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Polynomials and holomorphic functions on $\mathcal{A}$-compact sets in Banach spaces

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In this paper we study the behavior of holomorphic mappings on $\mathcal{A}$-compact sets. Motivated by the recent work of Aron, Çalişkan, García and Maestre (2016), we give several conditions (on the holomorphic mappings and on the λ-Banach operator ideal $\mathcal{A}$) under which $\mathcal{A}$-compact sets are preserved. Appealing to the notion of tensor stability for operator ideals, we first address the question in the polynomial setting. Then, we define a radius of $(\mathcal{A};\mathcal{B})$-compactification that permits us to tackle the analytic case. Our approach, for instance, allows us to show that the image of any (p,r)-compact set under any holomorphic function (defined on any open set of a Banach space) is again (p,r)-compact.

Introduction

Several classes of functions are described by the nature of their images on compact sets. For instance, linear operators or polynomials between Banach spaces are continuous if and only if they map compact sets into compact sets. In this paper we propose to study the behavior of certain classes of functions on $\mathcal{A}$-compact sets of Carl and Stephani [11], determined by an operator ideal $\mathcal{A}$. More precisely, given a class of continuous functions $\mathfrak{F}$ and two operator ideals $\mathcal{A}$ and $\mathcal{B}$, we are interested in studying those functions in $\mathfrak{F}$ mapping $\mathcal{A}$-compact sets into $\mathcal{B}$-compact sets. We denote this class by $\mathfrak{F}_{(\mathcal{A};\mathcal{B})}$ and say that an element in $\mathfrak{F}_{(\mathcal{A};\mathcal{B})}$ is $(\mathcal{A};\mathcal{B})$-compactifying.

In the recent years many authors studied different type of functions between Banach spaces (such as linear operators, polynomials, holomorphic and continuous functions) in relation with the class of p-compact sets of Sinha and Karn [35]. For instance, in [30], Pietsch considers the class of (s,p)-compactifying operators

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as those mapping s-compact to p-compact sets, for $1 \leq p \leq s < \infty$. This class also was treated by Delgado and Piñeiro in [31]. However, the class of $(\mathcal{A};\mathcal{B})$-compactifying linear operators $\mathcal{L}_{(\mathcal{A};\mathcal{B})}$, in a general setting, can be traced back to the article of Stephani (see [36, Theorem 4.1] for a full characterization of $\mathcal{L}_{(\mathcal{A};\mathcal{B})}$). On the other hand, Aron and Rueda show that continuous homogeneous polynomials preserve the class of p-compact sets [4, Theorem 3.2] and Aron, Çalişkan, García and Maestre give a partial result for holomorphic functions preserving p-compact sets [3, Theorem 3.5], see the paragraph preceding Example 4.6 for details. Also, Muñoz, Oja and Piñeiro [26] characterize the space $\mathcal{C}_{(\mathcal{K},\mathcal{K}_p)}$ of continuous functions from a compact Hausdorff space into a Banach space whose range is p-compact. In order to proceed, let us introduce some definitions and notation.

As usual, $\mathcal{L}, \mathcal{F}, \overline{\mathcal{F}}$ and $\mathcal{K}$ are the ideals of bounded, finite rank, approximable and compact linear operators, respectively; all considered with the supremum norm $\|\cdot\|$. Also, $\mathcal{A} = (\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ denotes a λ-Banach operator ideal, $0 < \lambda \leq 1$. When considering $\mathcal{A}$ and $\mathcal{B}$, we will assume that both are λ-Banach ideals with the same λ. Given a Banach space E over the real or complex field $\mathbb{K}$, B_E and E' denote its closed unit ball and its dual space, respectively. Now, we recall the basics of the Carl–Stephani theory. A subset K of E is *relatively $\mathcal{A}$-compact* if there exist a Banach space Z, an operator $T \in \mathcal{A}(Z;E)$ and a compact set $M \subset Z$ such that $K \subset T(M)$ [11, Lemma 1.1]. A sequence $(x_n)_n$ in E is *$\mathcal{A}$-null* if there exist a Banach space Z, an operator $R \in \mathcal{A}(Z;E)$ and a null sequence $(z_n)_n \subset Z$ such that $x_n = Rz_n$ for all $n \in \mathbb{N}$ [11, Lemma 1.2]. As in the case of compact sets, every $\mathcal{A}$-compact set is contained in the absolutely convex hull of an $\mathcal{A}$-null sequence [11, Theorem 1.1]. Several operator ideals may generate the same system of $\mathcal{A}$-compact sets. This is the case, for instance, of the surjective hull of $\mathcal{A}$, $\mathcal{A}^{sur}$ [11, p. 79] and also of $\mathcal{A} \circ \overline{\mathcal{F}}$ [24, Corollary 1.9].

Regarding linear operators, it is clear that $\mathcal{A} \subset \mathcal{L}_{(\mathcal{K};\mathcal{A})}$ and that $\mathcal{L}_{(\mathcal{A};\mathcal{A})} = \mathcal{L}$ for any $\mathcal{A}$. Also, for any class of continuous functions $\mathfrak{F}$ and any pair of ideals $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{A}$, $\mathfrak{F}_{(\mathcal{A};\mathcal{B})} \subset \mathfrak{F}_{(\mathcal{A};\mathcal{A})}$ holds trivially. Inspired by [3,4,30,31,36] we study when $\mathfrak{F}_{(\mathcal{A};\mathcal{B})} = \mathfrak{F}$ or when $\mathfrak{F}_{(\mathcal{A};\mathcal{A})} = \mathfrak{F}$ for different classes $\mathfrak{F}$ of homogeneous polynomials and holomorphic functions and different λ-Banach operator ideals $\mathcal{A}$ and $\mathcal{B}$. Before starting any discussion, notice that the class of continuous functions provides a negative result. The next example is an extension and uses the ideas of [3, Example 3.1].

Example. Let $\mathcal{A}$ be a λ-Banach operator ideal and E be a Banach space. Suppose that there exists a relatively compact set in E which is not relatively $\mathcal{A}$-compact. Then, there exists a continuous function $f\colon \mathbb{R} \to E$ such that $f([0,1])$ is not $\mathcal{A}$-compact. In particular, $\mathcal{C}_{(\mathcal{A};\mathcal{A})}(\mathbb{R};E) \subsetneq \mathcal{C}(\mathbb{R};E)$.

To see this, take a null sequence $(x_j)_j \subset E$ which is not $\mathcal{A}$-null. Now, consider

$$f(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ (j+1)(1-jt)x_{j+1} + j((j+1)t-1)x_j & \text{if } t \in [\frac{1}{j+1}, \frac{1}{j}] \text{ for } j \in \mathbb{N}, \\ x_1 & \text{if } t \geq 1. \end{cases}$$

Since $f(\frac{1}{j}) = x_j$ for all $j \in \mathbb{N}$, we conclude that $(x_j)_j \subset f([0,1])$ which implies that $f([0,1])$ is not relatively $\mathcal{A}$-compact and, clearly, $[0,1]$ is an $\mathcal{A}$-compact set for any $\mathcal{A}$.

The paper is organized as follows: In Section 1 we deal with the class of n-homogeneous $(\mathcal{A};\mathcal{B})$-compactifying polynomials, denoted by $\mathcal{P}^n_{(\mathcal{A};\mathcal{B})}$, which is a subclass of $\mathcal{P}^n$, the space of all n-homogeneous polynomials. We introduce a λ-norm on this class $\|\cdot\|_{(\mathcal{A};\mathcal{B})}$, under which $\mathcal{P}^n_{(\mathcal{A};\mathcal{B})}$ is a λ-Banach polynomial ideal. Then we focus on homogeneous polynomials preserving $\mathcal{A}$-compact sets, that is the class $\mathcal{P}^n_{(\mathcal{A};\mathcal{A})}$, and show that the property is hereditary on the degree (Proposition 1.5). Contrary to what happens in the linear case, or even in the p-compact setting for polynomials, we show that n-homogeneous polynomials ($n \geq 2$) do not preserve Π_p-compact sets (Examples 1.1 and Example 1.6). Here Π_p denotes the ideal of p-summing operators, $1 \leq p < \infty$. In Section 2, with the notions of (symmetric) tensor norms and tensor stability of λ-Banach operator ideals we show conditions under which polynomials preserve $\mathcal{A}$-compact sets (Theorem 2.2). We apply our results to provide several examples. For instance, we show that polynomials defined on $L_1(\mu)$ preserve Π_1-compact sets (Example 2.6). In Section 3 we present examples of $(\mathcal{A};\mathcal{B})$-compactifying polynomials for some classes of polynomials generated by composition. Our examples rely on classical ideals and show how several other examples may be constructed in an analogous way.

In Section 4 we pass to the holomorphic setting and show that each polynomial in the Taylor series expansion of any $(\mathcal{A};\mathcal{B})$-compactifying analytic function is also $(\mathcal{A};\mathcal{B})$-compactifying. Then we define a radius of $(\mathcal{A};\mathcal{B})$-compactification which allows us to obtain a reciprocal result and present several examples. For instance, we show that the image of any (p,r)-compact set under any holomorphic function, defined on any open set of a Banach space, is again (p,r)-compact. When $r = p'$ the latter result extends [3, Theorem 3.5].

The main examples we present are based on (p,r)-compact sets of Ain, Lillemets and Oja [1]. For $1 \le p < \infty$ and $1 \le r \le p'$ with p' the conjugate of p, a set K of E is relatively (p,r)-compact if there exists a p-summable sequence $(x_j)_j \in \ell_p(E)$ such that

$$K \subset \Big\{ \sum_{j=1}^{\infty} a_j x_j \colon (a_j)_j \in B_{\ell_r} \Big\},$$

where $(a_j)_j \in B_{c_0}$ if $r = \infty$. The (p,p')-compact sets are the p-compact sets of Sinha and Karn. If the sequence $(x_j)_j$ is unconditionally p-summing, that is $(x_j)_j \in \ell_p^{w,0}(E)$, the class of unconditionally (p,r)-compact sets, studied in [2], is obtained (for $r = p'$ see also [21]). These type of compactness are given in terms of the extended notion of nuclear operators $\mathcal{N}_{(t,u,v)}$ (see [29, 18.1.1] for the definition). Namely, the p-compact sets correspond with $\mathcal{N}^p$-compact sets, where $\mathcal{N}^p = \mathcal{N}_{(p,1,p)}$ is the ideal of right p-nuclear operators [24, Remark 1.3]. Also, (p,r)-compact sets are $\mathcal{N}_{(p,1,r')}$-compact sets [2, Proposition 2.4], and unconditionally (p,r)-compact sets are determined by $\mathcal{N}_{(\infty,p',r')}$ (see the paragraph above [2, Theorem 4.1]).

