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PERIODIC SOLUTIONS FOR INDEFINITE SINGULAR EQUATIONS WITH SINGULARITIES IN THE SPATIAL VARIABLE AND NON-MONOTONE NONLINEARITY

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Abstract. TWe prove the existence of $T-$periodic solutions for the second order non-linear equation

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = h(t)g(u),$$

where the non-linear term g has two singularities and the weight function h changes sign. We find a relation between the degeneracy of the zeroes of the weight function and the order of one of the singularities of the non-linear term. The proof is based on the classical Leray-Schauder continuation theorem. Some applications to important mathematical models are presented.

1. **Introduction and main results.** This paper is devoted to study the periodic problem associated to the equation

$$(\phi(u'))' = h(t)g(u), \tag{1}$$

where $\phi : (-1, 1) \to \mathbb{R}$ is an increasing odd homeomorphism, $g : I \to \mathbb{R}^+$ is continuous with $I := (\alpha, \beta) \subset \mathbb{R}$ a bounded interval, and the (nontrivial) weight function $h : \mathbb{R} \to \mathbb{R}$ is $T-$periodic and locally integrable. By $T-$periodic solution of (1) we shall understand a $T-$periodic function $u : \mathbb{R} \to \mathbb{R}$ which is locally absolutely continuous together with its first derivative and that satisfies the equation.

Our assumptions for the nonlinearity g read as follows:

(G1) There exists a point $t_\sharp \in (\alpha, \beta)$ such that $g\mid_{(\alpha,t_\sharp]}$ is non-increasing and $g\mid_{[t_\sharp,\beta)}$ is non-decreasing.

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(G2)

$$\lim_{x\to\alpha^+} g(x) = \lim_{x\to\beta^-} g(x) = +\infty, \qquad \int_\alpha^{t_\sharp} g(s)ds = +\infty.$$

Regarding the weight function h, a first elementary observation is that if u is any $T-$periodic solution of (1) then

$$\int_0^T h(t)g(u(t))dt = 0.$$

Thus, a necessary condition for the existence of a $T-$periodic solution is that h changes sign.

Based on this fact, we shall assume that the positivity set of h (restricted to the interval $[0,T]$) is a finite union of disjoint open sub-intervals. As we shall see, this generalizes the conditions of previous works on indefinite singular equations. Specifically, we shall assume the existence of a finite set of points $\{\ell_1 < \ell_1' < \dots < \ell_n < \ell_n'\} \subset [0,T]$ such that $h\mid_{(\ell_i,\ell_i')} > 0$ for all $i = 1,\dots,n$ and $h \le 0$ on $[0,T]\setminus \cup_{i=1}^n[\ell_i,\ell_i']$. Thus, writing as usual $h := h^+ - h^-$ the assumption reads:

$$h(t) = \begin{cases} h^+(t), & \text{if } t \in \cup_{i=1}^n[\ell_i,\ell_i'], \\ -h^-(t), & \text{otherwise.} \end{cases} \tag{2}$$

Furthermore, we shall assume that

$$\lim_{t\to 0^+} \inf_{s\in\cup_{i=1}^n[\ell_i+t,\ell_i'-t]} h(s)\int_{\alpha+4nt}^{t_\sharp} g(r)dr = +\infty. \tag{3}$$

It is worthy to notice that if $\liminf_{t\to\alpha^+} g(\alpha+4nt)/g(\alpha+t) > 0$ then, by Cauchy's mean value theorem, the value $4nt$ in the latter integral may be replaced by t, resulting in

$$\lim_{t\to 0^+} \inf_{s\in\cup_{i=1}^n[\ell_i+t,\ell_i'-t]} h(s)\int_{\alpha+t}^{t_\sharp} g(r)dr = +\infty. \tag{4}$$

In particular, (3) and (4) are equivalent when g behaves as $(t-\alpha)^{-k}$ near α, but not if the singularity is too strong, for example $g(t) \sim e^{1/(t-\alpha)}$.

The preceding condition imposes a relation between the order of one of the singularities of the nonlinear term (specifically, the first of them) and the degeneracy of the zeroes of the weight function. In other words, if the multiplicity of the zeroes of h is large, our condition applies when the singularity of the nonlinear term is sufficiently strong.

Theorem 1.1. *Assume that the previous conditions (G1), (G2), (2) and (3) hold and that $T < 2\min\{t_\sharp - \alpha, \beta - t_\sharp\}$. Then (1) has a $T-$periodic solution, provided that $\overline{h} := \frac{1}{T}\int_0^T h(t)\,dt < 0$.*

To better understand the relation given by (3) or (4), let us consider the work [20], in which the author studied the classical equation

$$u'' = \frac{h(t)}{u^\lambda}, \tag{5}$$

where $\lambda \ge 1$ and $h : \mathbb{R} \to \mathbb{R}$ is a locally integrable function satisfying necessary conditions: namely, h changes sign and $\overline{h} < 0$. The existence of a $T-$periodic solution of (5) was proved, distinguishing two possible situations:

Situation A. The weight function h is piecewise constant (with a finite number of pieces) and the singularity at 0 is strong (i.e. $\lambda \geq 1$).

Siuation B. The weight function h has only a finite number of nondegenerate zeroes and the singularity at 0 is very strong (i.e. $\lambda \geq 2$).

Our assumption (4) covers completely the situation A in [20] for the relativistic case. In fact, it is quite more general, because it does not require that the weight function h is piecewise constant neither that it has a finite number of pieces (only a finite number of pieces where it is positive).

If we take a look to situation B, assuming that h has only nondegenerate zeroes, our condition (4) applies when $\lambda > 2$. Although the result in [20] is more general since it allows $\lambda \geq 2$, it has two weak points with regard to Theorem 1.1:

- In our case, h may have a non-countable number of zeroes. Indeed, it is noticed that the zeroes of the weight function outside the intervals $[\ell_j, \ell'_j]$ determining the part where it is negative do not play a role in (3) or (4). Our weight function may vanish even on a positive measure set.
- Our case also applies when the weight function has degenerate zeroes. According to (3), this can be done by increasing the order of the singularity, which corresponds to λ in the particular case (5).

The related Neumann problem was recently studied by Boscaggin and Zanolin in [9]. They proved the existence of Neumann solutions in many situations in which the weight function changes sign exactly once over the given time interval. When their result is particularized to equation (5), one obtains the existence of an (increasing) solution u verifying that $u'(0) = u'(T) = 0$, under the following assumptions:

i): h verifies the necessary conditions ($\overline{h} < 0$ and h changes sign).

ii): There exist numbers $\gamma \in (0, \lambda)$ and $c \in (0, +\infty)$ such that $t^{-\gamma} \int_0^t h(s)ds \to c$ as $t \to 0^+$.

Although the results in [9] apply to several types of nonlinearities (including those with more than one singularity) it seems that the arguments cannot be easily adapted to prove the existence of Neumann or periodic solutions when the weight function has more than two pieces (see [9, Lemma 5]). Moreover, a detailed analysis of the main result of [9] shows that finding suitable conditions for the existence of Neumann (or periodic) solutions of non-linear differential equations with more than one singularity is a difficult task, taking into account that it requires to look for the intersection points between two curves of solutions that depart or arrive with zero velocity and which are not explicitly defined.

Concerning the previous condition ii), in the same line of our research, the authors of [9] found a relation between the degeneracy of the zeroes of the weight function and the order of the singularity of the nonlinear term. In particular, if the weight function has only nondegenerate zeroes (situation B), condition ii) applies when $\lambda > 2$.

