.$$

Then, $\mathbf{f}_I$ is the set of polynomials obtained by specializing the variables indexed by I to 0 in the polynomials in $\mathbf{f}$ and discarding the ones that vanish identically, and $\mathcal{A}^I = (\mathcal{A}_j^I)_{j \in J_I}$ is the family of supports of $\mathbf{f}_I$.

We will use an auxiliary polynomial system defined as follows:

$$\mathbf{f}(I) = (f_{1,I}, \dots, f_{n,I}), \text{ where } f_{j,I} = \begin{cases} (f_j)_I & \text{if } j \in J_I \\ f_j & \text{if } j \notin J_I \end{cases}.$$

Note that the family of supports of these polynomials is

$$\mathcal{A}(I) = (\mathcal{A}_{1,I}, \dots, \mathcal{A}_{n,I}), \text{ where } \mathcal{A}_{j,I} = \begin{cases} \mathcal{A}_j^I & \text{if } j \in J_I \\ \mathcal{A}_j & \text{if } j \notin J_I \end{cases}.$$

Lemma 18 *Under the previous assumptions and notation, if $\zeta \in \mathbb{C}^n$ is an isolated zero of $\mathbf{f}$ lying in O_I, then ζ is also an isolated zero of $\mathbf{f}(I)$ and $mult_\zeta(\mathbf{f}) \le mult_\zeta(\mathbf{f}(I))$.*

Proof: The fact that ζ is an isolated zero of $\mathbf{f}(I)$ follows from the facts that $\mathbf{f}(I)$ is a generic system supported on $\mathcal{A}(I)$ vanishing at ζ, and that, for every $\widetilde{I} \subset I$, $J_{\widetilde{I}}(\mathcal{A}(I)) = J_{\widetilde{I}}(\mathcal{A})$. The inequality between the multiplicities is a consequence of Lemma 3. □

We now focus on a special case of polynomial systems with the same structure as $\mathbf{f}(I)$, namely, systems of n polynomials in n variables which contain r polynomials depending only on r variables.

Proposition 19 *Let $\mathbf{h} = (h_1, \dots, h_n)$ be a system of polynomials in $\mathbb{C}[x_1, \dots, x_n]$ such that $h_1, \dots, h_r \in \mathbb{C}[x_1, \dots, x_r]$. Let $\xi \in \mathbb{C}^r$ be an isolated nondegenerate common zero of $h_1, \dots h_r$ such that $0 \in \mathbb{C}^{n-r}$ is an isolated zero of $\mathbf{h}_\xi := (h_{r+1}(\xi, x_{r+1}, \dots, x_n), \dots, h_n(\xi, x_{r+1}, \dots, x_n))$. Then, $\zeta = (\xi, 0) \in \mathbb{C}^n$ is an isolated zero of $\mathbf{h}$ satisfying:*

$$\text{mult}_\zeta(\mathbf{h}) = \text{mult}_0(\mathbf{h}_\xi).$$

Proof: Under our assumptions, it follows that $\zeta = (\xi, 0)$ is an isolated zero of the system $\mathbf{h}$: if there is an irreducible curve C passing through ζ, since ξ is an isolated common zero of $h_1, \dots, h_r \in \mathbb{C}[x_1, \dots, x_r]$, we have that $C \subset \{x_1 = \xi_1, \dots, x_r = \xi_r\}$ and so, $(\xi, 0) \in C \subset \{x_1 = \xi_1, \dots, x_r = \xi_r, h_{r+1}(x) = 0 \dots, h_n(x) = 0\} = \{\xi\} \times V(\mathbf{h}_\xi)$, contradicting the fact that 0 is an isolated zero of $\mathbf{h}_\xi$.

In order to prove the stated equality of multiplicities, we will compare the multiplicity matrices $S_k(\mathbf{h}, \zeta)$ and $S_k(\mathbf{h}_\xi, 0)$ for $k \in \mathbb{N}$ (see Section 2.3 for the definition of multiplicity matrices). To this end, we will analyze the structure of $S_k(\mathbf{h}, \zeta)$.