Working with $\mathcal{A}$-compact sets, those linear operators mapping bounded sets into $\mathcal{A}$-compact sets arise naturally. These operators form the ideal of *$\mathcal{A}$-compact operators* denoted by $\mathcal{K}_{\mathcal{A}}$, which were introduced and studied in [11]. In [24], it is shown that $\mathcal{K}_{\mathcal{A}}$ becomes a Banach operator ideal whenever $\mathcal{A}$ is Banach ideal. For this, a measure of the $\mathcal{A}$-compact sets K of E is defined as

$$m_{\mathcal{A}}(K;E) = \inf\{\|T\|_{\mathcal{A}} \colon K \subset T(M),\ T \in \mathcal{A}(X;E) \text{ and } M \subset B_X\},$$

where the infimum is taken considering all Banach spaces X, all operators $T \in \mathcal{A}(X;E)$ and all compact sets $M \subset B_X$ for which the inclusion $K \subset T(M)$ holds. When the context $K \subset E$ is understood, we simply write $m_{\mathcal{A}}(K)$ instead of $m_{\mathcal{A}}(K;E)$. If K is $\mathcal{A}$-compact, then its closed absolutely convex hull $\Gamma(K)$, is also $\mathcal{A}$-compact and $m_{\mathcal{A}}(K) = m_{\mathcal{A}}(\Gamma(K))$. Although the original definition of $m_{\mathcal{A}}$ was conceived in [24] for normed operator ideals, it is easy to see that it extends verbatim for λ-normed (Banach) operator ideals and all the properties remain valid with the obvious modifications. Now, $\mathcal{K}_{\mathcal{A}}$ is a λ-normed (Banach) operator ideal if we define for E and F Banach spaces and $T \in \mathcal{K}_{\mathcal{A}}(E;F)$ the following λ-norm [24]:

$$\|T\|_{\mathcal{K}_{\mathcal{A}}} = m_{\mathcal{A}}(T(B_E);F).$$

In particular, we denote by $\mathcal{K}_{(p,r)}$ and $\mathcal{U}_{(p,r)}$ the λ-Banach ideals of (p,r)-compact operators and of unconditionally (p,r)-compact operators, respectively. When it is convenient for $r = p'$ we write, as usual, $\mathcal{K}_p$ and $\mathcal{U}_p$ the respective Banach operator ideals.

We refer to [29] for the basics of λ-Banach operator ideals and [13] or [32] for definitions and results of tensor norms and operator ideals. Also, we refer to [15] for polynomials and holomorphic functions.

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1. On $(\mathcal{A};\mathcal{A})$-compactifying polynomials

From the definition of $\mathcal{A}$-compact sets, it is easily seen that any continuous linear operator is $(\mathcal{A};\mathcal{A})$-compactifying. The class of $(\mathcal{A};\mathcal{B})$-compactifying operators (those mapping $\mathcal{A}$-compact sets into $\mathcal{B}$-compact sets) were first studied, in a more general setting, by Stephani [36], while the particular case of the $(\mathcal{K}_s;\mathcal{K}_p)$-compactifying operators was treated in detail (under the name of (s,p)-compactifying) in [30] and [31]. For polynomials, in [4, Theorem 3.2], it is proved that any (homogeneous) polynomial is $(\mathcal{K}_p;\mathcal{K}_p)$-compactifying. On the other hand, in [3, Example 4.2], it is shown that n-homogeneous polynomials are not $(\mathcal{K}_p;\mathcal{K}_q)$-compactifying if $1 \leq q < p$.

Recall that for $n \in \mathbb{N}$, a mapping $P: E \to F$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear operator A from E to F such that $P(x) = A(x,\dots,x)$. The vector space of all continuous n-homogeneous polynomials from E to F, $\mathcal{P}^n(E;F)$, is a Banach space endowed with the supremum norm. Notice that for $n = 0$ we have the constant mappings and for $n = 1$, $\mathcal{L}(E;F)$ is obtained. As usual, when $F = \mathbb{K}$ we write $\mathcal{P}^n(E)$ instead of $\mathcal{P}^n(E;\mathbb{K})$. The ideal of all continuous polynomials, consisting of linear combinations of continuous homogeneous polynomials, will be denoted by $\mathcal{P}$.

Now we are in a position to show that the positive result for p-compact sets and polynomials is not true, in general, for $\mathcal{A}$-compact sets. As usual, $\mathcal{QN}_p$ denotes the ideal of quasi p-nuclear operators.

Example 1.1. Let $P \in \mathcal{P}^2(\ell_2;\ell_1)$ be the polynomial defined by $P(x) = (x_1^2, x_2^2, \dots)$ for $x = (x_1, x_2, \dots)$. Then, P is not $(\Pi_p;\Pi_p)$-compactifying for any $1 \leq p < \infty$.

Proof. Fix $1 \leq p < \infty$. It is enough to find a Π_p-compact set $K \subset \ell_2$ such that $P(K) \subset \ell_1$ is not Π_p-compact. Take $n \in \mathbb{N}$, $n \geq p$, and consider a sequence $(a_j)_j \in c_0$ which is not in ℓ_{2n}. Take the set $K = \{a_j e_j : j \in \mathbb{N}\} \subset \ell_2$, where e_j denotes the canonical unit vector for each $j \in \mathbb{N}$. As $L = \{a_j e_j : j \in \mathbb{N}\} \subset \ell_1$ is compact and the inclusion $\iota: \ell_1 \to \ell_2$ is absolutely summing (see e.g. [13, Ex. 11.5]), $K = \iota(L)$ is a relatively Π_1-compact set of ℓ_2 (and hence relatively Π_p-compact for all $p \geq 1$).

Let us suppose that $P(K) = \{a_j^2 e_j : j \in \mathbb{N}\}$ is a Π_p-compact set. Since the sequence $(a_j^2 e_j)_j$ is also null, by [24, Proposition 1.4], $(a_j^2 e_j)_j$ is a Π_p-null sequence. Then the operator $T: \ell_1 \to \ell_1$ defined by $T(e_j) = a_j^2 e_j$ (canonically extended to ℓ_1) is a Π_p-compact operator. By [24, Proposition 2.1], we know that $\mathcal{K}_{\Pi_p} = (\Pi_p \circ \overline{\mathcal{F}})^{sur}$. Also we have the inclusions

$$\mathcal{K}_{\Pi_p}(\ell_1;\ell_1) = (\Pi_p \circ \overline{\mathcal{F}})^{sur}(\ell_1;\ell_1) = (\Pi_p \circ \overline{\mathcal{F}})(\ell_1;\ell_1) = \mathcal{QN}_p(\ell_1;\ell_1) \subset \mathcal{QN}_n(\ell_1;\ell_1).$$

Then, T belongs to $\mathcal{QN}_n(\ell_1;\ell_1)$. The Persson–Pietsch multiplication table [27, Satz 48] gives that the composition operator $\widetilde{T} = T \circ \overset{n}{\cdots} \circ T$ belongs to $\mathcal{QN}(\ell_1;\ell_1)$. Now, consider $S = \widetilde{T} \circ \widetilde{T}$. By [28, Theorem 3.3.2], S belongs to $\mathcal{N}(\ell_1;\ell_1)$ and as $S(e_j) = a_j^{2n} e_j$ we conclude that $(a_j^{2n})_j \in \ell_1$ which is a contradiction. Therefore, $P(K)$ cannot be a Π_p-compact set. $\square$

In the next example we appeal to the existence of a relatively compact set in ℓ_1 which is not unconditionally p-compact for $1 < p < \infty$. If this were not the case, we would have $\mathcal{K} = \mathcal{U}_p$ which is a contradiction.

Example 1.2. Fix $1 < p < \infty$ and $n \in \mathbb{N}$ such that $n \geq p'$. Let $P \in \mathcal{P}^n(\ell_{p'};\ell_1)$ be the polynomial defined by $P(x) = (x_1^n, x_2^n, \dots)$ for $x = (x_1, x_2, \dots)$. Then, P is not $(\mathcal{U}_{(p,1)};\mathcal{U}_{(p,p')})$-compactifying. As a consequence, P is not $(\mathcal{U}_{(p,r)};\mathcal{U}_{(p,r)})$-compactifying for any $1 \leq r \leq p'$.

Proof. Take $L \subset \ell_1$ a compact set which is not unconditionally (p,p')-compact. Then, there exists a sequence $(a_j)_j \in c_0$ such that $L \subset \Gamma\{a_j e_j : j \in \mathbb{N}\}$ and therefore the set $M = \{a_j e_j : j \in \mathbb{N}\}$ is a compact set in ℓ_1 which is not unconditionally (p,p')-compact.

For each $j \in \mathbb{N}$, take e_j the canonical unit vector and let $K = \{a_j^{1/n} e_j : j \in \mathbb{N}\} \subset \ell_{p'}$. As $(a_j^{1/n} e_j)_j \in \ell_p^{w,0}(\ell_{p'})$, K is unconditionally $(p, 1)$-compact. With P as in the statement, $P(K) = M$ and the result follows. $\square$

The above examples motivate the definition of the distinguished class of n-homogeneous polynomials mapping $\mathcal{A}$-compact sets into $\mathcal{B}$-compact sets, for λ-Banach operator ideals $\mathcal{A}$ and $\mathcal{B}$. We denote by $\mathcal{P}^n_{(\mathcal{A};\mathcal{B})}$ the space of $(\mathcal{A};\mathcal{B})$-compactifying n-homogeneous polynomials which turns out to be a λ-Banach polynomial ideal with the λ-norm defined below. Recall that a *λ-normed* ideal $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ of polynomials is a subclass of $\mathcal{P}$ such that

(i) $\mathcal{Q}^n(E;F) = \mathcal{Q} \cap \mathcal{P}^n(E;F)$ is a linear subspace of $\mathcal{P}^n(E;F)$ for any Banach spaces E and F, and $\|\cdot\|_{\mathcal{Q}}$ is a λ-norm on it,

(ii) for any Banach spaces Z and W and operators $T \in \mathcal{L}(Z;E)$ and $S \in \mathcal{L}(F;W)$ and $P \in \mathcal{Q}^n(E;F)$, the polynomial $S \circ P \circ T : Z \longrightarrow W$ belongs to $\mathcal{Q}^n(Z;W)$ with $\|S \circ P \circ T\|_{\mathcal{Q}} \leq \|S\| \|P\|_{\mathcal{Q}} \|T\|^n$,

(iii) $z \mapsto z^n$ belongs to $\mathcal{Q}^n(\mathbb{K};\mathbb{K})$ and has norm one.

When $(\mathcal{Q}^n(E;F), \|\cdot\|_{\mathcal{Q}})$ is complete for all Banach spaces E and F, we say that it is λ-Banach polynomial ideal.

The following result sets the framework for our study, its proof is straightforward and is omitted.

