

Timoshenko theory effect on the vibration of axially functionally graded cantilever beams carrying concentrated masses

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Abstract. In this paper is studied the effect of considering the theory of Timoshenko in the vibration of AFG beams that support ground masses. As it is known, Timoshenko theory takes into account the shear deformation and the rotational inertia, provides more accurate results in the general study of beams and is mandatory in the case of high frequencies or non-slender beams. The Rayleigh-Ritz Method is employed to obtain approximated solutions of the problem. The accuracy of the procedure is verified through results available in the literature that can be represented by the model under study. The incidence of the Timoshenko theory is analyzed for different cases of beam slenderness, variation of its cross section and compositions of its constituent material, as well as different amounts and positions of the attached masses.

Keywords: vibration of beams; AFG beam; tapered beam; Timoshenko beam; attached masses; Rayleigh-Ritz method

1. Introduction

Recently, the authors presented a paper (Rossit *et al.* 2017) regarding dynamic behavior of cantilever tapered beams carrying concentrated masses. For the authors knowledge it was the first paper on the matter since, there has not been any previous attempt to solve the problem of an axially functionally graded beam carrying an attached mass.

Subsequently, Nikolić (2017) analyzed a non-uniform axially functionally graded cantilever beam with a tip body, by means of the rigid element method.

In those papers, beams flexural deformation is described by means of Bernoulli-Euler theory.

According to a recent literature survey, apparently, there are no papers on Timoshenko AFG beams carrying attached masses. As it is known Timoshenko theory, posed in 1921 (Timoshenko 1921, 1922), provides more accurate results in the general study of beams and is mandatory in the case of high frequencies or non-slender beams. Remarks of historical character on the subject can be found in the paper by Elishakoff *et al.* (2015).

Certainly, even papers about bare AFG Timoshenko beams are scarce. Among them, mention must be made of the paper of Shahba *et al.* (2011) who studied free vibration and stability of AFG Timoshenko beams through a finite element approach. Huang *et al.* (2013) presented a new

approach: by introducing an auxiliary function, they changed the coupled governing equations with variable coefficients for the deflection and rotation to a single governing equation. He *et al.* (2013) improved the traditional beam element to consider the variable axial parameters which were formulated in terms of a power series. Tang *et al.* (2014) obtained closed form solutions for uniform AFG Timoshenko beams whose bending stiffness and distributed mass density are assumed to obey a unified exponential law. Rajasekaran and Norouzzadeh Tochaei (2014) investigated the free vibration analysis of AFG Timoshenko beams carried out through the differential transformation element method and the quadrature element method. They introduce an element-based differential method that significantly improves the accuracy of results. They also introduced a lower order differential quadrature element based on differential quadrature element method. They showed the accuracy of both methods with several numerical examples.

Also, Sarkar and Ganguli (2014) founded closed form solutions for certain polynomial variations of the material mass density, elastic modulus and shear modulus, along the length of the beam. Gan *et al.* (2015) presented a finite element procedure for dynamic analysis of non-uniform AFG Timoshenko beams under multiple moving point loads. Sun *et al.* (2016) suggested a new initial value method to determine critical tip force and axial loading at buckling of a standing column with varying cross-section and variable material properties under self-weight and tip force. More recently, in 2017 Zhao *et al.* introduced a new approach based on Chebyshev polynomials theory and Tudjono *et al.* derived exact shape functions for both non-uniform (non-prismatic section) and inhomogeneous (functionally graded material) Timoshenko beam element formulation explicitly and Chen *et al.* (2017) developed the

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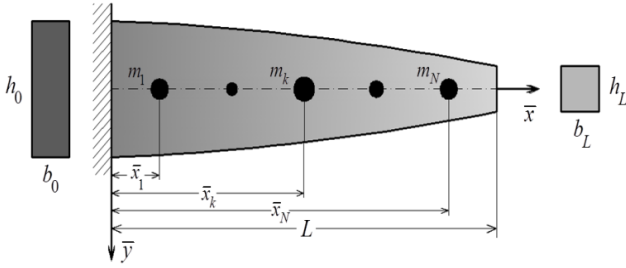


Fig. 1 AFG cantilever tapered beam with N masses attached at arbitrary points

initial value method of a system of ordinary differential equations to determine the resonance frequencies of functionally graded nanocantilevers carrying a nanoparticle. In the present paper, we describe the determination of the natural frequencies of vibration of a Timoshenko cantilever beam with varying rectangular cross section and made of axially functionally graded material, carrying attached masses at arbitrary positions, having into account their rotatory inertia. In view of the mathematical difficulties of the problem since variable coefficients appear in the governing differential equations, approximated methods are required to solve general cases. The well-known Rayleigh-Ritz method (Ilanko and Monterrubio 2014) is employed and its suitability to apply to Timoshenko beams with properties varying according to its axial axis and attached masses is verified by comparison with particular cases of the posed model published in the literature.

2. Analytical approach

According to the Timoshenko beam theory, the determination of the beam frequencies involves the effects of rotational inertia, shear deformation and their combined effects.

For the normal modes of beam vibration (Fig. 1) it can be expressed

$$\begin{aligned} v(\bar{x}, t) &= \bar{V}(\bar{x}) \cos(\omega t) \\ \psi(\bar{x}, t) &= \bar{\Psi}(\bar{x}) \cos(\omega t) \end{aligned} \tag{1}$$

where v is the transverse displacement of the mid-surface in the y -direction displacement, ψ is the angle of rotation of the normal to the mid-surface of the beam, t is the time and ω is the circular frequency in radians per second.