Recall that for the system $\mathbf{h}$, for $k \geq 1$, the columns of $S_k(\mathbf{h}, \zeta)$ are indexed by α for $|\alpha| \leq k$ and its rows are indexed by (β, j) for $|\beta| \leq k-1$ and $1 \leq j \leq n$; the entry corresponding to row (β, j) and column α is

$$(S_k(\mathbf{h}, \zeta))_{(\beta,j),\alpha} = \partial_\alpha((x-\zeta)^\beta h_j)(\zeta),$$

where ∂_α is defined in (1).

Note that, for $\gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{Z}_{\geq 0})^n$, we have

$$\frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x^\alpha}((x-\zeta)^\beta x^\gamma)(\zeta) = \begin{cases} \prod_{i=1}^{r} \binom{\gamma_i}{\alpha_i - \beta_i} \xi_i^{\gamma_i + \beta_i - \alpha_i} & \text{if } \beta_i \leq \alpha_i \leq \beta_i + \gamma_i \ \forall\, 1 \leq i \leq r \\ & \text{and } \alpha_i = \beta_i + \gamma_i \ \forall\, r+1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Then, an entry of $S_k(\mathbf{h}, \zeta)$ corresponding to a row indexed by (β, j) and a column indexed by α is 0 whenever $|\beta| \geq |\alpha|$.

We will first consider the columns of $S_k(\mathbf{h}, \zeta)$ indexed by vectors of the form $\alpha = (0, \ldots, 0, \alpha_{r+1}, \ldots, \alpha_n) \neq 0$. For $1 \leq j \leq r$, since the polynomial h_j does not depend on the variables $x_{r+1}, \ldots, x_n$, we have that $(S_k(\mathbf{h}, \zeta))_{(\beta,j),\alpha} = 0$ for every β. For $r+1 \leq j \leq n$ and β with $\beta_i \neq 0$ for some $1 \leq i \leq r$, we also have $(S_k(\mathbf{h}, \zeta))_{(\beta,j),\alpha} = 0$ since $\beta_i > \alpha_i = 0$ (see equation (7)). Finally, for $r+1 \leq j \leq n$ and $\beta = (0, \ldots, 0, \beta_{r+1}, \ldots, \beta_n)$,

$$\begin{aligned} (S_k(\mathbf{h}, \zeta))_{(\beta,j),\alpha} &= \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial x_{r+1}^{\alpha_{r+1}} \ldots \partial x_n^{\alpha_n}} x_{r+1}^{\beta_{r+1}} \ldots x_n^{\beta_n} h_j(\xi, x_{r+1}, \ldots, x_n)(0) \\ &= (S_k(\mathbf{h}_\xi, 0))_{((\beta_{r+1},\ldots,\beta_n),j),(\alpha_{r+1},\ldots,\alpha_n)}. \end{aligned} \quad (8)$$

We analyze now the remaining columns of the matrix.

Consider the submatrix of $S_k(\mathbf{h}, \zeta)$ given by the columns indexed by α such that $(\alpha_1, \ldots, \alpha_r) \neq 0$ and $|\alpha| = k$. From identity (7), we can observe that in every row indexed by (β, j) for $|\beta| = k-1$ and $1 \leq j \leq n$, the only columns with (possibly) non-zero coordinates are indexed by $\alpha = \beta + e_i$ where $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$; moreover,

$$(S_k(\mathbf{h}, \zeta))_{(\beta,j),\beta+e_i} = \frac{\partial h_j}{\partial x_i}(\zeta).$$