Proposition 1.3. *Let $\mathcal{A}, \mathcal{B}$ be λ-Banach operator ideals, E, F be Banach spaces and $n \in \mathbb{N}$. For $P \in \mathcal{P}^n_{(\mathcal{A};\mathcal{B})}(E;F)$ define*

$$\|P\|_{(\mathcal{A};\mathcal{B})} := \sup\{m_{\mathcal{B}}(P(K)) : K \subset E \textit{ is } \mathcal{A}\textit{-compact and } m_{\mathcal{A}}(K) = 1\}.$$

Then, $\|\cdot\|_{(\mathcal{A};\mathcal{B})}$ is a λ-norm on $\mathcal{P}^n_{(\mathcal{A};\mathcal{B})}$ and $\left(\mathcal{P}^n_{(\mathcal{A};\mathcal{B})}, \|\cdot\|_{(\mathcal{A};\mathcal{B})}\right)$ is a λ-Banach polynomial ideal.

Clearly, $\mathcal{P}^n_{(\mathcal{K};\mathcal{K})} = \mathcal{P}^n$ for all n. Also, by [4, Theorem 3.2], $\mathcal{P}^n_{(\mathcal{K}_p;\mathcal{K}_p)} = \mathcal{P}^n$ for all n. Moreover, from [4, Corollary 3.3], we see that $\|P\| \leq \|P\|_{(\mathcal{K}_p;\mathcal{K}_p)} \leq \frac{n^n}{n!}\|P\|$ for any $P \in \mathcal{P}^n$.

To initiate a systematic discussion, we first consider $(\mathcal{A};\mathcal{A})$-compactifying polynomials. We appeal to the definition of polynomial ideals coming from tensor norms. For a general background of symmetric tensor norms we refer to [17]. Let α_s be a finitely generated symmetric tensor norm (s-tensor norm, for short) of order n and let E, F be Banach spaces. We say that $P \in \mathcal{P}^n(E;F)$ is *α_s-continuous* if its linearization, denoted by L_P, belongs to $\mathcal{L}(\widehat{\otimes}^{n,s}_{\alpha_s} E; F)$. Then, considering the continuous n-homogeneous polynomial $\Delta^E_{n,\alpha_s} : E \to \widehat{\otimes}^{n,s}_{\alpha_s} E$ given by $\Delta^E_{n,\alpha_s}(x) = x \otimes \cdots \otimes x$, we have the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "P"] \arrow[d, "\Delta^E_{n,\alpha_s}"'] & F. \\ \widehat{\otimes}^{n,s}_{\alpha_s} E \arrow[ur, "L_P"'] & \end{tikzcd} \tag{1}$$

We denote by $\mathcal{P}^n_{\alpha_s}(E;F)$ the class of all α_s-continuous n-homogeneous polynomials, which is a Banach ideal endowed with the norm given by $\|P\|_{\alpha_s} = \|L_P\|_{\mathcal{L}(\widehat{\otimes}^{n,s}_{\alpha_s} E;F)}$. This type of polynomials was first considered, in a more general setting, in [18, Section 4.2]. As usual, we denote by π (π_s) the projective (symmetric) tensor norm and by ε (ε_s) the injective (symmetric) tensor norm. As it is well-known $\mathcal{P}^n_{\pi_s}(E;F) = \mathcal{P}^n(E;F)$ isometrically. Also, for an operator $T \in \mathcal{L}(E;F)$ we denote by $\otimes^n T : \otimes^{n,s} E \to \otimes^{n,s} F$ the operator defined on the elementary symmetric tensors by $T(x \otimes \cdots \otimes x) = Tx \otimes \cdots \otimes Tx$ and canonically extended. Besides, regarding $\mathcal{A}$-compact sets, from [24, Corollary 1.9] and [24, Proposition 2.1] we can infer, for a λ-Banach

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operator ideal $\mathcal{A}$, that a set $K \subset E$ is relatively $\mathcal{A}$-compact if and only if there exists $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ such that $K \subset T(B_{\ell_1})$ and $m_{\mathcal{A}}(K) = \inf\{\|T\|_{\mathcal{K}_{\mathcal{A}}}\}$, where the infimum is taken over all the operators $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ such that $K \subset T(B_{\ell_1})$.

Proposition 1.4. *Let $\mathcal{A}$ be a λ-Banach operator ideal and E be a Banach space. Fix $n \in \mathbb{N}$ and α_s an s-tensor norm on $\otimes^{n,s} E$. The following are equivalent.*

(i) *$\mathcal{P}^n_{\alpha_s}(E; F) \subset \mathcal{P}^n_{(\mathcal{A};\mathcal{A})}(E; F)$, for any Banach space F.*
(ii) *The polynomial $\Delta^E_{n,\alpha_s} \colon E \to \widehat{\otimes}^{n,s}_{\alpha_s} E$ is $(\mathcal{A}; \mathcal{A})$-compactifying.*
(iii) *For any T in $\mathcal{K}_{\mathcal{A}}(\ell_1; E)$, the operator $\otimes^n T$ is in $\mathcal{K}_{\mathcal{A}}\big(\widehat{\otimes}^{n,s}_{\pi_s} \ell_1; \widehat{\otimes}^{n,s}_{\alpha_s} E\big)$.*

Moreover,

$$\|P\|_{(\mathcal{A};\mathcal{A})} \leq \|\Delta^E_{n,\alpha_s}\|_{(\mathcal{A};\mathcal{A})} \|P\|_{\alpha_s} \quad \text{and} \quad \|\otimes^n T\|_{\mathcal{K}_{\mathcal{A}}} \leq \|\Delta^E_{n,\alpha_s}\|_{(\mathcal{A};\mathcal{A})} \|T\|^n_{\mathcal{K}_{\mathcal{A}}},$$

for all $P \in \mathcal{P}^n_{\alpha_s}(E; F)$ and $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$.

Proof. Notice that with $F = \widehat{\otimes}^{n,s}_{\alpha_s} E$ and $P = \Delta^E_{n,\alpha_s} \colon E \to \widehat{\otimes}^{n,s}_{\alpha_s} E$ in (1), L_P is the identity operator on $\widehat{\otimes}^{n,s}_{\alpha_s} E$. Hence Δ^E_{n,α_s} belongs to $\mathcal{P}^n_{\alpha_s}(E; \widehat{\otimes}^{n,s}_{\alpha_s} E)$ and (i) implies (ii). The converse is straightforward since continuous linear operators preserve $\mathcal{A}$-compact sets.

For any $T \in \mathcal{L}(\ell_1; E)$ we have the diagram

$$\begin{tikzcd}
\ell_1 \arrow[r, "T"] \arrow[d, "\Delta^{\ell_1}_{n,\pi_s}"'] & E \arrow[d, "\Delta^E_{n,\alpha_s}"] \\
\widehat{\otimes}^{n,s}_{\pi_s} \ell_1 \arrow[r, "\otimes^n T"'] & \widehat{\otimes}^{n,s}_{\alpha_s} E.
\end{tikzcd}$$

Then the following inclusions are clear:

$$\Delta^E_{n,\alpha_s}(T(B_{\ell_1})) = \otimes^n T \circ \Delta^{\ell_1}_{n,\pi_s}(B_{\ell_1}) \subset \otimes^n T\Big(B_{\widehat{\otimes}^{n,s}_{\pi_s} \ell_1}\Big), \tag{2}$$

$$\otimes^n T\Big(B_{\widehat{\otimes}^{n,s}_{\pi_s} \ell_1}\Big) \subset \otimes^n T\Big(\Gamma(\Delta^{\ell_1}_{n,\pi_s}(B_{\ell_1}))\Big) = \Gamma\Big(\Delta^E_{n,\alpha_s}(T(B_{\ell_1}))\Big). \tag{3}$$

Now, fix $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ and consider the inclusions in (3). As $T(B_{\ell_1})$ is $\mathcal{A}$-compact, (ii) implies that $\otimes^n T\Big(B_{\widehat{\otimes}^{n,s}_{\pi_s} \ell_1}\Big)$ is an $\mathcal{A}$-compact set and then $\otimes^n T$ is an $\mathcal{A}$-compact operator. Hence, (iii) holds. On the other hand, given an $\mathcal{A}$-compact set $K \subset E$, there exists an operator $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ such that $K \subset T(B_{\ell_1})$. Then, $\Delta^E_{n,\alpha_s}(K) \subset \Delta^E_{n,\alpha_s}(T(B_{\ell_1}))$ and, by (2), being $\otimes^n T$ an $\mathcal{A}$-compact operator (ii) holds. Finally, with simple calculations the inequalities of the norms are obtained and the proof is complete. $\square$

The proof of (iii) implies (i) in the above proposition uses the same ideas of [5, Theorem 3.5]. Preservation of $\mathcal{A}$-compact sets is hereditary on the degree of homogeneity as the next result shows.

Proposition 1.5. *Let $\mathcal{A}$ be a λ-Banach operator ideal, E, F be Banach spaces and let $n \in \mathbb{N}$. If $\mathcal{P}^n_{(\mathcal{A};\mathcal{A})}(E; F) = \mathcal{P}^n(E; F)$ then $\mathcal{P}^m_{(\mathcal{A};\mathcal{A})}(E; F) = \mathcal{P}^m(E; F)$ for all $m < n$.*

Proof. As $\mathcal{P}^m(E; F) = \mathcal{P}^m_{\pi_s}(E; F)$ for any m, by Proposition 1.4, it is enough to show that for any $T \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$, the operator $\otimes^m T \colon \widehat{\otimes}^{m,s}_{\pi_s} \ell_1 \to \widehat{\otimes}^{m,s}_{\pi_s} E$ is $\mathcal{A}$-compact.

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As in [6, Proposition 11], for each $m < n$ there exist continuous operators $j_m : \widehat{\otimes}^{m,s}_{\pi_s} \ell_1 \to \widehat{\otimes}^{(m+1),s}_{\pi_s} \ell_1$ and $p_m : \widehat{\otimes}^{(m+1),s}_{\pi_s} E \to \widehat{\otimes}^{m,s}_{\pi_s} E$ such that the following diagram commutes

$$\begin{tikzcd} \widehat{\otimes}^{m,s}_{\pi_s} \ell_1 \arrow[r, "\otimes^m T"] \arrow[d, "j_m"'] & \widehat{\otimes}^{m,s}_{\pi_s} E \\ \widehat{\otimes}^{(m+1),s}_{\pi_s} \ell_1 \arrow[r, "\otimes^{(m+1)} T"'] & \widehat{\otimes}^{(m+1),s}_{\pi_s} E. \arrow[u, "p_m"'] \end{tikzcd} \tag{4}$$

Thus, $\otimes^m T = p_m \circ \otimes^{(m+1)} T \circ j_m$, for all $m < n$. As $\otimes^n T$ is $\mathcal{A}$-compact, we see that $\otimes^{n-1} T$ is $\mathcal{A}$-compact. The result follows by a recursive reasoning. $\square$

From Examples 1.1 and 1.2 and the above proposition we have the following:

Example 1.6.