In contrast with all the above-mentioned references on periodic solutions for indefinite singular equations in the classical case, there are no related results concerning the relativistic operator. Thus, our results represent a reasonable progress in this theory, complementing the literature on the periodic problem with singular $\phi-$Laplacian operator (see for example [2, 3, 4, 5, 6, 16, 18, 19, 22]). Since no results are available for the relativistic equation, we considered previous works on the classical case in order to get an idea about what kind of conditions should be accurate for our problem. As a general rule, when dealing with the relativistic operator one expects to obtain better results than in the classical case; however, this is not necessarily true when a singular nonlinear term is involved. For example, in [5] the authors proved the existence of a $T-$periodic solution of

$$(\phi(u'))' = \pm\frac{1}{u^\lambda} + e(t), \tag{6}$$

with $\lambda \geq 1$ and $\overline{e} < 0$ or $\overline{e} > 0$ for the respective repulsive and attractive cases. In the same manner as it was done in [17] for the classical operator, it was proved in [2] that the condition $\lambda \geq 1$ is essential in the repulsive case of (6). On the other hand, when the singularity is attractive, the existence of a $T-$periodic solution also requires that $\lambda \geq 1$ when $e \in L^1([0,T];\mathbb{R})$. This was shown by using the same argument as in [14], which allowed to find an example of an external force e with $\overline{e} > 0$ such that (6) has no $T-$periodic solution when $\lambda \in (0,1)$. Therefore, the same optimal results are obtained both in the classical and the relativistic cases.

An important particular case is the equation

$$(\phi(u'))' = \frac{h(t)}{\cos^2 u}, \tag{7}$$

where h is defined as above. This equation can be related to the dynamic on the sphere of a relativistic particle subjected to the influence of an electric field created by a charge of a time-depending magnitude fixed in the north pole. Here h is an integrable $T-$periodic function corresponding to the magnetic interaction between the charges. In other words, this problem models the dynamical behaviour of a relativistic particle moving on $\mathbb{S}^2$ under the influence of Newton's law (Kepler's problem on $\mathbb{S}^2$). For more details see [7, 8, 15].

By virtue of Theorem 1.1 the following consequence is deduced:

Corollary 1. *Suppose that $T < \pi$. Then (7) has a $T-$periodic solution provided that one of the following situations occurs:*

- $\overline{h} < 0$ *and there exists* $\{\ell_1 < \ell_1' < \ldots < \ell_n < \ell_n'\} \subseteq [0,T]$ *such that* (2) *holds and*
$$\lim_{t\to 0^+} \inf_{s\in[\ell_i+t,\ell_i'-t]} h(s)\tan(\pi/2 - t) = +\infty, \qquad \forall \quad i = 1,\ldots,n; \tag{8}$$
- $\overline{h} > 0$ *and there exists* $\{s_1 < s_1' < \ldots < s_n < s_n'\} \subseteq [0,T]$ *such that* $h\mid_{(s_i,s_i')} < 0$ *for all* $i = 1,\ldots,n$,
$$h(t) = \begin{cases} -h^-(t), & \text{if } t \in \cup_{i=1}^n [s_i, s_i'], \\ h^+(t), & \text{otherwise}, \end{cases}$$
for $t \in [0,T]$ *and*
$$\lim_{t\to 0^+} \inf_{s\in[s_i+t,s_i'-t]} |h(s)|\tan(\pi/2 - t) = +\infty, \qquad \forall \quad i = 1,\ldots,n; \tag{9}$$
- $\overline{h} \neq 0$ *and there exists* $\{r_1 < r_1' < \ldots < r_n < r_n'\} \subseteq [0,T]$ *such that* h *has constant sign over* (r_i, r_i') *for all* $i = 1,\ldots,n$,
$$h(t) = \begin{cases} h(t), & \text{if } t \in \cup_{i=1}^n [r_i, r_i'], \\ 0, & \text{otherwise}, \end{cases}$$
for $t \in [0,T]$ *and*
$$\lim_{t\to 0^+} \inf_{s\in[r_i+t,r_i'-t]} |h(s)|\tan(\pi/2 - t) = +\infty, \qquad \forall \quad i = 1,\ldots,n. \tag{10}$$

In particular (8) *(resp.* (9) *or* (10)*) holds when* $|h(t)| \geq c > 0$ *for all* $t \in [\ell_i, \ell_i']$ *(resp.* $t \in [s_i, s_i']$ *or* $t \in [r_i, r_i']$*) for all* $i = 1, \ldots, n$.

Our result can be seen as a generalization of the main theorem in [15] considering relativistic effects in the interaction of the particles.

Another interesting equation has the form

$$(\phi(u'))' = \frac{h(t)}{u^\lambda (1-u)^\mu}, \tag{11}$$

where $\lambda > 0$ and $\mu > 0$. In this case, our main result applies as follows:

Corollary 2. *Suppose that* $T < 2\min\{\lambda, \mu\}/(\lambda + \mu)$. *Then* (11) *has a* $T-$*periodic solution provided that one of the following situations occurs:*

- $\overline{h} < 0$ *and there exists* $\{\ell_1 < \ell_1' < \ldots < \ell_n < \ell_n'\} \subseteq [0, T]$ *such that* (2) *holds and*

$$\lim_{t\to 0^+} \frac{\inf_{s\in[\ell_i+t, \ell_i'-t]} h(s)}{t^{\lambda-1}} = +\infty, \quad \text{or} \quad \lim_{t\to 0^+} \inf_{s\in[\ell_i+t,\ell_i'-t]} h(s) \ln\left(\frac{1}{t}\right) = +\infty, \tag{12}$$

  respectively if $\lambda > 1$ *or* $\lambda = 1$, *for all* $i = 1, \ldots, n$;
- $\overline{h} > 0$ *and there exist* $\{s_1 < s_1', \ldots, s_n < s_n'\} \subseteq [0, T]$ *such that* $h\mid_{(s_i, s_i')} < 0$ *for all* $i = 1, \ldots, n$,

$$h(t) = \begin{cases} -h^-(t), & \text{if } t \in \cup_{i=1}^n [s_i, s_i'], \\ h^+(t), & \text{otherwise}, \end{cases}$$

  for $t \in [0, T]$ *and*

$$\lim_{t\to 0^+} \frac{\inf_{s\in[s_i+t, s_i'-t]} |h(s)|}{t^{\mu-1}} = +\infty, \quad \text{or} \quad \lim_{t\to 0^+} \inf_{s\in[s_i+t,s_i'-t]} |h(s)| \ln\left(\frac{1}{t}\right) = +\infty, \tag{13}$$

  respectively if $\mu > 1$ *or* $\mu = 1$, *for all* $i = 1, \ldots, n$;
- $\overline{h} \neq 0$ *and there exists* $\{r_1 < r_1', \ldots, r_n < r_n'\} \subseteq [0, T]$ *such that* h *has constant sign over* (r_i, r_i') *for all* $i = 1, \ldots, n$,

$$h(t) = \begin{cases} h(t), & \text{if } t \in \cup_{i=1}^n [r_i, r_i'], \\ 0, & \text{otherwise}, \end{cases}$$

  for $t \in [0, T]$ *and*

$$\lim_{t\to 0^+} \frac{\inf_{s\in[r_i+t, r_i'-t]} |h(s)|}{t^{\gamma-1}} = +\infty, \quad \text{or} \quad \lim_{t\to 0^+} \inf_{s\in[r_i+t,r_i'-t]} |h(s)| \ln\left(\frac{1}{t}\right) = +\infty, \tag{14}$$

  respectively if $\gamma := \min\{\lambda, \mu\} > 1$ *or* $\gamma = 1$, *for all* $i = 1, \ldots, n$.