Note that, for $1 \leq j \leq r$ and $r+1 \leq i \leq n$, we have $\frac{\partial h_j}{\partial x_i} \equiv 0$. Then, for every β with $|\beta| = k-1$, in the rows indexed by (β, j) for $1 \leq j \leq r$ we have a copy of the Jacobian matrix $\mathcal{J} := \left(\frac{\partial h_j}{\partial x_i}(\xi)\right)_{1 \leq j, i \leq r}$ in the columns indexed by $\beta + e_1, \ldots, \beta + e_r$, and all other entries of the matrix $S_k(\mathbf{h}, \zeta)$ in these rows are zero. We remark that $\mathcal{J}$ is an invertible matrix since ξ is a nonsingular common zero of $h_1, \ldots h_r$. Note that, for every α with $|\alpha| = k$ and $\alpha_i \geq 1$ for some $1 \leq i \leq r$, there is at least one $\beta = \alpha - e_i$ with $|\beta| = k-1$; so, all the columns indexed by α with $|\alpha| = k$ and $(\alpha_1, \ldots, \alpha_r) \neq 0$ are involved in at least one of the copies of $\mathcal{J}$.

Therefore, by performing row operations in $S_k(\mathbf{h}, \zeta)$ we can obtain a matrix such that each column indexed by a vector α with $|\alpha| = k$ and $(\alpha_1, \ldots, \alpha_r) \neq 0$ contains all zero entries except for a unique coordinate equal to 1 in a row indexed by (β, j) for some β

with $|\beta| = k-1$ and $1 \leq j \leq r$, and all these 1's lie in different rows. Moreover, these row operations do not modify the remaining columns of $S_k(\mathbf{h}, \zeta)$.

Then, the dimension of the kernel of $S_k(\mathbf{h}, \zeta)$ is the same as the dimension of the kernel of the matrix obtained by removing the columns indexed by α with $(\alpha_1, \dots, \alpha_r) \neq 0$ and $|\alpha| = k$. We repeat this procedure for $s = k, k-1, \dots, 1$ (in this order) and we conclude that the dimension of the kernel of $S_k(\mathbf{h}, \zeta)$ is the same as the dimension of the kernel of the submatrix obtained by removing all columns indexed by α with $(\alpha_1, \dots, \alpha_r) \neq 0$. This submatrix consists of the first column of $S_k(\mathbf{h}, \zeta)$, which is identically zero, and all columns indexed by $\alpha = (0, \dots, 0, \alpha_{r+1}, \dots, \alpha_n) \neq 0$. Due to our previous considerations on the matrix formed by these columns, we have that the only rows that are not zero are those indexed by (β, j) with $r+1 \leq j \leq n$ and $\beta = (0, \dots, 0, \beta_{r+1}, \dots, \beta_n)$ and these are exactly the rows of $S_k(\mathbf{h}_\xi, 0)$ (see identity (8)). Therefore,

$$\dim(\ker(S_k(\mathbf{h}, \zeta))) = \dim(\ker(S_k(\mathbf{h}_\xi, 0))) \text{ for every } k \geq 1.$$

The result follows. □

Now we can prove Theorem 16.

Proof: Without loss of generality, we may assume that $I = \{r+1, \dots, n\}$ for some $r \in \{1, \dots, n\}$ and $J_I = \{1, \dots, r\}$.

We will first prove that $\text{mult}_\zeta(\mathbf{f}) \geq \text{mult}_0(\mathbf{g})$.

We make the change of variables

$$\begin{array}{ll} x_1 := \sum_{i=1}^r c_{1i} y_i + \zeta_1 & \quad x_{r+1} := y_{r+1} \\ \vdots & \quad \vdots \\ x_r := \sum_{i=1}^r c_{ri} y_i + \zeta_r & \quad x_n := y_n \end{array}$$

where $(c_{ki})_{1 \leq k, i \leq r} \subset \mathbb{Q}$ are generic constants and obtain the polynomial system $\widetilde{\mathbf{f}} = (\widetilde{f}_1, \dots, \widetilde{f}_n)$ in $\mathbb{C}[y_1, \dots, y_n]$ from the system $\mathbf{f}$. Note that $\text{mult}_0(\widetilde{\mathbf{f}}) = \text{mult}_\zeta(\mathbf{f})$.

For every $1 \leq j \leq n$, let $\widetilde{\mathcal{A}}_j$ be the support of $\widetilde{f}_j$.