The non-dimensional coordinate is defined as

$$x = \bar{x}/L \tag{2}$$

where L is the length of the beam and

$$\begin{aligned} V(x) &= \frac{\bar{V}(\bar{x})}{L}; \\ \Psi(x) &= \bar{\Psi}(\bar{x}) \end{aligned} \tag{3}$$

are the dimensionless expressions of the transverse displacement and bending angle, respectively.

If the cross section varies smoothly, the energy functional J for the vibrating beam of length L carrying

attached N masses m_k at positions x_k (see Fig. 1) is given by

$$\begin{aligned} J(V, \Psi) &= \frac{1}{2} \int_0^1 \left[\frac{E(x)I(x)}{L^2} \left(\frac{d\Psi(x)}{dx} \right)^2 + \right. \\ &+ \frac{1}{2} \kappa G(x)A(x) \left(\frac{dV(x)}{dx} - \Psi(x) \right)^2 \Big] L dx - \\ &- \frac{1}{2} \omega^2 \int_0^1 \rho(x) \left[A(x)(V(x))^2 L^2 + I(x)(\Psi(x))^2 \right] L dx - \\ &- \frac{1}{2} \omega^2 \sum_{k=1}^N m_k (V(x_k))^2 + m_k r_k^2 (\Psi(x_k))^2 \end{aligned} \tag{4}$$

where $A(x)$ is the varying cross section and $I(x)$ its second moment of area, the FGM density is $\rho(x)$, the Young's modulo is $E(x)$ and the shear modulus is $G(x)$. r_k defines the radius of gyration of the mass m_k with respect to the neutral axis of the beam and κ is the shear coefficient.

As the material and geometric characteristics of the beam may be general, one can define

$$\begin{aligned} E(x) &= E_0 f_E(x) \\ G(x) &= G_0 f_G(x) \\ \rho(x) &= \rho_0 f_\rho(x) \\ A(x) &= A_0 f_A(x) \\ I(x) &= I_0 f_I(x) \\ b(x) &= b_0 f_b(x) \\ h(x) &= h_0 f_h(x) \end{aligned} \tag{5}$$

It was considered

$$G(x) = \frac{E(x)}{2(1+\nu)} \tag{6}$$

where ν is the Poisson ratio.

Obviously $f_A = f_b \times f_h$, $f_I = f_b \times f_h^3$, $f_G = f_E$.

The subscript "0" refers to the cross section of the beam adopted as the reference section.

To apply the Rayleigh-Ritz method, it is necessary to approximate the spatial component of the solution

$$\begin{aligned} V(x) &\cong V_a(x) = \sum_{j=1}^{Nq} D_j q_j(x) \\ \Psi(x) &\cong \Psi_a(x) = \sum_{i=1}^{Np} C_i p_i(x) \end{aligned} \tag{7}$$

where p_i and q_j are coordinate functions that satisfy the essential boundary conditions, C_i and D_j are arbitrary constants, Np and Nq are the number of terms.

Following Rayleigh-Ritz procedure, the functional is minimized with respect to every arbitrary constant

$$\begin{aligned} \frac{\partial J[V_a(x), \Psi_a(x)]}{\partial C_i} &= 0 \quad ; \quad i = 1, 2, \dots, Np. \\ \frac{\partial J[V_a(x), \Psi_a(x)]}{\partial D_j} &= 0 \quad ; \quad j = 1, 2, \dots, Nq. \end{aligned} \tag{8}$$

Then a linear system of equations is formed

$$\mathbf{R} \begin{Bmatrix} C_i \\ D_j \end{Bmatrix} = \{0\} \tag{9}$$

which results in the following eigenvalue equation

$$\mathbf{A} = \mathbf{K} - \Omega^2 \mathbf{M} \tag{10}$$

with $\Omega = \omega L \sqrt{\frac{\rho_0 A_0}{E_0 I_0}}$ the frequency coefficient and where

$$\mathbf{K} = \begin{bmatrix} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{bmatrix} \tag{11}$$

$$\begin{aligned} k_{ii} &= \int_0^1 \left(f_E f_I p_i' p_i' + \frac{s^2 \kappa}{2(1+\nu)} f_E f_A p_i p_i \right) dx \\ k_{ij} &= - \int_0^1 \left(\frac{s^2 \kappa}{2(1+\nu)} f_E f_A p_i q_j' \right) dx \\ k_{ji} &= - \int_0^1 \left(\frac{s^2 \kappa}{2(1+\nu)} f_E f_A q_j' p_i \right) dx \\ k_{jj} &= \int_0^1 \left(\frac{s^2 \kappa}{2(1+\nu)} f_E f_A q_j' q_j' \right) dx \end{aligned} \tag{12}$$

$$\mathbf{M} = \begin{bmatrix} m_{ii} & m_{ij} \\ m_{ji} & m_{jj} \end{bmatrix} \tag{13}$$

$$\begin{aligned} m_{ii} &= \frac{1}{s^2} \int_0^1 (f_\rho f_I p_i p_i) dx + \sum_{k=1}^N M_k c_k^2 p_i(x_k) p_i(x_k) \\ m_{ij} &= 0 \\ m_{ji} &= 0 \end{aligned} \tag{14}$$

$$m_{jj} = \int_0^1 (f_\rho f_A q_j q_j) dx + \sum_{k=1}^N M_k q_j(x_k) q_j(x_k)$$

are the elements of matrices \mathbf{K} and \mathbf{M} , respectively; with

$$\begin{aligned} M_k &= \frac{m_k}{\rho_0 A_0 L}; \quad c_k = \frac{r_k}{L}; \quad s = L \sqrt{\frac{A_0}{I_0}}; \quad q_j' = \frac{dq_j}{dx} \\ p_j' &= \frac{dp_j}{dx} \end{aligned} \tag{15}$$

