Characterizing Acceptability Semantics of Argumentation Frameworks with Recursive Attack and Support Relations

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Abstract

Over the last decade, several extensions of Dung's Abstract Argumentation Frameworks (AFs) have been introduced in the literature. Some of these extensions concern the nature of the attack relation, such as the consideration of recursive attacks, whereas others incorporate additional interactions, such as a support relation. Recently, the *Attack-Support Argumentation Framework* ($ASAF$) was proposed, which accounts for recursive attacks and supports, attacks to supports and supports to attacks, at any level, where the support relation is interpreted as *necessity*. Currently, to determine the accepted elements of an $ASAF$ (which may be arguments, attacks, and supports) it is required to translate such an $ASAF$ into a Dung's AF. In this work, we provide a formal characterization of acceptability semantics directly on the $ASAF$, without requiring such a translation. We prove that our characterization is sound since it satisfies different results from Dung's argumentation theory; moreover, we formally show that the approach proposed here for addressing acceptability is equivalent to the preexisting one, in which the $ASAF$ was translated into an AF. Also, we formalize the relationship between the $ASAF$ and other frameworks on which it is inspired: the Argumentation Framework with Recursive Attacks ($AFRA$) and the Argumentation Framework with Necessities (AFN).

Keywords: abstract argumentation, bipolar argumentation, recursive interactions, acceptability semantics.

1. Introduction

Argumentation has been shown as an important and fertile topic of research in Artificial Intelligence [11, 14, 52, 42, 6], for instance, by providing a reasoning model for rational decision making [37, 39, 12, 7], for decision making under uncertainty and/or dynamic environments [5, 35], for handling inconsistencies in logic-based environments [53, 15, 13, 36], Artificial Intelligence and Law [1, 51], and for reaching agreements [40, 4, 55]. In particular, within the community of argumentation, there has been considerable interest in studying *Abstract Argumentation Frameworks* (AFs) [30] since they allow to abstract from the way in which arguments and interactions are obtained, while enabling to explore different theoretical properties on arguments and their relationships, as well as providing various ways for characterizing their acceptability semantics [8].

Briefly, Dung's AFs are directed graphs in which the nodes are arguments and the edges represent attacks (conflicts) between them; starting from that work, in the last decade, several extensions of AFs were introduced. On the one hand, we can consider extensions that regard the attack relation, including the formalization of collective attacks from sets of arguments [43], the consideration of arguments that attack attacks [41], and a generalization of [41]'s ideas by characterizing *Argumentation Frameworks With Recursive Attacks* ($AFRAs$) [9, 10]. On the other hand, we can consider extensions to AFs by adding new forms of interactions. Among these, we can add a preference relation to decide the effects of attacks in AFs [3], or consider a support relation between arguments leading to the characterization of *Bipolar Argumentation Frameworks* (BAFs) [23].

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Following the work of [23], where the support relation was just considered as a positive interaction between arguments, different interpretations for the support relation were proposed in the literature. In particular, the most well-known interpretations are evidential support [47], deductive support [57], and necessary support [46]. Each one of these perspectives establishes some acceptability constraints on the arguments related by the support relation. Then, the different approaches characterize a series of complex attacks [24], which enforce those acceptability constraints by taking into account the coexistence of attacks and supports in the framework.

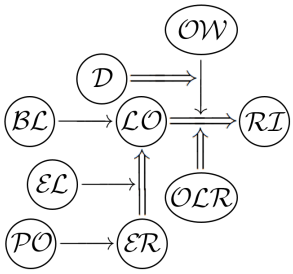
Beginning with the works presented in [24] and [27], where different interpretations of support were compared and discussed, the interest in studying AFs that consider a support relation has grown. Furthermore, recent works have focused on a deeper study of the necessity interpretation of the support relation (see [44, 50, 48, 25]). Among these we can distinguish [50], where the author gives an instantiation of necessary support in ASPIC+ using sub-arguments; and [25], where an axiomatization of necessary support was proposed through different frameworks.

In [29], two lines of work for extending Dung's AFs were combined by defining the *Attack-Support Argumentation Framework* ($ASAF$). Specifically, the $ASAF$ extends the AF by incorporating a necessary support relation and allowing for attacks and supports between arguments, as well as attacks and supports from an argument to the attack and support relations, at any level. The intuition behind the existence of a high-order support (*i.e.,* a support targeting an attack/support link) is that the supporting argument provides the context under which the target interaction holds. Hence, for instance, given a support α from an argument $\mathcal{A}$ to an attack or a support β, argument $\mathcal{A}$ should be accepted in order for the interaction β to hold. Similarly, extending the intuition behind the existence of a recursive attack relation (*e.g.,* in [41] to model preferences), high-order attacks in an $ASAF$ (*i.e.,* attacks targeting an attack/support link) capture the intuition that the attacking argument provides a context under which the target interaction should not hold. Next, we introduce an example that illustrates the capabilities of the $ASAF$, providing representational tools that facilitate the modeling through the use of high-order attacks and supports.

Let us consider a scenario where there is a room that has one lamp. Then, for the room to be illuminated, it is necessary that the lamp is on; in particular, this makes sense in the given context, where there is no other lamp in the room. Also, for the lamp to be on, it is necessary that the room has electricity. However, if the lamp in the room is an emergency lamp, its light can be on without receiving electricity. Furthermore, the fact that in the room there is an open window with no curtains provides a context where the fact that the lamp is on is not necessary for the room to be illuminated; nevertheless, this observation only helps in a context in which there is daylight coming from the outside. Finally, let us suppose that the lamp in the room is broken and that the building in which the room is located is suffering from a power outage. In this scenario we can identify the following arguments, whose conclusions are detailed next:

- $\mathcal{RI}$: "The room is illuminated"
- $\mathcal{LO}$: "The lamp in the room is on"
- $\mathcal{OLR}$: "There is no other lamp in the room (the lamp is the only one)"
- $\mathcal{ER}$: "There is electricity in the room"
- $\mathcal{EL}$: "The lamp in the room is an emergency lamp"
- $\mathcal{OW}$: "There is an open window with no curtains in the room"
- $\mathcal{D}$: "There is daylight on the outside"
- $\mathcal{BL}$: "The lamp in the room is broken"
- $\mathcal{PO}$: "The building where the room is located is affected by a power outage"

These arguments and the interactions involving them can be represented by the $ASAF$ illustrated below, where a single arrow $\longrightarrow$ denotes an attack and a double arrow $\Longrightarrow$ denotes a support, adopting a necessity interpretation.



Given the characteristics of the attack and support relations of the *ASAF*, it is possible to have interactions (attacks or supports) that do not hold since they are somehow made ineffective by other interactions. For instance, given the example introduced above, the necessity support relation between arguments $\mathcal{LO}$ and $\mathcal{RI}$ is made ineffective by the attack from $\mathcal{OW}$. This is because, as expressed before, having an open window through which daylight comes into the room provides a context under which the necessity relation expressing that the lamp should be on in order for the room to be illuminated does not hold. These complex interactions, which require accounting for the "validity" of attacks and supports, are captured in [29] by allowing for extensions of the *ASAF* to contain not only arguments but also attacks and supports; hence, the attacks and supports present in the corresponding extensions are those that hold. In particular, acceptability calculus in [29] is addressed by translating the *ASAF* into a Dung's *AF*, by means of an intermediate *AFN*, and then obtaining the extensions of the *ASAF* in terms of the corresponding extensions of its associated *AF*.

The first contribution of this paper is to obtain a full characterization of acceptability semantics directly from the *ASAF*[1]. Before characterizing semantic notions directly on the *ASAF*, we will formally define the conditions under which defeats on the elements of an *ASAF* occur, given the coexistence of the recursive attack and support relations. As highlighted in the literature of abstract argumentation, having a characterization of the semantics is significant because it yields a natural and intuitive mechanism for studying different characteristics of a framework, as well as for determining the acceptability status of its elements under different criteria. In particular, having such a characterization allows the user to focus on the semantic aspects of the *ASAF*, exploring and identifying features of its arguments, attacks and supports directly. That is, the direct approach we propose in this paper allows the user to abstract from the intrinsic details of a translation such as the one given in [29], while also avoiding the need for understanding the particularities related to the formalization of an *AFN* or an *AF*, and the process carried out for obtaining its extensions. Furthermore, if new semantics for the *ASAF* were to be defined in the future, they could exploit specific aspects leading to the existence of defeats, as well as other intermediate semantic notions involved in the direct characterization proposed in this paper.

We will show that our characterization of the *ASAF* semantics is sound in the sense that it satisfies different theoretical results established in [30] for Dung's *AF*s. On the other hand, as the second and main contribution of this paper, we will formally show that the extension-based approach we introduced here is equivalent to the one proposed in [29] in the sense that they both lead to obtaining the same extensions of the framework under the same semantics. Furthermore, given our characterization of the acceptability semantics for the *ASAF*, we will formally determine the relationship between the *ASAF* and the frameworks on which its foundations rely on: the *Argumentation Framework with Recursive Attacks* (*AFRA*) [9, 8] and the *Argumentation Framework with Necessities* (*AFN*) [45, 46, 16].

The rest of this paper is organized as follows. Section 2 starts by giving some background, including notions from Dung's theory [30], the *AFRA* [10], the *AFN* [46], and the formal definition of the *ASAF*. Then, Section 3 identifies the different conflicts that may arise between the elements of the *ASAF*, leading to the characterization of diverse forms of defeat. In particular, since attacks and supports in an *ASAF* may be affected by other interactions, we will also account for the conditions under which the attacks and supports of the *ASAF* are defeated. Section 4 formally defines the acceptability semantics of the *ASAF*. We start by defining some basic semantic notions and then follow Dung's extension-based approach for defining the extensions of the *ASAF* under different semantics. Later, in Section 5, we show that the approach for obtaining the extensions of an *ASAF* proposed in Section 4 is equivalent to the one given in [29], in the sense that they lead to obtaining the same extensions of a given framework.

[1] This work extends the preliminary results of [28].

Section 6 discusses related work and provides a formal account of the relationship between the *ASAF* and the *AFRA*, and between the *ASAF* and the *AFN*. Finally, in Section 7, we draw some conclusions and comment on future lines of work.

2. Essential Background

In this section, we include the background required for characterizing the acceptability semantics of the *ASAF*. We first present some basic notions related to Dung's *AF*s [30]. Then, we include some background corresponding to the Argumentation Framework with Recursive Attacks [9, 10] and the Argumentation Framework with Necessities [45, 46, 16]. Finally, we introduce the *ASAF* proposed in [29].

2.1. Abstract Argumentation Framework (AF)

The Abstract Argumentation Framework defined in [30] consists of a set of arguments and a set of conflicts between them:

Definition 1 (*AF*)**.** *An* Abstract Argumentation Framework (*AF*) *is a pair* $\langle \mathbb{A}, \mathbb{R} \rangle$*, where* $\mathbb{A}$ *is a finite and non-empty set of arguments and* $\mathbb{R} \subseteq \mathbb{A} \times \mathbb{A}$ *is an attack relation.*

Given an *AF*, in [30] a series of semantic notions are defined leading to the characterization of collectively acceptable sets of arguments. In particular, we will use the prefix 'D' when referring to semantic notions for Dung's *AF*s. As will become clear in Section 5, this will help to distinguish whether the semantic notions apply to an *AF* or to an *ASAF*.

Definition 2 (D-conflict-freeness, D-acceptability, D-admissibility)**.** *Let* $\langle \mathbb{A}, \mathbb{R} \rangle$ *be an AF and* $\mathbf{S} \subseteq \mathbb{A}$*.*

- $\mathbf{S}$ *is* D-conflict-free *iff* $\nexists \mathcal{A}, \mathcal{B} \in \mathbf{S}$ *s.t.* $(\mathcal{A}, \mathcal{B}) \in \mathbb{R}$.
- $\mathcal{A} \in \mathbb{A}$ *is* D-acceptable w.r.t. $\mathbf{S}$ *iff* $\forall \mathcal{B} \in \mathbb{A}$ *s.t.* $(\mathcal{B}, \mathcal{A}) \in \mathbb{R}$, $\exists \mathcal{C} \in \mathbf{S}$ *s.t.* $(\mathcal{C}, \mathcal{B}) \in \mathbb{R}$.
- $\mathbf{S}$ *is* D-admissible *iff it is D-conflict-free and* $\forall \mathcal{A} \in \mathbf{S}$, $\mathcal{A}$ *is D-acceptable w.r.t.* $\mathbf{S}$.

Then, by adding restrictions to the notion of admissibility, the complete, preferred, stable, and grounded extensions of an *AF* are defined as follows:

Definition 3 (*AF* D-Extensions)**.** *Let* $AF = \langle \mathbb{A}, \mathbb{R} \rangle$ *and* $\mathbf{S} \subseteq \mathbb{A}$.

- $\mathbf{S}$ *is a* D-complete extension *of AF iff it is D-admissible and* $\forall \mathcal{A} \in \mathbb{A}$*, if* $\mathcal{A}$ *is D-acceptable w.r.t.* $\mathbf{S}$*, then* $\mathcal{A} \in \mathbf{S}$.
- $\mathbf{S}$ *is a* D-preferred extension *of AF iff it is a maximal (w.r.t.* $\subseteq$*) D-admissible set of AF.*
- $\mathbf{S}$ *is a* D-stable extension *of AF iff it is D-conflict-free and* $\forall \mathcal{A} \in \mathbb{A} \backslash \mathbf{S}$, $\exists \mathcal{B} \in \mathbf{S}$ *s.t.* $(\mathcal{B}, \mathcal{A}) \in \mathbb{R}$.
- $\mathbf{S}$ *is the* D-grounded extension *of AF iff it is the smallest (w.r.t.* $\subseteq$*) D-complete extension of AF.*

2.2. Argumentation Framework with Recursive Attacks (AFRA)

Next, we briefly review the Argumentation Framework with Recursive Attacks (*AFRA*) [9, 10], which extends Dung's approach by allowing for attacks to the attack relation.

Definition 4 (*AFRA*)**.** *An* Argumentation Framework with Recursive Attacks (*AFRA*) *is a pair* $\langle \mathbb{A}, \mathbb{R} \rangle$ *where* $\mathbb{A}$ *is a set of arguments and* $\mathbb{R} \subseteq \mathbb{A} \times (\mathbb{A} \cup \mathbb{R})$ *is an attack relation.*

Given an attack $\alpha = (\mathcal{A}, X) \in \mathbb{R}$, $\mathcal{A}$ is called the source of α, denoted $\mathsf{src}(\alpha) = \mathcal{A}$, and X is called the target of α, denoted $\mathsf{trg}(\alpha) = X$. The authors in [10] consider that attacks, rather than their source arguments, are the subjects able to defeat arguments and other attacks. Then, they provide a characterization of defeats, distinguishing between direct and indirect defeats. On the one hand, direct defeats to an argument or an attack are obtained directly from the *AFRA*'s attack relation. On the other hand, defeats on arguments are propagated to the attacks they originate, and are identified as indirect defeats.

Definition 5 (Defeat in *AFRA*)**.** *Let* $\langle \mathbb{A}, \mathbb{R} \rangle$ *be an AFRA,* $\alpha, \beta \in \mathbb{R}$ *and* $X \in \mathbb{A} \cup \mathbb{R}$:

- α directly defeats X *iff* $\mathsf{trg}(\alpha) = X$.
- α indirectly defeats β *iff* $\mathsf{trg}(\alpha) = X$*, and* $X = \mathsf{src}(\beta)$.

In general, given $\alpha \in \mathbb{R}$ *and* $Y \in \mathbb{A} \cup \mathbb{R}$*, we say that* α defeats Y *iff* α *directly defeats or indirectly defeats* Y*.*

In [9, 10] the authors characterize the acceptability semantics of the *AFRA* in a similar way to [30]. First, they define the following basic semantic notions.

Definition 6 (*AFRA* Semantic Notions)**.** *Let* $\langle \mathbb{A}, \mathbb{R} \rangle$ *be an AFRA,* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R}$ *and* $X \in \mathbb{A} \cup \mathbb{R}$*:*

- $\mathbf{S}$ *is* conflict-free *iff* $\nexists\, \alpha, Y \in \mathbf{S}$ *s.t.* α *defeats* Y*.*
- X *is* acceptable *w.r.t.* $\mathbf{S}$ *iff* $\forall\, \alpha \in \mathbb{R}$ *s.t.* α *defeats* X*,* $\exists\, \beta \in \mathbf{S}$ *s.t.* β *defeats* α*.*
- $\mathbf{S}$ *is* admissible *iff it is conflict-free and* $\forall\, Y \in \mathbf{S}$*,* Y *is acceptable w.r.t.* $\mathbf{S}$.

Then, the complete, preferred, stable, and grounded[2] extensions of an *AFRA* are defined as follows.