In particular (12) *(resp.* (13) *or* (14)*) holds when* $|h(t)| \geq c > 0$ *for all* $t \in [\ell_i, \ell_i']$ *(resp.* $t \in [s_i, s_i']$ *or* $t \in [r_i, r_i']$*) for all* $i = 1, \ldots, n$.

At first sight, Corollary 2 does not seem to be valid if one considers (11) with $\mu = 0$, that is:

$$(\phi(u'))' = \frac{h(t)}{u^\lambda}. \tag{15}$$

However, a detailed analysis of the proof of Theorem 1.1 yields the following:

Corollary 3. *Assume that* (2) *and* (12) *hold. Then* (15) *has a* $T-$*periodic solution if and only if* $\overline{h} < 0$. *In particular* (12) *holds when* $\lambda \geq 1$ *and* $|h(t)| \geq c > 0$ *for all* $t \in [\ell_i, \ell_i']$, $i = 1, \ldots, n$.

Since, in this case, the nonlinearity is (strictly) decreasing, the condition on the period of h included in the above theorems (small $T > 0$) can be omitted whether we can find an uniform bound for the maximum value of the (possible) $T-$periodic solutions of (15) (this happens if $\overline{h} < 0$). The fact that Corollary 2 cannot be applied when $\mu = 0$ means that arguing as above it is not possible to prove that any $T-$periodic solution u of (15) verifies the estimation $\|u\|_\infty < 1$.

We conclude this introduction with a brief list of notations used throughout the paper. For every locally integrable function h we denote

$$H_+ := \int_0^T [h(t)]_+ dt, \quad H_- := \int_0^T [h(t)]_- dt, \quad H := H_+ - H_-,$$

where for any real number a we write $[a]_+ := \max\{0, a\}$, $[a]_- := \max\{0, -a\}$, and $\overline{h} := H/T = \frac{1}{T}\int_0^T h(t)\, dt$.

Before jumping into the mathematical details of the paper, it is worthy to notice that the change of variable $v := u - t_\sharp$ allows us to assume, without loss of generality, that $\alpha < 0$, $\beta > 0$ and $t_\sharp = 0$.

2. **A continuum of $T-$periodic solutions to a modified problem.** For each $\delta > 0$, we define the truncation function

$$g_\delta(u) = \begin{cases} g(\beta - \delta), & \text{if } u \geq \beta - \delta, \\ g(u), & \text{if } \alpha + \delta \leq u \leq \beta - \delta, \\ g(\alpha + \delta) & \text{if } u \leq \alpha + \delta, \end{cases}$$

and consider the modified equation

$$(\phi(u'))' = h(t) g_\delta(u). \tag{16}$$

In what follows, the (real) Banach spaces $C := C([0, T]; \mathbb{R})$ and $\widetilde{C} := \{u \in C : u(0) = 0\}$ are considered with their usual norms. The $1-$dimensional subspace of C composed by the constant functions will be identified with $\mathbb{R}$; under this identification, we define the projection over this subspace

$$Q : C \to \mathbb{R}, \qquad Q[u] := \frac{1}{T}\int_0^T u(t)\, dt.$$

Moreover, for any $u \in \text{Ker} Q$ we denote by $K[u](t) := \int_0^t u(s) ds$. By using the Lyapunov-Schmidt decomposition $u = \widetilde{u} + \mu$ we shall rewrite the problem of finding $T-$periodic solutions of (16) in an abstract form. For the reader's convenience, let us briefly sketch the procedure. In the first place, recall (see [5, Proposition 2]) that there exists a continuous operator $Q_\phi : C \to \mathbb{R}$ verifying

$$\int_0^T \phi^{-1}\left(-Q_\phi[\varphi] + \varphi(t)\right) dt = 0, \qquad \forall\, \varphi \in C.$$

Next, write $N[u] := h(t) g_\delta(u)$ and observe that u is $T-$periodic and satisfies (16) if and only if $QN[u] = 0$ and $\phi(u') = c + \varphi$, where $\varphi := K(N[u] - QN[u]) = K(I - Q)N[u]$, and the constant c is chosen in such a way that

$$\int_0^T \phi^{-1}(c + \varphi(t))\, dt = 0.$$

This yields $c = -Q_\phi(\varphi)$, that is: $c + \varphi = (I - Q_\phi)\varphi$. Finally, from the identity $u' = \phi^{-1}(c + \varphi)$ we deduce that (16) can be rewritten as

$$\widetilde{u} = K\phi^{-1}(I - Q_\phi)K(I - Q)N[\mu + \widetilde{u}], \qquad QN[\mu + \widetilde{u}] = 0, \tag{17}$$

Define $M : \mathbb{R} \times \widetilde{C} \to \widetilde{C}$ the operator

$$M[\mu, \widetilde{u}] := K\phi^{-1}(I - Q_\phi)K(I - Q)N[\mu + \widetilde{u}]. \tag{18}$$

The fixed point problem (depending on the parameter μ)

$$\widetilde{u} = M[\mu, \widetilde{u}], \qquad (\mu, \widetilde{u}) \in \mathbb{R} \times \widetilde{C} \tag{19}$$

defines a problem whose solutions are related to the $T-$periodic solutions of

$$(\phi(u'))' = h(t)g_\delta(u) - \frac{1}{T}\int_0^T h(s)g_\delta(u)ds, \tag{20}$$

where $u = \mu + \widetilde{u}$. Thus, in order to solve (17) (namely, to find $T-$periodic solutions of (16)) it is sufficient to find a solution $u = \mu + \widetilde{u}$ of (19) such that $QN[u] = 0$.

Observe that, since by definition we have $\phi^{-1} : \mathbb{R} \to (-1, 1)$, the solutions of (19) verify $\|\widetilde{u}'\|_\infty < 1$. Thus, it follows from [2, Lemma 6] that $\|\widetilde{u}\|_\infty < T/2$. This implies that $d_{LS}(I - M[0, \cdot], U_0, 0) = 1$, where $U_0 := \{\widetilde{u} \in \widetilde{C} : \|\widetilde{u}\|_\infty < T/2\}$. Applying the Leray-Schauder Continuation Theorem, we find a continuum set $\mathcal{C}$ of solutions to (19) such that $\mathcal{C}_\mu \neq \emptyset$ for all $\mu \in \mathbb{R}$, where $\mathcal{C}_\mu := \{\tilde{u} \in \tilde{C} : (\mu, \tilde{u}) \in \mathcal{C}\}$.

Let us establish some important properties of this set that shall be used in the proof of our main result.