Then, the eigenvalue problem can be expressed as

$$|\mathbf{KM}^{-1} - \Omega^2 \mathbf{I}| = |\mathbf{B} - \lambda \mathbf{I}| = 0 \tag{16}$$

where $\lambda = \Omega^2$ are the eigenvalues of matrix \mathbf{B} and \mathbf{I} the identity matrix.

For the cantilever beam, the following coordinate functions are chosen

$$\{p_i\}_{i=1}^{N_p} = \{x^i\}_{i=1}^{N_p} \tag{17}$$

$$\{q_j\}_{j=1}^{N_q} = \{x^j\}_{j=1}^{N_q} \tag{18}$$

Table 1 Frequency coefficients for a tapered AFG Timoshenko beam ($\Omega_i = \omega_i L \sqrt{(\rho_0 A_0)/(E_0 I_0)}$)

	Ω_1	Ω_2	Ω_3	Ω_4
Shahba <i>et al.</i>	3.9359	15.1577	31.2638	47.7164
Rajasekaran <i>et al.</i>	3.9358	15.1532	31.2236	47.5830
Huang <i>et al.</i>	3.93579	15.1533	31.2239	47.5857
Zhao <i>et al.</i>	3.93585	15.1540	31.2257	47.5871
Present	3.93579	15.1533	31.2239	47.5836

which satisfy essential boundary conditions.

3. Numerical results

Since there were not found, in the technical literature, values of natural frequencies of vibration of AFG Timoshenko beams with attached masses in order to verify the accuracy of the proposed model, comparisons are made with particular cases available in the literature.

First, Table 1 compares values for a tapered Timoshenko beam made of axially functionally graded material studied by Shahba *et al.* (2011), Huang *et al.* (2013), Rajasekaran and Norouzzadeh Tochaei (2014) and Zhao *et al.* (2017). They obtained values for the first four natural frequency coefficients for a case that can be represented in the present model.

The two constituent materials are Zirconia (ZrO₂) and Aluminum (Al).

$$\text{ZrO}_2: E_0 = 200 \text{ GPa}; \rho_0 = 5700 \frac{\text{kg}}{\text{m}^3} \tag{19}$$

$$\text{Al}: E_1 = 70 \text{ GPa}; \rho_1 = 2702 \frac{\text{kg}}{\text{m}^3} \tag{20}$$

and properties of AFG material, like mass density ρ , Young's modulus E , shear modulus G , continuously vary in the axial direction with a power law relation. Then, a generic material property $P(x)$ is assumed to vary along the beam axis x

$$P(x) = P_0 \left[1 + \frac{(P_1 - P_0)}{P_0} x^n \right] \tag{21}$$

In order to compare results, it is taken $n=2$ in Eq. (21). The geometry of the varying section is defined adopting in Eqs. (5)

$$f_A = 1 - 0.1x, f_I = (1 - 0.1x)^3 \tag{22}$$

They adopted

$$s = 10; \kappa = \frac{5}{6}; \nu = 0.30 \tag{23}$$

Then, Table 2 compares values with the case studied by Torabi *et al.* (2013): A wedge cantilever homogeneous Timoshenko beam ($\nu= 0.25, \kappa=2/3$) with an attached tip mass ($M=0.32$) and whose cross-sectional properties are

Table 2 Frequency coefficients for a tapered homogeneous Timoshenko beam with a tip mass ($\Omega_i = \omega_i L^2 \sqrt{(\rho_0 A_0) / (E_0 I_0)}$)

	$s=10$	$s=25$	Solution
Ω_1	1.997	2.117	Torabi <i>et al.</i>
	1.9977	2.0957	Present
Ω_2	10.695	13.420	Torabi <i>et al.</i>
	10.6947	13.4311	Present
Ω_3	24.388	36.109	Torabi <i>et al.</i>
	24.3869	36.1016	Present
Ω_4	40.174	66.697	Torabi <i>et al.</i>
	40.1487	66.6219	Present
Ω_5	56.739	102.451	Torabi <i>et al.</i>
	56.7489	102.108	Present

defined by: $f_A = 1 - 0.4x$, $f_I = (1 - 0.4x)^3$. Two different slenderness ratios are considered and the first five non-dimensional natural frequencies are presented.

In all cases the calculations were done with $N_p = N_q = 20$ in Eqs. (17)-(18), and the cross section at the clamped edge ($x=0$) is taken as the reference cross section. As it can be seen, the agreement is excellent in both cases. It is worth mentioning that the Rayleigh-Ritz method gives upper bounds of the wanted values.

Next, an example of the convenience of using Timoshenko theory in certain situations: in the paper that preceded and gave rise to the present (Rossit *et al.* 2017) are compared frequency values with a previous work (Chen and Liu, 2006) of a case of a Bernoulli-Euler beam with attached masses. Taking into account the geometry of the analyzed structure, the use of Timoshenko theory to describe the flexural behavior of the beam would have yielded more accurate results.