(a) For each $n \in \mathbb{N}$ and $1 \le p < \infty$ there is a polynomial in $\mathcal{P}^n(\ell_2; \ell_1)$ which is not $(\Pi_p; \Pi_p)$-compactifying.
(b) For $1 < p < \infty$, $n \in \mathbb{N}$ and $1 \le r \le p' \le n$ there is a polynomial in $\mathcal{P}^n(\ell_{p'}; \ell_1)$ which is not $(\mathcal{U}_{(p,r)}; \mathcal{U}_{(p,r)})$-compactifying.

We observe that Proposition 1.5 can be restated with $\mathcal{P}^n_{\alpha_s}$ instead of $\mathcal{P}^n$ provided that the diagram (4) remains commutative with continuity if we change $\widehat{\otimes}^{m,s}_{\pi_s} E$ by $\widehat{\otimes}^{m,s}_{\alpha_s} E$ for every $m < n$. This happens for instance for ε_s or for any s-norm being part of a family of complemented symmetric tensor norms (see [6] for definition).

2. Tensor stability and $(\mathcal{A}; \mathcal{A})$-compactifying polynomials

The factorization technique used in (1) involving tensor products and the idea of preserving classes of sets determined by operator ideals, lead us to the notion of tensor stability. Based on the definition given in [10], fixed two tensor norms α and β we say that a λ-Banach operator ideal $\mathcal{A}$ is (α, β)-*tensorstable* if for any Banach spaces E, F, X, Y and any $S \in \mathcal{A}(E; F)$ and $T \in \mathcal{A}(X; Y)$ the tensor product operator $S \otimes_{(\alpha,\beta)} T : E \widehat{\otimes}_\alpha X \to F \widehat{\otimes}_\beta Y$ belongs to $\mathcal{A}$. If $\alpha = \beta$ the definition of a β-tensorstable ideal is covered (see [10] or [13, 34.1]). When the Banach spaces E and F are fixed we say that $\mathcal{A}$ is (α, β)-*tensorstable for* $(E; F)$. If in addition there is a constant $C \ge 1$ satisfying $\|S \otimes_{(\alpha,\beta)} T\|_{\mathcal{A}} \le C \|S\|_{\mathcal{A}} \|T\|_{\mathcal{A}}$, we say that $\mathcal{A}$ is (α, β)-*tensorstable* for $(E; F)$ *with constant* C. Such a constant always exists if the Banach spaces are not fixed (see [13, Sec. 34]). For $C = 1$ the term *metrically* (α, β)-*tensorstable* is used. Notice that when $\widetilde{\alpha} \le \alpha$ and $\beta \le \widetilde{\beta}$ are tensor norms, if $\mathcal{A}$ is $(\widetilde{\alpha}, \widetilde{\beta})$-tensorstable for $(E; F)$ (with constant C), then $\mathcal{A}$ is (α, β)-tensorstable for $(E; F)$ (with constant C).

As $\mathcal{A}$-compact sets of a Banach space E are determined by operators in $\mathcal{K}_{\mathcal{A}}(\ell_1; E)$ the next lemma will be of use.

Lemma 2.1. *Let $\mathcal{A}$ be a λ-Banach operator ideal and β a tensor norm. Let E be a Banach space and suppose that $\mathcal{A}$ is (π, β)-tensorstable for $(\ell_1; E)$ (with constant C), then $\mathcal{K}_{\mathcal{A}}$ is (π, β)-tensorstable for $(\ell_1; E)$ (with constant C).*

Proof. Let X, Y be Banach spaces. Take $S \in \mathcal{K}_{\mathcal{A}}(\ell_1; E)$ and $T \in \mathcal{K}_{\mathcal{A}}(X; Y)$. As $S(B_{\ell_1})$ and $T(B_X)$ are relatively $\mathcal{A}$-compact sets, for $\epsilon > 0$, there exist $L_1, L_2 \subset B_{\ell_1}$ compact sets and operators $\widetilde{S} \in \mathcal{A}(\ell_1; E)$ and $\widetilde{T} \in \mathcal{A}(\ell_1; Y)$ such that $S(B_{\ell_1}) \subset \widetilde{S}(L_1)$ and $T(B_X) \subset \widetilde{T}(L_2)$ with $\|\widetilde{S}\|_{\mathcal{A}} \le (1+\epsilon)\|S\|_{\mathcal{K}_{\mathcal{A}}}$ and $\|\widetilde{T}\|_{\mathcal{A}} \le (1+$

$\epsilon)\|T\|_{\mathcal{K}_{\mathcal{A}}}$. To see that $S\otimes T\colon \ell_1\widehat{\otimes}_\pi X \to E\widehat{\otimes}_\beta Y$ belongs to $\mathcal{K}_{\mathcal{A}}$, note that the operator $\widetilde{S}\otimes\widetilde{T}\colon \ell_1\widehat{\otimes}_\pi \ell_1 \to E\widehat{\otimes}_\beta Y$ is in $\mathcal{A}$ and that

$$S\otimes T(B_{\ell_1\widehat{\otimes}_\pi X}) = \Gamma\Big(S\otimes T(B_{\ell_1}\otimes B_X)\Big) \subset \Gamma\Big(\widetilde{S}\otimes\widetilde{T}(L_1\otimes L_2)\Big).$$

Since the tensor product of relatively compact sets is relatively compact and $L_1\otimes L_2 \subset B_{\ell_1\widehat{\otimes}_\pi \ell_1}$ we have $S\otimes T \in \mathcal{K}_{\mathcal{A}}(\ell_1\widehat{\otimes}_\pi X; E\widehat{\otimes}_\beta Y)$. Moreover, $\|S\otimes T\|_{\mathcal{K}_{\mathcal{A}}} \le \|\widetilde{S}\otimes\widetilde{T}\|_{\mathcal{A}} \le C\|\widetilde{S}\|_{\mathcal{A}}\|\widetilde{T}\|_{\mathcal{A}} \le C(1+\epsilon)^2\|S\|_{\mathcal{K}_{\mathcal{A}}}\|T\|_{\mathcal{K}_{\mathcal{A}}}$, and the proof follows. $\square$

Observe that, with almost the same proof, Lemma 2.1 remains valid if we replace $\mathcal{K}_{\mathcal{A}}$ with $\mathcal{A}^{sur}$. Indeed, if $S\in\mathcal{A}^{sur}(\ell_1;E)$ and $T\in\mathcal{A}^{sur}(X;Y)$, $S\in\mathcal{A}(\ell_1;E)$ and $T\circ q_X\in\mathcal{A}(\ell_1(B_X);Y)$ for $q_X\colon \ell_1(B_X)\to X$ the usual quotient map. The assertion follows from the inclusion

$$S\otimes T(B_{\ell_1\widehat{\otimes} X}) \subset \Gamma\Big(S\otimes(T\circ q_X)(B_{\ell_1}\otimes B_{\ell_1(B_X)})\Big).$$

The next theorem shows the relation between tensor stability and the preservation of $\mathcal{A}$-compact sets under polynomials. As usual, $\sigma_n\colon \otimes^n E\to\otimes^{n,s}E$ denotes the symmetrization mapping.

Theorem 2.2. *Let $\mathcal{A}$ be a λ-Banach operator ideal, E be Banach space and suppose that $\mathcal{A}$ is (π,π)-tensorstable for $(\ell_1;E)$. Then, every polynomial in $\mathcal{P}^n(E;F)$ is $(\mathcal{A};\mathcal{A})$-compactifying for any Banach space F and any $n\in\mathbb{N}$. Moreover, if $\mathcal{A}$ is (π,π)-tensorstable for $(\ell_1;E)$ with constant C, then*

$$\|P\| \le \|P\|_{(\mathcal{A};\mathcal{A})} \le C^{n-1}\left\|\sigma_n\colon \widehat{\otimes}_\pi^n E\to\widehat{\otimes}_{\pi_s}^{n,s}E\right\|\|P\|.$$

Proof. Let us prove that (iii) of Proposition 1.4 holds. Fix $n\in\mathbb{N}$ and $T\in\mathcal{K}_{\mathcal{A}}(\ell_1;E)$ with $\|T\|_{\mathcal{K}_{\mathcal{A}}}=1$. We shall show that $\otimes^n T$ belongs to $\mathcal{K}_{\mathcal{A}}(\widehat{\otimes}_{\pi_s}^{n,s}\ell_1;\widehat{\otimes}_{\pi_s}^{n,s}E)$ and $\|\otimes^n T\|_{\mathcal{K}_{\mathcal{A}}} \le C^{n-1}\left\|\sigma_n\colon\widehat{\otimes}_\pi^n E\to\widehat{\otimes}_{\pi_s}^{n,s}E\right\|$. Denote by $(\otimes T)^n\colon \widehat{\otimes}_\pi^n\ell_1\to\widehat{\otimes}_\pi^n E$ the operator defined on the elementary tensors of the full tensor product by $x_1\otimes x_2\otimes\cdots\otimes x_n\mapsto Tx_1\otimes Tx_2\otimes\cdots\otimes Tx_n$ (extended by linearity and completion).

We claim that $(\otimes T)^n\in\mathcal{K}_{\mathcal{A}}(\widehat{\otimes}_\pi^n\ell_1;\widehat{\otimes}_\pi^n E)$ and $\|(\otimes T)^n\|_{\mathcal{K}_{\mathcal{A}}}\le C^{n-1}$. Let us reason by induction. First, note that $(\otimes T)^2 = T\otimes_{(\pi,\pi)}T\colon \ell_1\widehat{\otimes}_\pi\ell_1\to E\widehat{\otimes}_\pi E$. By the hypothesis and Lemma 2.1, we know that $(\otimes T)^2$ is $\mathcal{A}$-compact with norm at most C. Suppose that the operator $(\otimes T)^{n-1}$ is $\mathcal{A}$-compact and $\left\|(\otimes T)^{n-1}\right\|_{\mathcal{K}_{\mathcal{A}}}\le C^{n-2}$. As π is an associative tensor norm, $\widehat{\otimes}_\pi^n F \overset{1}{=} F\widehat{\otimes}_\pi(\widehat{\otimes}_\pi^{n-1}F)$ for every Banach space F. As $T\in\mathcal{K}_{\mathcal{A}}(\ell_1;E)$, $(\otimes T)^{n-1}$ is $\mathcal{A}$-compact and $\mathcal{A}$ is (π,π)-tensorstable for $(\ell_1;E)$, the claim follows from the diagram

$$\begin{tikzcd}
\widehat{\otimes}_\pi^n\ell_1 \arrow[r, "(\otimes T)^n"] \arrow[d, equal] & \widehat{\otimes}_\pi^n E \arrow[d, equal] \\
\ell_1\widehat{\otimes}_\pi(\widehat{\otimes}_\pi^{n-1}\ell_1) \arrow[r, "T\otimes(\otimes T)^{n-1}"] & E\widehat{\otimes}_\pi(\widehat{\otimes}_\pi^{n-1}E).
\end{tikzcd}$$

Now, the commutative diagram, where ι_n is the norm one inclusion,

$$\begin{tikzcd}
\widehat{\otimes}_{\pi_s}^{n,s}\ell_1 \arrow[r, "\otimes^n T"] \arrow[d, "\iota_n"'] & \widehat{\otimes}_{\pi_s}^{n,s}E \\
\widehat{\otimes}_\pi^n\ell_1 \arrow[r, "(\otimes T)^n"] & \widehat{\otimes}_\pi^n E \arrow[u, "\sigma_n"']
\end{tikzcd}$$

shows that $\otimes^n T\in\mathcal{K}_{\mathcal{A}}$. The proof follows from Proposition 1.4. $\square$

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We have a similar result for (π, ε)-tensorstable ideals where the class of ε_s-continuous polynomials appear. With the proof of [7, Proposition 3.11] as prototype, we can see that the class $\mathcal{P}_{\varepsilon_s}$ corresponds to the ideal of weakly integrable polynomials. An n-homogeneous polynomial $P\colon E \to F$ is weakly integrable if for every linear functional $y' \in F'$, the scalar valued n-homogeneous polynomial $y' \circ P \in \mathcal{P}^n(E)$ is integral (for definition see [15, Definition 2.23]).