Lemma 2.1. *Let $I \subseteq [0, T]$ be compact. Then*

$$\left\{\mu + \min_{t \in I} \widetilde{u}(t) : (\mu, \widetilde{u}) \in \mathcal{C}\right\} = \mathbb{R}$$

Proof. Let $\mu \in \mathbb{R}$ and fix $\widetilde{u}_1, \widetilde{u}_2 \in \widetilde{C}$ such that

$$(\mu - T, \widetilde{u}_1) \in \mathcal{C}, \qquad (\mu + T, \widetilde{u}_2) \in \mathcal{C}.$$

Define the continuous operator

$$\widehat{\varphi} : \mathcal{C} \to \mathbb{R}, \qquad \widehat{\varphi}[\mu, \widetilde{u}] := \mu + \min_{t \in I} \widetilde{u}(t).$$

Since $\widehat{\varphi}[\mu - T, \widetilde{u}_1] \leq \mu \leq \widehat{\varphi}[\mu + T, \widetilde{u}_2]$, the connection property of $\mathcal{C}$ implies the existence of $(\mu_*, \widetilde{u}_*) \in \mathcal{C}$ such that $\widehat{\varphi}[\mu_*, \widetilde{u}_*] = \mu$. □

When the mean value of the Nemystkii operator is non-positive we deduce the following property:

Lemma 2.2. *For all $\delta > 0$, each $T-$periodic solution u of (20) verifies that*

$$\left|\frac{\phi(u')}{g_\delta(u)}\right| \leq \|h\|_1 \qquad \forall \quad t \in [0, T],$$

whenever $QN[u] \leq 0$ and $u(t) \leq 0$ for all $t \in [0, T]$.

Proof. Let u be a $T-$periodic solution of (20) such that $u(t) \leq 0$ for all $t \in [0, T]$ and $QN[u] \leq 0$. Let $t_m \in [0, T]$ be the point where u attains its minimum value on $[0, T]$. Multiplying both sides of (20) by $g_\delta(u)^{-1}$ and integrating by parts in the left hand side of the identity, one arrives to

$$\frac{\phi(u')}{g_\delta(u)} = \int_{t_m}^t h(s)ds + \int_{t_m}^t \left(\phi(u')\frac{d}{ds}\left[\frac{1}{g_\delta(u)}\right] - \frac{QN[u]}{g_\delta(u)}\right)ds \tag{21}$$

for all $t \in \mathbb{R}$ Taking into account that g_δ is non-increasing on $(-\infty, 0]$, since $u(t) \leq 0$ for all $t \in [0,T]$, one can easily check that

$$\phi(u')\frac{d}{ds}\left[\frac{1}{g_\delta(u)}\right] \geq 0$$

for almost all $s \in \mathbb{R}$. Now, from (21) we have

$$\frac{\phi(u')}{g_\delta(u)} \leq \|h\|_1 \qquad \forall \quad t \in [t_m - T, t_m], \quad \frac{\phi(u')}{g_\delta(u)} \geq -\|h\|_1 \qquad \forall \quad t \in [t_m, t_m + T].$$

Therefore, the proof follows from the periodicity of u. □

3. **Periodic solutions to a modified equation.** In this section, we deal with the existence of $T-$periodic solutions of (16). To this end, it will be important to employ our hypothesis $T < 2\min\{-\alpha, \beta\}$, which guarantees that the solutions passing close to the singularity cannot change sign. In more detail, this hypothesis allows to take $\varepsilon_0 > 0$ sufficiently small such that

$$\frac{T}{2} + \varepsilon < \min\{-\alpha, \beta\}, \qquad \forall \quad \varepsilon \in (0, \varepsilon_0].$$

Thus, if u is a $T-$periodic solution of (20) then by [2, Lemma 6]

$$\max_{t\in[0,T]} u(t) - \min_{t\in[0,T]} u(t) < T/2,$$

and we conclude that

$$\max_{t\in[0,T]} u(t) < 0 \qquad \text{if} \quad \min_{t\in[0,T]} u(t) \leq \alpha + \varepsilon$$

or

$$\min_{t\in[0,T]} u(t) > 0 \qquad \text{if} \quad \max_{t\in[0,T]} u(t) \geq \beta - \varepsilon$$

for $\varepsilon \in (0, \varepsilon_0]$. In this sense, our hypothesis allows us to control the monotonicity of our nonlinearity, which shall be essential in our proofs.

Let us define $I_i := [\bar{\ell}_i, \bar{\ell}'_i] \subseteq (\ell_i, \ell'_i)$ and $I := \cup_{i=1}^n I_i$. The main goal of this section consists in proving the following result:

Proposition 1. *There exists $0 < \varepsilon < \varepsilon_0$ (depending on ℓ_i, $\bar{\ell}_i$, $\bar{\ell}'_i$, $\bar{\ell}_i$ for all $i = 1, \ldots, n$) such that for all $0 < \delta \leq \varepsilon$ there exists a $T-$periodic solution of (16) with $\min_{t\in I} u(t) > \alpha + \varepsilon$.*

Intuitively, the key to prove this result is to show that there are no $T-$periodic solutions of (20) whose minimum value over I is close to α whenever $QN[u] \leq 0$. Our strategy is inspired in the classical argument of Lazer and Solimini to find a priori bounds over the set of possible periodic solutions. In our situation, we fix the subintervals of $[0,T]$ where the sign of the potential is well-defined and we adapt the mentioned argument on each one of these intervals. Here, the requirements on a priori estimates are weakened, obtaining a priori bounds not over the set of all possible periodic solutions but, instead, over a subset of solutions connected with these subintervals.

Lemma 3.1. *There exists $0 < \varepsilon < \varepsilon_0$ (depending on ℓ_i, $\bar{\ell}_i$, $\bar{\ell}'_i$, $\bar{\ell}_i$ for all $i = 1, \ldots, n$) such that for all $0 < \delta \leq \varepsilon$, each $T-$periodic solution u of (20) with $QN[u] \leq 0$ verifies that $\min_{t\in I} u(t) \neq \alpha + \varepsilon$.*

Proof. We choose $a_i \in (\ell_i, \bar{\ell}_i)$ and $a'_i \in (\bar{\ell}'_i, \ell'_i)$ for arbitrary $i = 1, \ldots, n$. Since

$$\lim_{x \to 0^+} \int_{\bar{\ell}'_i}^{a'_i} \phi^{-1}\left(h_0(t - \bar{\ell}'_i) g(\alpha + x)\right) dt - x > 0,$$

$$\lim_{x \to 0^+} \int_{a_i}^{\bar{\ell}_i} \phi^{-1}\left(h_0(\bar{\ell}_i - t) g(\alpha + x)\right) dt - x > 0,$$

for all $i = 1, \ldots, n$ with $h_0 := \min\{\inf_{t \in [a_i, a'_i]} h(t) : i = 1, \ldots, n\} > 0$, there exists x_0 (depending only on a_i, $\bar{\ell}_i$, $\bar{\ell}'_i$ and a'_i for $i = 1, \ldots, n$) such that

$$\int_{\bar{\ell}'_i}^{a'_i} \phi^{-1}\left(h_0(t - \bar{\ell}'_i) g(\alpha + x)\right) dt > x, \qquad \forall \quad i = 1, \ldots, n \tag{22}$$

$$\int_{a_i}^{\bar{\ell}_i} \phi^{-1}\left(h_0(\bar{\ell}_i - t) g(\alpha + x)\right) dt > x, \qquad \forall \quad i = 1, \ldots, n, \tag{23}$$

for all $x \in (0, x_0]$. We define $\varepsilon \in (0, \varepsilon_0)$ (depending only on a_i, $\bar{\ell}_i$, $\bar{\ell}'_i$ and a'_i for $i = 1, \ldots, n$) such that

$$\|h\|_1 < h_0 g(\alpha + x_0)^{-1} \int_{\alpha+\varepsilon}^{\alpha+x_0} g(s) ds. \tag{24}$$