The physical properties and dimensions of the beam studied are: Young's modulus $E = 2.051 \times 10^{11}$ N/m², mass density $\rho = 7850$ kg/m³, constant beam width $b = 0.1$ m, beam length $L = 1.60$ m, lineal variable beam depth: 0.08 m at the free end and 0.40 m at the clamped end.

Each attached mass has a magnitude of one-fifth of the actual total mass of the beam: 60.288 kg.

In the present model, according to the definition used for the relative magnitude of the mass, it must be adopted

$$M_k = \frac{12}{100}; c_k = 0; k = 1 \text{ to } 5. \tag{24}$$

$$\omega_i = \frac{\Omega_i}{L^2} \sqrt{\frac{EI_0}{\rho A_0}}$$

where, as indicated previously, ω_i is the natural frequency in radians per second.

Two particular situations were considered:

a) One point mass attached at the free end (Table 3) (Fig. 2).

Five equal point masses attached at coordinates: $0.125L$, $0.3125L$, $0.5L$, $0.6875L$ and $0.875L$ respectively (Table 4) (Fig. 1).

Table 3 Frequency values in radians per second for a tapered homogeneous cantilever beam with one attached mass

	Chen and Liu	Rossit <i>et al.</i>	Present
ω_1	569.6279	569.3747	557.5622
ω_2	2508.895	2503.714	2297.209
ω_3	6743.232	6710.268	5548.167
ω_4	13408.53	13289.00	9823.906
ω_5	22570.17	22240.74	14743.28
	Bernoulli-Euler		Timoshenko

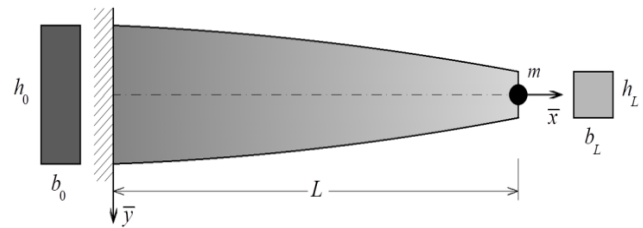


Fig. 2 AFG cantilever tapered beam with one mass m attached at the free end

Table 4 Frequency values in radians per second for a tapered homogeneous cantilever beam with five attached masses

	Chen and Liu	Rossit <i>et al.</i>	Present
ω_1	613.220	613.194	594.523
ω_2	2525.54	2524.88	2259.80
ω_3	6366.50	6356.55	5056.30
ω_4	12184.0	12116.4	8716.58
ω_5	16089.9	15928.0	11551.5
	Bernoulli-Euler		Timoshenko

When the free end of the beam has no attached mass, the summation in Eq. (18) starts with $j=2$ to partially satisfy the natural edge conditions and accelerate convergence of Rayleigh-Ritz method.

Next, some cases are evaluated to show the influence of the application of Timoshenko theory on an AFG vibrating beam with attached masses. The effect of the slenderness of the beam: $L/h_0 = s/\sqrt{12}$ on such influence is evidenced.

Among the infinity of cases that could be evaluated due to the quantity and variability of the parameters involved in the description of the behavior of the posed model, just the case of a beam whose height varies quadratically is considered: $h(x) = h_0(1 - 0.5x^2)$. In all cases, there are taken into account the quantity and positions of masses of Tables 3 and 4.

Properties of AFG materials, like mass density ρ , Young's modulus E , shear modulus G , continuously vary in the axial direction according to Eq. (21).

Three cases are considered in Eq. (21), $n=0$ (homogeneous material), $n=1$ (AFG material with linear variation) and $n=2$ (AFG material with quadratic variation).

Table 5 Frequency coefficients for a homogeneous tapered beam with a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	2.61342	12.4385	25.7434	42.1832	66.2152
10	2.65308	13.1585	27.9641	50.8473	87.2670
50	2.66619	13.4136	28.8753	55.3339	100.861
100	2.66661	13.4218	28.9059	55.4978	101.411
500	2.66674	13.4244	28.9157	55.5506	101.589
1000	2.66674	13.4245	28.9160	55.5522	101.595
B-E	2.66674	13.4245	28.9161	55.5528	101.597

Table 6 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	3.37453	15.5940	30.9115	54.5280	88.4008
10	3.42831	16.2669	33.4780	66.0183	116.311
50	3.44611	16.4974	34.5370	71.8186	133.950
100	3.44667	16.5047	34.5726	72.0287	134.657
500	3.44685	16.5071	34.5840	72.0964	134.886
1000	3.44686	16.5072	34.5844	72.0985	134.893
B-E	3.44686	16.5072	34.5845	72.0992	134.896

Table 7 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	3.57172	16.9323	33.1276	59.9104	97.620
10	3.62692	17.5915	35.9643	72.8383	128.876
50	3.64517	17.8147	37.1361	79.3762	148.734
100	3.64574	17.8218	37.1755	79.6132	149.531
500	3.64592	17.8240	37.1881	79.6896	149.789
1000	3.64593	17.8241	37.1885	79.6919	149.797
B-E	3.64593	17.8241	37.1886	79.6927	149.800