Definition 7 (*AFRA* Extensions)**.** *Let* $\Gamma = \langle \mathbb{A}, \mathbb{R} \rangle$ *be an AFRA and* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R}$*:*

- $\mathbf{S}$ *is a* complete extension *of* Γ *iff it is admissible and* $\forall\, X \in (\mathbb{A} \cup \mathbb{R})$*, if* X *is acceptable w.r.t.* $\mathbf{S}$*, then* $X \in \mathbf{S}$.
- $\mathbf{S}$ *is a* preferred extension *of* Γ *iff it is a maximal* (*w.r.t.* $\subseteq$) *admissible set of* Γ*.*
- $\mathbf{S}$ *is a* stable extension *of* Γ *iff it is conflict-free and* $\forall\, X \in (\mathbb{A} \cup \mathbb{R}) \backslash \mathbf{S}$*,* $\exists \alpha \in \mathbf{S}$ *s.t.* α *defeats* X*.*
- $\mathbf{S}$ *is the* grounded extension *of* Γ *iff it is the least fixed point of* $\mathbb{F}_\Gamma : 2^{\mathbb{A} \cup \mathbb{R}} \mapsto 2^{\mathbb{A} \cup \mathbb{R}}$*, where* $\mathbb{F}_\Gamma(\mathbf{S}) = \{X \mid X$ *is acceptable w.r.t.* $\mathbf{S}\}$*. Equivalently,* $\mathbf{S}$ *is the grounded extension of* Γ *iff it is the smallest* (*w.r.t.* $\subseteq$) *complete extension of* Γ*.*

Note that in Definitions 1–3, $\mathbf{S}$ was used to denote a set of arguments. In contrast, in Definitions 6 and 7, it was used to denote a set of arguments and attacks. This difference relies on the context where the reference to the set is made. In the context of an *AF*, where arguments are the only elements to be accounted for when characterizing semantic notions, a set $\mathbf{S}$ will only contain arguments of the *AF*. Then, in the context of an *AFRA*, where arguments and attacks are relevant for the semantics of the framework, a set $\mathbf{S}$ may contain attacks as well as arguments. Furthermore, as will become evident in Sections 3–5, in the context of an *ASAF* a set $\mathbf{S}$ may also contain supports, in addition to arguments and attacks. In general, when we want to refer to a set of elements of a given framework (whether it is an *AF*, an *AFRA*, or an *ASAF*), we will use an uppercase bold letter, regardless of the elements the set may contain (*i.e.,* arguments, attacks and/or supports); moreover, a set will be generally named using the letter 'S', being the initial letter of 'set', resulting in $\mathbf{S}$. Finally, it should be noted that uppercase non-bold letters like 'X' or 'Y' are used to denote arbitrary elements within a set of elements of a given framework. Thus, elements like X or Y are not to be confused with sets like $\mathbf{S}$, $\mathbf{S'}$, or $\mathbf{T}$.

2.3. Argumentation Framework with Necessities (*AFN*)

Here, we will introduce some background notions of the Argumentation Framework with Necessities (*AFN*) [45, 46, 16], which extends Dung's framework by incorporating a support relation between arguments. In particular, the support relation of the *AFN* has a necessity interpretation, where the meaning of $\mathcal{A}$ supporting $\mathcal{B}$ is that the acceptance of $\mathcal{A}$ is necessary to obtain the acceptance of $\mathcal{B}$. In other words, the necessary support relation in the *AFN* imposes the following acceptability constraints on the arguments it relates: the acceptance of $\mathcal{B}$ implies the acceptance of $\mathcal{A}$ or, equivalently, the non-acceptance of $\mathcal{A}$ implies the non-acceptance of $\mathcal{B}$.

[2] In [9, 10] there is also a characterization of semi-stable and ideal extensions. However, they are not included here since, as mentioned before, in this paper we will focus on the four basic semantics of [30].

Definition 8 (*AFN*)**.** *An* Argumentation Framework with Necessities (*AFN*) *is a tuple* $\langle \mathbb{A}, \mathbb{R}, \mathbb{N} \rangle$ *where* $\mathbb{A}$ *is a set of arguments,* $\mathbb{R} \subseteq \mathbb{A} \times \mathbb{A}$ *is an attack relation, and* $\mathbb{N} \subseteq \mathbb{A} \times \mathbb{A}$ *is an irreflexive and transitive necessity relation.*

The attack relation $\mathbb{R}$ of the *AFN* coincides with the analogous relation in Dung's *AF*. On the other hand, given the acceptability constraints associated with the support relation of the *AFN*, the authors characterize the notion of *extended defeat*[3] in order to reinforce such constraints. Specifically, an extended defeat will occur when having an attack followed by a necessary support.

Definition 9 (*AFN* Extended Defeat)**.** *Let* $\langle \mathbb{A}, \mathbb{R}, \mathbb{N} \rangle$ *be an AFN and* $\mathcal{A}, \mathcal{B} \in \mathbb{A}$*. There is an* extended defeat *from* $\mathcal{A}$ *to* $\mathcal{B}$*, noted as* $\mathcal{A}\,\mathbb{R}^+\mathcal{B}$*, iff* $\mathcal{A}\,\mathbb{R}\,\mathcal{B}$ *or* $\exists \mathcal{C} \in \mathbb{A}$ *s.t.* $\mathcal{A}\,\mathbb{R}\,\mathcal{C}\,\mathbb{N}\mathcal{B}$*.*

In [46] the authors show that every *AFN* has an associated *AF*, as characterized by the following definition. Then, they show that the extensions of an *AFN* and its associated *AF* under the complete, preferred, stable and grounded semantics coincide.

Definition 10 (*AF* associated with an *AFN*)**.** *Given an AFN* $\langle \mathbb{A}, \mathbb{R}, \mathbb{N} \rangle$*, its* associated *AF is* $\langle \mathbb{A}, \mathbb{R}^+ \rangle$*.*

In particular, the *AF* associated with an *AFN* takes the extended defeat relation $\mathbb{R}^+$ into account. Then, since direct attacks are a particular case of extended defeats (see Definition 9), the associated *AF* contemplates the original attacks and the additional extended defeats on the arguments of the *AFN*.

2.4. Attack-Support Argumentation Framework (ASAF)

Next, we will present the fundamental background notions regarding the Attack-Support Argumentation Framework (*ASAF*) [29]. The *ASAF* formalism extends the *AFRA* presented in Section 2.2 by incorporating a support relation enabling to express support not only for arguments but also for attacks and for the support relation itself, and extending the *AFRA*'s attack relation by allowing for attacks to the support relation. In particular, the support relation of the *ASAF* follows the necessity interpretation of the *AFN* introduced in Section 2.3. As a result, the *ASAF* also extends the *AFN* by allowing for recursive attacks and supports, as well as attacks to supports, and vice-versa. Below, we include the definition of the *ASAF*, corresponding to an *AF* with recursive attack and support relations.

Definition 11 (*ASAF*)**.** *An* Attack-Support Argumentation Framework (*ASAF*) *is a tuple* $\langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *where* $\mathbb{A}$ *is a set of arguments,* $\mathbb{R} \subseteq \mathbb{A} \times (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *is an attack relation and* $\mathbb{S} \subseteq \mathbb{A} \times (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *is a necessary support relation. We assume that* $\mathbb{S}$ *is acyclic and* $\mathbb{R} \cap \mathbb{S} = \emptyset$*.*

As stated before, attacks and supports in an *ASAF* can also be attacked and supported. Thus, for instance, an attack from an argument $\mathcal{A}$ to a support from $\mathcal{B}$ to X will be represented by a pair $(\mathcal{A}, (\mathcal{B}, X))$ in the attack relation $\mathbb{R}$ of the *ASAF*, where the pair $(\mathcal{B}, X)$ belongs to the support relation $\mathbb{S}$ of the *ASAF*. In cases like this one, to simplify the notation, we will denote the attack originated by $\mathcal{A}$ with the pair $(\mathcal{A}, \alpha)$, where $\alpha = (\mathcal{B}, X)$. In general terms, the attack from an argument $\mathcal{C}$ to an attack or a support $\alpha = (\mathcal{A}, X)$ will be referred to as $(\mathcal{C}, \alpha)$; similarly, the support from an argument $\mathcal{D}$ to an attack or a support $\beta = (\mathcal{B}, Y)$ will be referred to as $(\mathcal{D}, \beta)$. Moreover, as in [9, 10], given an attack $\alpha = (\mathcal{A}, X) \in \mathbb{R}$, $\mathcal{A}$ is called the source of α, denoted $\mathsf{src}(\alpha) = \mathcal{A}$, and X is called the target of α, denoted $\mathsf{trg}(\alpha) = X$. Analogously, given a support $\beta = (\mathcal{B}, Y) \in \mathbb{S}$, $\mathcal{B}$ is called the source of β, denoted $\mathsf{src}(\beta) = \mathcal{B}$, and Y is called the target of β, denoted $\mathsf{trg}(\beta) = Y$.

It can be noted that the support relation $\mathbb{S}$ of an *ASAF* is required to be acyclic. As expressed by the authors in [16], such cycles are undesirable because they correspond to a type of fallacy known as *begging the question* (Latin *petitio principii*) also accurately referred to as *arguing in a cycle*. Furthermore, necessity cycles are excluded in the *AFN* by requiring the support relation $\mathbb{N}$ to be irreflexive and transitive. Even though, as also mentioned by the authors in [16] one could generalize the results to an arbitrary necessity relation by filtering out the extensions containing cycles of necessities, the formalization of the *ASAF* in [29] followed the direction of [16] requiring the support relation of the *ASAF* to be acyclic. Finally, note that the acyclicity requirement captures the non-reflexivity of [16]. Then, even though the support relation $\mathbb{S}$ is not required to be transitive, as will become clear later, the

[3]In [45, 46, 16] the term 'attack' is used; however, for uniformity purposes, we will use the term 'defeat' instead.

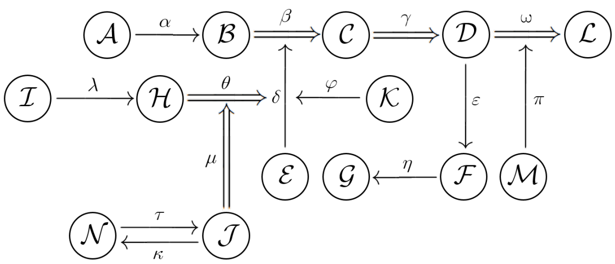
transitivity of support will be captured by explicitly accounting for the existence of chains of support when characterizing defeats in the *ASAF*.

In [29] the authors state that, given the combination of attack and support relations in an *ASAF*, it is required to identify attacks and supports unequivocally. This is because, when an argument $\mathcal{A}$ is attacking (respectively, supporting) a pair $\alpha = (\mathcal{B}, X)$, one needs to know whether $\mathcal{A}$ is attacking (respectively, supporting) an attack from $\mathcal{B}$ to X or a support from $\mathcal{B}$ to X. If $\alpha = (\mathcal{B}, X)$ is such that it may correspond to *both* an attack and a support from $\mathcal{B}$ to X, when $\mathcal{A}$ attacks (respectively, supports) α it would not be possible to determine whether $\mathcal{A}$ is attacking (respectively, supporting) the attack from $\mathcal{B}$ to X or the support from $\mathcal{B}$ to X. As a result, the attack and support relations of the *ASAF* are assumed to be disjoint (*i.e.,* $\mathbb{R} \cap \mathbb{S} = \emptyset$) and thus, a pair $\gamma = (\mathcal{E}, Z)$ in the attack relation *or* the support relation will be unequivocally identified by γ.

Another reason why it is required that the attack and support relations of an *ASAF* to be disjoint relates to the situations under which extended defeats occur in the *AFN*. If $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{A} \Longrightarrow \mathcal{B}$, the attack encodes the intuition that if $\mathcal{A}$ is accepted, then $\mathcal{B}$ should not be accepted (analogously, if $\mathcal{B}$ is accepted, $\mathcal{A}$ should not be accepted); in contrast, the support encodes the intuition that if $\mathcal{B}$ is accepted, then $\mathcal{A}$ should be accepted. Therefore, by simultaneously having an attack and a support from $\mathcal{A}$ to $\mathcal{B}$ in a scenario where $\mathcal{B}$ is accepted, $\mathcal{A}$ would have to be simultaneously accepted and not accepted, reflecting some kind of inconsistency within argument $\mathcal{B}$. As a result, to avoid situations like this one, it is assumed that $\mathbb{R} \cap \mathbb{S} = \emptyset$.

An *ASAF* can be graphically represented using a graph-like notation: an argument $\mathcal{A} \in \mathbb{A}$ will be denoted as a node in the graph, an attack $\alpha = (\mathcal{A}, X) \in \mathbb{R}$ will be denoted as $\mathcal{A} \xrightarrow{\alpha} X$, and a support $\beta = (\mathcal{B}, Y) \in \mathbb{S}$ will be denoted as $\mathcal{B} \stackrel{\beta}{\Longrightarrow} Y$. To illustrate this, let us consider the following example.

Example 1. *Let us consider the ASAF* Δ_1 *with the following graphical representation:*



We have the first-level attacks $\alpha = (\mathcal{A}, \mathcal{B})$, $\varepsilon = (\mathcal{D}, \mathcal{F})$, $\eta = (\mathcal{F}, \mathcal{G})$, $\lambda = (\mathcal{I}, \mathcal{H})$, $\tau = (\mathcal{N}, \mathcal{J})$ *and* $\kappa = (\mathcal{J}, \mathcal{N})$. *The first-level supports are* $\beta = (\mathcal{B}, \mathcal{C})$, $\gamma = (\mathcal{C}, \mathcal{D})$ *and* $\omega = (\mathcal{D}, \mathcal{L})$. *The second-level interactions are the attacks* $\delta = (\mathcal{E}, \beta)$ *and* $\pi = (\mathcal{M}, \omega)$. *Then, we have the third-level attack and support on* δ*: respectively,* $\varphi = (\mathcal{K}, \delta)$ *and* $\theta = (\mathcal{H}, \delta)$. *Finally, the only fourth-level interaction is the support* $\mu = (\mathcal{J}, \theta)$.

3. Different forms of Defeat in the *ASAF*

To characterize the acceptability semantics of the *ASAF* we first need to identify the conflicts that may occur between its elements. The set of all conflicts between the elements of the *ASAF* will be called the set of *defeats*, in order to distinguish them from the conflicts explicitly given in the attack relation. Similarly to [10], we consider a notion of defeat which regards attacks, rather than their source arguments, as the subjects able to defeat arguments, attacks or supports.

In the following, we will distinguish between two groups of defeats: those that are inferred directly from the attack relation of the *ASAF*, and those that result from the combination of the attack and support relations of the *ASAF*. The former will be referred to as *unconditional defeats*, and are defined in Section 3.1, whereas the latter are the *conditional defeats*, defined in Section 3.2.

3.1. Unconditional Defeats

Unconditional defeats in the *ASAF* are obtained in two cases. The first case corresponds to conflicts already captured by the attack relation of the *ASAF*, which we call *direct defeats*.

Definition 12 (Direct Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$ *and* $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$*. We say that* α directly defeats X*, denoted* α d-def X*, iff* $\mathsf{trg}(\alpha) = X$*.*

Recall that attacks are the entities able to effect defeat on other elements in the *ASAF*. However, as pointed out in [10] attacks are strictly related to their source arguments, meaning that an attack does not make sense if its source argument does not hold. To capture this intuition, next we will characterize the second kind of unconditional defeat which can be inferred directly from the attack relation of the *ASAF*: the *indirect defeat.*

Definition 13 (Indirect Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ *be an ASAF and* $\alpha, \beta \in \mathbb{R}$*. We say that* α indirectly defeats β*, denoted* α i-def β*, iff* α d-def $\mathsf{src}(\beta)$*.*

Note that supports in an *ASAF* cannot be indirectly defeated. This is because, differently from attacks, a support $\gamma = (\mathcal{A}, \mathcal{B}) \in \$$ still makes sense when its source ($\mathcal{A}$) is not accepted. Moreover, in such a case where $\mathcal{A}$ is not accepted and the support γ holds, the acceptability constraint stating that $\mathcal{B}$ should not be accepted either must be enforced (and, as will be shown in Section 3.2, this will be achieved through the characterization of conditional defeats).

Then, both forms of unconditional defeat are grouped together in the following definition.

Definition 14 (Unconditional Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$ *and* $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$*. We say that* α unconditionally defeats X*, denoted* α u-def X*, iff* α d-def X *or* α i-def X*.*

Example 2. *Given the ASAF* Δ_1 *from Example 1, the following unconditional defeats occur. The direct defeats are:* α d-def $\mathcal{B}$, ε d-def $\mathcal{F}$, η d-def $\mathcal{G}$, λ d-def $\mathcal{H}$, τ d-def $\mathcal{J}$, κ d-def $\mathcal{N}$, δ d-def β, φ d-def δ, π d-def ω*; and the indirect defeats are:* ε i-def η, τ i-def κ, κ i-def τ*.*

3.2. Conditional Defeats

As we mentioned, the coexistence of attacks and supports may lead to having conflicts that are not captured by the attack relation of the *ASAF* alone. These conflicts will be identified as *conditional defeats* since, unlike the defeats defined in Section 3.1, their existence depends on the consideration of the support relation (in addition to the attack relation) of the *ASAF*. Following the interpretation of support as necessity, such conflicts are handled in [46] by characterizing the notion of extended defeat, which reinforces the acceptability constraints presented in Section 2: given an attack $\mathcal{A} \longrightarrow \mathcal{B}$ and a sequence of necessary supports $\mathcal{B} \Longrightarrow \ldots \Longrightarrow \mathcal{C}$, there is an extended defeat from $\mathcal{A}$ to $\mathcal{C}$.