Let $\delta > 0$ be fixed with $\delta \in (0, \varepsilon]$. Assume there exists a $T-$periodic solution of (20) such that $\min_{t \in I} u(t) = \alpha + \varepsilon$ and $QN[u] \le 0$. Then there exists $t_* \in I_i$ for some $i = 1, \ldots, n$ such that $u(t_*) = \alpha + \varepsilon$ and we distinguish two cases:

Case I. $[t_* \in [\bar{\ell}_i, \bar{\ell}'_i)]$. Observe that $u'(t_*) \ge 0$. We claim that $\max_{t \in [t_*, a'_i]} u(t) \ge \alpha + x_0$. Indeed, if we suppose $u(t) < \alpha + x_0$ for all $t \in [t_*, a'_i]$ then the inequality $QN[u] \le 0$ implies that

$$(\phi(u'))' \ge h_0 g_\delta(u) \qquad \forall \quad t \in [a_i, a'_i]. \tag{25}$$

Since $u'(t_*) \ge 0$ then $u'(t) \ge 0$, $\alpha + \varepsilon \le u(t)$ for $t \in [t_*, a'_i]$ and $u(a'_i) = \max_{t \in [t_*, a'_i]} u(t)$. Multiplying both sides of (25) by $g(u)^{-1}$ and integrating from t_* to $t \in [t_*, a'_i]$ we arrive to

$$\phi(u') \ge h_0(t - t_*) g(u) \qquad \forall \quad t \in [t_*, a'_i].$$

Since ϕ is an increasing homeomorphism we have

$$u'(t) \ge \phi^{-1}\left(h_0(t - t_*) g(\alpha + x_0)\right) \qquad \forall \quad t \in [t_*, a'_i].$$

Integrating the latter inequality over $[t_*, a'_i]$ we obtain a contradiction with (22).

Since $\max_{t \in [t_*, a'_i]} u(t) \ge \alpha + x_0$, we can define $b_* \in (t_*, a'_i]$ such that $u(b_*) = \alpha + x_0$ satisfying $\alpha + \varepsilon \le u(t) \le \alpha + x_0$ for all $t \in [t_*, b_*]$ (see Figure 1, case I). From (25), $u' \ge 0$ (since $u'(t_*) \ge 0$) on $[t_*, b_*]$. Moreover,

$$(\phi(u'))' u' \ge h_0 g(u) u' \qquad \forall \quad t \in [t_*, b_*].$$

By integrating between t_* and $t \in [t_*, b_*]$ we arrive to

$$\phi(u') \ge h_0 \int_{\alpha+\varepsilon}^{u(t)} g(s) ds \qquad \forall \quad t \in [t_*, b_*].$$

Multiplying both sides of the above inequality by $g(u)^{-1}$ and applying Lemma 2.2 we have

$$\|h\|_1 \ge h_0 g(u(t))^{-1} \int_{\alpha+\varepsilon}^{u(t)} g(s) ds \qquad \forall \quad t \in [t_*, b_*],$$

which contradicts (24) for $t = b_*$.

Case II. $[t_* \in (\bar{\ell}_i, \bar{\ell}_i']]$. Here, $u'(t_*) \leq 0$ and we claim that $\max_{t\in[a_i,t_*]} u(t) \geq \alpha + x_0$. Assume on the contrary that $u(t) < \alpha + x_0$ for all $t \in [a_i, t_*]$. Since (25) remains valid, it is seen that $(\phi(u'))' > 0$ on that interval. Because $u'(t_*) \leq 0$, we deduce that $u'(t) \leq 0$ and $\alpha + \varepsilon \leq u(t)$ for all $t \in [a_i, t_*]$. Arguing as in Case I., multiplying both sides of (25) by $g(u)^{-1}$, integrating from $t \in [a_i, t_*]$ to t_* and using that $u(a_i) = \max_{t\in[a_i,t_*]} u(t)$ we arrive to

$$-u'(t) \geq \phi^{-1}\left(h_0(t_* - t)g(\alpha + x_0)\right) \qquad \forall \quad t \in [a_i, t_*].$$

Integrating the latter inequality over $[a_i, t_*]$ we obtain a contradiction with (23).

Finally, since $\max_{t\in[a_i,t_*]} u(t) \geq \alpha + x_0$ we can define $a_* \in [a_i, t_*)$ such that $u(a_*) = \alpha + x_0$ and $\alpha + \varepsilon \leq u(t) \leq \alpha + x_0$ for all $t \in [a_*, t_*]$ (see Figure 1, case II). Then $(\phi(u'))' > 0$ and $u'(t) \leq 0$ (since $u'(t_*) \leq 0$) on $[a_*, t_*]$. Moreover,

$$(\phi(u'))'u' \leq h_0 g(u)u' \qquad \forall \quad t \in [a_*, t_*].$$

By integrating from $t \in [a_*, t_*]$ to t_* we observe that

$$\phi(u') \leq -h_0 \int_{\alpha+\varepsilon}^{u(t)} g(s)ds \qquad \forall \quad t \in [a_*, t_*].$$

Multiplying both sides of the above inequality by $g(u)^{-1}$ and applying Lemma 2.2 we obtain

$$-\|h\|_1 \leq -h_0 g(u)^{-1} \int_{\alpha+\varepsilon}^{u(t)} g(s)ds,$$

which contradicts (24) taking $t = a_*$. □

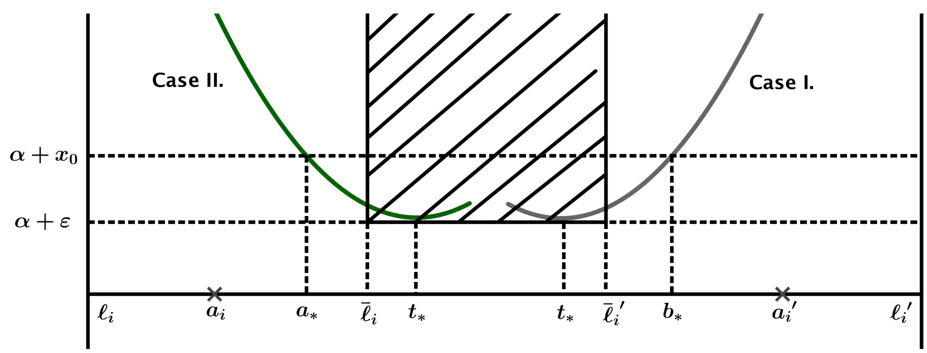

Figure 1. The figure illustrates a possible behaviour of any $T-$periodic solution u of (20) when t_* is included on $[\bar{\ell}_i, \bar{\ell}_i')$ (Case I.) or if $t_* \in (\bar{\ell}_i, \bar{\ell}_i']$ (Case II.).

Let us consider now the function $\varphi : \Sigma \to \mathbb{R}$ given by $(\mu, \widetilde{u}) \mapsto \mu + \min_{t\in I} \widetilde{u}(t)$, where Σ denotes the set of solutions of problem (19). It is observed that $\mathcal{C} \subseteq \Sigma$, $\varphi\mid_{\mathcal{C}} = \widehat{\varphi}$ and hence $\varphi(\Sigma) = \mathbb{R}$ (see Lemma 1). Before giving a proof of Proposition 1, let us establish a preliminary result.

Lemma 3.2. *Let $a < b$ be real numbers. Then there exists $A_{[a,b]}$ a connected component of $\varphi^{-1}([a,b])$ linking $\varphi^{-1}(a)$ with $\varphi^{-1}(b)$.*

Proof. In the first place, notice that the operator φ has the following 'nice' property:

- φ^{-1} maps unbounded (resp. bounded) sets into unbounded (resp. bounded) sets.