In the calculations, the AFG material made of steel and aluminum oxide Al_2O_3 (alumina) proposed by Su *et al.* (2013) is used. Their Young's modulus and density are

$$E_{St} = 210GPa; \rho_{St} = 7800kg / m^3; E_{Alum} = 390GPa; \rho_{Alum} = 3960kg / m^3; \nu_{St} = \nu_{Alum} = 0.30 \quad (25)$$

Note that the alumina, more rigid, is lighter than steel. It is adopted $\kappa = 5/6$ and the masses are all the same

$$M_k = 0.20, c_k = \frac{r_k}{L} = 0.10 \quad (26)$$

Tables 5-7 show the values of the frequency coefficient $\Omega_i = \omega_i L^2 \sqrt{(\rho_{st} A_0) / (E_{st} I_0)}$ for the three compositions of

Table 8 Frequency coefficients for a homogeneous tapered beam with 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	2.54668	11.4422	25.0435	40.2884	55.9815
10	2.59584	12.3499	28.1789	47.0494	70.4052
50	2.61217	12.6836	29.4089	49.8045	76.3278
100	2.61269	12.6944	29.4493	49.8951	76.5124
500	2.61285	12.6979	29.4623	49.9241	76.5714
1000	2.61286	12.6980	29.4627	49.9250	76.5733
B-E	2.61286	12.6980	29.4628	49.9253	76.5739

Table 9 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	3.34776	14.4042	31.0773	49.7108	68.3974
10	3.41463	15.4125	34.4194	57.6441	89.0157
50	3.43689	15.7767	35.6846	60.7844	97.5336
100	3.43759	15.7884	35.7258	60.8869	97.7912
500	3.43782	15.7922	35.7390	60.9198	97.8734
1000	3.43783	15.7923	35.7394	60.9208	97.8759
B-E	3.43783	15.7923	35.7395	60.9211	97.8768

Table 10 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	3.53034	15.4729	33.2349	53.0152	72.9484
10	3.59840	16.5480	36.7570	61.5285	94.8568
50	3.62101	16.9345	38.0812	64.8727	103.644
100	3.62173	16.9470	38.1242	64.9813	103.905
500	3.62196	16.9510	38.1380	65.0161	103.988
1000	3.62196	16.9511	38.1384	65.0172	103.990
B-E	3.62196	16.9511	38.1386	65.0176	103.991

the cantilever beam considered when a mass is attached at the free end.

Different values of the slenderness of the beam are taken into account. The value for the Bernoulli-Euler beam is indicated at the last row of every table for comparison

Tables 8 to 10 show the results when five masses are attached to the beam.

As can be seen and as might be expected, as the beam's slenderness increases the frequency values tend to the values obtained using the simpler theory of Bernoulli-Euler.

Figs. 3 to 7 show the differences of the values of the first five frequencies of the different beam cases analyzed with respect to the respective value of the Bernoulli-Euler beam. The graphs indicate the decrease in absolute value $|\Delta\%|$, according to the slenderness of the beams for the analyzed cases:

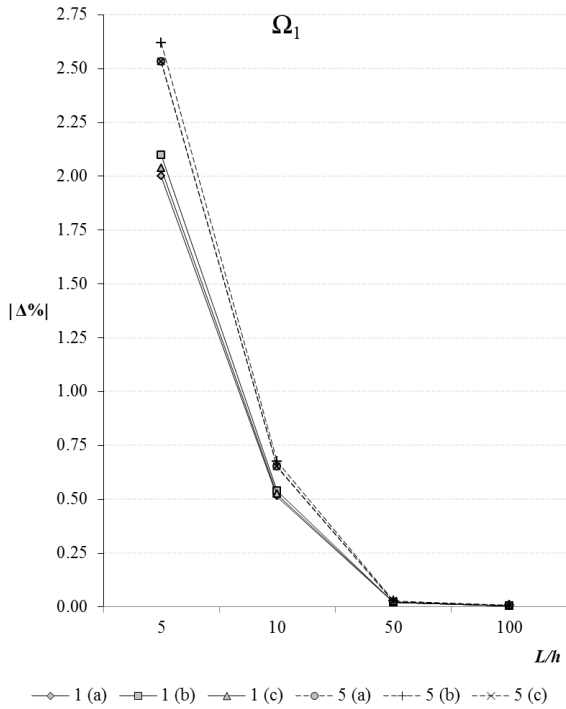


Fig. 3 Variations of the first frequency by the use of theory of Timoshenko

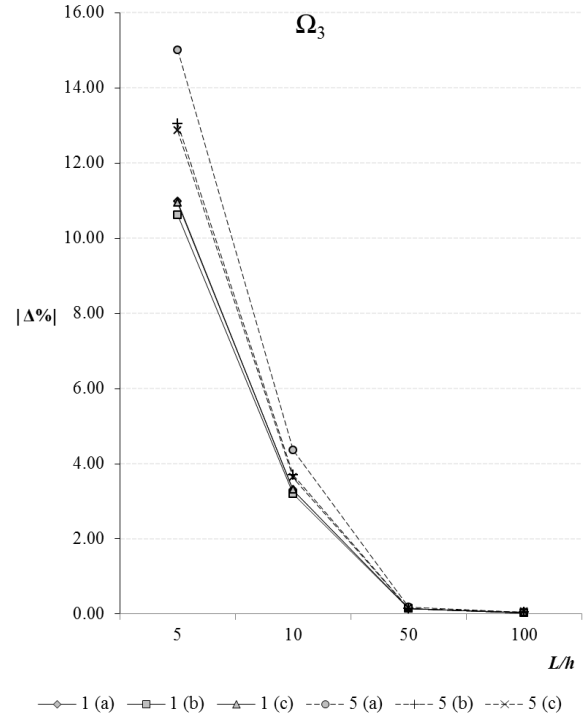


Fig. 5 Variations of the third frequency by the use of theory of Timoshenko

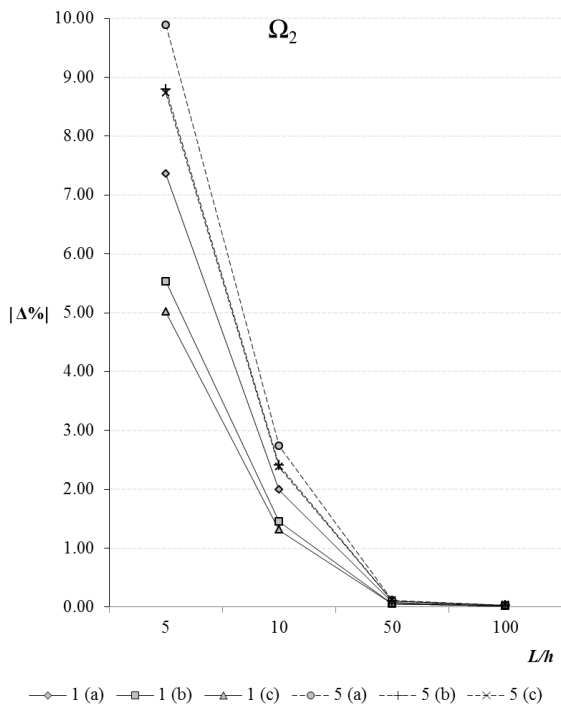


Fig. 4 Variations of the second frequency by the use of theory of Timoshenko

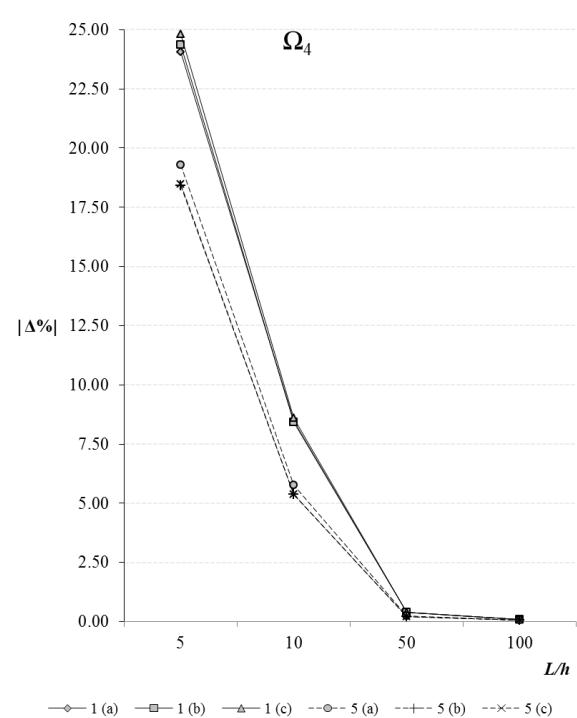


Fig. 6 Variations of the fourth frequency by the use of theory of Timoshenko

- 1(a): Homogeneous beam with one mass attached.
 - 1(b): AFG (linear) beam with one mass attached.
 - 1(c): AFG (quadratic) beam with one mass attached.
 - 5(a): Homogeneous beam with five masses attached
 - 5(b): AFG (linear) beam with five masses attached.
 - 5(c): AFG (quadratic) beam with five masses attached.
- It is plotted up to the ratio $L/h = 100$ to improve the

representation of trends, since for the slenderest beams the differences are very small.

It is observed that for the first three frequencies, the influence of considering the theory of Timoshenko is greater in the beams with five masses for all the material compositions. However, for the fourth and fifth frequencies, the influence is greatest on beams carrying just one mass.

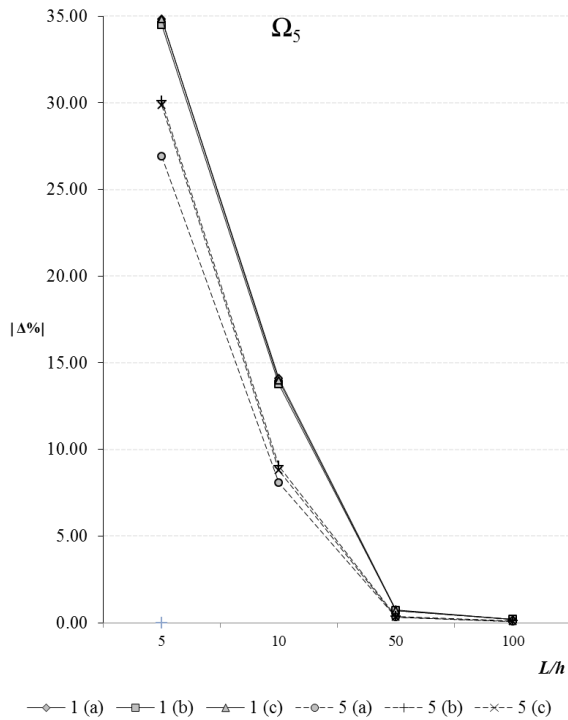


Fig. 7 Variations of the fifth frequency by the use of theory of Timoshenko

Table 11 Convergence study (case of the first row in Table 10)