Theorem 2.3. *Let $\mathcal{A}$ be a λ-Banach operator ideal, E be Banach space and suppose that $\mathcal{A}$ is (π, ε)-tensorstable for $(\ell_1; E)$. Then, every polynomial in $\mathcal{P}^n_{\varepsilon_s}(E; F)$ is $(\mathcal{A}; \mathcal{A})$-compactifying for any Banach space F and any $n \in \mathbb{N}$. Moreover, if $\mathcal{A}$ is (π, ε)-tensorstable for $(\ell_1; E)$ with constant C,*

$$\|P\|_{(\mathcal{A};\mathcal{A})} \leq C^{n-1} \|P\|_{\varepsilon_s}.$$

Proof. The result follows by mimicking the proof of the above theorem considering the pair (π, ε) instead of (π, π). For the norm inequality also use that $\left\|\sigma_n \colon \widehat{\otimes}^n_\varepsilon E \to \widehat{\otimes}^{n,s}_{\varepsilon_s} E\right\| = 1$ (see e.g. [16, Proposition 3.1]). □

Remark 2.4. Theorem 2.2 can be enunciated in a more general form. For instance, if we consider a family of symmetric tensor norms α_s and a tensor norm β such that for a Banach space E, the operator $\sigma_n \colon (E \otimes_\beta (E \otimes_\beta (E \otimes_\beta \ldots (E \otimes_\beta E) \ldots))) \to \otimes^{n,s}_{\alpha_s} E$ is continuous for every $n \in \mathbb{N}$. Under this assumption, if a λ-Banach operator ideal $\mathcal{A}$ is (π, β)-tensorstable for $(\ell_1; E)$, then $\mathcal{P}^n_{\alpha_s}(E; F) \subset \mathcal{P}^n_{(\mathcal{A};\mathcal{A})}(E; F)$ for every Banach space F and any $n \in \mathbb{N}$.

Now, we give more examples of $\mathcal{A}$-compact sets which are preserved under polynomials.

Example 2.5. Every polynomial is $(\mathcal{N}; \mathcal{N})$-compactifying. Moreover, for any Banach spaces E and F, $n \in \mathbb{N}$ and $P \in \mathcal{P}^n(E; F)$,

$$\|P\| \leq \|P\|_{(\mathcal{N};\mathcal{N})} \leq \left\|\sigma_n \colon \widehat{\otimes}^n_\pi E \to \widehat{\otimes}^{n,s}_{\pi_s} E\right\| \|P\|.$$

Proof. By [13, 34.1], $\mathcal{N}$ is a metrically (π, π)-tensorstable ideal. Then the result follows by Theorem 2.2. □

The above example can be reformulated in terms of the ideal of (Grothendieck) integral operators, $\mathcal{I}$, since $\mathcal{I}$- and $\mathcal{N}$-compact set coincide [25, Proposition 2.2].

Example 1.1 shows the existence of a 2-homogeneous polynomial which is not $(\Pi_p; \Pi_p)$-compactifying for any $1 \leq p < \infty$. The following examples show positive partial results if we restrict the domain or the class of polynomials.

Example 2.6. Every polynomial in $\mathcal{P}(L_1(\mu); F)$ is $(\Pi_1; \Pi_1)$-compactifying for any Banach space F. Moreover, for any $n \in \mathbb{N}$ and $P \in \mathcal{P}^n(L_1(\mu); F)$,

$$\|P\| \leq \|P\|_{(\Pi_1;\Pi_1)} \leq \frac{n^n}{n!} \|P\|.$$

Proof. By [20, Theorem 3], Π_1 is (π, π)-tensorstable for $(\ell_1; L_1(\mu))$ and one may check that this holds with constant $C = 1$. As $\left\|\sigma_n \colon \widehat{\otimes}^n_\pi L_1(\mu) \to \widehat{\otimes}^{n,s}_{\pi_s} L_1(\mu)\right\| \leq \dfrac{n^n}{n!}$ (in fact, the bound is attained if the dimension of $L_1(\mu)$ is at least n), an application of Theorem 2.2 completes the proof. □

Example 2.7. Every polynomial in $\mathcal{P}_{\varepsilon_s}$ is $(\Pi_p; \Pi_p)$-compactifying for every $1 \leq p < \infty$. Moreover, for any Banach spaces E and F, $n \in \mathbb{N}$ and $P \in \mathcal{P}^n_{\varepsilon_s}(E; F)$,

$$\|P\|_{(\Pi_p;\Pi_p)} \leq \|P\|_{\varepsilon_s}.$$

Proof. By [19, Theorem 3.2] (see also [13, Corollary 34.5.2]), Π_p is metrically $(\varepsilon,\varepsilon)$-tensorstable. As $\varepsilon \leq \pi$, Π_p is metrically (π,ε)-tensorstable. An application of Theorem 2.3 completes the proof. □

The class of weakly extendible polynomials, introduced and studied by Carando [7] and Kirwan and Ryan [22], is another classical ideal associated to an s-tensor norm. Next, we show that weakly extendible polynomials also preserve Π_1-compact sets. Following [7, Definition 3.10] an n-homogeneous polynomial $P\colon E \to F$ is weakly extendible if for every linear functional $y' \in F'$, the scalar n-homogeneous polynomial $y' \circ P \in \mathcal{P}^n(E)$ can be extended to any superspace. That is, for any Banach space X with $X \supset E$ there exists $\widetilde{P} \in \mathcal{P}^n(X)$ such that $\widetilde{P}(x) = y' \circ P(x)$ for every $x \in E$. As shown in [7, Proposition 3.11] (see also [22]), the class of weakly extendible polynomials coincides with η_s-continuous polynomials for η_s, the s-tensor norm defined as follows. Fix a Banach space E and let $J_E\colon E \hookrightarrow \ell_\infty(B_{E'})$ be the canonical (isometric) inclusion, for $u \in \otimes^{n,s}E$, $\|u\|_{\eta_s} = \|\otimes^{n,s}J_E(u)\|_{\pi_s}$. Before proceeding, we need a technical result. Recall that an operator ideal $\mathcal{A}$ is right-accessible if $\mathcal{A} \circ \overline{\mathcal{F}} = \mathcal{A}^{min}$, where $\mathcal{A}^{min}$ denotes the minimal kernel of $\mathcal{A}$ (see [13, Proposition 25.2]). Also, we denote by $\mathcal{A}^{inj}$ the injective hull of $\mathcal{A}$.

Lemma 2.8. *Let $\mathcal{A}$ be a λ-Banach operator ideal. Then $\mathcal{K}_{\mathcal{A}}^{inj} = \mathcal{K}_{\mathcal{A}^{inj}}$ isometrically.*

Proof. Let E, F be Banach spaces. Consider $q_E\colon \ell_1(B_E) \to E$ the canonical quotient and J_F as above. As, by [24, Proposition 2.1], $\mathcal{K}_{\mathcal{A}} = (\mathcal{A} \circ \overline{\mathcal{F}})^{sur}$ we have the following:

$$T \in \mathcal{K}_{\mathcal{A}}^{inj}(E;F) \Leftrightarrow J_F \circ T \circ q_E \in \mathcal{A} \circ \overline{\mathcal{F}}(\ell_1(B_E);\ell_\infty(B_{F'})) \Leftrightarrow T \circ q_E \in \mathcal{A}^{inj\ min}(\ell_1(B_E);F),$$

where the last equivalence follows from a combination of [13, Corollary 25.2.2] and [13, Proposition 25.11], since both $\ell_\infty(B_{F'})$ and $\ell_1(B_E)'$ have the metric approximation property. Now,

$$T \circ q_E \in \mathcal{A}^{inj\ min}(\ell_1(B_E);F) \Leftrightarrow T \in (\mathcal{A}^{inj\ min})^{sur}(E;F) \Leftrightarrow T \in (\mathcal{A}^{inj} \circ \overline{\mathcal{F}})^{sur}(E;F),$$

where the last equivalence follows from the fact that any injective λ-Banach ideal is right-accessible [13, 21.2]. Another application of [24, Proposition 2.1] completes the proof. □

Proposition 2.9. *Let $\mathcal{A}$ be a λ-Banach operator ideal and E be a Banach space. Suppose that every polynomial in $\mathcal{P}^n(E;F)$ is $(\mathcal{A};\mathcal{A})$-compactifying for any Banach space F and any $n \in \mathbb{N}$. Then, every polynomial in $\mathcal{P}^n_{\eta_s}(E;F)$ is $(\mathcal{A}^{inj};\mathcal{A}^{inj})$-compactifying for any Banach space F and any $n \in \mathbb{N}$. Moreover, if there exists $C > 0$ such that for every $P \in \mathcal{P}^n(E;F)$, $\|P\|_{(\mathcal{A};\mathcal{A})} \leq C\,\|P\|$, then $\|P\|_{(\mathcal{A}^{inj};\mathcal{A}^{inj})} \leq C\|P\|_{\mathcal{P}_{\eta_s}}$ for every $P \in \mathcal{P}^n_{\eta_s}(E;F)$.*

Proof. Since $\mathcal{K}_{\mathcal{A}^{inj}} = \mathcal{K}_{\mathcal{A}}^{inj}$, by Proposition 1.4 (iii), it is enough to show that the tensor operator $\otimes^n T$ belongs to $\mathcal{K}_{\mathcal{A}}^{inj}(\widehat{\otimes}^{n,s}_{\pi_s}\ell_1;\widehat{\otimes}^{n,s}_{\eta_s}E)$ for any $T \in \mathcal{K}_{\mathcal{A}}^{inj}(\ell_1;E)$. As $J_E \circ T$ is in $\mathcal{K}_{\mathcal{A}}(\ell_1;\ell_\infty(B_{E'}))$, by Proposition 1.4, we get that the operator $\otimes^n(J_E \circ T)\colon \widehat{\otimes}^{n,s}_{\pi_s}\ell_1 \to \widehat{\otimes}^{n,s}_{\pi_s}\ell_\infty(B_{E'})$ is $\mathcal{A}$-compact and satisfies

$$\|\otimes^n(J_E \circ T)\|_{\mathcal{K}_{\mathcal{A}}} \leq C\,\|J_E \circ T\|^n_{\mathcal{K}_{\mathcal{A}}} = C\,\|T\|^n_{\mathcal{K}_{\mathcal{A}}^{inj}}\,.$$