This can be easily observed taking into account that for every $x = (\mu, \widetilde{u}) \in \Sigma$ the inequality $\|\widetilde{u}'\|_\infty < 1$ holds (i.e., $\Sigma \subseteq \mathbb{R} \times B(0, T/2)$). This property will be crucial in order to prove the statement.

It has been already mentioned that $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$ are non-empty sets. By contradiction, we assume they are not joined by a connected component of $\varphi^{-1}([a,b])$. By Whyburn's lemma, there exist $C_a \supseteq \varphi^{-1}(a)$ and $C_b \supseteq \varphi^{-1}(b)$ two disjoint closet sets such that $\varphi^{-1}([a,b]) = C_a \cup C_b$, and an open bounded set $\Omega \supseteq C_a$ such that $\Omega \cap C_b = \emptyset = \partial\Omega \cap \varphi^{-1}([a,b])$ (see ref. [1, 12, 21]). We define $\Theta[\mu, \widetilde{u}] := (\mu - \min_{t\in I} \widetilde{u}(t), \widetilde{u})$ and consider the open set

$$\Omega_\mu := \{\widetilde{u} \in \widetilde{C} : (\mu, \widetilde{u}) \in \Theta^{-1}(\Omega)\}$$

and the homotopy operator

$$F : \mathbb{R} \times \widetilde{C} \to \widetilde{C}, \qquad F[\mu, \widetilde{u}] := \widetilde{u} - M\left(\Theta[\mu, \widetilde{u}]\right),$$

where M is defined by (18). Now we claim that

$$F[\mu, \widetilde{u}] \neq 0 \qquad \text{for} \quad \widetilde{u} \in \partial\Omega_\mu, \quad \mu \in [a,b]. \tag{26}$$

Indeed, if $F[\mu, \widetilde{u}] = 0$ for some $\widetilde{u} \in \partial\Omega_\mu$ and $\mu \in [a,b]$, then $(\mu - \min_{t\in I} \widetilde{u}(t), \widetilde{u}) \in \partial\Omega \cap K = \emptyset$, a contradiction. By virtue of the generalized homotopy invariance of the degree, we conclude that

$$d_{LS}(F[\mu, \cdot], \Omega_\mu, 0) \text{ is constant for every } \mu \in [a,b]. \tag{27}$$

Moreover, if $F[a, \widetilde{u}] = 0$, then $\varphi[\mu_0, \widetilde{u}] = a$ with $\mu_0 := a - \min_{t\in I} \widetilde{u}(t)$, whence it follows that $(\mu_0, \widetilde{u}) \in C_a \subseteq \Omega$ and hence $\widetilde{u} \in \Omega_a$. Consequently,

$$d_{LS}(F[a, \cdot], \Omega_a, 0) = d_{LS}(F[a, \cdot], B(0, T/2), 0).$$

Taking into account that

$$\widetilde{u} \neq \sigma M\left(\Theta[\mu, \widetilde{u}]\right) \qquad \forall \quad \sigma \in [0,1], \quad \|\widetilde{u}\|_\infty = T/2,$$

by using the latter identity we obtain

$$d_{LS}(F[a, \cdot], \Omega_a, 0) = 1. \tag{28}$$

Finally, observe that if $F[b, \widetilde{u}] = 0$ then $(\mu_1, \widetilde{u}) \in C_b$, where $\mu_1 := b - \min_{t\in I} \widetilde{u}(t)$. Since $\Omega \cap C_b = \emptyset$, it follows that $\widetilde{u} \notin \Omega_b$ and hence

$$d_{LS}(F[b, \cdot], \Omega_b, 0) = 0.$$

This contradicts (27) and (28). □

Remark 1. It is worth mentioning that the proof of the above result is fundamental because, in general, the sets $\varphi^{-1}(a)$ and $\varphi^{-1}(b)$ themselves might not be connected in $\varphi^{-1}([a,b])$.

Proof of Proposition 1. Take $\varepsilon > 0$ verifying Lemma 3.1 and $\delta \in (0, \varepsilon]$. Set $A := A_{[\alpha+\varepsilon,\beta+T]}$ a connected component of $\varphi^{-1}([\alpha+\varepsilon,\beta+T])$ connecting $\varphi^{-1}(\alpha+\varepsilon)$ with $\varphi^{-1}(\beta+T)$ (see Lemma 3.2). Let $(\mu, \widetilde{u}_\mu) \in A$ such that $\varphi(\mu, u_\mu) = \alpha+\varepsilon$, by Lemma 3.1 we have that $QN[\mu + \widetilde{u}_\mu] > 0$. On the other hand, we can take $(\widehat{\mu}, \widetilde{u}_{\widehat{\mu}}) \in A$ such that $\varphi(\widehat{\mu}, \widetilde{u}_{\widehat{\mu}}) = \beta + T$, then $QN[\widehat{\mu} + \widetilde{u}_{\widehat{\mu}}] < 0$. From the connectedness of A, by applying the mean value theorem we conclude that there exists $(\mu, \widetilde{u}) \in A$ such that $QN[\mu + \widetilde{u}] = 0$. Thus, the function $u := \mu + \widetilde{u}$ is a $T-$periodic solution of (16) verifying that $\varphi[\mu, \widetilde{u}] = \min_{t\in I} u(t) > \alpha + \varepsilon$ (we remark that, from Lemma 3.1, the equality cannot happen). □

4. **Proof of the main results.** For the sake of simplicity we suppose the validity of all the assumptions of Theorem 1.1. As a consequence of assuming that $T < 2\beta$, the solutions of (16) that cross the modified region from above cannot change sign. However, as we shall see, this situation cannot occur.

Lemma 4.1. *For all $0 < \delta \leq \varepsilon_0$, each $T-$periodic solution u of (16) verifies that $\max_{t\in[0,T]} u(t) \leq \beta - \delta$.*

Proof. Suppose, on the contrary, that there exists $\delta \in (0, \varepsilon_0]$ such that (16) admits a $T-$periodic solution u verifying that $\max_{t\in[0,T]} u(t) > \beta - \delta$. Since $\delta \leq \varepsilon_0$, we know that $u(t) > 0$ for all $t \in [0, T]$. By multiplying by $g_\delta(u)^{-1}$ in both sides of (16) and integrating over $[0, T]$ we obtain

$$H = -\int_0^T \phi(u')\frac{d}{ds}\left[\frac{1}{g_\delta(u)}\right] ds.$$

However, it is clear that

$$\phi(u')\frac{d}{dt}\left[\frac{1}{g_\delta(u)}\right] \leq 0 \qquad \text{for almost} \quad t \in [0, T],$$

contradicting that $H < 0$. □

The rest of the section is devoted to prove that if $|\ell_i - \bar{\ell}_i|$ and $|\ell'_i - \bar{\ell}'_i|$ are sufficiently small for all $i = 1, \ldots, n$ and ε is given by Proposition 1, then we can find $\delta \in (0, \varepsilon)$ for which the $T-$periodic solution of (16) is in fact a solution of (1).