$N_{p,q}$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	3.53362	15.5960	34.2454	62.0190	97.4123
10	3.53129	15.5170	33.5176	53.8933	74.6629
15	3.53053	15.4876	33.3527	53.3795	73.5660
20	3.53034	15.4729	33.2349	53.0152	72.9484
25	3.53022	15.4690	33.1829	52.6410	72.1216
30	3.53014	15.4651	33.1366	52.5256	71.8635
35	3.53010	15.4601	33.0987	52.4371	71.6952

Table 12 Frequency coefficients for a homogeneous tapered beam with a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	1.27563	2.14165	8.58443	21.5106	39.2133
10	1.27843	2.15684	8.87121	23.0863	43.8814
50	1.27932	2.16182	8.96939	23.6722	45.8118
100	1.27935	2.16197	8.97252	23.6913	45.8767
500	1.27936	2.16202	8.97352	23.6974	45.8975
1000	1.27936	2.16202	8.97355	23.6976	45.8981
B-E	1.27936	2.16202	8.97354	23.6976	45.8983

In terms of the composition, it can be said that the influence of using Timoshenko theory in general is similar for the two cases of AFG material studied and for the homogeneous beam. Only in the case of the beam with 5 masses and for the fifth frequency the influence of the theory of Timoshenko is greater for the two cases with AFG

Table 13 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	1.36309	2.39179	11.2089	28.1562	51.5514
10	1.36512	2.41198	11.5495	29.9232	56.7953
50	1.36576	2.41860	11.6656	30.5684	58.8905
100	1.36578	2.41880	11.6692	30.5893	58.9602
500	1.36579	2.41887	11.6704	30.5960	58.9826
1000	1.36579	2.41887	11.6705	30.5963	58.9833
B-E	1.36579	2.41887	11.6704	30.5962	58.9833

Table 14 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and a mass at the free end

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	1.40096	2.48318	12.3504	31.2174	57.0748
10	1.40280	2.50489	12.7168	33.1502	62.8409
50	1.40338	2.51200	12.8413	33.8538	65.1354
100	1.40340	2.51222	12.8453	33.8766	65.2117
500	1.40341	2.51229	12.8465	33.8839	65.2361
1000	1.40341	2.51230	12.8466	33.8841	65.2369
B-E	1.40341	2.51229	12.8466	33.8841	65.2369

materials than for the homogeneous beam.

4. Convergence of the procedure

4.1 Selection of number of terms

The situation of the first row of Table 10 will be used to evaluate the convergence of the approach because it is the most complex of the analyzed ones: greater shear effect, height of the cross section varying quadratically and five attached masses. Table 11 indicates the results for different numbers of terms in the summations of Eqs. (17)-(18).

The values in Table 11 prove the convergence of the method (it is recalled that Rayleigh-Ritz gives upper bounds of the wanted values). The computation time is significantly increased for $N_p, N_q > 20$. Therefore, considering that the results are sufficiently indicative from an engineering viewpoint and evaluating the time of data processing, was taken $N_p = N_q = 20$ for all cases.

4.2 Stability of the solution in a case of large taper ratio

For large taper ratios, the model moves further away from the classical theory of strength of materials, so it is important to verify whether the proposed solution tend steadily to the values obtained using the simpler theory of Bernoulli-Euler as the beam's slenderness increases.

Table 15 Frequency coefficients for a homogeneous tapered beam with 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	1.97783	5.36874	10.3193	19.4102	27.3144
10	1.99942	5.48667	10.7394	21.2467	29.5836
50	2.00644	5.48576	10.8857	21.9391	30.4221
100	2.00666	5.48667	10.8904	21.9617	30.4493
500	2.00673	5.48696	10.8919	21.9689	30.4580
1000	2.00673	5.48697	10.8919	21.9691	30.4583
B-E	2.00674	5.48697	10.8919	10.8919	30.4584

Table 16 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying linearly and 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	2.45983	6.23187	12.3419	23.6946	31.9500
10	2.48622	6.31773	12.8358	25.8668	34.3836
50	2.49480	6.34590	13.0068	26.6812	35.2872
100	2.49507	6.34678	13.0123	26.7077	35.3165
500	2.49515	6.34707	13.0140	26.7162	35.3259
1000	2.49515	6.34708	13.0141	26.7165	35.3262
B-E	2.49516	6.34708	13.0141	26.7165	35.3263

Table 17 Frequency coefficients for a tapered beam with AFG material (Alum-St) varying quadratically and 5 masses attached

L/h_0	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
5	2.60538	6.58682	13.1136	25.2506	34.0783
10	2.63303	6.67367	13.6388	27.5666	36.6714
50	2.64202	6.70212	13.8203	28.4326	37.6315
100	2.64231	6.70301	13.8261	28.4607	37.6626
500	2.64240	6.70330	13.8280	28.4698	37.6726
1000	2.64240	6.70331	13.8280	28.4700	37.6729
B-E	2.64240	6.70331	13.8280	28.4701	37.6730

The case $h(x) = h_0(1 - 0.95x)$ is evaluated for the same materials composition, quantity and positions of masses of Tables 5 to 10.

As can be seen in Tables 12 to 17 the trend of the model is suitable.

It is required to clarify that in the cases in which no natural condition is satisfied at the free edge (Tables 12 to 14) it has been necessary to take $Np = Nq = 25$ in Eqs. (17) and (18) to obtain stability in the trend.