For $1 \leq j \leq r$, since $f_j(x_1, \dots, x_r, 0, \dots, 0) \neq 0$ and has a non-constant term (since it vanishes at $(\zeta_1, \dots, \zeta_r) \in (\mathbb{C}^*)^r$), due to the genericity of the coefficients and the change of variables, we have that the monomials $y_1, \dots, y_r$ appear with non-zero coefficients in $\widetilde{f}_j(y)$. On the other hand, again, for the genericity of coefficients and change of variables, for $r+1 \leq j \leq n$,

$$\pi_I(\widetilde{\mathcal{A}}_j) = \pi_I(\mathcal{A}_j); \tag{9}$$

moreover, taking into account that

$$\widetilde{f}_j(0, \dots, 0, y_{r+1}, \dots, y_n) = f_j(\zeta_1, \dots, \zeta_r, x_{r+1}, \dots, x_n),$$

we conclude that

$$\{\beta \in (\mathbb{Z}_{\geq 0})^{n-r} \mid (\mathbf{0}, \beta) \in \widetilde{\mathcal{A}}_j)\} = \pi_I(\mathcal{A}_j). \tag{10}$$

Let $\mathbf{h} = (h_1, \dots, h_n)$ be a generic polynomial system with supports $\widetilde{\mathcal{A}} = (\widetilde{\mathcal{A}}_1, \dots, \widetilde{\mathcal{A}}_n)$. Note that condition (H1) holds for $\widetilde{\mathcal{A}}$. Let us see that $\widetilde{\mathcal{A}}$ also satisfies condition (H2), which implies that 0 is an isolated zero of $\mathbf{h}$. For $\widetilde{I} \subset \{1, \dots, n\}$, if $\#\widetilde{I} + \#J_{\widetilde{I}}(\widetilde{\mathcal{A}}) < n$, when setting $y_i = 0$ in $\widetilde{\mathbf{f}}$ for every $i \in \widetilde{I}$, we obtain a system in $n - \#\widetilde{I}$ unknowns with $\#J_{\widetilde{I}}(\widetilde{\mathcal{A}}) < n - \#\widetilde{I}$ equations. This system vanishes at 0 and defines a positive dimensional variety, contradicting the fact that $0 \in \mathbb{C}^n$ is an isolated common zero of $\widetilde{\mathbf{f}}$. By Lemma 3, the inequality $\mathrm{mult}_0(\widetilde{\mathbf{f}}) \geq \mathrm{mult}_0(\mathbf{h})$ holds.

Applying Proposition 14 to the polynomials in the system $\mathbf{h}$, since the monomials $y_1, \dots, y_r$ appear with non-zero coefficients in $h_1, \dots, h_r$ and, for $r+1 \leq j \leq n$, the supports $\mathrm{supp}(h_j) = \mathrm{supp}(\widetilde{f}_j)$ satisfy conditions (9) and (10), it follows that $\mathrm{mult}_0(\mathbf{h}) = \mathrm{mult}_0(\widetilde{\mathbf{g}}, \mathbf{g})$, where $\widetilde{\mathbf{g}} = (\widetilde{g}_1, \dots, \widetilde{g}_r)$ with $\widetilde{g}_j = \sum_{i=1}^{r} \vartheta_{ji} y_i + p_j(y_{r+1}, \dots, y_n)$ for $1 \leq j \leq r$, and $\mathbf{g} = (g_{r+1}, \dots, g_n)$ with $g_j \in \mathbb{C}[y_{r+1}, \dots, y_n]$ a generic polynomial with support $\pi_I(\mathcal{A}_j)$ for $r+1 \leq j \leq n$.

Then, if A is the inverse of the matrix (ϑ_{ji}) and

$$A.\,(\widetilde{g}_1, \dots, \widetilde{g}_r)^t = (y_1 + q_1(y_{r+1}, \dots, y_n), \dots, y_r + q_r(y_{r+1}, \dots, y_n))^t,$$

the following is an isomorphism:

$$\begin{array}{rcl} \mathbb{Q}[y_1, \dots, y_n]/(\widetilde{\mathbf{g}}, \mathbf{g}) & \to & \mathbb{Q}[y_{r+1}, \dots, y_n]/(\mathbf{g}) \\ \overline{y_i} & \mapsto & \overline{-q_i} \quad \text{for all } 1 \leq i \leq r \\ \overline{y_i} & \mapsto & \overline{y_i} \quad \text{for all } r+1 \leq i \leq n \end{array}$$

and hence $\mathrm{mult}_0(\widetilde{\mathbf{g}}, \mathbf{g}) = \mathrm{mult}_0(\mathbf{g})$.