Proof of Theorem 1.1. From (3), define $a_* > 0$ sufficiently small such that

$$\inf_{s\in\cup_{i=1}^n[\ell_i+a_*,\ell'_i-a_*]} h(s)g(\alpha+\varepsilon)^{-1}\int_{\alpha+4na_*}^{\alpha+\varepsilon} g(r)dr > \|h\|_1,$$

where ε is given by Proposition 1. If we set $a_i := \ell_i + a_* \in (\ell_i, \bar{\ell}_i)$, $a'_i := \ell'_i - a_* \in (\bar{\ell}'_i, \ell'_i)$ then

$$\inf_{s\in\cup_{i=1}^n[a_i,a'_i]} h(s)g(\alpha+\varepsilon)^{-1}\int_{\alpha+4na_*}^{\alpha+\varepsilon} g(r)dr > \|h\|_1.$$

Defining $\delta := \sum_{i=1}^n |\ell_i - a_i| + \sum_{i=1}^n |\ell'_i - a'_i|$, since $2na_* = \delta$, the latter inequality implies that

$$\inf_{s\in[a_i,a'_i]} h(s)g(\alpha+\varepsilon)^{-1}\int_{\alpha+2\delta}^{\alpha+\varepsilon} g(r)dr > \|h\|_1, \qquad \forall \quad i = 1, \ldots, n. \tag{29}$$

According to Proposition 1 (for ε and δ already fixed) there exists a $T-$periodic solution of (16) with $\min_{t\in I} u(t) > \alpha + \varepsilon$ (and from Lemma 4.1, $\max_{t\in[0,T]} u(t) < \beta - \delta$). The remaining part of the proof is devoted to check that $\min_{t\in[0,T]} u(t) \geq \alpha + \delta$. We assume, on the contrary that there exists $t_* \in [0, T]$ such that $u(t_*) =$

$\min_{t\in[0,T]} u(t) < \alpha+\delta$. First of all, notice that we can assume without loss of generality that $t_* \in \cup_{i=1}^n [\ell_i, \ell_i']$. Indeed, this happens because the minimum value of u cannot occur in any open interval where $h<0$. Thus, since $u(t_*)<\alpha+\delta$, we obtain that $t_* \in \cup_{i=1}^n [\ell_i,\ell_i'] \setminus I$. For instance, we assume that $t_* \in [\ell_i,\ell_i']\setminus I_i$. Observe that

$$(\phi(u'))' \geq h_0 g_\delta(u) \qquad \forall \quad t\in[a_i,a_i'], \tag{30}$$

where $h_0 := \inf_{t\in[a_i,a_i']} h(t)$. Now, we distinguish three cases.

Case I. $[t_* \in [a_i,a_i']]$. We consider two sub-cases:

a) $[t_* \in [a_i,\bar{\ell}_i)]$. Since $u(\bar{\ell}_i) > \alpha+\varepsilon$ we can define $t_1 \in (t_*,\bar{\ell}_i)$ such that $u(t_1)=\alpha+\varepsilon$ (see Figure 2). Moreover, since $u'(t_*)=0$, (30) implies that $u'(t)\geq 0$ for all $t\in[t_*,\bar{\ell}_i]$ and we deduce that

$$(\phi(u'(t)))'u'(t) \geq h_0 g_\delta(u(t))u'(t) \qquad \forall \quad t\in[t_*,\bar{\ell}_i].$$

We integrate from t_* to $t\in[t_*,\bar{\ell}_i]$ in both sides of the inequality to obtain

$$\phi(u'(t)) \geq h_0 \int_{u(t_*)}^{u(t)} g_\delta(s)ds \qquad \forall \quad t\in[t_*,\bar{\ell}_i].$$

Multiplying by $g_\delta(u)^{-1}$ we have

$$\frac{\phi(u')}{g_\delta(u)} \geq \frac{h_0}{g_\delta(u(t))} \int_{\delta+\alpha}^{u(t)} g_\delta(s)ds \qquad \forall \quad t\in[t_*,\bar{\ell}_i].$$

According to Lemma 2.2 we obtain a contradiction with (29) when $t=t_1$.

b) $[t_* \in (\bar{\ell}_i', a_i']]$. Since $u(\bar{\ell}_i') > \alpha+\varepsilon$ we can define $\widetilde{t} \in (\bar{\ell}_i', t_*)$ such that $u(\widetilde{t})=\alpha+\varepsilon$ (see Figure 2). Moreover, since $u'(t_*)=0$, (30) implies that $u'(t)\leq 0$ for all $t\in[\bar{\ell}_i',t_*]$ and we have that

$$(\phi(u'))'u' \leq h_0 g_\delta(u)u' \qquad \forall \quad t\in[\bar{\ell}_i',t_*].$$

We integrate from $t\in[\bar{\ell}_i',t_*]$ to t_* in both sides of the inequality to verify that

$$\phi(u'(t)) \leq -h_0 \int_{u(t_*)}^{u(t)} g_\delta(s)ds \qquad \forall \quad t\in[\bar{\ell}_i',t_*].$$

Arguing as in **a)**, multiplying by $g_\delta(u)^{-1}$ in the latter inequality and applying Lemma 2.2 we deduce:

$$-\|h\|_1 \leq \frac{-h_0}{g_\delta(u(t))} \int_{\alpha+\delta}^{u(t)} g_\delta(s)ds \qquad \forall \quad t\in[\bar{\ell}_i',t_*],$$

which contradicts (29) taking $t=\widetilde{t}$.

Case II. $[t_* \in (a_i',\ell_i']]$. Our first task will consist in verifying that $u(a_i')\geq\alpha+2\delta$. By contradiction, assume that $u(a_i')<\alpha+2\delta$. Since u is convex on $[\ell_i,\ell_i']$ and $u'(t_*)=0$ then $u'(t)\leq 0$ for all $t\in[\ell_i,t_*]$. Multiplying both sides of (30) by u' and integrating from $t\in[a_i,a_i']$ to a_i' we obtain that

$$\phi(u'(t)) \leq -h_0 \int_{u(a_i')}^{u(t)} g_\delta(s)ds \qquad \forall \quad t\in[a_i,a_i']. \tag{31}$$

On the other hand, since $u(\bar{\ell}_i') > \alpha+\varepsilon$ we can define $\widetilde{t} \in (\bar{\ell}_i', a_i')$ such that $u(\widetilde{t})=\alpha+\varepsilon$ (see Figure 3). Multiplying by $g_\delta(u)^{-1}$ in both sides of (31) and applying Lemma

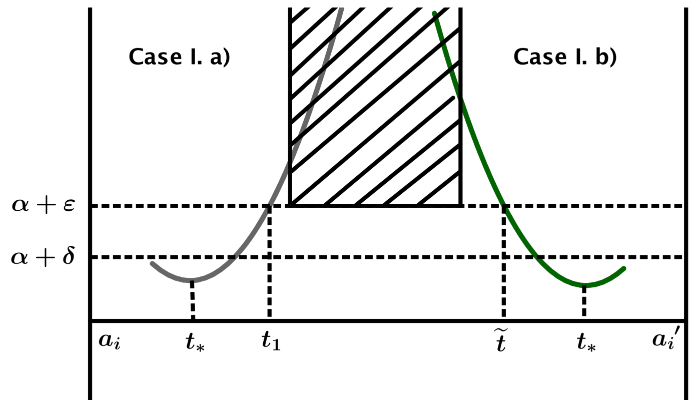

FIGURE 2. The figure illustrates a possible behaviour of the $T-$periodic solution u of (16) when t_* is included on $[a_i, a_i']$, distinguishing if $t_* \in [a_i, \bar{\ell}_i)$ (Case I. a)) or $t_* \in (\bar{\ell}_i', a_i']$ (Case I. b)).