These intuitions are captured in the *ASAF* by characterizing the situations under which an *extended defeat* occurs. In particular, we will distinguish the *support sequence* involved in this form of defeat, and the corresponding supports will be referred to as the *support set.*

Definition 15 (Support Sequence and Support Set)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ *be an ASAF and* $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$*. We say that* $\Sigma = [\mathcal{A}_1, \ldots, \mathcal{A}_n = X]$ *is a* support sequence for X $(n \geq 2)$ *iff for every* $\mathcal{A}_i$ $(1 \leq i \leq n-1)$ *it holds that* $(\mathcal{A}_i, \mathcal{A}_{i+1}) \in \$$*. We define the* support set of Σ *as* $\mathbf{S} = \bigcup_{i=1}^{n-1} \{S_i\}$*, with* $S_i = (\mathcal{A}_i, \mathcal{A}_{i+1})$*, where* $\mathcal{A}_i$ *and* $\mathcal{A}_{i+1}$ *are consecutive elements of* Σ*.*

Note that since the support relation $\$$ is defined so that the source of a support link in an *ASAF* is an argument (by Definition 11, $\$ \subseteq \mathbb{A} \times (\mathbb{A} \cup \mathbb{R} \cup \$)$), all elements in a support sequence but the last one (*i.e.,* the $\mathcal{A}_i$ in Definition 15) will be arguments. Furthermore note that, as expressed by the condition $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$ in Definition 15, this does not prevent the last element in a support sequence from being an argument as well.

Definition 16 (Extended Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$*,* $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$ *and* $\mathbf{S} \subseteq \$$*. We say that* α extendedly defeats X given $\mathbf{S}$*, denoted* α e-def X given $\mathbf{S}$*, iff there exists a support sequence* $\Sigma = [\mathcal{A}_1, \ldots, \mathcal{A}_n = X]$ *for* X *s.t.* $\mathsf{trg}(\alpha) = \mathcal{A}_1$ *and* $\mathbf{S}$ *is the support set of* Σ*.*

Extended defeats in the *ASAF* are illustrated by the following example.

Example 3. *Given the ASAF* Δ_1 *from Example 1, the following extended defeats occur:* α e-def $\mathcal{C}$ given $\{\beta\}$, α e-def $\mathcal{D}$ given $\{\beta, \gamma\}$, α e-def $\mathcal{L}$ given $\{\beta, \gamma, \omega\}$, λ e-def δ given $\{\theta\}$*, and* τ e-def θ given $\{\mu\}$*.*

It can be noted that Definition 16 explicitly identifies the support sequence originating an extended defeat. Therefore, as shown by the following proposition, adding a support link to a support sequence results in a new extended defeat. The proof of this proposition, together with the proofs of every other formal result of the paper are included in the Appendix.

Proposition 1. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$*,* $\beta \in \mathbb{S}$ *and* $\mathbf{S} \subseteq \mathbb{S}$*. If* α *e-def* $\mathsf{src}(\beta)$ *given* $\mathbf{S}$*, then* α *e-def* $\mathsf{trg}(\beta)$ *given* $\mathbf{S} \cup \{\beta\}$*.*

Moreover, the following proposition shows that the existence of an extended defeat implies the existence of a direct defeat.

Proposition 2. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$*,* $X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *and* $\mathbf{S} \subseteq \mathbb{S}$*. If* α *e-def* X *given* $\mathbf{S}$*, then* $\exists \beta \in \mathbf{S}$ *s.t.* α *d-def* $\mathsf{src}(\beta)$*.*

Given that the *ASAF* combines intuitions and results from the *AFRA* [10] and the *AFN* [46], it is reasonable to combine the intuitions behind the notions of indirect defeat and extended defeat to characterize additional conflicts between the elements of the *ASAF*. In other words, similarly to the indirect defeat, we define the notion of *extended-indirect defeat* where an extended defeat on an argument is propagated to the attacks it originates. This kind of defeat is also conditional since it relies on the existence of an extended defeat, hence on the existence of supports.

Definition 17 (Extended-Indirect Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha, \beta \in \mathbb{R}$ *and* $\mathbf{S} \subseteq \mathbb{S}$*. We say that* α extended-indirectly defeats β given $\mathbf{S}$*, denoted* α *ei-def* β *given* $\mathbf{S}$*, iff* α *e-def* $\mathsf{src}(\beta)$ *given* $\mathbf{S}$*.*

Example 4. *Given the ASAF* Δ_1 *from Example 1, the only extended-indirect defeat is* α *ei-def* ε *given* $\{\beta, \gamma\}$*. This is because, as shown in Example 3,* α *e-def* $\mathcal{D}$ *given* $\{\beta, \gamma\}$ *and, as illustrated in Example 1,* $\mathcal{D} = \mathsf{src}(\varepsilon)$*.*

Here, similarly to Proposition 2, the following proposition shows that the existence of an extended-indirect defeat implies the existence of a direct defeat.

Proposition 3. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha, \gamma \in \mathbb{R}$ *and* $\mathbf{S} \subseteq \mathbb{S}$*. If* α *ei-def* γ *given* $\mathbf{S}$*, then* $\exists \beta \in \mathbf{S}$ *s.t.* α *d-def* $\mathsf{src}(\beta)$*.*

Finally, the extended and extended-indirect defeats are grouped together under the general notion of *conditional defeat*.

Definition 18 (Conditional Defeat)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$*,* $X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *and* $\mathbf{S} \subseteq \mathbb{S}$*. We say that* α conditionally defeats X given $\mathbf{S}$*, denoted* α *c-def* X *given* $\mathbf{S}$*, iff* α *e-def* X *given* $\mathbf{S}$ *or* α *ei-def* X *given* $\mathbf{S}$*.*

4. The Acceptability Semantics of the *ASAF*

After identifying the situations under which defeats between the elements of the *ASAF* occur, in this section we will characterize the extensional semantics of the *ASAF*. In particular, as stated in [29], the extensions of the *ASAF* may not only include arguments, but also attacks and supports. This is to reflect the fact that attacks and supports can be affected by other interactions; hence, the presence of an attack or a support in an extension of the *ASAF* will imply that the corresponding relationship holds.

Following the methodology of [30], in Section 4.1 we will first define some basic semantic notions for the *ASAF*. Then, we will show that the notion of acceptability complies with the constraints imposed by the attack and support relations of the *ASAF*. Moreover, we will show that important results from [30] regarding the notions of acceptability and admissibility also hold in the context of the *ASAF*. Then, in Section 4.2, we will define the acceptability semantics of the *ASAF* by characterizing its complete, preferred, stable, and grounded extensions; furthermore, we will show that the *ASAF* satisfies the relationships between the complete, preferred, stable and grounded extensions given in [30].

4.1. Semantic Notions

As in [30], the notion of conflict-freeness establishes the minimum requirements a set of elements of an $ASAF$ should satisfy in order to be collectively accepted. In particular, since the $ASAF$ allows for unconditional and conditional defeats, a set will be conflict-free if it does not contain all the elements leading to the occurrence of one of those defeats.

Definition 19 (Conflict-freeness)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an* $ASAF$ *and* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$*. We say that* $\mathbf{S}$ is conflict-free *iff:*

- $\nexists\, \alpha, X \in \mathbf{S}$ *s.t.* α *u-def* X*; and*
- $\nexists\, \beta, Y \in \mathbf{S}$, $\nexists\, \mathbf{S}' \subseteq \mathbf{S}$ *s.t.* β *c-def* Y *given* $\mathbf{S}'$*.*

Recall that, as mentioned at the end of Section 2.2, bold uppercase letters like $\mathbf{S}$ are used to denote sets of elements of a given framework. For instance, in Definitions 15–18, $\mathbf{S}$ denotes a set of supports of an $ASAF$ (*i.e.,* $\mathbf{S} \subseteq \mathbb{S}$). In contrast, in Definition 19, $\mathbf{S}$ is used to denote a set of arguments, attacks and/or supports of an $ASAF$ (*i.e.,* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$). That is, notwithstanding the naming similarity between $\mathbf{S}$ and $\mathbb{S}$, the use of $\mathbf{S}$ to denote a set of elements of an $ASAF$ has no implication whatsoever of $\mathbf{S}$ being a set of supports; rather, the nature of the elements in a set $\mathbf{S}$ will be explicitly indicated, and will depend on the context in which the set is characterized.

Example 5. *Let* Δ_1 *be the ASAF from Example 1. Some conflict-free sets of* Δ_1 *are:* $\emptyset$*,* $\{\mathcal{M}, \omega\}$*,* $\{\mathcal{N}, \mathcal{J}\}$*,* $\{\lambda, \delta\}$*,* $\{\mu, \mathcal{E}, \delta\}$*,* $\{\alpha, \beta, \varepsilon\}$*,* $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \beta, \gamma, \omega, \theta, \mu\}$ *and* $\{\mathcal{A}, \alpha, \gamma, \mathcal{M}, \pi, \mathcal{L}, \mathcal{I}, \lambda, \mathcal{K}, \varphi, \beta, \mathcal{F}, \eta, \mathcal{E}, \mu\}$*. In contrast, the sets* $\{\alpha, \mathcal{B}\}$*,* $\{\lambda, \theta, \delta\}$*,* $\{\pi, \omega\}$ *and* $\{\tau, \kappa\}$*, among others, are not conflict-free.*

According to Definition 19, any set of elements of an $ASAF$ which does not include an attack will be conflict-free. This is the case of the set $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{I}, \mathcal{J}, \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \beta, \gamma, \omega, \theta, \mu\}$ illustrated in Example 5, which includes every argument and support of the $ASAF$ Δ_1 but none of its attacks. Moreover, when considering conditional defeats, all the elements required for the existence of a defeat must be included in a non-conflict-free set. Hence, if one of the supports in the corresponding support set is missing, the resulting set is conflict-free. This situation is illustrated by the conflict-free sets $\{\lambda, \delta\}$ and $\{\alpha, \beta, \varepsilon\}$ in Example 5.

Next, we define the notion of acceptability in the context of an $ASAF$, which characterizes the defense by a set of arguments, attacks and supports against the occurrence of defeats on its elements. Since the $ASAF$ allows for unconditional and conditional defeats, we need to consider all the defeats that may occur, as well as the different ways for providing defense against them.

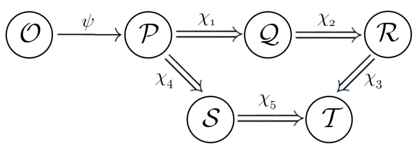
Definition 20 (Acceptability)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an* $ASAF$*,* $X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *and* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$*. We say that* X is acceptable w.r.t. $\mathbf{S}$ *iff it holds that:*

1. $\forall\, \alpha \in \mathbb{R}$ *s.t.* α *u-def* X*, either:*
   (*a*) $\exists\, \beta \in \mathbf{S}$ *s.t.* β *u-def* α*; or*
   (*b*) $\exists\, \beta \in \mathbf{S}$, $\exists\, \mathbf{S}' \subseteq \mathbf{S}$ *s.t.* β *c-def* α *given* $\mathbf{S}'$*.*
2. $\forall\, \alpha \in \mathbb{R}$, $\forall\, \mathbf{T} \subseteq \mathbb{S}$ *s.t.* α *c-def* X *given* $\mathbf{T}$*, either:*
   (*a*) $\exists\, \beta \in \mathbf{S}$ *s.t.* β *u-def* α*;*
   (*b*) $\exists\, \beta \in \mathbf{S}$, $\exists\, \gamma \in \mathbf{T}$ *s.t.* β *u-def* γ*;*
   (*c*) $\exists\, \beta \in \mathbf{S}$, $\exists\, \mathbf{S}' \subseteq \mathbf{S}$ *s.t.* β *c-def* α *given* $\mathbf{S}'$*; or*
   (*d*) $\exists\, \beta \in \mathbf{S}$, $\exists\, \mathbf{S}' \subseteq \mathbf{S}$, $\exists\, \gamma \in \mathbf{T}$ *s.t.* β *c-def* γ *given* $\mathbf{S}'$*.*

As the preceding definition shows, defense against an unconditional defeat may only be achieved by defeating the corresponding attack. On the other hand, a conditional defeat may be repelled by defeating the corresponding attack or one of the supports in the support set; in either case, defense can be provided by both unconditional and conditional defeats. Moreover, it should be noted that, although Definition 20 accounts for a set $\mathbf{S}$ of arguments, attacks and supports of an $ASAF$, the only elements contributing to the defense are the attacks and supports. This is because attacks and supports are ones leading to the existence of defeats (see Definitions 14 and 18). In other words, similarly to the *AFRA*, defense through an unconditional defeat can only be provided by an attack. In contrast, defense through a conditional defeat is provided by an attack and a support set. These intuitions are illustrated in the following example.

Example 6. *If we consider the ASAF* Δ_1 *from Example 1, e.g.,* $\mathcal{A}$ *and* φ *are acceptable w.r.t.* $\emptyset$ *since they are not defeated (neither unconditionally nor conditionally) in* Δ_1*. Also,* $\mathcal{N}$ *is acceptable w.r.t.* $\{\tau\}$*; this is because* κ d-def $\mathcal{N}$ *and* τ i-def κ *(case* 1.*a in Definition 20). Similarly,* β *is acceptable w.r.t.* $\{\lambda, \theta\}$ *since* δ d-def β *is the only defeat on* β*, and* λ e-def δ given $\{\theta\}$ *(case* 1.*b). As another example,* θ *is acceptable w.r.t.* $\{\kappa\}$ *because even though* τ e-def θ given $\{\mu\}$*, it holds that* κ i-def τ *(case* 2.*a). Then, for instance,* $\mathcal{D}$ *is acceptable w.r.t.* $\{\delta\}$*. This is because* α e-def $\mathcal{D}$ given $\{\beta, \gamma\}$ *is the only defeat on* $\mathcal{D}$*, and it holds that* δ d-def β *(case* 2.*b). Finally,* $\mathcal{F}$ *and* η *are acceptable w.r.t.* $\{\alpha, \beta, \gamma\}$*. In particular,* ε d-def $\mathcal{F}$ *and* ε i-def η*, whereas* α ei-def ε given $\{\beta, \gamma\}$ *(case* 2.*c). In contrast, for example,* $\mathcal{B}$ *is not acceptable w.r.t.* $\emptyset$*, since* $\mathcal{B}$ *is directly defeated by* α *and the empty set does not have elements leading to the existence of a defeat (neither unconditional nor conditional) on* α*. In addition,* δ *is not acceptable w.r.t.* $\{\tau, \mu\}$*. This is because* λ e-def δ given $\{\theta\}$ *and* φ d-def δ*; then, even though* τ e-def θ given $\{\mu\}$ *(case* 2.*d), no elements in the set* $\{\tau, \mu\}$ *allow to obtain an unconditional or conditional defeat on* φ*.*

Note that, as established by Definition 20 and illustrated in Example 6, in order for an element X of an *ASAF* to be acceptable w.r.t. a set $\mathbf{S}$, this set should defend X against *every* defeat it receives. In particular, this should also be the case given the existence of multiple conditional defeats on X; furthermore, this applies to the case where multiple conditional defeats are originated by combining the same attack with alternative support sequences. For example, let us consider an *ASAF* where the relations in the graph below hold:



Given such an *ASAF*, it is the case that ψ e-def $\mathcal{T}$ given $\{\chi_1, \chi_2, \chi_3\}$ and ψ e-def $\mathcal{T}$ given $\{\chi_4, \chi_5\}$. As a result, argument $\mathcal{T}$ will be acceptable w.r.t. a set $\mathbf{S}$ that is able to defeat (either unconditionally or conditionally) the attack ψ or at least one element of *each* support set $\{\chi_1, \chi_2, \chi_3\}$ and $\{\chi_4, \chi_5\}$.

The following proposition shows that, like in the *AFRA*, the acceptability of an attack implies the acceptability of its source.

Proposition 4. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\alpha \in \mathbb{R}$ *and* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$*. If* α *is acceptable w.r.t.* $\mathbf{S}$*, then* $\mathsf{src}(\alpha)$ *is acceptable w.r.t.* $\mathbf{S}$*.*

The following proposition shows that the notion of acceptability meets the constraints imposed by the necessity interpretation of support adopted by the *ASAF*.

Proposition 5. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *a conflict-free set and* $\alpha \in \mathbb{S}$ *acceptable w.r.t.* $\mathbf{S}$*. If* $\mathsf{trg}(\alpha)$ *is acceptable w.r.t.* $\mathbf{S}$*, then* $\mathsf{src}(\alpha)$ *is acceptable w.r.t.* $\mathbf{S}$*; equivalently, if* $\mathsf{src}(\alpha)$ *is not acceptable w.r.t.* $\mathbf{S}$*, then* $\mathsf{trg}(\alpha)$ *is not acceptable w.r.t.* $\mathbf{S}$*.*

The following proposition shows that the notion of acceptability is monotonic with respect to set inclusion.

Proposition 6. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *and* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$*. If* X *is acceptable w.r.t.* $\mathbf{S}$*, then* $\forall \mathbf{S}' \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ *s.t.* $\mathbf{S} \subseteq \mathbf{S}'$: X *is acceptable w.r.t.* $\mathbf{S}'$*.*

Next, like in [30], admissible sets of the *ASAF* are defined by combining the notions of conflict-freeness and acceptability.

Definition 21 (Admissibility)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$*. We say that* $\mathbf{S}$ is admissible *iff it is conflict-free and* $\forall X \in \mathbf{S}$: X *is acceptable w.r.t.* $\mathbf{S}$*.*

Example 7. *Let Δ_1 be the ASAF from Example 1. Some admissible sets of Δ_1 are $\emptyset$, $\{\alpha, \beta, \gamma, \varphi, \mathcal{F}, \mathcal{M}\}$ and $\{\mathcal{A}, \alpha, \gamma, \mathcal{M}, \pi, \mathcal{L}, \mathcal{I}, \lambda, \mathcal{K}, \varphi, \beta, \mathcal{F}, \eta, \mathcal{E}, \mu, \tau, \mathcal{N}\}$. In contrast, for instance, the sets $\{\beta, \theta, \mathcal{J}, \kappa\}$ and $\{\varepsilon, \mathcal{G}\}$ are not admissible; the former because no defeat on δ (which directly defeats β) can be obtained from the set $\{\beta, \theta, \mathcal{J}, \kappa\}$, whereas the latter because α ei-def ε given $\{\beta, \gamma\}$ and the set $\{\varepsilon, \mathcal{G}\}$ provides no way of defeating α, β nor γ.*

The following lemma shows that the notions of acceptability and admissibility recently introduced for the *ASAF* allow to extend Dung's fundamental lemma to this framework.