Therefore,

$$\mathrm{mult}_\zeta(\mathbf{f}) = \mathrm{mult}_0(\widetilde{\mathbf{f}}) \geq \mathrm{mult}_0(\mathbf{h}) = \mathrm{mult}_0(\widetilde{\mathbf{g}}, \mathbf{g}) = \mathrm{mult}_0(\mathbf{g}).$$

To prove the other inequality, note that, by Lemma 18, we have that

$$\mathrm{mult}_\zeta(\mathbf{f}) \leq \mathrm{mult}_\zeta(\mathbf{f}(I)).$$

Then, applying Proposition 19 to the system $\mathbf{f}(I)$ and $\xi = (\zeta_1, \dots, \zeta_r)$, we deduce that

$$\mathrm{mult}_\zeta(\mathbf{f}(I)) = \mathrm{mult}_0(\mathbf{f}(I)_\xi).$$

By the genericity of the coefficients of $\mathbf{f}$ and the triangular structure of $\mathbf{f}(I)$, the system $\mathbf{f}(I)_\xi$ turns to be a generic system supported on $\mathcal{B}^I_{r+1}, \dots, \mathcal{B}^I_n$.

We conclude that $\mathrm{mult}_\zeta(\mathbf{f}) \leq \mathrm{mult}_0(\mathbf{g})$. □

Taking into account that the results in Section 3 enable us to express the multiplicity of the origin as an isolated zero of a generic sparse system in terms of mixed volumes and mixed integrals, we can now state a similar result regarding the multiplicity of any affine isolated zero of a generic sparse system of n equations in n unknowns.

Theorem 20 *Let $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_n)$ be a family of finite sets in $(\mathbb{Z}_{\geq 0})^n$ and $\mathbf{f} = (f_1, \dots, f_n)$ be a generic sparse system of polynomials in $\mathbb{C}[x_1, \dots, x_n]$ supported on $\mathcal{A}$. Let $I \subset \{1, \dots, n\}$ satisfying conditions (A1), (A2) and (A3). For $j \notin J_I$, let $\mathcal{B}^I_j = \pi_I(\mathcal{A}_j)$, where $\pi_I : \mathbb{Z}^n \to \mathbb{Z}^{\#I}$ is the projection to the coordinates indexed by I. Let $M_I := MV_{\#I}\big((\mathcal{B}^I_j \cup \{0\})_{j \notin J_I}\big) - MV_{\#I}\big((\mathcal{B}^I_j)_{j \notin J_I}\big) + 1$.*

Then, for every isolated zero $\zeta \in \mathbb{C}^n$ of $\mathbf{f}$ such that $\zeta_i = 0$ if and only if $i \in I$, we have

$$\mathrm{mult}_\zeta(\mathbf{f}) = MV_{\#I}((\mathcal{B}_j^I \cup \{0, M_I e_i\}_{i=1}^{\#I})_{j \notin J_I}) - MV_{\#I}((\mathcal{B}_j^I \cup \{M_I e_i\}_{i=1}^{\#I})_{j \notin J_I}).$$

Moreover, if $(\rho_j)_{j \notin J_I}$ are the convex functions that parameterize the lower envelopes of the polytopes $\mathrm{conv}(\mathcal{B}_j^I \cup \{M_I e_i\}_{i=1}^{\#I})$ and $(\overline{\rho}_j)_{j \notin J_I}$ are their restrictions as defined in (4), then $\mathrm{mult}_\zeta(\mathbf{f}) = MI'_{\#I}((\overline{\rho}_j)_{j \notin J_I})$.

Note that the previous formula for multiplicities can be refined applying Proposition 15 instead of Proposition 12.