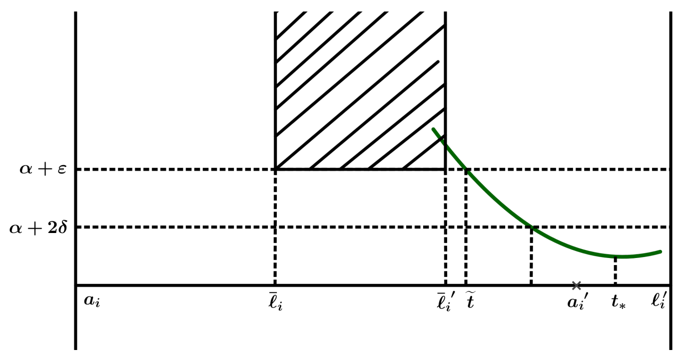

FIGURE 3. The figure illustrates a possible behaviour of the $T-$periodic solution u of (16) on the interval $[\bar{\ell}_i', \ell_i']$, assuming that $u(a_i') < \alpha + 2\delta$.

2.2 we deduce that

$$-\|h\|_1 \leq \frac{-h_0}{g_\delta(u(t))} \int_{2\delta}^{u(t)} g_\delta(s)ds \qquad \forall \quad t \in [a_i, a_i'].$$

This contradicts (29) taking $t = \widetilde{t}$.

At this point, our situation corresponds to Figure 4. Thus, let $t_0 \in [a_i', t_*)$ be such that $u(t_0) = \alpha + 2\delta$. The contradiction follows immediately because $\|u'\|_\infty < 1$ and the length of the interval (a_i', l_i') is less than δ. Thus, $u(t_*) \geq \alpha + \delta$.

Case III. $[t_* \in [\ell_i, a_i)]$. It is analogous to Case II. □

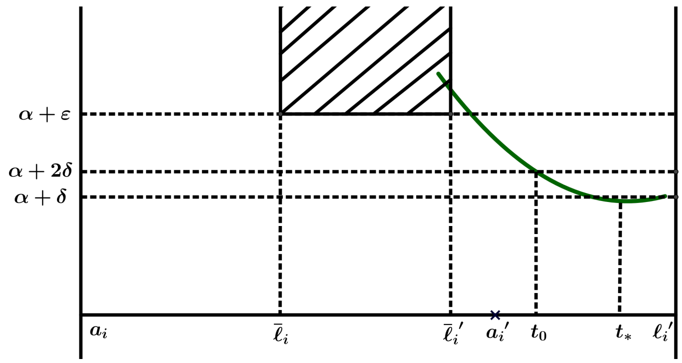

FIGURE 4. The figure illustrates a possible behaviour of the $T-$periodic solution u of (16) on the interval $[\bar{\ell}_i', \ell_i']$, assuming that $u(a_i') \geq \alpha + 2\delta$.

At the end of this section we shall prove some consequences of Theorem 1.1. Firstly, we note that if our nonlinearity is defined by

$$g(x) := \frac{1}{\cos^2 x} \qquad \text{or} \qquad g(x) := \frac{1}{x^\lambda(1-x)^\mu},$$

then $t_\sharp = 0$ and $\alpha = -\pi/2$, $\beta = \pi/2$ and $t_\sharp = \lambda/(\lambda+\mu)$, $\alpha = 0$, $\beta = 1$, respectively.

As already mentioned in the introduction, if $g(\alpha+4nt)/g(\alpha+t) \not\to 0$ as $t \to 0^+$, then condition (3) can be replaced by (4). This is the case when the nonlinearity is defined as above. On the other hand, (4) is equivalent to

$$\lim_{t\to 0^+} \inf_{s\in[\ell_i+t,\ell_i'-t]} h(s) \int_{\alpha+t}^{t_\sharp} g(s)ds = +\infty \qquad \forall \quad i = 1,\dots,n. \tag{32}$$

Proof of Corollary 1. If $\overline{h} < 0$, then the result is deduced from the fact that (8) implies (32). For $\overline{h} > 0$, we may consider the change of variable $v := -u$. Thus, combining both cases we conclude the proof for $\overline{h} \neq 0$. □

Proof of Corollary 2. If $\overline{h} < 0$, then the result is obtained by observing that (12) implies (32). For $\overline{h} > 0$, we consider the change of variable $v := 1-u$ and the proof follows. □

Proof of Corollary 3. After multiplying both sides of (15) by u^λ and integrating by parts, it is seen that a necessary condition for the existence of $T-$periodic solutions is that $\overline{h} < 0$. Conversely, since

$$\lim_{x\to+\infty} \left(\frac{x-T/2}{x}\right)^\lambda = 1,$$

we can find $x_0 > 0$ such that

$$H_+ - \left(\frac{x-T/2}{x}\right)^\lambda H_- < 0 \qquad \forall \quad x \geq x_0. \tag{33}$$

Thus, it suffices to prove the following

Claim: for every $\delta \in (0,1)$, each $T-$periodic solution u of (16) for $\beta := x_0 + 1$,

$\alpha = 0$ and $g(x) = 1/x^\lambda$ verifies that $\|u\|_\infty < \beta - \delta$.

Indeed, the same argument done in Theorem 1.1 applies, replacing Lemma 4.1 by the previous claim. To prove the latter, we may argue by contradiction. Assume there exists a $T-$periodic solution u of (16) such that $\|u\|_\infty > \beta - \delta$. Direct integration in (16) over $[0, T]$ gives

$$\begin{aligned} 0 &\leq \frac{H_+}{\left(\min_{t\in[0,T]} u(t)\right)^\lambda} - \frac{H_-}{(\beta-\delta)^\lambda} \\ &\leq \frac{H_+}{(\beta-\delta-T/2)^\lambda} - \frac{H_-}{(\beta-\delta)^\lambda} \\ &\leq \frac{1}{(\beta-\delta-T/2)^\lambda}\left[H_+ - \left(\frac{\beta-\delta-T/2}{\beta-\delta}\right)^\lambda H_-\right]. \end{aligned}$$

From (33) we deduce that $\beta - \delta < x_0 = \beta - 1$; which is a contradiction. □

5. **Conclusions and final remarks.** Some mathematical models justify the need of studying the existence of periodic solutions for indefinite singular differential equations having one or more singularities in the non-linear term. Sometimes, these models also present a non-monotone nonlinearity. In this work, we have found periodic solutions with small period when the external forcing term has negative mean value and the degeneracy of its zeroes is small with respect to the order of the (left-hand side) singularity of our non-linear term. When we particularize our results to the physical models given by (7) and (11), the symmetrical properties of the non-linearities allow to show in different situations that such results can be extended if the mean value of the forcing term is not zero. However, if we were able to prove Lemma 4.1 under the hypothesis $\overline{h} \leq 0$ (instead of $\overline{h} < 0$) it would have been easy to extend our results without supposing that $\overline{h} \neq 0$. This remains as an open question motivated by the results obtained in [15] for the classical case.

To the best of our knowledge, the results on the existence of $T-$periodic solutions for indefinite singular equations have required that the non-linear term has a strong singularity (even in the classical case). Therefore, a natural problem would be to find sufficient conditions on the external forcing term guaranteeing the existence of T-periodic solutions in the weak case.

On the other hand, our theorems ensure the existence of at least one periodic solution but they do not deal with the question of uniqueness (or multiplicity). According to [15] it is reasonable to think that several periodic solutions could appear in (7) (see the case when the forcing term has two weights in [15]).

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