Lemma 1. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF, $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ an admissible set of Δ, and $X, Y \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ s.t. X and Y are acceptable w.r.t. $\mathbf{S}$. Then, it holds that (1) $\mathbf{S}' = \mathbf{S} \cup \{X\}$ is admissible, and (2) Y is acceptable w.r.t. $\mathbf{S}'$.*

4.2. Extensional Semantics of the ASAF

From the semantic notions defined in Section 4.1, we are now able to characterize the complete, preferred, stable, and grounded extensions of the *ASAF*.

Definition 22 (*ASAF* Extensions)**.** *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$.*

- $\mathbf{S}$ *is a* complete extension *of Δ iff it is admissible and $\forall X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$, if X is acceptable w.r.t. $\mathbf{S}$, then $X \in \mathbf{S}$.*
- $\mathbf{S}$ *is a preferred extension of Δ iff it is a maximal (w.r.t. $\subseteq$) admissible set of Δ.*
- $\mathbf{S}$ *is a stable extension of Δ iff it is conflict-free and $\forall X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S}) \backslash \mathbf{S}$, $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α u-def X or α c-def X given $\mathbf{S}'$.*
- $\mathbf{S}$ *is the grounded extension of Δ iff it is the smallest (w.r.t. $\subseteq$) complete extension of Δ.*

Example 8. *Let us consider the ASAF Δ_1 from Example 1 and the grounded and preferred semantics. The grounded extension of Δ_1 is $\mathbf{G}_\Delta = \{\mathcal{A}, \alpha, \gamma, \mathcal{M}, \pi, \mathcal{L}, \mathcal{I}, \lambda, \mathcal{K}, \varphi, \beta, \mathcal{F}, \eta, \mathcal{E}, \mu\}$. Note that although* src(μ) *is involved in an attack cycle that is not resolved when considering the grounded semantics, the support μ holds and thus, $\mu \in \mathbf{G}_\Delta$. Then, when considering the preferred semantics, there are two alternatives for resolving the attack cycle involving* src(μ)*, leading to the existence of two preferred extensions of Δ_1: $\mathbf{P}_{1\Delta} = \mathbf{G}_\Delta \cup \{\tau, \mathcal{N}\}$ and $\mathbf{P}_{2\Delta} = \mathbf{G}_\Delta \cup \{\kappa, \mathcal{J}, \theta\}$. In particular, even though λ e-def δ given $\{\theta\}$ and $\{\tau, \mu\} \subseteq \mathbf{P}_{1\Delta}$ is such that τ e-def θ given $\{\mu\}$, as discussed in Example 6, it is also the case that φ d-def δ but no elements in $\mathbf{P}_{1\Delta}$ allow to obtain a defeat (either unconditional or conditional) on φ; therefore, $\delta \notin \mathbf{P}_{1\Delta}$.*

Example 9. *Let us consider again the example given in the introduction, where the depicted ASAF is $\Delta_9 = \langle \mathbb{A}_9, \mathbb{R}_9, \mathbb{S}_9 \rangle$, with $\mathbb{A}_9 = \{\mathcal{RI}, \mathcal{LO}, \mathcal{OLR}, \mathcal{ER}, \mathcal{EL}, \mathcal{OW}, \mathcal{D}, \mathcal{BL}, \mathcal{PO}\}$, $\mathbb{R}_9 = \{\delta = (\mathcal{EL}, \gamma), \varepsilon = (\mathcal{OW}, \alpha), \lambda = (\mathcal{BL}, \mathcal{LO}), \mu = (\mathcal{PO}, \mathcal{ER})\}$, and $\mathbb{S}_9 = \{\alpha = (\mathcal{LO}, \mathcal{RI}), \beta = (\mathcal{OLR}, \alpha), \gamma = (\mathcal{ER}, \mathcal{LO}), \kappa = (\mathcal{D}, \varepsilon)\}$. Then, for instance, given the grounded and preferred semantics, the only grounded and preferred extension of Δ_9 is $\mathbf{GP}_\Delta = \{\mathcal{D}, \kappa, \mathcal{OW}, \varepsilon, \mathcal{BL}, \lambda, \mathcal{EL}, \delta, \mathcal{PO}, \mu, \mathcal{OLR}, \beta, \mathcal{RI}\}$. As a result, since the support α does not belong to the grounded and preferred extension (meaning that it is not necessary for the lamp to be on in order for the room to be illuminated), argument $\mathcal{RI}$ is accepted (i.e., the room is illuminated) even though argument $\mathcal{LO}$ is not (i.e., the lamp in the room is not on).*

Next, we will show that the *ASAF* semantics from Definition 22 fullfil the relationships between the homonymous semantics proposed in [30]. The following lemma illustrates the relationship between the preferred and complete extensions of an *ASAF*.

Lemma 2. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF. Every preferred extension of Δ is also a complete extension of Δ, but not vice-versa.*

Similarly, the following lemma relates the stable and preferred extensions of an *ASAF*.

Lemma 3. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF. Every stable extension of Δ is also a preferred extension of Δ, but not vice-versa.*

Then, by combining the two preceding results, we obtain the following corollary.

Corollary 1. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF. Every stable extension of* Δ *is also a complete extension of* Δ*, but not vice-versa.*

Finally, in [30] it was shown that the grounded extension of an *AF* is, in particular, a complete extension of the *AF*. It can be noted that this result is trivially satisfied in the context of the *ASAF* since, by Definition 22, the grounded extension of an *ASAF* is characterized as its smallest (w.r.t. $\subseteq$) complete extension.

5. Equivalent Approaches for the Acceptability Calculus in the *ASAF*

Now, we will set about studying the relationship between the form in which acceptability was determined in the *ASAF* proposed in [29] and the one proposed in Section 4. We will show that these two approaches are equivalent in the sense that they lead to obtaining the same outcome when considering the complete, preferred, stable, or grounded semantics[4].

We will start introducing the proposal in [29], which requires transforming the *ASAF* into a Dung's *AF*, obtaining the extensions from this *AF*, and then getting the *ASAF*'s extensions from them. This transformation requires two steps: first, the *ASAF* is translated into an *AFN*; and then, the *AF* associated with that *AFN* is obtained.

The following definition characterizes the first step of the translation showing how to translate an *ASAF* into its associated *AFN*[5].

Definition 23 (*AFN* associated with an *ASAF*)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF. The AFN* associated with Δ *is* $\Delta_{AFN} = \langle \mathbb{A}_{AFN}, \mathbb{R}_{AFN}, \mathbb{N}_{AFN} \rangle$*, where:*

- $\mathbb{A}_{AFN} = \mathbb{A} \cup \mathbb{R} \cup \mathbb{S} \cup \{\beta^+, \beta^- \mid \beta \in \mathbb{S}\}$.
- $\mathbb{R}_{AFN}$ *and* $\mathbb{N}_{AFN}$ *are such that:*
    1. $\forall\, \alpha = (\mathcal{A}, X) \in \mathbb{R}$*:*
        - $(\mathcal{A}, \alpha) \in \mathbb{N}_{AFN}$*; and*
        - $(\alpha, X) \in \mathbb{R}_{AFN}$.
    2. $\forall\, \beta = (\mathcal{B}, Y) \in \mathbb{S}$*:*
        - $(\beta, \beta^+) \in \mathbb{N}_{AFN}$*;*
        - $(\beta, \beta^-) \in \mathbb{N}_{AFN}$*;*
        - $(\mathcal{B}, \beta^+) \in \mathbb{N}_{AFN}$*;*
        - $(\mathcal{B}, \beta^-) \in \mathbb{R}_{AFN}$*; and*
        - $(\beta^-, Y) \in \mathbb{R}_{AFN}$.

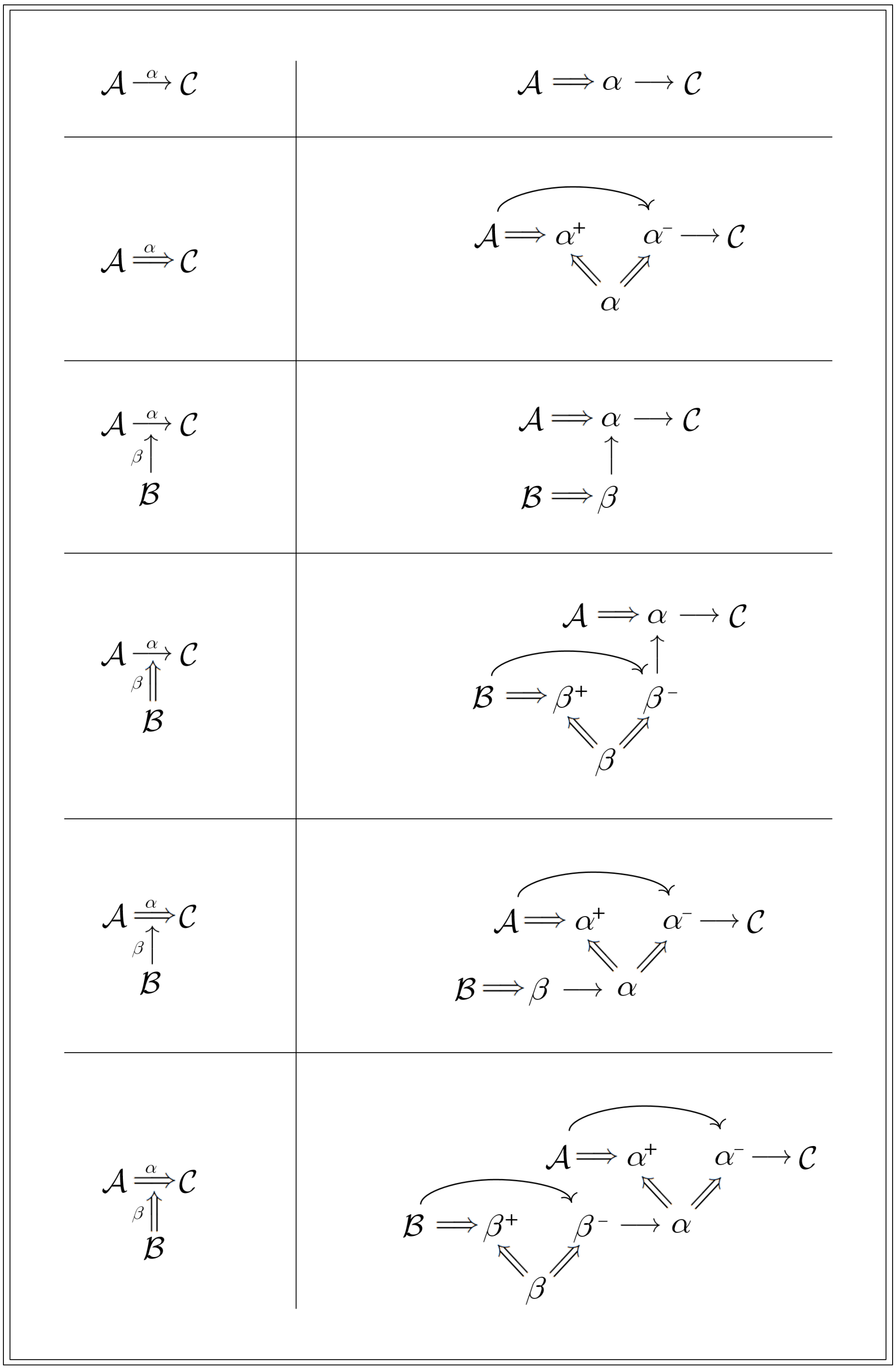
This translation is such that arguments, attacks and supports of the *ASAF* are mapped into arguments in the associated *AFN*. Since, similarly to the *AFRA*, an attack $\alpha = (\mathcal{A}, X)$ in the *ASAF* is dependent on its source argument $\mathcal{A}$, the associated *AFN* includes a necessary support from $\mathcal{A}$ to α. On the other hand, recall that the support relation of the *ASAF* follows a necessity interpretation, thus establishing constraints on the elements it relates. Given a support $\beta = (\mathcal{B}, Y)$, the constraint relating $\mathcal{B}$ and Y can be seen (and satisfied) from two points of view. From a positive point of view, it establishes that "if Y is accepted, then $\mathcal{B}$ should also be accepted"; we will refer to this as the *positive constraint*. From a negative point of view, the constraint indicates that "if $\mathcal{B}$ is not accepted, then Y should not be accepted either"; we refer to this as the *negative constraint*[6].

The translation incorporates the arguments β^+ and β^- in the associated *AFN*, which are used for encoding the positive and negative constraints. Then, since these arguments express constraints relating

[4] As mentioned before, we focus on the same semantics studied in [29].

[5] The translation presented here is an improved characterization of the one given in [29]; the differences between them will be discussed later in Section 6.

[6] It should be noted that the *positive constraint* and the *negative constraint* are complementary; they provide alternative ways of expressing the constraint associated with a support following the necessity interpretation, from a positive and from a negative point of view, respectively.



(a) (b)

Figure 1: Different cases illustrating the translation from an $ASAF$ (a) into its associated AFN (b).

the source and target of a support β, either one of them should only be enforced when the support β holds; this is captured in the associated *AFN* by including necessary supports from β to β^+ and β^-. Since the positive constraint corresponds to the case where the source $\mathcal{B}$ (also, the target Y) is accepted (otherwise, if $\mathcal{B}$ was not accepted, the negative constraint would have held), a necessary support from $\mathcal{B}$ to β^+ is included in the associated *AFN*. Similarly, the attack from $\mathcal{B}$ to β^- models the intuition that if the source $\mathcal{B}$ is accepted (thus, the positive constraint would hold), then the negative constraint is not satisfied. Finally, the attack from β^- to Y in the associated *AFN* captures the fact that if the negative constraint holds (in which case the source $\mathcal{B}$ would not be accepted, in contrast with the β^- argument), then the target Y cannot be accepted.

A graphical representation of the translation given in Definition 23 is illustrated in Figure 1. The first two cases in Figure 1 consider the translation of an attack and a support between arguments. The remaining four cases correspond to translations where a combination of the attack and support relations occur: respectively, attack to attack, support to attack, attack to support, and support to support. It can be noted that the translation of attacks and supports is the same in all cases. In other words, the translation of attacks and supports from an *ASAF* into its associated *AFN* does not depend on the nature of the targets of interactions (*i.e.,* it does not matter whether the targets are arguments, attacks or supports).

To characterize the second step of the translation from an *ASAF* into its associated *AF*, [29] makes use of the existing translation of an *AFN* into its associated *AF* (see Definition 10). Hence, the *AF* associated with an *ASAF* is obtained as follows.

Definition 24 (*AF* associated with an *ASAF*)**.** *Let* Δ *be an ASAF and* Δ_{AFN} *the AFN associated with* Δ*. The AF* associated with Δ *is* Δ_{AF}*, where* Δ_{AF} *is the AF associated with* Δ_{AFN}*.*

As mentioned before, the support relation of an *ASAF* is required to be acyclic, whereas the support relation of an *AFN* is required to be irreflexive and transitive. However, this difference does not interfere with the translation of an *ASAF* into an AF which involves an intermediate translation into an AFN. This is because support chains in the *ASAF* cease to exist when translated into the associated *AFN*, given the incorporation of the β, β^+ and β^- arguments corresponding to a support β of the *ASAF*.

Finally, in [29] the extensions of an *ASAF* are obtained by mapping the extensions of its associated *AF* into sets of arguments, attacks and supports of the *ASAF*.

Definition 25 (*AF-ASAF* Individual Mapping Function)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF. Given* $X \in \mathbb{A}_{AF}$*, the AF-ASAF* individual mapping function $\mathsf{D\text{-}IMap} : \mathbb{A}_{AF} \longmapsto \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is defined as follows:*

- *If* $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$*, then* $\mathsf{D\text{-}IMap}(X) = X$*.*
- *If* $X = \beta^+$ *or* $X = \beta^-$*, with* $\beta \in \mathbb{S}$*, then* $\mathsf{D\text{-}IMap}(X) = \beta$*.*

Definition 26 (*AF-ASAF* Mapping Function)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF. Given* $\mathbf{S} \subseteq \mathbb{A}_{AF}$*, the AF-ASAF* mapping function $\mathsf{D\text{-}Map} : 2^{\mathbb{A}_{AF}} \longmapsto 2^{\mathbb{A} \cup \mathbb{R} \cup \mathbb{S}}$ *is defined as* $\mathsf{D\text{-}Map}(\mathbf{S}) = \{\mathsf{D\text{-}IMap}(X) \mid X \in \mathbf{S}\}$*.*

As a result, following the approach of [29], extensions of an *ASAF* Δ under a semantics σ do not exist "on their own". That is, they are obtained by applying the $\mathsf{D\text{-}Map}$ function to the extensions of its associated *AF* Δ_{AF} under σ.