Values of Tables 15 to 17 are obtained taken $Np = Nq = 20$ in Eqs. (17) and (18).

5. Conclusions

The study of the use of FG materials in resistant

structures has had an important development due to the ability of these materials to resist great efforts and to reduce the weight of structures.

For this purpose, it is important to have tools that allow carrying out studies of the performance of structures in different work situations. Among them, the behavior of structural elements that support motors or machines attached to them is usual in many applications and their operation can cause severe efforts in the structure.

In that case, the knowledge of the dynamic parameters of the beam-mass system is fundamental for what is necessary to have a model that correctly describes its behavior. In the case of short beams or higher frequencies, Timoshenko theory should be used to describe the flexural behavior of the beam.

The well-known Rayleigh-Ritz method provides an accurate and convenient procedure to tackle the problem of a vibrating Timoshenko AFG beam carrying attached masses, a problem of which the authors have not found data in the literature.

As a general conclusion, one may say that the influence of the use of Timoshenko theory on the frequencies of cantilever AFG beams supporting masses follows the same trends as for homogeneous beams.

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References

- Chen, D.W. and Liu, T.L. (2006), "Free and forced vibrations of a tapered cantilever beam carrying multiple point masses", *Struct. Eng. Mech.*, **23**(2), 209-216.
- Chen, D.W., Sun, D.L. and Li, X.F. (2017), "Surface effects on resonance frequencies of axially functionally graded Timoshenko nanocantilevers with attached nanoparticle", *Compos. Struct.*, **173**(1), 116-126.
- Elishakoff, I., Kaplunov, J. and Nolde, E. (2015), "Celebrating the centenary of Timoshenko's study of effects of shear deformation and rotary inertia", *Appl. Mech. Rev.*, **67**(6), 060802.
- Gan, B.S., Trinh, T.H., Le, T.H. and Nguyen, D.K. (2015), "Dynamic response of non-uniform Timoshenko beams made of axially FGM subjected to multiple moving point loads", *Struct. Eng. Mech.*, **53**(5), 981-995.
- He, P., Liu, Z. and Li, C. (2013), "An improved beam element for beams with variable axial parameters", *Shock Vibr.*, **20**(4), 601-617.
- Huang, Y., Yang, L. and Luo, Q. (2013), "Free vibration of axially functionally graded Timoshenko beams with non-uniform cross-section", *Compos.: Part B*, **45**(1), 1493-1498.
- Ilanko, S. and Monterrubio, L.E. (2014), *The Rayleigh-Ritz Method for Structural Analysis*, John Wiley & Sons Inc., London, U.K.

- Nikolić, A. (2017), "Free vibration analysis of a non-uniform axially functionally graded cantilever beam with a tip body", *Arch. Appl. Mech.*, **87**(7), 1227-1241.
- Rajasekaran, S. and Norouzzadeh Tochaei, E. (2014), "Free vibration analysis of axially functionally graded tapered Timoshenko beams using differential transformation element method and differential quadrature element method of lowest-order", *Meccan.*, **49**(4), 995-1009.
- Rossit, C.A., Bambill, D.V. and Gilardi, G.J. (2017), "Free vibrations of AFG cantilever tapered beams carrying attached masses", *Struct. Eng. Mech.*, **61**(5), 685-691.
- Sarkar, K. and Ganguli, R. (2014), "Closed-form solutions for axially functionally graded Timoshenko beams having uniform cross-section and fixed-fixed boundary condition", *Compos.: Part B*, **58**, 361-370.
- Shahba, A., Attarnejad, R., Marvi, M.T. and Hajilar, S. (2011), "Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions", *Compos.: Part B*, **42**(4), 801-808.
- Su, H., Banerjee, J.R. and Cheung, C.W. (2013), "Dynamic stiffness formulation and free vibration analysis of functionally graded beams", *Compos. Struct.*, **106**, 854-862.
- Sun, D.L., Li, X.F. and Wang, C.Y. (2016), "Buckling of standing tapered Timoshenko columns with varying flexural rigidity under combined loadings", *Int. J. Struct. Stab. Dyn.*, **16**(6), 1550017.
- Tang, A.Y., Wu, J.X., Li, X.F. and Lee, K.Y. (2014), "Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams", *Int. J. Mech. Sci.*, **89**, 1-11.
- Timoshenko, S.P. (1921), "On the correction for shear of the differential equation for transverse vibration of prismatic bar", *Philosoph. Mag.*, **41**(245), 744-746.
- Timoshenko, S.P. (1922), "On the transverse vibrations of bars of uniform cross section", *Philosoph. Mag.*, **43**(253), 125-131.
- Torabi, K., Afshari, H. and Heidari-Rarani M. (2013), "Free vibration analysis of a non-uniform cantilever Timoshenko beam with multiple concentrated masses using DQEM", *Eng. Sol. Mech.*, **1**(1), 9-20.
- Tudjono, S., Han, A., Nguyen, D., Kiryu, S. and Gan, B. (2017), "Exact shape functions for Timoshenko beam element", *IOSR J. Comput. Eng.*, **19**(3), 12-20.
- Zhao, Y., Huang, Y. and Guo, M. (2017), "A novel approach for free vibration of axially functionally graded beams with non-uniform cross-section based on Chebyshev polynomials theory", *Compos. Struct.*, **168**, 277-284.