Now, notice that $\otimes^n(J_E \circ T) = \otimes^n J_E \circ \otimes^n T$ with $\otimes^n J_E\colon \widehat{\otimes}^{n,s}_{\eta_s}E \to \widehat{\otimes}^{n,s}_{\pi_s}\ell_\infty(B_{E'})$ a linear isometry. Therefore, $\otimes^n T\colon \widehat{\otimes}^{n,s}_{\pi_s}\ell_1 \to \widehat{\otimes}^{n,s}_{\eta_s}E$ belongs to the injective hull of the ideal of $\mathcal{K}_{\mathcal{A}}$ (see e.g. [29, Proposition 8.4.4]). Also,

$$\|\otimes^n T\|_{\mathcal{K}_{\mathcal{A}}^{inj}} \leq \|\otimes^n(J_E \circ T)\|_{\mathcal{K}_{\mathcal{A}}} \leq C\,\|T\|^n_{\mathcal{K}_{\mathcal{A}}^{inj}}\,,$$

and the proof is complete. □

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Example 2.10. Every polynomial in $\mathcal{P}_{\eta_s}$ is $(\Pi_1;\Pi_1)$-compactifying. Moreover, for any Banach spaces E and F, $n\in\mathbb{N}$ and $P\in\mathcal{P}^n_{\eta_s}(E;F)$,

$$\|P\|_{(\Pi_1;\Pi_1)} \le \left\|\sigma_n\colon \widehat{\otimes}^n_\pi E \to \widehat{\otimes}^{n,s}_{\pi_s} E\right\| \|P\|_{\mathcal{P}_{\eta_s}}.$$

Proof. By Example 2.5, every polynomial preserves $\mathcal{N}$-compact sets, which coincide with $\mathcal{I}$-compact sets. Since $\mathcal{I}^{inj}=\Pi_1$, Proposition 2.9 gives the result. The inequality of the norms follows by combining those of Example 2.5 and Proposition 2.9. $\square$

To study the behavior of polynomials on (p,r)-compact sets and unconditionally (p,r)-compact sets, we give conditions on the tensor norms α and β under which the ideal $\mathcal{N}_{(t,p,q)}$ is (α,β)-tensorstable. Let $1\le p\le\infty$ and denote by d_p the Chevet–Saphar tensor norm [32, p. 135]. For the sequence space ℓ_p, $1\le p<\infty$, by [13, Corollary 15.10.2], the result stated in [13, 12.7] reads as follows: $\ell_p\widehat{\otimes}_{d_p}\ell_p=\ell_p(\ell_p)$. Thus, $\ell_p\widehat{\otimes}_{d_p}\ell_p$ is identified with ℓ_p via the mapping Λ_p defined on the elementary tensors as $\Lambda_p(a\otimes b)=(a_ib_j)_{(i,j)}$ where the indexes (i,j) are considered, for instance, with the square ordering. Also, by [33, Theorem 4.6], $c_0\otimes_{d_\infty}c_0=c_0\otimes_\varepsilon c_0$. Therefore, the corresponding identification $c_0\otimes_{d_\infty}c_0=c_0$ is also true.

Proposition 2.11. *Let $1\le p,q\le\infty$ and $0<t\le\infty$ such that $1+\frac{1}{t}\ge\frac{1}{p}+\frac{1}{q}$. Let α,β be tensor norms such that $d_{q'}\le\alpha$ on $\ell_{q'}\otimes\ell_{q'}$ and $\beta\le d_p$ on $\ell_p\otimes\ell_p$. Then, the ideal $\mathcal{N}_{(t,p,q)}$ is metrically (α,β)-tensorstable.*

Proof. For the proof, we borrow some ideas of [13, Proposition 34.5] and use the usual convention that $\ell_p=c_0$ when $p=\infty$. Fix E_i,F_i Banach spaces and $T_i\in\mathcal{N}_{(t,p,q)}(E_i;F_i)$ for $i=1,2$. As it can be inferred from the factorization of (t,p,q)-nuclear operators [29, Theorem 18.1.3], given $\epsilon>0$ there exist operators $S_i\in\overline{\mathcal{F}}(E_i;\ell_{q'})$, $R_i\in\overline{\mathcal{F}}(\ell_p;F_i)$ and diagonal operators $D_{\lambda^i}\colon\ell_{q'}\to\ell_p$ with $\lambda^i\in\ell_t$ such that $T_i=R_iD_{\lambda^i}S_i$, and $\|R_i\|=\|S_i\|=1$ and $\|\lambda^i\|_{\ell_t}\le(1+\epsilon)\|T_i\|_{\mathcal{N}_{(t,p,q)}}$ for $i=1,2$.

Now, define $\lambda\in\ell_t$, indexed on pairs (i,j) with the square ordering, by $\lambda_{(i,j)}=\lambda^1_i\lambda^2_j$. Clearly, $\|\lambda\|_{\ell_t}=\|\lambda^1\|_{\ell_t}\|\lambda^2\|_{\ell_t}$. Also, note that as approximable operators are α-tensorstable for any α [13, 34.1] and $d_{q'}\le\alpha$ on $\ell_{q'}\otimes\ell_{q'}$, the operator $S_1\otimes_{(\alpha,d_{q'})}S_2$ is approximable and $\|S_1\otimes_{(\alpha,d_{q'})}S_2\|\le1$. The same reasoning is valid for $R_1\otimes_{(d_p,\beta)}R_2$. Thus, we have the following commutative diagram

$$\begin{tikzcd}
E_1\widehat{\otimes}_\alpha E_2 \arrow[r, "T_1\otimes T_2"] \arrow[d, "S_1\otimes S_2"'] & F_1\widehat{\otimes}_\beta F_2 \\
\ell_{q'}\widehat{\otimes}_{d_{q'}}\ell_{q'} \arrow[r, "D_{\lambda^1}\otimes D_{\lambda^2}"] \arrow[d, "\Lambda_{q'}"'] & \ell_p\widehat{\otimes}_{d_p}\ell_p \arrow[u, "R_1\otimes R_2"'] \\
\ell_{q'} \arrow[r, "D_\lambda"] & \ell_p. \arrow[u, "\Lambda_p^{-1}"']
\end{tikzcd}$$

Another application of [29, Theorem 18.1.3] gives that $T_1\otimes T_2$ is in $\mathcal{N}_{(t,p,q)}$ and

$$\begin{aligned}\|T_1\otimes T_2\|_{\mathcal{N}_{(t,p,q)}} &\le \|R_1\otimes R_2\|\|\Lambda_p^{-1}\|\|\lambda\|_{\ell_t}\|\Lambda_{q'}\|\|S_1\otimes S_2\| \\ &\le \|\lambda^1\|_{\ell_t}\|\lambda^2\|_{\ell_t}\le(1+\epsilon)^2\|T_1\|_{\mathcal{N}_{(t,p,q)}}\|T_2\|_{\mathcal{N}_{(t,p,q)}}.\end{aligned}$$

Therefore, the proof is complete. $\square$

Notice that the above result for $1+\frac{1}{t}=\frac{1}{p}+\frac{1}{q}$ also can be obtained by combining [13, Proposition 34.4(2)] and [13, Proposition 34.5].

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Example 2.12. For $1 \le p \le \infty$ and $1 \le r \le p'$, every polynomial is $(\mathcal{K}_{(p,r)}; \mathcal{K}_{(p,r)})$-compactifying. Moreover, for any Banach spaces E and F, $n \in \mathbb{N}$ and $P \in \mathcal{P}^n(E;F)$,

$$\|P\|_{(\mathcal{K}_{(p,r)};\mathcal{K}_{(p,r)})} \le \left\| \sigma_n \colon \widehat{\otimes}_\pi^n E \to \widehat{\otimes}_{\pi_s}^{n,s} E \right\| \|P\| .$$

Proof. By [2, Proposition 2.4], (p,r)-compact sets are $\mathcal{N}_{(p,1,r')}$-compact sets. As $\frac{1}{p} \ge \frac{1}{r'}$ and $d_r \le \pi = d_1$; a direct application of Proposition 2.11 gives that $\mathcal{N}_{(p,1,r')}$ is metrically (π,π)-tensorstable. Now, the result follows by Theorem 2.2. $\square$

When $r = p'$ in the above example, we cover [4, Theorem 3.2] and [3, Corollary 3.3]. Example 1.2 shows that if $1 < p < \infty$ and $n \ge p'$, there exists an n-homogeneous polynomial which is not $(\mathcal{U}_{(p,r)}; \mathcal{U}_{(p,r)})$-compactifying for any $1 \le r \le p'$. The following example shows a positive partial result if we restrict the class of polynomials.

Example 2.13. For $1 \le p < \infty$ and $1 \le r \le p'$, every polynomial in $\mathcal{P}_{\varepsilon_s}$ is $(\mathcal{U}_{(p,r)}; \mathcal{U}_{(p,r)})$-compactifying. Moreover, for any Banach spaces E and F, $n \in \mathbb{N}$ and $P \in \mathcal{P}^n_{\varepsilon_s}(E;F)$,

$$\|P\|_{(\mathcal{U}_{(p,r)};\mathcal{U}_{(p,r)})} \le \|P\|_{\varepsilon_s} .$$

Proof. By the paragraph above [2, Theorem 4.1], $\mathcal{U}_{(p,r)}$-compact sets are $\mathcal{N}_{(\infty,p',r')}$-compact sets. As $\frac{1}{p'} + \frac{1}{r'} \le 1$, $d_r \le \pi$ and $\varepsilon \le d_{p'}$; a direct application of Proposition 2.11 gives that $\mathcal{N}_{(\infty,p',r')}$ is metrically (π,ε)-tensorstable. Hence, the result follows by Theorem 2.3. $\square$

3. On $(\mathcal{A};\mathcal{B})$-compactifying polynomials

There are two classical types of polynomial ideals generated by a λ-Banach operator ideal $\mathcal{A}$. Namely, if $n \in \mathbb{N}$, $\mathcal{P}^n_{\mathcal{A}} = \mathcal{A} \circ \mathcal{P}^n$ and $\mathcal{P}^n_{[\mathcal{A}]} = \mathcal{P}^n \circ \mathcal{A}$, both ideals of homogeneous polynomials are considered with the usual composition λ-norm. Given the nature of their definitions, $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{[\mathcal{A}]}$ have an expected behavior on different type of compact sets. Here we present some examples which involve well-known polynomial ideals and $\mathcal{A}$-compact sets. We give an example of each type. Once this is done, it will be clear how to proceed with other examples.