Definition 27 (*ASAF* Mapped Extensions)**.** *Let* Δ *be an ASAF and* Δ_{AF} *its associated AF. If* E *is an extension of* Δ_{AF} *under a D-*σ *semantics, with* $\sigma \in \{$*complete, preferred, stable, grounded*$\}$*, then* E_Δ *is an extension of* Δ *under the semantics* σ*, where* $\mathsf{D\text{-}Map}(\mathsf{E}) = \mathsf{E}_\Delta$*.*

Example 10. *Let* Δ_1 *be the ASAF from Example 1. The AFN associated with* Δ_1 *is* $\Delta_{1_{AFN}}$*, and is illustrated in in Figure 2. Then, the AF associated with* Δ_1 *(and with* $\Delta_{1_{AFN}}$*) is* $\Delta_{1_{AF}}$*, and is depicted in Figure 3. If we consider the D-grounded semantics, the D-grounded extension of* $\Delta_{1_{AF}}$ *is* $\mathbf{G}_{AF} = \{\mathcal{A}, \alpha, \beta^-, \gamma^-, \mathcal{L}, \beta, \gamma, \mathcal{I}, \lambda, \varphi, \mathcal{K}, \mathcal{F}, \pi, \mathcal{E}, \eta, \mathcal{M}, \mu\}$*. Then,* $\mathsf{D\text{-}Map}(\mathbf{G}_{AF}) = \{\mathcal{A}, \alpha, \beta, \gamma, \mathcal{L}, \mathcal{I}, \lambda, \varphi, \mathcal{K}, \mathcal{F}, \pi, \mathcal{E}, \eta, \mathcal{M}, \mu\}$*; therefore,* $\mathsf{D\text{-}Map}(\mathbf{G}_{AF}) = \mathbf{G}_\Delta$*, where* $\mathbf{G}_\Delta$ *is the grounded extension of* Δ_1 *obtained in Example 8.*

If we now consider the D-preferred semantics, the D-preferred extensions of $\Delta_{1_{AF}}$ *are* $\mathbf{P}_{1AF} = \mathbf{G}_{AF} \cup \{\mathcal{N}, \tau, \mu^-\}$ *and* $\mathbf{P}_{2AF} = \mathbf{G}_{AF} \cup \{\kappa, \mathcal{J}, \mu^+, \theta, \theta^-\}$*. Then, we have* $\mathsf{D\text{-}Map}(\mathbf{P}_{1AF}) = \mathsf{D\text{-}Map}(\mathbf{G}_{AF} \cup$

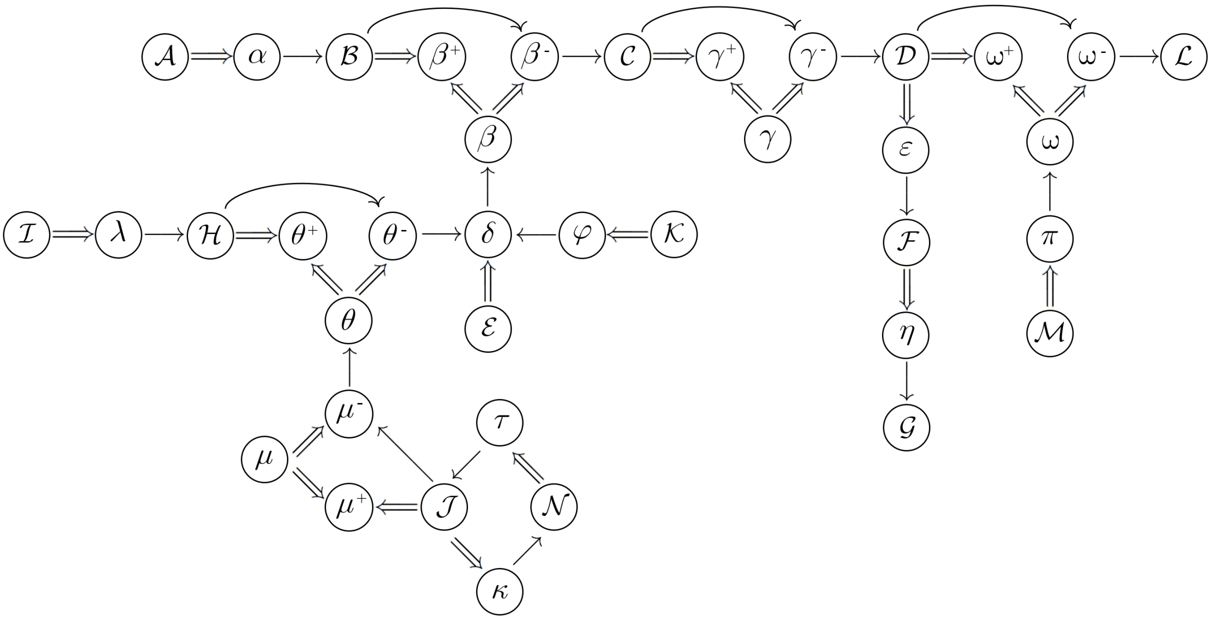


Figure 2: AFN $\Delta_{1_{AFN}}$ associated with the *ASAF* Δ_1 from Example 1.

$\{\mathcal{N}, \tau, \mu^-\}) = \mathbf{G}_\Delta \cup \{\mathcal{N}, \tau\}$*; therefore,* D-Map$(\mathbf{P}_{1AF}) = \mathbf{P}_{1\Delta}$*, where* $\mathbf{P}_{1\Delta}$ *is the first preferred extension of* Δ_1 *obtained in Example 8. Similarly,* D-Map$(\mathbf{P}_{2AF})$ = D-Map$(\mathbf{G}_{AF} \cup \{\kappa, \mathcal{J}, \mu^+, \theta, \theta^-\}) = \mathbf{G}_\Delta \cup \{\kappa, \mathcal{J}, \theta\}$*; therefore,* D-Map$(\mathbf{P}_{2AF}) = \mathbf{P}_{2\Delta}$*, where* $\mathbf{P}_{2\Delta}$ *is the second preferred extension of* Δ_1 *obtained in Example 8.*

As it can be observed from Examples 8 and 10, the grounded and preferred extensions of the *ASAF* Δ_1 obtained by mapping the extensions of its associated *AF* are the same as those obtained directly from the *ASAF*, in Example 8. The remainder of this section is devoted to formally show the equivalence between these two approaches. In order to prove this equivalence, we will use some intermediate results, showing the relationship between the semantic notions characterized directly on the *ASAF*, and those corresponding to its associated *AF*. As mentioned before, we will use the prefix 'D' to distinguish the semantic notions corresponding to an *AF* (in this case, the *AF* associated with an *ASAF*) from those corresponding to the *ASAF*.

5.1. Intermediate Results

The intermediate results developed in this section will be used later for proving the equivalence between the approach proposed in Section 4 for obtaining the complete, preferred, stable and grounded extensions of the *ASAF* and the one given in [29]. Specifically, we will show that the complete, preferred, stable, and grounded extensions of the *ASAF* after applying the corresponding mapping function (respectively, Map or D-Map) coincide with the D-complete, D-preferred, D-stable, and D-grounded extensions of its associated *AF*. Thus, in some cases, we will start by assuming that the set of elements of the associated *AF* (respectively, of the *ASAF*) under analysis is a D-complete (respectively, complete) extension.

The following lemma establishes a correspondence between D-conflict-free sets of the *AF* associated with an *ASAF* and conflict-free sets of the *ASAF*.

Lemma 4. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF. If* $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a* D*-complete extension of* Δ_{AF}*, then* D-Map$(\mathbf{S})$ *is a conflict-free set of* Δ*.*

To show the reverse of Lemma 4, we need to characterize a mapping function that, given a set of arguments, attacks and supports of an *ASAF*, leads to obtaining a set of arguments of its associated *AF*.

Definition 28 (*ASAF-AF* Mapping Function)**.** *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF. Given* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$*, we define* Map : $2^{\mathbb{A} \cup \mathbb{R} \cup \mathbb{S}} \longmapsto 2^{\mathbb{A}_{AF}}$ *the ASAF-AF* mapping function *s.t.* Map$(\mathbf{S}) = \mathbf{S}_{\Delta_{AF}}$ *and for every* $X \in \mathbf{S}$ *it holds that:*

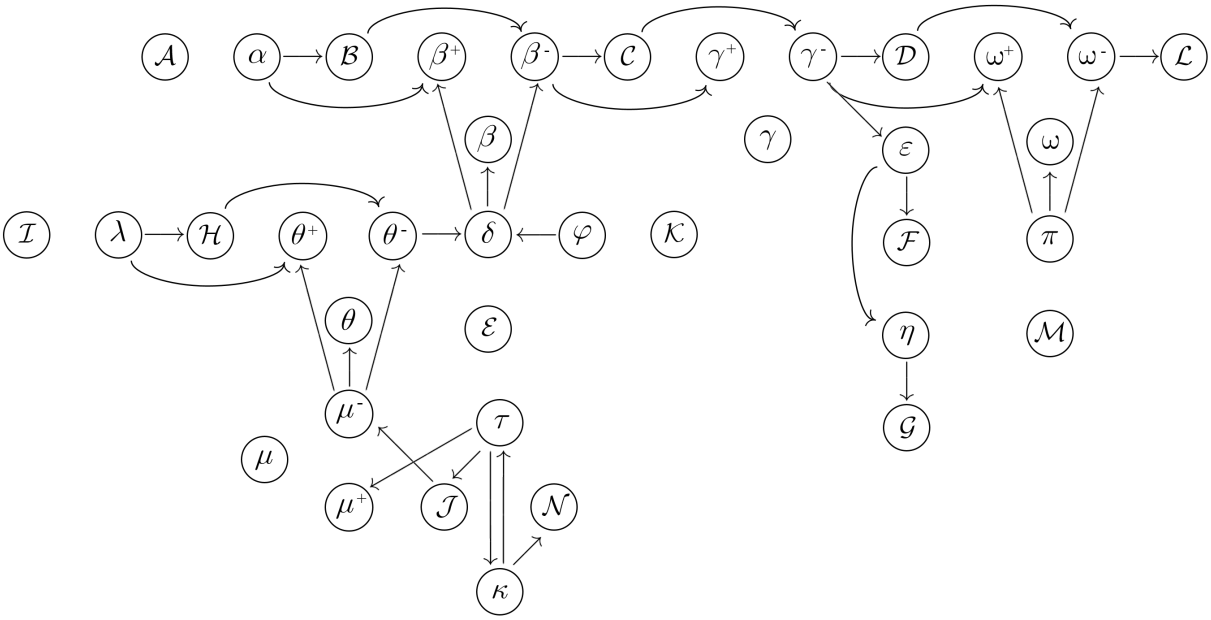


Figure 3: AF $\Delta_{1_{AF}}$ associated with the $ASAF$ Δ_1 from Example 1.

1. *If* $X \in \mathbb{A} \cup \mathbb{R}$, *then* $X \in \mathbf{S}_{\Delta_{AF}}$.
2. *If* $X \in \mathbb{S}$, *then* $X \in \mathbf{S}_{\Delta_{AF}}$ *and:*
   (a) *If* $\mathsf{src}(X) \in \mathbf{S}$ *and* $\mathsf{trg}(X) \in \mathbf{S}$, *then* $X^+ \in \mathbf{S}_{\Delta_{AF}}$.
   (b) *If* $\mathsf{src}(X) \notin \mathbf{S}$, $\mathsf{trg}(X) \notin \mathbf{S}$ *and* $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ *s.t.* α *u-def* $\mathsf{src}(X)$ *or* α *c-def* $\mathsf{src}(X)$ *given* $\mathbf{S}'$, *then* $X^- \in \mathbf{S}_{\Delta_{AF}}$.

The example below shows how the $\mathsf{Map}(\cdot)$ function provides a way of mapping extensions obtained directly from an $ASAF$ into extensions of its associated AF.

Example 11. *Let us consider the ASAF* Δ_1 *from Example 1 and its grounded and preferred extensions illustrated in Example 8; respectively, the grounded extension* $\mathbf{G}_\Delta = \{\mathcal{A}, \alpha, \gamma, \mathcal{M}, \pi, \mathcal{L}, \mathcal{I}, \lambda, \mathcal{K}, \varphi, \beta, \mathcal{F}, \eta, \mathcal{E}, \mu\}$ *and the preferred extensions* $\mathbf{P}_{1\Delta} = \mathbf{G}_\Delta \cup \{\tau, \mathcal{N}\}$ *and* $\mathbf{P}_{2\Delta} = \mathbf{G}_\Delta \cup \{\kappa, \mathcal{J}, \theta\}$. *Then, we have* $\mathsf{Map}(\mathbf{G}_\Delta) = \{\mathcal{A}, \alpha, \mathcal{L}, \mathcal{I}, \lambda, \varphi, \mathcal{K}, \mathcal{F}, \pi, \mathcal{E}, \eta, \mathcal{M}, \beta, \beta^-, \gamma, \gamma^-, \mu\}$; *therefore,* $\mathsf{Map}(\mathbf{G}_\Delta) = \mathbf{G}_{AF}$. *Similarly,* $\mathsf{Map}(\mathbf{P}_{1\Delta}) = \mathsf{Map}(\mathbf{G}_\Delta \cup \{\mathcal{N}, \tau\})$ *and* $\mathsf{Map}(\mathbf{P}_{2\Delta}) = \mathsf{Map}(\mathbf{G}_\Delta \cup \{\kappa, \mathcal{J}, \theta\})$. *Thus,* $\mathsf{Map}(\mathbf{P}_{1\Delta}) = \mathbf{G}_{AF} \cup \{\mathcal{N}, \tau, \mu^-\} = \mathbf{P}_{1AF}$, *whereas* $\mathsf{Map}(\mathbf{P}_{2\Delta}) = \mathbf{G}_{AF} \cup \{\kappa, \mathcal{J}, \theta, \mu^+, \theta^-\} = \mathbf{P}_{2AF}$.

The following lemma formalizes the fact that applying the mapping function $\mathsf{Map}(\cdot)$ to a complete extension of an $ASAF$ leads to a D-conflict-free set of arguments from its associated AF.

Lemma 5. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF. If* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a complete extension of* Δ, *then* $\mathsf{Map}(\mathbf{S})$ *is a D-conflict-free set of* Δ_{AF}.

Next, Lemma 6 characterizes a correspondence between the notions of acceptability and D-acceptability when considering an $ASAF$ and its associated AF.

Lemma 6. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) *If* $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-complete extension of* Δ_{AF} *and* $X \in \mathbb{A}_{AF}$ *is D-acceptable w.r.t.* $\mathbf{S}$, *then* $\mathsf{D\text{-}IMap}(X)$ *is acceptable w.r.t.* $\mathsf{D\text{-}Map}(\mathbf{S})$.

B) *If* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a complete extension of* Δ *and* $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is acceptable w.r.t.* $\mathbf{S}$, *then* X *is D-acceptable w.r.t.* $\mathsf{Map}(\mathbf{S})$.

The following lemma complements Lemma 6 by addressing the acceptability of the arguments X^+ and X^- of the associated AF, where X is a support in the $ASAF$.

Lemma 7. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF,* $X \in \mathbb{S}$ *and* $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *a complete extension of* Δ*.*

A) If X *and* $\mathsf{src}(X)$ *are acceptable w.r.t.* $\mathbf{S}$*, then* X^+ *is D-acceptable w.r.t.* $\mathsf{Map}(\mathbf{S})$*.*

B) If X *is acceptable w.r.t.* $\mathbf{S}$ *and* $\exists \alpha \in \mathbb{R}$*,* $\exists \mathbf{T} \subseteq \mathbb{S}$ *s.t.* α *is acceptable w.r.t.* $\mathbf{S}$*,* $\forall S_i \in \mathbf{T}$ *it holds that* S_i *is acceptable w.r.t.* $\mathbf{S}$*, and* α *u-def* $\mathsf{src}(X)$ *or* α *c-def* $\mathsf{src}(X)$ *given* $\mathbf{T}$*, then* X^- *is D-acceptable w.r.t.* $\mathsf{Map}(\mathbf{S})$*.*

Finally, by combining the results from Lemmas 4–7, we can establish a correspondence between the notions of D-admissibility and admissibility, as shown by Lemma 8.

Lemma 8. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-complete extension of* Δ_{AF}*, then* $\mathsf{D\text{-}Map}(\mathbf{S})$ *is admissible.*

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a complete extension of* Δ*, then* $\mathsf{Map}(\mathbf{S})$ *is D-admissible.*

5.2. Correspondence between extensions of an ASAF and its associated AF

After establishing a correspondence between the basic semantic notions for an *ASAF* and its associated *AF*, we will now turn to formalize a correspondence between the complete, preferred, stable, and grounded extensions of an *ASAF*, and the D-complete, D-preferred, D-stable, and D-grounded extensions of its associated *AF*. Theorem 1 starts by establishing the correspondence for the case of D-complete and complete extensions.

Theorem 1. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-complete extension of* Δ_{AF}*, then* $\mathsf{D\text{-}Map}(\mathbf{S})$ *is a complete extension of* Δ*.*

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a complete extension of* Δ*, then* $\mathsf{Map}(\mathbf{S})$ *is a D-complete extension of* Δ_{AF}*.*

The correspondence between the preferred extensions of an *ASAF* and the D-preferred extensions of its associated *AF* is addressed by Theorem 2.

Theorem 2. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-preferred extension of* Δ_{AF}*, then* $\mathsf{D\text{-}Map}(\mathbf{S})$ *is a preferred extension of* Δ*.*

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a preferred extension of* Δ*, then* $\mathsf{Map}(\mathbf{S})$ *is a D-preferred extension of* Δ_{AF}*.*

Then, the correspondence between D-stable and stable extensions is shown in Theorem 3.