4.2 Examples

The following examples illustrate the result in the previous section.

Example 4 *Consider the generic polynomial system*

$$\begin{cases} c_{11}x_1^2 + c_{12}x_1^2x_2^2 + c_{13}x_1x_3 + c_{14}x_1x_2^2x_3 + c_{15}x_3^4 + c_{16}x_2^2x_3^4 = 0 \\ c_{21}x_1^4 + c_{22}x_1^4x_2^2 + c_{23}x_1^2x_3 + c_{24}x_1^2x_2^2x_3 + c_{25}x_3^4 + c_{26}x_2^2x_3^4 = 0 \\ c_{31}x_1 + c_{32}x_1x_2^2 + c_{33} + c_{34}x_2^2 + c_{35}x_3 + c_{36}x_2^2x_3 = 0 \end{cases}$$

taken from [9, Example 3]. There is a unique nonempty set $I = \{1, 3\}$ satisfying conditions (A1), (A2) and (A3), which leads to two isolated solutions with $x_1 = 0$, $x_2 \neq 0$ and $x_3 = 0$. Since $J_I = \{3\}$, Theorem 16 tell us that the multiplicity of each of these solutions equals the multiplicity of $(0, 0)$ as an isolated root of a generic sparse system supported on $\mathcal{B}_1^I = \{(2, 0), (1, 1), (0, 4)\}$ and $\mathcal{B}_2^I = \{(4, 0), (2, 1), (0, 4)\}$, namely a system of the type

$$\begin{cases} a_1x_1^2 + b_1x_1x_3 + c_1x_3^4 = 0 \\ a_2x_1^4 + b_2x_1^2x_3 + c_3x_3^4 = 0 \end{cases}$$

This multiplicity can be computed, by Proposition 5, as $MV_2(\mathcal{B}_1^I \cup \{(0, 0)\}, \mathcal{B}_2^I \cup \{(0, 0)\}) - MV_2(\mathcal{B}_1^I, \mathcal{B}_2^I) = 7$ or, alternatively, by Theorem 10, as $MI'_2(\overline{\rho}_1, \overline{\rho}_2) = 7$, where $\overline{\rho}_1$ and $\overline{\rho}_2$ are the functions whose graphs are given in Example 2.

Example 5 *Consider the generic polynomial system*

$$\begin{cases} a_{11}x_1 + a_{12}x_1x_2 = 0 \\ a_{21}x_2^2 + a_{22}x_1^2x_2^4 + a_{23}x_1^3 = 0 \\ a_{31}x_3 + a_{32}x_1x_3 + a_{33}x_3^2x_4^2 + a_{34}x_3^3x_4 = 0 \\ a_{41}x_4^3 + a_{42}x_2^3x_4^3 + a_{43}x_3^2x_4^3 + a_{44}x_4^5 + a_{45}x_3^2x_4^5 = 0 \end{cases}$$

Using [10, Proposition 5] we can check that all zeros of the system are isolated. Moreover, all the subsets $I \subset \{1, 2, 3, 4\}$ satisfying conditions (A1), (A2) and (A3) are

$$I_1 = \emptyset, \quad I_2 = \{3\}, \quad I_3 = \{1, 2\}, \quad I_4 = \{3, 4\}, \quad I_5 = \{1, 2, 3\} \text{ and } I_6 = \{1, 2, 3, 4\}.$$

By Bernstein's theorem, the system has 24 different simple zeros with all non-zero coordinates (associated to I_1) and, by Theorem 20, we can see that there are

- 6 *simple zeros associated to I_2,*

- 8 *zeros with multiplicity* 2 *associated to* I_3*,*
- 3 *zeros with multiplicity* 3 *associated to* I_4*,*
- 2 *zeros with multiplicity* 2 *associated to* I_5*,*

and that the origin is an isolated zero of multiplicity 6*.*

That is, the system has a total of 65 *(isolated) zeros counting multiplicities. Note that, in this case,* $SM_4(\mathcal{A}) = 65$ *is smaller than* $MV_4(\mathcal{A} \cup \{0\}) = 85$*.*

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