We start with the class of (Grothendieck) integral homogeneous polynomials $\mathcal{P}^n_{\mathcal{I}}$. By [12, Proposition 2.5] and [9, Proposition 1], it is the composition Banach polynomial ideal $\mathcal{P}^n_{\mathcal{I}} = \mathcal{I} \circ \mathcal{P}^n$.

Example 3.1. Every polynomial in $\mathcal{P}_{\mathcal{I}}$ is $(\mathcal{K};\mathcal{N})$-compactifying and therefore is $(\mathcal{K};\mathcal{A})$-compactifying for every Banach operator ideal $\mathcal{A}$. Moreover, if $n \in \mathbb{N}$ and $P \in \mathcal{P}^n_{\mathcal{I}}$,

$$\|P\|_{(\mathcal{K};\mathcal{A})} \le \|P\|_{(\mathcal{K};\mathcal{N})} \le \|P\|_{\mathcal{I}} .$$

Proof. Continuous mappings preserve compact sets and also $\mathcal{I} \subset \mathcal{L}_{(\mathcal{K};\mathcal{I})}$. As $\mathcal{I}$- and $\mathcal{N}$-compact sets coincide, the isometric identity $\mathcal{P}^n_{\mathcal{I}} = \mathcal{I} \circ \mathcal{P}^n$ shows that every polynomial in $\mathcal{P}^n_{\mathcal{I}}$ is $(\mathcal{K};\mathcal{N})$-compactifying. As $\mathcal{N} \subset \mathcal{A}$ for any Banach operator ideal $\mathcal{A}$, every polynomial in $\mathcal{P}^n_{\mathcal{I}}$ is $(\mathcal{K};\mathcal{A})$-compactifying. The norm inequalities follow from the norm one inclusions

$$\mathcal{P}^n_{\mathcal{I}} = \mathcal{I} \circ \mathcal{P}^n \subset \mathcal{L}_{(\mathcal{K};\mathcal{I})} \circ \mathcal{P}^n \subset \mathcal{P}^n_{(\mathcal{K};\mathcal{I})} = \mathcal{P}^n_{(\mathcal{K};\mathcal{N})} \subset \mathcal{P}^n_{(\mathcal{K};\mathcal{A})}. \quad \square$$

Notice that Example 3.1 remains valid if instead of $\mathcal{P}_{\mathcal{I}}$ we consider the subclasses of nuclear or Pietsch integral polynomials. The next example deals with the ideal of p-dominated polynomials of Matos, which in fact is the polynomial composition ideal $\mathcal{P}^n_{[\Pi_p]} = \mathcal{P}^n \circ \Pi_p$ (see [34, Proposition 3.6] for multilinear mappings). For the ideal of p-summing operators we have the following:

Lemma 3.2. *Let $1 \le p, r \le \infty$ and $0 < t \le \infty$ such that $\frac{1}{t} \ge \frac{1}{r} - \frac{1}{p}$. Then, every operator in Π_p is $(\mathcal{N}_{(t,p',r')}; \mathcal{K}_{(s,r)})$-compactifying for $\frac{1}{s} = \frac{1}{t} + \frac{1}{p}$.*

Proof. By [29, Remark 20.2.2], $\Pi_p \circ \mathcal{N}_{(t,p',r')} \subset \mathcal{N}_{(s,1,r')}$. The result is immediate from the definition of compact sets given by operator ideals and the fact that $\mathcal{N}_{(s,1,r')}$ generates the (s,r)-compact sets [2, Proposition 2.4]. □

Example 3.3. Let $1 \le p, r \le \infty$ and $0 < t \le \infty$. Then,

(a) If $\frac{1}{t} \ge \frac{1}{r} - \frac{1}{p}$, every polynomial in $\mathcal{P}_{[\Pi_p]}$ is $(\mathcal{N}_{(t,p',r')}; \mathcal{K}_{(s,r)})$-compactifying for $\frac{1}{s} = \frac{1}{t} + \frac{1}{p}$.
(b) If $1 \le r \le p'$, every polynomial in $\mathcal{P}_{[\Pi_p]}$ is $(\mathcal{U}_{(p,r)}; \mathcal{K}_{(p,r)})$-compactifying.

Proof. Statement (a) follows from the above lemma and Example 2.12 while (b) is a particular case of (a) where $t = \infty$ is considered. □

4. On $(\mathcal{A}; \mathcal{B})$-compactifying holomorphic functions

In this section we focus on some classes of holomorphic functions. For E and F complex Banach spaces and U an open subset of E, we denote by $\mathcal{H}(U; F)$ the space of all holomorphic functions from U to F. Our aim is to understand to what extent the results obtained in the previous sections pass to the analytic setting. This type of study was initiated by Aron, Çalişkan, García and Maestre [3] where they treat the case of p-compact sets.

Recall that given E, F Banach spaces and an open set $U \subseteq E$, a function $f: U \to F$ is holomorphic if for each $x_0 \in U$ there exists a sequence of polynomials $P_n f(x_0) \in \mathcal{P}^n(E; F)$ such that

$$f(x) = \sum_{n=0}^{\infty} P_n f(x_0)(x - x_0),$$

uniformly for all x in some neighborhood of x_0. We say that $\sum_{n=0}^{\infty} P_n f(x_0)$, is the Taylor series expansion of f at x_0 and that $P_n f(x_0)$ is the n-component of the series at x_0.

Proposition 4.1. *Let $\mathcal{A}, \mathcal{B}$ be λ-Banach operator ideals, let E, F be Banach spaces and $U \subset E$ an open set. If $f \in \mathcal{H}(U; F)$ is $(\mathcal{A}; \mathcal{B})$-compactifying, then for each $x_0 \in U$ and every $n \in \mathbb{N}$, the n-component of the series of f at x_0 is $(\mathcal{A}; \mathcal{B})$-compactifying.*

Proof. Fix $x_0 \in U$ and take $\sum_{n=0}^{\infty} P_n f(x_0)$ the Taylor series expansion of f at x_0. Take $K \subset E$ an absolutely convex $\mathcal{A}$-compact set and let us show that $P_n f(x_0)(K)$ is a $\mathcal{B}$-compact set for each n. Set $\Delta = \{z \in \mathbb{C}\colon |z| \le 1\}$ and denote by Δ° its interior. There is $\delta > 0$ so that $L = \{x_0 + \delta t x\colon t \in (1+\delta)\Delta, x \in K\}$ is included in U. As f is $(\mathcal{A}; \mathcal{B})$-compactifying and L is $\mathcal{A}$-compact it suffices to prove that

$$\{P_n f(x_0)(x)\colon x \in \delta K\} \subset \Gamma(f(L)).$$

Suppose this is not true and take $z = P_n f(x_0)(\widetilde{x}) \notin \Gamma(f(L))$ for some $\widetilde{x} \in \delta K$. By the Hahn–Banach theorem, there is $\varphi \in F'$ so that $|\varphi(z)| > 1$ and $|\varphi(\Gamma(f(L))| \le 1$. Now, defining $g: (1+\delta)\Delta^\circ \to \mathbb{C}$ by $g(t) = \varphi(f(x_0 + t\widetilde{x}))$ we have a holomorphic function. By the Cauchy inequality, we obtain a contradiction since

$$1 < |\varphi(z)| = \left| \frac{g^{(n)}(0)}{n!} \right| \le \sup\{|g(t)|\colon |t| = 1\} \le 1. \quad \square$$

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In virtue of Proposition 4.1, it is natural to inspect under which conditions a holomorphic function whose components in the Taylor series expansion are all $(\mathcal{A};\mathcal{B})$-compactifying is itself $(\mathcal{A};\mathcal{B})$-compactifying. In order to do so, we define for f in $\mathcal{H}(U;F)$ its radius of $(\mathcal{A};\mathcal{B})$-compactification at $x_0 \in U$ as

$$r_{(\mathcal{A};\mathcal{B})}(f;x_0) = 1/\limsup \|P_n f(x_0)\|_{(\mathcal{A};\mathcal{B})}^{1/n}.$$

As usual, the radius is infinite if $\limsup \|P_n f(x_0)\|_{(\mathcal{A};\mathcal{B})}^{1/n} = 0$ and the radius is zero if $P_n f(x_0)$ fails to be $(\mathcal{A};\mathcal{B})$-compactifying for some n. Notice that for $r(f;x_0) = 1/\limsup \|P_n f(x_0)\|^{1/n}$, the radius of uniform convergence of f at x_0, we have $r_{(\mathcal{A};\mathcal{B})}(f;x_0) \leq r(f;x_0)$. In what follows we will need the next result which is the λ-Banach version of [23, Lemma 3.1] (see also [37, Lemma 3]). We omit the proof.

Lemma 4.2. *Let $\mathcal{A}$ be a λ-Banach operator ideal and E be a Banach space. Consider $(K_n)_n \subset E$ a sequence of $\mathcal{A}$-compact sets such that $\sum_{n=1}^{\infty} m_{\mathcal{A}}(K_n)^\lambda < \infty$. Then, the set $K = \{\sum_{n=1}^{\infty} x_n \colon x_n \in K_n\}$ is $\mathcal{A}$-compact and $m_{\mathcal{A}}(K)^\lambda \leq \sum_{n=1}^{\infty} m_{\mathcal{A}}(K_n)^\lambda$.*

We first give a positive result for holomorphic functions mapping $\mathcal{A}$-compact sets of small size into $\mathcal{B}$-compact sets.