Theorem 3. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-stable extension of* Δ_{AF}*, then* $\mathsf{D\text{-}Map}(\mathbf{S})$ *is a stable extension of* Δ*.*

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a stable extension of* Δ*, then* $\mathsf{Map}(\mathbf{S})$ *is a D-stable extension of* Δ_{AF}*.*

Finally, completing the characterization of the semantic relations between extensions, the correspondence between the grounded extension of an *ASAF* and the D-grounded extension of its associated *AF* is given by Theorem 4.

Theorem 4. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is the D-grounded extension of* Δ_{AF}*, then* $\mathsf{D\text{-}Map}(\mathbf{S})$ *is the grounded extension of* Δ*.*

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is the grounded extension of* Δ*, then* $\mathsf{Map}(\mathbf{S})$ *is the D-grounded extension of* Δ_{AF}*.*

The preceding theorems show that, given an *ASAF* Δ and its associated *AF* Δ_{AF}, for each extension E of Δ under a given semantics $\sigma \in \{$*complete*, *preferred*, *stable*, *grounded*$\}$, there exists a corresponding extension in Δ_{AF}; conversely, for each extension E' of Δ_{AF} under a semantics D-σ there exists a corresponding extension in Δ. That is, we have shown that each extension of the *ASAF* has a corresponding extension in the associated *AF* and *vice-versa*. Nonetheless, to show that the two approaches are equivalent it is necessary to show that there exists a one-to-one correspondence between the extensions of the *ASAF* and those of its associated *AF*; that is to say, that computing the extensions at the level of the *ASAF* leads to obtaining the same results as the ones obtained when computing the extensions through its associated *AF*. To do this, the following lemma shows that the functions $\mathsf{Map}(\cdot)$ and $\mathsf{D\text{-}Map}(\cdot)$ are the inverse of each other when considering, respectively, D-complete and complete extensions.

Lemma 9. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF and* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ *is a D-complete extension of* Δ_{AF}*, then* $\mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S})) = \mathbf{S}$.

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ *is a complete extension of* Δ*, then* $\mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S})) = \mathbf{S}$.

As a result, the following theorem establishes a one-to-one correspondence between the extensions of the *ASAF* and those of its associated *AF* under the complete, preferred, stable, and grounded semantics.

Theorem 5. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF,* $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ *its associated AF and a semantics* $\sigma \in \{$*complete, preferred, stable, grounded*$\}$*. It holds that* E *is an extension of* Δ *under the* σ *semantics iff* $\mathsf{Map}(\mathsf{E})$ *is an extension of* Δ_{AF} *under the D-*σ *semantics. Equivalently,* E' *is an extension of* Δ_{AF} *under the D-*σ *semantics iff* $\mathsf{D\text{-}Map}(\mathsf{E}')$ *is an extension of* Δ *under the* σ *semantics .*

6. Related Work and Discussion

This paper extends the work in [28], characterizing the acceptability semantics of the *Attack-Support Argumentation Framework* (*ASAF*) proposed in [29]. We have shown that several properties originally proposed in [30] for Dung's Abstract Argumentation Framework (*AF*) also hold for the *ASAF*. Additionally, in Section 5 we showed that the approach proposed here (specifically, in Section 4) is equivalent to the one proposed in [29], since they lead to obtaining the same extensions of the *ASAF* under the complete, preferred, stable and grounded semantics.

Differently from [29], the approach proposed here for obtaining the extensions of an *ASAF* does not make use of a translation into a Dung's *AF*. As mentioned in the introduction, having a characterization of semantic notions directly on the *ASAF* provides various advantages. In particular, the understanding of such semantic notions can be of help to achieve better models when encoding knowledge. This is because, initially, the knowledge engineer may consider a given set of arguments and interactions (attacks and supports) to make up an *ASAF*. Then, after analyzing the outcome of the framework, it might be the case that undesired results were obtained as a consequence of some interactions being wrongfully represented.

For instance, if we account for the example given in the introduction, one could consider an alternative representation where the argument expressing that there is an open window in the room ($\mathcal{OW}$) provides support for the argument claiming that the room is illuminated ($\mathcal{RI}$). However, such a representation would not be correct since the consideration of the support link $\mathcal{OW} \Longrightarrow \mathcal{RI}$, together with the existence of the support $\mathcal{LO} \Longrightarrow \mathcal{RI}$, would imply that *both* $\mathcal{LO}$ and $\mathcal{OW}$ have to be accepted in order for $\mathcal{RI}$ to be accepted[7]. Similarly, one could think that the argument expressing that there is only one lamp in the room ($\mathcal{OLR}$) provides additional support for $\mathcal{RI}$, meaning that there exists a support $\mathcal{OLR} \Longrightarrow \mathcal{RI}$ in the *ASAF*. However, again, this representation would not be correct since, together with the support $\mathcal{LO} \Longrightarrow \mathcal{RI}$, it would imply that both $\mathcal{LO}$ and $\mathcal{OLR}$ have to be accepted in order for $\mathcal{RI}$ to be accepted. Then, if there was additional information indicating that the lamp was not the only lamp in the room (in which case argument $\mathcal{OLR}$ would not be accepted), it would imply that $\mathcal{RI}$ is not accepted either. Furthermore, that undesired outcome would be obtained even though it could be the case that there were other reasons for the room being illuminated (such as the fact that there is daylight coming from an open window in the room).

As a result, having a characterization of semantic notions directly on the *ASAF* (that is, not only the definition of acceptability semantics but also basic notions like defeat, conflict-freeness, and acceptability) helps to easily and rapidly identify undesired situations like the ones illustrated above. In contrast, upon the absence of such notions, the knowledge engineer would have to constantly translate the *ASAF* into its associated *AFN* and then into its associated *AF* to then be able to check the outcome against undesired results. Furthermore, the advantage becomes more evident at the level of basic notions, as undesired results can be detected at an early stage without requiring the computation of extensions, *e.g.,* by detecting that a given set of arguments, attacks, and supports is not admissible when it was assumed to be.

It should be noted that, as mentioned in Section 5, the translation of an *ASAF* into its associated *AFN* given in Definition 23 differs from the one introduced in [29]. The difference relies on the way in

[7] This is because the *ASAF* adopts a necessity interpretation for the support relation.

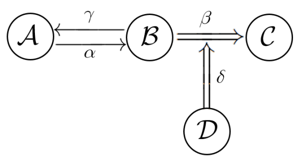
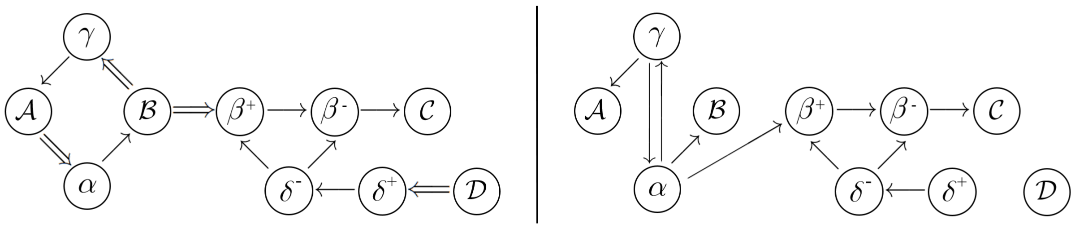


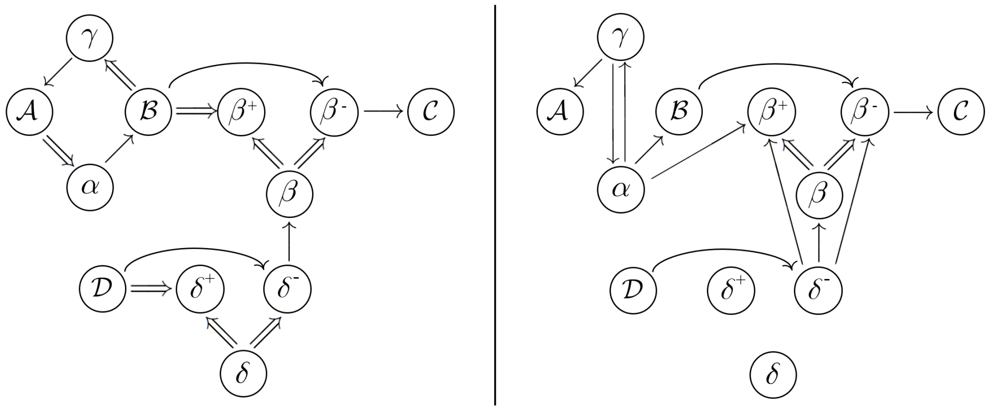
Figure 4: Example of an $ASAF$.

which supports are encoded in the AFN associated with an $ASAF$. Given a scenario where a support $\alpha = (\mathcal{A}, \mathcal{B})$ holds but the chosen semantics σ cannot determine whether the positive or the negative constraint associated with α holds, the translation proposed in [29] is such that neither α^+ nor α^- would belong to the corresponding extension of the associated AF under the semantics σ. In contrast, the translation given in Definition 23 is such that, in cases like the one mentioned above, neither α^+ nor α^- would belong to the corresponding extension of the associated AF; notwithstanding this, argument α would belong to that extension. To illustrate this difference, let us consider the $ASAF$ depicted in Figure 4.

If we consider the $ASAF$ from Figure 4 and translation proposed in [29], the associated AFN and the associated AF would be as depicted below on the left and below on the right, respectively.



Then, under the grounded semantics, the only extension of the associated AF would be $\{\mathcal{D}, \delta^+\}$. Hence, by Definition 27, we would get the set $\{\mathcal{D}, \delta\}$ as the grounded extension of the $ASAF$. In contrast, if we consider the translation given in Definition 23, the associated AFN and the associated AF will be as depicted below on the left and below on the right, respectively.



Given this translation, the grounded extension of the associated AF is $\{\mathcal{D}, \delta, \delta^+, \beta\}$; as a result, the mapped grounded extension of the $ASAF$ is $\{\mathcal{D}, \delta, \beta\}$. That is, differently from before, argument β belongs to the grounded extension meaning that, even though the attack cycle between arguments $\mathcal{A}$ and $\mathcal{B}$ is not resolved by the grounded semantics, the support β holds. Notwithstanding this, it should be remarked that we do not regard the translation given in [29] as incorrect; rather, we believe the translation given in Definition 23 is more fine-grained, allowing to explicitly distinguish between cases where a support β is not included in an extension of the $ASAF$ because it does not hold, from cases where it was not included in the extension (using the translation of [29]) because the chosen semantics was not able to determine whether the positive or the negative constraint associated with β held. As a result, by incorporating this distinction into the translation given in Definition 23, as shown in Section 5, the two approaches for the acceptability calculus of the *ASAF* are equivalent.

As pointed out in [29], the *ASAF* relates to other approaches in abstract argumentation that account for the existence of a support relation, like the *Bipolar Argumentation Framework* (*BAF*) [23], the *Evidential Argumentation System* (*EAS*) [47], the meta-argumentation approach for representing deductive support proposed in [57] and the *Argumentation Framework with Necessities*(*AFN*) [46]. However, the main difference between the *ASAF* and all these approaches is that they do not account for recursive supports (*i.e.,* supporting a support at any level), nor account for the possibility of supporting attacks. On the other hand, similarly to the *ASAF*, [57] defines a second-order attack relation which allows for attacks to the attack and support relations. Nevertheless, in contrast with the *ASAF*, the second-order attack relation proposed in [57] only allows for attacks to first-order supports and attacks.

From the above mentioned approaches, the one that most closely relates to the *ASAF* is the *AFN* [46]. This is mainly because both frameworks adopt a necessity interpretation for the support relation. Also, when identifying the defeats that may occur in the *ASAF* in Section 3, we accounted for the existence of extended defeats, like the *AFN*. Moreover, it can be shown that an *ASAF* without high level interactions (*i.e.,* an *ASAF* where the targets of every attack and support are arguments) is compatible with an *AFN* having the same attack and support relations. In particular, since the *ASAF* also accounts for the acceptability of attacks and supports, the extensions of the *ASAF* and the corresponding *AFN* will not coincide. Rather, as shown by the following proposition, the extensions of the *AFN* are equivalent to the corresponding extensions of the *ASAF* after filtering out attacks and supports.

Proposition 7. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF s.t. $\forall \alpha \in (\mathbb{R} \cup \mathbb{S})$, $\mathsf{trg}(\alpha) \in \mathbb{A}$, $\Phi = \langle \mathbb{A}, \mathbb{R}, \mathbb{S}^+ \rangle$ an AFN (where $\mathbb{S}^+$ is the transitive closure of $\mathbb{S}$) and a semantics $\sigma \in$ {complete, preferred, stable, grounded}. It holds that:*

(1) *If E_Δ is an extension of Δ under the semantics σ, then E_Φ is an extension of Φ under the semantics σ, where $\mathsf{E}_\Phi = \{\mathcal{A} \in \mathsf{E}_\Delta \mid \mathcal{A} \in \mathbb{A}\}$.*

(2) *If E_Φ is an extension of Φ under the semantics σ, then there exists an extension E_Δ of Δ under the semantics σ s.t. $\mathsf{E}_\Phi = \{\mathcal{A} \in \mathsf{E}_\Delta \mid \mathcal{A} \in \mathbb{A}\}$.*

It could be noted that, according to Definition 8, the necessary support relation of an *AFN* should be irreflexive and transitive. In contrast, as stated by Definition 11, the *ASAF* requires the necessary support relation to be acyclic. On the one hand, by being acyclic, the support relation of the *ASAF* is also irreflexive. On the other hand, it could be noted that the transitive nature of necessary support (as required in the *AFN*) is captured in the *ASAF* by explicitly considering a sequence of supports in the characterization of extended and extended-indirect defeats (see Definitions 16 and 17). Given a support sequence $[\mathcal{B}, \ldots, \mathcal{C}]$, the *AFN* is such that there is also a necessary support from $\mathcal{B}$ to every element in the sequence (in particular, to $\mathcal{C}$). In the *ASAF*, given the support sequence $[\mathcal{B}, \ldots, \mathcal{C}]$, by Proposition 5 the acceptability constraints derived from each support link in the sequence are combined to establish the fact that the acceptability of $\mathcal{C}$ implies the acceptability of $\mathcal{B}$ (equivalently, the non-acceptability of $\mathcal{B}$ implies the non-acceptability of $\mathcal{C}$), thus capturing the behavior of the necessary support link from $\mathcal{B}$ to $\mathcal{C}$ in the *AFN*.

Our work also relates to [10] in various aspects. On the one hand, the characterization of the semantic notions for the *ASAF* given in Section 4.1 and the characterization of extensions of the *ASAF* proposed in Section 4.2 follow the methodology adopted by the *Argumentation Framework with Recursive Attacks* (*AFRA*). On the other hand, a relationship between the *ASAF* and the *AFRA* can be observed in Section 3, where different kinds of defeat that may occur in the *ASAF* were identified. In particular, the characterization of direct and indirect defeats in the *ASAF* coincides with their characterization in the *AFRA*, as proposed in [10]. As a result, we can show that the *ASAF* is an extension of the *AFRA* in the sense that an *ASAF* with an empty support relation leads to obtaining the same extensions (under the complete, preferred, grounded, and stable semantics) as the resulting *AFRA* defined by the arguments and the attack relation of the *ASAF*. The relationship between the *ASAF* and the *AFRA* is formalized by the following proposition.

Proposition 8. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \emptyset \rangle$ be an ASAF, $\Gamma = \langle \mathbb{A}, \mathbb{R} \rangle$ an AFRA and let σ be a semantics such that $\sigma \in$ {complete, preferred, stable, grounded}. It holds that E is an extension of Δ under the σ semantics iff E is an extension of Γ under the σ semantics.*

Another formalism that can be used to model support in abstract argumentation is the *Abstract Dialectical Framework* (*ADF*) [18, 17]. An *ADF* is a directed graph, whose nodes represent arguments which can be accepted or not, and the links between the nodes represent dependencies. Each argument $\mathcal{A}$ in the graph is associated with an acceptance condition $C_{\mathcal{A}}$, which is some propositional function whose truth status is determined by the corresponding values of the acceptance conditions for those arguments $\mathcal{B}$ such that $(\mathcal{B}, \mathcal{A})$ is link in the *ADF* (*i.e.,* $\mathcal{B}$ is a parent of $\mathcal{A}$). The revisited approach to *ADFs* proposed in [17] is such that the links between the nodes are not specified explicitly, but they are inferred from the acceptance conditions within the nodes. This implies that links between the nodes (namely, interactions between arguments) in the *ADF* are somehow represented by the acceptance conditions of the nodes involved in the corresponding interactions.

If we wanted to represent an *ASAF* using an *ADF* we would need to be able to determine the acceptance status of interactions, as well as the acceptance status of arguments. Therefore, we would need to come up with a way of determining the acceptance status of acceptance conditions, in addition to that of nodes in the *ADF*. Nevertheless, since acceptance conditions are defined only for nodes (arguments) in the *ADF*, we would need to model interactions as nodes. Moreover, this would be necessary in order to allow for interactions to target other interactions, as in the *ASAF*. As a result, in order to model an *ASAF* using an *ADF*, we could include additional nodes for representing the interactions similarly to the approach proposed in [29], where the *AF* associated with an *ASAF* was used in order to determine the acceptance status of arguments and interactions of an *ASAF*. Furthermore, in order to model an *ASAF* using an *ADF* one could start by obtaining the *AF* associated with the *ASAF*, and then model that *AF* using an *ADF*.

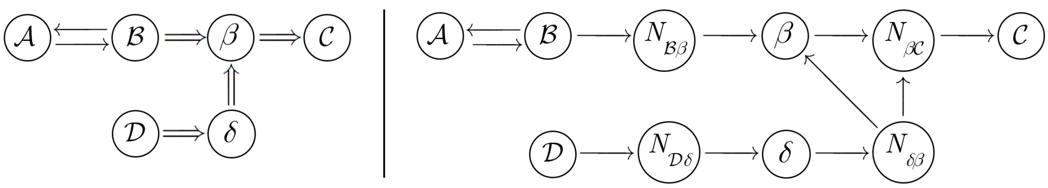
An alternative characterization of an *ASAF* as an *ADF* could be provided by making use of the results presented in [49]. There, the author presents several translations starting from different abstract argumentation frameworks that extend Dung's *AF* into *ADFs*. Among the frameworks studied in [49] is the extended version of the *AFN* proposed in [44], where the necessity support relation accounts for sets of arguments, with the requirement that an argument is accepted if at least one element in each of its supporting sets is accepted. Then, one could translate the *ASAF* into its associated *AFN* using the translation given in Definition 23 and then use the translation of an *AFN* into its associated *ADF* given in [49]. However, since, differently from [49], the *AFN* considered in Definition 23 is such that the necessary support relation is defined over pairs of arguments, in order to make use of the translation given in [49] an additional step is required: to characterize an *AFN* like the one in [44], where each set of supporting arguments contains just one argument (the argument being the source of the corresponding support link in the *AFN* obtained by Definition 23).

Although an *ASAF* could be represented as an *ADF* following any of the alternatives discussed above, the creation of new arguments and interactions is required in either case. Therefore, for each step of the chosen translation, the user would have to clearly understand the meaning of the new elements to be introduced, as well as the semantics of the underlying frameworks. As a result, one could consider a complementary approach using the *ASAF* and the *ADF* for modeling recursive interactions, in the following sense. The use of an *ASAF*, as well as the direct characterization of its semantics proposed in this paper provides an intuitive and natural representational tool for modeling recursive interactions. Then, in order to exploit the higher expressive power and the theoretical results associated with the use of *ADFs* and, for instance, its software support [32, 33], a translation of an *ASAF* into an *ADF* can be made. Finally, it is important to note that the translation of an *AFN* into an *ADF* proposed in [49] relies on two features of the *AFN* of [44]. On the one hand, it requires the *AFN* to be strongly consistent, meaning that no argument simultaneously attacks and supports another argument. On the other hand, as stated by the author in [49], the *AFN* semantics in [44] are built around the notion of coherence, which requires all relevant arguments to be (support-wise) derived in an acyclic manner. Thus, the *AFN* considered in the translation of [49] shares the constraints imposed on the support relation of an *ASAF*, as expressed in Definition 11; namely, that the attack and support relations are disjoint, and that the support relation is acyclic.

In [21] the authors propose an alternative approach for addressing the acceptability calculus in the context of an *ASAF*. In that work, an alternative translation of an *ASAF* into a Dung's *AF* by making use of an intermediate *AFN* is provided, drawing on [25] where a translation from an *AFN* into a Dung's *AF* was proposed. The translation given in [21] is driven by three features that can be identified in interactions involved in a recursion: groundness, validity, and activation, where interactions have to be active in order to be included in the extensions of an *ASAF*. In particular, as proposed in [21], an interaction is considered to be grounded if its source is accepted. The validity of an interaction is determined by looking at the interactions that may affect it, that is, interactions attacking and supporting it; for instance, an interaction that is attacked by another interaction that is active will not be considered as valid. Then, an interaction is considered to be active if it is grounded and valid.

It should be noted that the translation given in [21] is such that it might lead to obtaining different results (regarding acceptability) than both the translation proposed in [29] and the semantics proposed here (see Section 4). The fact that the activation of interactions is the feature determining their inclusion in the extensions of an $ASAF$ is what makes the results from [21] differ from ours. Specifically, the difference relies on the acceptance of support relations. By requiring a support to be active in order to be accepted, [21] requires that support to be both grounded and valid. This implies that, in order to be accepted in [21], the constraints associated with a necessary support should be satisfied from a positive point of view[8]. In contrast, the approach for calculating the acceptability of arguments and interactions of an $ASAF$ proposed in this paper (as well as the one given in [29]) accounts for the possibility of satisfying the constraints of a necessary support either from a positive or a negative point of view. To relate this back to the features of interactions identified in [21], the approach to acceptability calculus proposed in Section 4 is such that an interaction will be part of an extension of an $ASAF$ iff it is (at least) valid.

To illustrate the above mentioned difference, let us consider again the $ASAF$ depicted in Figure 4. Then, if we consider the translation given in [21], the associated AFN and the associated AF will be as depicted below on the left and below on the right, respectively[9].



Given this translation, the grounded extension of the associated AF is $\{\mathcal{D}, \delta\}$, meaning that the only active interaction is δ, leaving β outside because it is not grounded. In particular, argument $N_{\mathcal{B}\beta}$ — which expresses "the support β is not grounded" — does not belong to the grounded extension either. This is because of the attack cycle between arguments $\mathcal{A}$ and $\mathcal{B}$, which is not resolved by the grounded semantics. As a result, argument β — expressing that "the support β is active" — and argument $N_{\beta\mathcal{C}}$ — expressing that "the support β is valid but not grounded" — do not belong to the grounded extension either. In contrast, as shown before, the direct approach proposed in this paper (Section 4) and the translation given in Definition 23 are such that the grounded extension of the $ASAF$ also includes β. It is important to note that, even though the approach from [21] leads to obtaining the same grounded extension as [29] when considering the $ASAF$ illustrated in Figure 4, this would not have been the case if argument $\mathcal{D}$ was not "grounded" according to the terminology proposed in [21] (for instance, because $\mathcal{D}$ was attacked by an unattacked argument $\mathcal{E}$).

There exists another difference between the approach from [21] and the one proposed here, as well as the one given in [29]. As it can be observed in the above example, the approach from [21] does not account for the acceptability of every interaction. That is, the authors in [21] distinguish between "labeled" and "unlabeled" interactions, where labeled interactions are those involved in a recursion (either as a target, or as targeting another interaction). Hence, the associated AFN is such that arguments for encoding the groundness, validity, and activation of interactions (*e.g.,* $N_{\mathcal{B}\beta}, N_{\beta\mathcal{C}}$ and β) are only generated for labeled interactions. In contrast, interactions such as the attacks between $\mathcal{A}$ and $\mathcal{B}$ are encoded directly in [21], without introducing additional arguments. On the other hand, the approach from [29] and the one proposed in this paper are such that the acceptability status of *every* interaction is explicitly accounted

[8]Recall that, given a necessary support $\alpha = (\mathcal{A}, X)$, from a positive point of view the constraints derived from α establish that if X is accepted, then $\mathcal{A}$ should also be accepted; on the other hand, the constraints could be satisfied from a negative point of view, where if $\mathcal{A}$ is not accepted, then X should not be accepted either.

[9]The authors in [21] refer to these as the associated *Bipolar Argumentation System* with necessary support (BAS) and the associated Dung's *Meta Argumentation System* (MAS); however, for uniformity purposes, here we refer to them as AFN and AF, respectively.

for. As a result, interactions not involved in a recursion will always hold and thus, they will belong to the extensions of their corresponding *ASAF*.

To conclude the discussion, let us consider other works that shed light on practical applications of the *ASAF*. In [22] the authors proposed logical encodings of argumentation frameworks with attack and support relations. For that purpose, the authors considered existing logical encodings of *metabolic networks* [2], which may be graphically represented by *Molecular Interaction Maps (MIMs)*. Briefly, a *MIM* is a graph whose nodes are proteins and edges are either relations involving those proteins (*e.g.,* protein p_1 induces the production of —respectively, inhibits— protein p_2), or relations from a protein to another relation (*e.g.,* protein p_3 activates —respectively, inhibits— the reaction relating proteins p_1 and p_2).

Given their characteristics, it is possible to draw a parallel between a *MIM* and an *ASAF*. Briefly, each protein in a *MIM* can be represented as an argument in an *ASAF*. Then, reactions involving two proteins can be modeled as attacks or supports between arguments in the *ASAF*, depending on their nature. Specifically, the notion of inhibition of a protein p_2 by another protein p_1 in a *MIM* has the following associated meaning: "if p_1 is present then p_2 is not present"; this can be represented by the existence of an attack $p_1 \longrightarrow p_2$ in the *ASAF*. The notion of production of a protein p_2 by another protein p_1 means that "if p_1 is present then p_2 is also present". In this case, the relationship between p_1 and p_2 can be represented with a necessary support $p_2 \Longrightarrow p_1$ in the *ASAF*.[10]

The modeling of reactions that target other reactions in a *MIM* requires some additional considerations. On the one hand, the fact that a protein p inhibits the reaction r could be represented with a high-order attack $p \longrightarrow r$, where r is an attack or a support in the *ASAF*, depending on its nature (respectively, an inhibition or a production reaction). On the other hand, a reaction expressing that a protein p activates the reaction r could be represented in the *ASAF* through a high-order support $p \Longrightarrow r$. Note that this representation would be accurate since, as pointed out in [22], the context associated with each reaction in a *MIM* is reduced to only one activation and one inhibition. Hence , since there would be at most one attack and, in particular, one support targeting each reaction, the activation of reactions can be encoded through the necessary support relation of the *ASAF* without problems. As a result, by modeling *MIMs* as *ASAFs*, we would be able to reason about the behavior of proteins and reactions in these networks. Specifically, this would be achieved by looking at the extensions of the corresponding *ASAFs* under different semantics to determine the accepted arguments and interactions.

Let us now consider [56] and [38], where different extensions of argumentation frameworks were used to reason about the trustworthiness of information sources. On the one hand, [56] proposes to model the sources of information and provide the means to attack untrustworthy sources. Also, the authors provide a representation of trust about those sources, which concerns not only the sources but also the information items and the relation with other information. The model they propose accounts for support relations, which are used to represent the links between an information source and the pieces of information it provides. In particular, it can be noted that this support links may target not only arguments but also interactions between them, meaning that the corresponding source of information provides evidence towards the existence of a given argument, attack, *etc.* On the other hand, [38] uses the *AFRA* [10] to address a trust model in a collaborative open multi-agent system. The proposed model is such that information sources (the agents in the system) share information about the trust they have assigned to their peers. Furthermore, this information is taken into consideration when evaluating the trust associated with a given piece of information. This leads to a recursive setting where the reliability of a certain credibility information depends on the credibility of other pieces of information that should be subject to the same analysis. In order to capture the recursion involved in this reasoning process, and the conflicts that may be derived from all the available information at different levels, the authors propose to use the *AFRA*.

The work of [56] and [38] suggests that the *ASAF* is suitable for modelling the dynamics associated with the consideration of trust relationships, and to reason with them accordingly. Furthermore, we could also take into consideration the notion of distrust, which can be associated with the notion of attacks. As a result, by using the *ASAF*, existing proposals could be extended to model and reason

[10]Note that this reaction can be naturally associated with a deductive support from p_1 to p_2. Thus, by relying on the duality between the necessary and deductive interpretations of support (see [24, 27]), the reaction can be represented using the necessary support relation of the *ASAF*.

with trust and distrusts relationships at different levels, involving elements of different nature. Briefly, information sources and pieces of information may be represented as arguments in an *ASAF*. Then, for instance, the fact that agent Ag_i distrusts the piece of information i could be modeled by the existence of an attack from Ag_i to i. As another example, the trust agent Ag_i has on another agent Ag_j could be represented by a support link in the *ASAF* relating Ag_i and Ag_j. Furthermore, if an agent Ag_k distrusts the fact that agent Ag_i trusts agent Ag_j, then the *ASAF* could include a high-order attack from Ag_k to the support link relating Ag_i and Ag_j. Analogously, if an agent Ag_m trusts the assessment of Ag_k regarding the relationship between Ag_i and Ag_j, a support link from Ag_m to the attack relating Ag_k and the support between Ag_i and Ag_j could be included in the *ASAF*.

7. Conclusions and Future Work

In this work, we have proposed a characterization of acceptability semantics for the *Attack-Support Argumentation Framework* (*ASAF*) [29]. On the one hand, similarly to [29], we adopted an extension-based approach. On the other hand, differently from [29], we characterized the acceptability semantics directly on the *ASAF*, without making use of a translation into a Dung's *AF*.

Before characterizing the acceptability semantics of the framework, we first identified the different situations under which the elements of the *ASAF* are in conflict; as a result, we identified two groups of defeats that may occur: the unconditional defeats and the conditional defeats. The former are defeats that can be inferred from the attack relation since they correspond to conflicts that do not involve the existence of supports. Specifically, the unconditional defeats are the direct defeats (which correspond to attacks expressed in the attack relation of the *ASAF*) and indirect defeats (which capture the intuition that attacks on an argument also affect the attacks that argument originates). The latter group corresponds to conditional defeats: the extended defeats and extended-indirect defeats. In particular, conditional defeats aim at capturing conflicts that arise from the coexistence of the attack and support relations of the *ASAF*. Moreover, given that the *ASAF* adopts a necessity interpretation for the support relation, the extended defeats coincide with their homonymous attacks in the *Argumentation Framework with Necessities* (*AFN*) [46] and thus, they allow to enforce the acceptability constraints derived from the necessary support relation.

As a first step towards defining the acceptability semantics of the *ASAF*, following Dung's methodology [30], we defined the notions of conflict-freeness, acceptability, and admissibility in the context of the *ASAF*. When defining these notions, it was necessary to account for all the kinds of defeat that may occur between the elements of an *ASAF*. In particular, since the conditional defeats require the consideration of a set of supports, when characterizing the notion of acceptability it was necessary to account for all the ways in which defense against a defeat can be provided: either by defeating the corresponding attack or, in the case of conditional defeats, by defeating one of the involved supports. Then, starting from the basic semantic notions, we provided a full characterization of complete, preferred, grounded and stable semantics for the *ASAF*.

Alongside the formalization of the acceptability semantics, we provided a series of formal results, which can be divided into different categories. The first group of results regards the relationship between the different forms of defeat that may occur in the *ASAF*. The second group of results shows that the *ASAF* complies with the acceptability constraints derived from its attack and necessary support relations. Then, the final group of results corresponds to properties shown for Dung's *AF*s in [30], which involve the monotonicity of the notion of acceptability, Dung's Fundamental Lemma, and the relationship between the complete, preferred, stable, and grounded semantics of the framework.

Later, we showed that the approach for obtaining the extensions of the *ASAF* that we proposed in this paper is equivalent to the one given in [29], in the sense that they lead to obtaining the same extensions under the complete, preferred, stable and grounded semantics. To prove the equivalence between the approach of [29] and ours, we made use of intermediate results that prove the equivalence of the two approaches in terms of the basic semantic notions of conflict-freeness, acceptability, and admissibility. Then, we effectively showed that there exists a one-to-one correspondence between the complete, preferred, stable, and grounded extensions obtained directly on the *ASAF* and those obtained via its associated *AF*.

We also provided a formal account of the relationship between the *ASAF* and the two formalisms it is inspired on, namely, the *Argumentation Framework with Recursive Attacks* (*AFRA*) [10] and the *Argumentation Framework with Necessities* (*AFN*) [46]. On the one hand, we showed that an *ASAF* where attacks and supports occur only at the argument level is equivalent to an AFN having the same arguments, attacks and supports as the $ASAF$, where the support relation is closed under transitivity. On the other hand, we proved that the $ASAF$ is indeed an extension of the $AFRA$ by formally showing that an $ASAF$ with an empty support relation is equivalent to the $AFRA$ specified by the arguments and the attack relation of the $ASAF$.

As it was mentioned before, the results in this paper regard the complete, preferred, stable, and grounded semantics. However, these results could be extended to other semantics, such as semi-stable [19], or ideal [31]; we plan to take additional semantics into account in our future work. Another issue that we aim at addressing in the future is the study of the computational complexity of the approach for computing the extensions of the $ASAF$ proposed in this paper. In particular, we are aware that the characterization of acceptability given in Definition 20 may encompass a high computational cost since it requires the consideration of many combinations of attacks and supports leading to the existence of defeats.

As another line of future work, we seek to empirically contrast the approach of [29] for computing the $ASAF$ extensions through the use of translations against the direct computation via the semantics characterized in this paper. For the first part, by using the AF associated with an $ASAF$, we can make use of the existing implementations for Dung's AFs, like the developments based on SAT solvers [54] or implementations for *Abstract Dialectical Frameworks* (*ADF*s) [34, 32, 33]. Thus, for implementing the approach of [29] we would just need to codify the translation resulting from Definitions 23 and 10, which can be easily done as it involves the creation of arguments and interactions that is linear on the size the $ASAF$. On the other hand, as a first step towards implementing the direct computation of the $ASAF$ semantics, we plan to characterize the complete labelling of the $ASAF$ similarly to [20]. Then, we aim at following the approach of [26], which relies on encoding the constraints of the complete labelling as a SAT problem and then make use of existing strategies like the ones discussed in [54] to iteratively produce and solve modified versions of the initial SAT problem according to the needs of the search process. Specifically, the constraints corresponding to a complete labelling would be encoded in conjunctive normal form, with different sets of clauses capturing different aspects of the semantics' behavior.