Lemma 4.3. *Let $\mathcal{A}, \mathcal{B}$ be λ-Banach operator ideals, let E, F be Banach spaces and $U \subset E$ an open set. Fix $x_0 \in U$ and $f \in \mathcal{H}(U;F)$ whose Taylor series expansion at x_0 is $\sum_{n=0}^{\infty} P_n f(x_0)$. Suppose that $P_n f(x_0)$ is $(\mathcal{A};\mathcal{B})$-compactifying for all n and $r_{(\mathcal{A};\mathcal{B})}(f;x_0) > 0$. If $K \subset U$ is an $\mathcal{A}$-compact set and $m_{\mathcal{A}}(K - x_0) < r_{(\mathcal{A};\mathcal{B})}(f;x_0)$, then $f(K)$ is $\mathcal{B}$-compact.*

Proof. As $r_{(\mathcal{A};\mathcal{B})}(f;x_0) \leq r(f;x_0)$ for an $\mathcal{A}$-compact set $K \subset U$ such that $m_{\mathcal{A}}(K - x_0) < r_{(\mathcal{A};\mathcal{B})}(f;x_0)$, we have

$$f(K) \subset \Big\{ \sum_{n=1}^{\infty} x_n \colon x_n \in P_n f(x_0)(K - x_0) \Big\}.$$

Also

$$\sum_{n=1}^{\infty} m_{\mathcal{B}}(P_n f(x_0)(K - x_0))^\lambda \leq \sum_{n=1}^{\infty} \Big(\|P_n f(x_0)\|_{(\mathcal{A};\mathcal{B})} m_{\mathcal{A}}(K - x_0)^n \Big)^\lambda.$$

As $\limsup (\|P_n f(x_0)\|_{(\mathcal{A};\mathcal{B})}^{1/n} m_{\mathcal{A}}(K - x_0))^\lambda < 1$, the series is convergent. Then, by Lemma 4.2, $f(K)$ is $\mathcal{B}$-compact and the proof is complete. □

In order to deal with $\mathcal{A}$-compact sets of arbitrary size we will need the following:

Lemma 4.4. *Let $\mathcal{A}$ be a λ-Banach operator ideal, let E be a Banach space and $K \subset E$ be a relatively $\mathcal{A}$-compact set such that $0 \in K$. Then, given $\epsilon > 0$, there exist $\delta > 0$ such that $m_{\mathcal{A}}(K \cap \delta B_E) \leq \epsilon$.*

Proof. Take $\epsilon > 0$ and $K \subset E$ as in the statement. There exist a Banach space Z, a compact set $L \subset B_Z$ and an operator $T \in \mathcal{A}(Z;E)$ such that $K \subset T(L)$ and $\|T\|_{\mathcal{A}} \leq (1+\epsilon) m_{\mathcal{A}}(K)$. Consider the quotient map $q \colon Z \to Z/\ker(T)$ and the injective operator $\widetilde{T}$ such that $T = \widetilde{T} \circ q$. Then, $\widetilde{T} \in \mathcal{A}^{sur}(Z/\ker(T);E)$, $\|\widetilde{T}\|_{\mathcal{A}^{sur}} \leq \|T\|_{\mathcal{A}}$ (see e.g. [29, Proposition 8.5.4]) and $K \subset \widetilde{T}(q(L))$ with $q(L)$ compact. As $0 \in K$, $0 \in q(L)$ and there exists $\delta > 0$ such that

$$K \cap \delta B_E \subset \widetilde{T}(q(L)) \cap \delta B_E \subset \widetilde{T}(q(L) \cap \epsilon B_Z) = \epsilon \widetilde{T}(\tfrac{1}{\epsilon} q(L) \cap B_Z).$$

Since $\frac{1}{\epsilon}q(L) \cap B_Z$ is relatively compact then $K \cap \delta B_E$ is $\mathcal{A}^{sur}$-compact. Now, we use that relatively $\mathcal{A}^{sur}$- and $\mathcal{A}$-compact sets coincide [11, p. 79] with the same measure [24, Proposition 1.8], then

$$m_{\mathcal{A}}(K \cap \delta B_E) \leq \epsilon \left\|\widetilde{T}\right\|_{\mathcal{A}^{sur}} \leq \epsilon(1+\epsilon) m_{\mathcal{A}}(K),$$

and the proof follows. □

Below we give the main theorem of this section from which all the examples we present are deduced.

Theorem 4.5. *Let $\mathcal{A}, \mathcal{B}$ be λ-Banach operator ideals, let E, F be Banach spaces and $U \subset E$ an open set. Let $f \in \mathcal{H}(U;F)$ whose Taylor series expansion at $x_0 \in U$ is $\sum_{n=0}^{\infty} P_n f(x_0)$. Suppose that for each $x_0 \in U$, $P_n f(x_0)$ is $(\mathcal{A};\mathcal{B})$-compactifying for every n and $r_{(\mathcal{A};\mathcal{B})}(f;x_0) > 0$. Then f is $(\mathcal{A};\mathcal{B})$-compactifying.*

Proof. Let $K \subset U$ be an $\mathcal{A}$-compact set. By Lemma 4.4, for each $x \in K$ we may choose $\delta_x > 0$ such that $m_{\mathcal{A}}((K-x) \cap \delta_x B_E) < r_{(\mathcal{A};\mathcal{B})}(f;x)$. Take $x_1, \ldots, x_k \in K$ such that $K = \bigcup_{j=1}^{k} K_j$ with $K_j = K \cap (x_j + \delta_{x_j} B_E)$. We claim that $f(K_j)$ is relatively $\mathcal{B}$-compact for $j = 1, \ldots, k$.

Indeed, for each j, $m_{\mathcal{A}}(K_j - x_j) = m_{\mathcal{A}}((K - x_j) \cap \delta_{x_j} B_E) < r_{(\mathcal{A};\mathcal{B})}(f;x_j)$. Then, the claim follows from Lemma 4.3.

To end the proof, notice that $f(K) = \bigcup_{j=1}^{k} f(K_j)$ and finite union of $\mathcal{B}$-compact sets is $\mathcal{B}$-compact. □

The case on p-compact sets was treated in [3]. Let E, F be Banach spaces and $U \subset E$ be a balanced open set. In [3, Theorem 3.5] the authors prove that if $K \subset U$ is a p-compact set such that there exists a sequence $(x_n)_n$ in $\ell_p(E) \cap U$ with $K \subset \left\{ \sum_{k=1}^{\infty} \alpha_k x_k \colon (\alpha_k)_k \in B_{\ell_{p'}} \right\} \subset U$, then $f(K)$ is p-compact for any $f \in \mathcal{H}(U;F)$. As a consequence, any entire function between Banach spaces is $(\mathcal{K}_p;\mathcal{K}_p)$-compactifying [3, Corollary 3.6]. For a discussion about the restriction considered on the p-compact sets K see [3, Remark 3.7]. Now, Theorem 4.5 allows us to prove in full generality [3, Theorem 3.5] as the next example shows.

Example 4.6. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p'$. Let E, F be Banach spaces, $U \subset E$ an open set. Then, every function in $\mathcal{H}(U;F)$ is $(\mathcal{K}_{(p,r)};\mathcal{K}_{(p,r)})$-compactifying.

Proof. By Example 2.12, every $P \in \mathcal{P}^n(E;F)$ is $(\mathcal{K}_{(p,r)};\mathcal{K}_{(p,r)})$-compactifying and satisfies $\|P\| \leq \|P\|_{(\mathcal{K}_{(p;r)};\mathcal{K}_{(p;r)})} \leq e^n \|P\|$. For each $x_0 \in U$, write the Taylor series expansion of f at x_0 as $\sum_{n=0}^{\infty} P_n f(x_0)$. As

$$1/\limsup \|P_n f(x_0)\|_{(\mathcal{K}_{(p;r)};\mathcal{K}_{(p;r)})}^{1/n} \geq \tfrac{1}{e} r(f, x_0) > 0,$$

the conclusion follows from Theorem 4.5. □

With a similar proof of the above example, using Example 2.6 instead of Example 2.12, we obtain the next result.

Example 4.7. Let $U \subset L_1(\mu)$ be an open set. Every function in $\mathcal{H}(U;F)$ is $(\Pi_1;\Pi_1)$-compactifying for any Banach space F.

Now we apply Theorem 4.5 to the class of weakly extendible holomorphic functions. Given E, F Banach spaces and $U \subset E$ an open set, $f \in \mathcal{H}(U;F)$ is weakly extendible if for every $y' \in F'$, $y' \circ f \in \mathcal{H}(U)$ is extendible.

Lemma 4.8. *Given E, F Banach spaces and $U \subset E$ an open set. Let $f \in \mathcal{H}(U; F)$ whose Taylor series expansion at $x_0 \in U$ is $\sum_{n=0}^{\infty} P_n f(x_0)$. Then f is weakly extendible if and only if for each $x_0 \in U$, $P_n f(x_0)$ belongs to $\mathcal{P}^n_{\eta_s}(E; F)$ and $\limsup \|P_n f(x_0)\|_{\eta_s}^{\frac{1}{n}} < \infty$.*

Proof. Fix $x_0 \in U$ and write f as $f(x) = \sum_{n=0}^{\infty} P_n f(x_0)(x - x_0)$. Since for every $y' \in F'$, $y' \circ f$ is extendible, by [8, Proposition 3.1] and the uniqueness of the Taylor series expansion of an holomorphic function, we get that for every $y' \in F'$ and every $n \in \mathbb{N}$, $y' \circ P_n f(x_0)$ is an extendible scalar valued polynomial and $\limsup \|y' \circ P_n f(x_0)\|_e < \infty$ (here, $\|y' \circ P_n f(x_0)\|_e$ is the extendible norm of the polynomial, see below [7, Proposition 3.2]). Thus, for every $n \in \mathbb{N}$, $P_n f(x_0) \in \mathcal{P}^n_{\eta_s}(E; F)$ and, by the Principle of Uniform Boundedness, $\limsup \|P_n f(x_0)\|_{\eta_s}^{\frac{1}{n}} < \infty$. □

With a similar proof of Example 4.6 and using Example 2.10 instead of Example 2.12, we obtain the next result.

Example 4.9. Let E, F be Banach spaces, $U \subset E$ an open set. Then, every function $\mathcal{H}(U; F)$ which is weakly extendible is $(\Pi_1; \Pi_1)$-compactifying.

Our final example deals with the class of weakly integral holomorphic functions in the sense of Dimant, Galindo, Maestre and Zalduendo [14]. Given E, F Banach spaces we say that $f \in \mathcal{H}(B_E^\circ; F)$ is weakly integral if for every $y' \in F'$, $y' \circ f \in \mathcal{H}(B_E^\circ)$ is scalar valued integral as defined in [14, p. 86].

Lemma 4.10. *Given E, F Banach spaces and $f \in \mathcal{H}(B_E^\circ; F)$ whose Taylor series expansion at $x_0 \in B_E^\circ$ is $\sum_{n=0}^{\infty} P_n f(x_0)$. Suppose that f is weakly integral, then for each $x_0 \in B_E^\circ$, $P_n f(x_0)$ belongs to $\mathcal{P}^n_{\varepsilon_s}(E; F)$ and* $\limsup \|P_n f(x_0)\|_{\varepsilon_s}^{\frac{1}{n}} < \infty$.

Proof. Recall that $P \in \mathcal{P}^n_{\varepsilon_s}(E; F)$ if and only if for every $y' \in F'$, $y' \circ P$ is an integral scalar valued polynomial (see the comment above Theorem 2.3). Then, the proof is analogous to that of Lemma 4.8 using [14, Proposition 2] instead of [8, Proposition 3.1]. □

Now, with a similar proof of Example 4.6 and using Example 2.13 instead of Example 2.12, we obtain the next result.

Example 4.11. Let $1 \leq p \leq \infty$ and $1 \leq r \leq p'$. Let E, F be Banach spaces. Every function in $\mathcal{H}(B_E; F)$ which is weakly integral is $(\mathcal{U}_{(p,r)}; \mathcal{U}_{(p,r)})$-compactifying.

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