Finally, it should be noted that, even though the extensions of an $ASAF$ could be efficiently obtained by means of existing implementations for other argumentation formalisms, the direct characterization of semantic notions proposed in this paper provides advantages from a modeling point of view. Furthermore, they provide intuitive means for the theoretical analysis of the $ASAF$, while also allowing for a better understanding of the impact of having recursive bipolar interactions in abstract argumentation frameworks.

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Appendix

Proposition 1. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{S}$ and $\mathbf{S} \subseteq \mathbb{S}$. If α e-def $\mathsf{src}(\beta)$ given $\mathbf{S}$, then α e-def $\mathsf{trg}(\beta)$ given $\mathbf{S} \cup \{\beta\}$.*

Proof. If α e-def $\mathsf{src}(\beta)$ given $\mathbf{S}$, then, by Definition 16, there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots, \mathsf{src}(\beta)]$ for $\mathsf{src}(\beta)$ s.t. $\mathbf{S}$ is the support set of Σ. Since by hyp. $\beta = (\mathsf{src}(\beta), \mathsf{trg}(\beta)) \in \mathbb{S}$, by Definition 15, $\Sigma' = [\mathcal{A}_1, \ldots, \mathsf{src}(\beta), \mathsf{trg}(\beta)]$ is a support sequence for $\mathsf{trg}(\beta)$ and $\mathbf{S} \cup \{\beta\}$ is the support set of Σ'. Thus, by Definition 16, α e-def $\mathsf{trg}(\beta)$ given $\mathbf{S} \cup \{\beta\}$. □

Proposition 2. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF, $\alpha \in \mathbb{R}$, $X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ and $\mathbf{S} \subseteq \mathbb{S}$. If α e-def X given $\mathbf{S}$, then $\exists \beta \in \mathbf{S}$ s.t. α d-def $\mathsf{src}(\beta)$.*

Proof. If α e-def X given $\mathbf{S}$, then, by Definition 16, there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots, X]$ for X, where $\mathsf{trg}(\alpha) = \mathcal{A}_1$ and $\mathbf{S}$ is the support set of Σ. Also, by Definition 16, $\exists \beta \in \mathbf{S}$ s.t. $\mathsf{src}(\beta) = \mathcal{A}_1$. Thus, by Definition 12, α d-def $\mathsf{src}(\beta)$. □

Proposition 3. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF, $\alpha, \gamma \in \mathbb{R}$ and $\mathbf{S} \subseteq \$$. If α ei-def γ given $\mathbf{S}$, then $\exists \beta \in \mathbf{S}$ s.t. α d-def $\mathsf{src}(\beta)$.*

Proof. It follows directly from Definition 17 and Proposition 2. □

Proposition 4. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF, $\alpha \in \mathbb{R}$ and $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$. If α is acceptable w.r.t. $\mathbf{S}$, then $\mathsf{src}(\alpha)$ is acceptable w.r.t. $\mathbf{S}$.*

Proof. Suppose by contradiction that α is acceptable w.r.t. $\mathbf{S}$ and $A = \mathsf{src}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$. Then, either (a) $\exists \beta \in \mathbb{R}$ s.t. β u-def A, and $\nexists \gamma \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$ s.t. γ u-def β or γ c-def β given $\mathbf{S}'$; or (b) $\exists \beta \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \$$ s.t. β c-def A given $\mathbf{T}$, and $\nexists \gamma \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$, $\nexists \delta \in \mathbf{T}$ s.t. γ u-def β, γ c-def β given $\mathbf{S}'$, γ u-def δ or γ c-def δ given $\mathbf{S}'$.

- (a) By Definition 11, it holds that $A = \mathsf{src}(\alpha) \in \mathbb{A}$. Then, if β u-def A, by Definitions 14 and 12, it must be the case that β d-def A. Therefore, by Definition 13, β i-def α.
- (b) By Definition 11, it holds that $A = \mathsf{src}(\alpha) \in \mathbb{A}$. Then, if β c-def A given $\mathbf{T}$, by Definitions 18 and 16, β e-def A given $\mathbf{T}$. Therefore, by Definition 17, β ei-def α given $\mathbf{T}$.

Then, by Definition 20, α would not be acceptable w.r.t. $\mathbf{S}$, contradicting the hypothesis. □

Proposition 5. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF, $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$ a conflict-free set and $\alpha \in \$$ acceptable w.r.t. $\mathbf{S}$. If $\mathsf{trg}(\alpha)$ is acceptable w.r.t. $\mathbf{S}$, then $\mathsf{src}(\alpha)$ is acceptable w.r.t. $\mathbf{S}$; equivalently, if $\mathsf{src}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$, then $\mathsf{trg}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$.*

Proof. If $A = \mathsf{src}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$, then it holds that either (a) $\exists \beta \in \mathbb{R}$ s.t. β u-def A, and $\nexists \gamma \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$ s.t. γ u-def β or γ c-def β given $\mathbf{S}'$; or (b) $\exists \beta \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \$$ s.t. β c-def A given $\mathbf{T}$, and $\nexists \gamma \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$, $\nexists \delta \in \mathbf{T}$ s.t. γ u-def β, γ c-def β given $\mathbf{S}'$, γ u-def δ or γ c-def δ given $\mathbf{S}'$.

- (a) By Definition 11, it holds that $A = \mathsf{src}(\alpha) \in \mathbb{A}$. Then, if β u-def A, by Definitions 14 and 12, it must be the case that β d-def A. Therefore, by Definition 16, β e-def $\mathsf{trg}(\alpha)$ given $\{\alpha\}$.
- (b) By Definition 11, it holds that $A = \mathsf{src}(\alpha) \in \mathbb{A}$. Then, if β c-def A given $\mathbf{T}$, by Definitions 18 and 16, it must be the case that β e-def A given $\mathbf{T}$. Therefore, by Proposition 1, β e-def $\mathsf{trg}(\alpha)$ given $\mathbf{T} \cup \{\alpha\}$.

Since by hypothesis α is acceptable w.r.t. $\mathbf{S}$ and $\mathbf{S}$ is conflict-free, $\nexists \lambda \in \mathbf{S}$, $\nexists \mathbf{S}'' \subseteq \mathbf{S}$ s.t. λ u-def α or λ c-def α given $\mathbf{S}''$. As a result, by Definition 20, $\mathsf{trg}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$. □

Proposition 6. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF, $X \in (\mathbb{A} \cup \mathbb{R} \cup \$)$ and $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$. If X is acceptable w.r.t. $\mathbf{S}$, then $\forall \mathbf{S}' \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$ s.t. $\mathbf{S} \subseteq \mathbf{S}' : X$ is acceptable w.r.t. $\mathbf{S}'$.*

Proof. Suppose by contradiction that X is acceptable w.r.t. $\mathbf{S}$ and $\exists \mathbf{S}' \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$ s.t. $\mathbf{S} \subseteq \mathbf{S}'$ and X is not acceptable w.r.t. $\mathbf{S}'$. Then, it holds that either (a) $\exists \alpha \in \mathbb{R}$ s.t. α u-def X and $\nexists \beta \in \mathbf{S}'$, $\nexists \mathbf{S}'' \subseteq \mathbf{S}'$ s.t. β u-def α or β c-def α given $\mathbf{S}''$; or (b) $\exists \alpha \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \$$ s.t. α c-def X given $\mathbf{T}$ and $\nexists \beta \in \mathbf{S}'$, $\nexists \mathbf{S}'' \subseteq \mathbf{S}'$, $\nexists \gamma \in \mathbf{T}$ s.t. β u-def α, β c-def α given $\mathbf{S}''$, β u-def γ or β c-def γ given $\mathbf{S}''$. Thus, since $\mathbf{S} \subseteq \mathbf{S}'$, by Definition 20, X would not be acceptable w.r.t. $\mathbf{S}$, contradicting the hypothesis. □

Lemma 1. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF, $\mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \$)$ an admissible set of Δ, and $X, Y \in (\mathbb{A} \cup \mathbb{R} \cup \$)$ s.t. X and Y are acceptable w.r.t. $\mathbf{S}$. Then, it holds that (1) $\mathbf{S}' = \mathbf{S} \cup \{X\}$ is admissible, and (2) Y is acceptable w.r.t. $\mathbf{S}'$.*

Proof.

(1) To prove that $\mathbf{S'}$ is admissible we have to prove that X is acceptable w.r.t. $\mathbf{S'}$ and $\mathbf{S'}$ is conflict-free. Since $\mathbf{S} \subseteq \mathbf{S'}$ and, by hypothesis, X is acceptable w.r.t. $\mathbf{S}$, by Proposition 6, X is acceptable w.r.t. $\mathbf{S'}$. Now, suppose by contradiction that $\mathbf{S'}$ is not conflict-free. Then, since by hypothesis $\mathbf{S}$ is admissible, it must be the case that $\exists W, Z \in \mathbf{S}$, $\exists \mathbf{T} \subseteq \mathbf{S}$ s.t. either (a) X u-def W; (b) W u-def X; (c) X c-def W given $\mathbf{T}$; (d) W c-def X given $\mathbf{T}$; or (e) W c-def Z given $\mathbf{T} \cup \{X\}$.

(1.a) If X u-def W, since by hypothesis $\mathbf{S}$ is admissible, it must be the case that $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S_1} \subseteq \mathbf{S}$ s.t. α u-def X or α c-def X given $\mathbf{S_1}$. Furthermore, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it must be the case that $\exists \beta \in \mathbf{S}$, $\exists \mathbf{S_2} \subseteq \mathbf{S}$, $\exists \gamma \in \mathbf{S_1}$ s.t. β u-def α, β c-def α given $\mathbf{S_2}$, β u-def γ, or β c-def γ given $\mathbf{S_2}$. As a result, the set $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible.

(1.b) If W u-def X, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, then $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S_1} \subseteq \mathbf{S}$ s.t. α u-def W or α c-def W given $\mathbf{S_1}$. As a result, in each case, the set $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible.

(1.c) If X c-def W given $\mathbf{T}$, since by hypothesis $\mathbf{S}$ is admissible, it must be the case that $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S_1} \subseteq \mathbf{S}$, $\exists \gamma \in \mathbf{T}$ s.t. either (i) α u-def X, (ii) α c-def X given $\mathbf{S_1}$, (iii) α u-def γ or (iv) α c-def γ given $\mathbf{S_1}$. Cases $(c.i)$ and $(c.ii)$ are analogous to case (b) and thus, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible. In cases $(c.iii)$ and $(c.iv)$, since $\alpha \in \mathbf{S}$, $\gamma \in \mathbf{T} \subseteq \mathbf{S}$ and $\mathbf{S_1} \subseteq \mathbf{S}$, the set $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible.

(1.d) This case is analogous to case (b) and thus, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible.

(1.e) If W c-def Z given $\mathbf{T} \cup \{X\}$, since by hypothesis $\mathbf{S}$ is admissible, then $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S_1} \subseteq \mathbf{S}$, $\exists \gamma \in \mathbf{T}$ s.t. either (i) α u-def W, (ii) α c-def W given $\mathbf{S_1}$, (iii) α u-def γ, (iv) α c-def γ given $\mathbf{S_1}$, (v) α u-def X or (vi) α c-def X given $\mathbf{S_1}$. Thus, in cases $(e.i)$-$(e.iv)$, the set $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible. In cases $(e.v)$ and $(e.vi)$, similarly to case (a), since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it would be the case that $\exists \beta \in \mathbf{S}$, $\exists \mathbf{S_2} \subseteq \mathbf{S}$, $\exists \lambda \in \mathbf{S_1}$ s.t. β u-def α, β c-def α given $\mathbf{S_2}$, β u-def λ or β c-def λ given $\mathbf{S_2}$; in all cases, the set $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is admissible.

(2) Since $\mathbf{S} \subseteq \mathbf{S'}$ and, by hypothesis, Y is acceptable w.r.t. $\mathbf{S}$, by Proposition 6, Y is acceptable w.r.t. $\mathbf{S'}$.

□

Lemma 2. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF. Every preferred extension of Δ is also a complete extension of Δ, but not vice-versa.*

Proof. Suppose that $\exists \mathbf{S} \subseteq (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ s.t. $\mathbf{S}$ is a preferred extension of Δ but not a complete extension of Δ. Then, by Definition 22, it would be the case that $\exists X \in (\mathbb{A} \cup \mathbb{R} \cup \mathbb{S})$ s.t. X is acceptable w.r.t. $\mathbf{S}$ and $X \notin \mathbf{S}$. By Lemma 1, $\mathbf{S} \cup \{X\}$ is admissible. Therefore, $\mathbf{S}$ would not be a maximal admissible set, contradicting the assumption that $\mathbf{S}$ is a preferred extension of Δ. To show that the reverse does not hold let us consider the ASAF $\Delta = \langle \mathbb{A}, \mathbb{R}, \emptyset \rangle$, with $\mathbb{A} = \{\mathcal{A}, \mathcal{B}\}$ and $\mathbb{R} = \{(\mathcal{A}, \mathcal{B}), (\mathcal{B}, \mathcal{A})\}$. By Definition 22, $\emptyset$ is a complete extension of Δ, whereas the only preferred extensions of Δ are $\{\mathcal{A}\}$ and $\{\mathcal{B}\}$. □

Lemma 3. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF. Every stable extension of Δ is also a preferred extension of Δ, but not vice-versa.*

Proof. It is clear that every stable extension of Δ is a maximal (w.r.t. $\subseteq$) admissible set of Δ, hence a preferred extension of Δ. To show that the reverse does not hold, let us consider the ASAF $\Delta = \langle \mathbb{A}, \mathbb{R}, \emptyset \rangle$, with $\mathbb{A} = \{\mathcal{A}\}$ and $\mathbb{R} = \{(\mathcal{A}, \mathcal{A})\}$. By Definition 22, $\emptyset$ is a preferred extension of Δ but not a stable extension of Δ. □

Corollary 1. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF. Every stable extension of Δ is also a complete extension of Δ, but not vice-versa.*

Proof. It follows directly from Lemmas 2 and 3. □

Lemma 4. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF. If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-complete extension of Δ_{AF}, then $\mathsf{D\text{-}Map}(\mathbf{S})$ is a conflict-free set of Δ.*

Proof. Suppose $\mathbf{S}$ is a D-complete extension of Δ_{AF} and $\mathsf{D\text{-}Map}(\mathbf{S})$ is not conflict-free. Then, it must be the case that $\exists \alpha, X \in \mathsf{D\text{-}Map}(\mathbf{S}), \exists \mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. either (a) α $\mathsf{u\text{-}def}$ X or (b) α $\mathsf{c\text{-}def}$ X given $\mathbf{S}'$. Also, in both cases, by Definition 26, it holds that $\alpha, X \in \mathbf{S}$.

(a) If α $\mathsf{u\text{-}def}$ X, by Definition 14, either α $\mathsf{d\text{-}def}$ X or α $\mathsf{i\text{-}def}$ X. However, by Definition 24, in both cases $(\alpha, X) \in \mathbb{R}_{AF}$. Thus, $\mathbf{S}$ would not be D-conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.

(b) If α $\mathsf{c\text{-}def}$ X given $\mathbf{S}'$, then by Definition 18, it must be the case that either (i) α $\mathsf{e\text{-}def}$ X given $\mathbf{S}'$ or (ii) α $\mathsf{ei\text{-}def}$ X given $\mathbf{S}'$.

- $(b.i)$ If α $\mathsf{e\text{-}def}$ X given $\mathbf{S}'$, by Definition 16 there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots, \mathcal{A}_n = X]$ for X s.t. $\mathbf{S}'$ is the support set of Σ and $\mathsf{trg}(\alpha) = \mathcal{A}_1$. By Definition 26, for every $S_i = (\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbf{S}'$ (with $1 \leq i \leq n$) it holds that $S_i \in \mathbf{S}$ or $S_i^+ \in \mathbf{S}$ or $S_i^- \in \mathbf{S}$. In the last two cases, since by hypothesis $\mathbf{S}$ is a D-complete extension, if $S_i^+ \in \mathbf{S}$ or $S_i^- \in \mathbf{S}$, then it holds that $S_i \in \mathbf{S}$. Also, by Definition 24, for every S_i it holds that $(\mathcal{A}_i, S_i^-) \in \mathbb{R}_{AF}$ and $(S_i^-, S_{i+1}^+) \in \mathbb{R}_{AF}$; additionally, $(\alpha, \mathcal{A}_1) \in \mathbb{R}_{AF}$, $(\alpha, S_1^+) \in \mathbb{R}_{AF}$ and $(S_{n-1}^-, X) \in \mathbb{R}_{AF}$. Therefore, since $\alpha \in \mathbf{S}$, and in particular $S_1 \in \mathbf{S}$, it must be the case that $S_1^+ \notin \mathbf{S}$ and $S_1^- \in \mathbf{S}$ (because by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}). Hence, for every $S_i \in \mathbf{S}'$ it must be the case that $S_i^- \in \mathbf{S}$ and $S_i^+ \notin \mathbf{S}$. In particular, this would imply that $S_{n-1}^- \in \mathbf{S}$, and therefore, $\mathbf{S}$ would not be D-conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.
- $(b.ii)$ If α $\mathsf{ei\text{-}def}$ X given $\mathbf{S}'$, then, by Definition 17, it holds that α $\mathsf{e\text{-}def}$ $\mathsf{src}(X)$ given $\mathbf{S}''$, with $\mathbf{S}'' \subseteq \mathbb{S}$ and $X \in \mathbb{R}$. Then, by Definition 16, there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots, \mathcal{A}_n = \mathsf{src}(X)]$ for $\mathsf{src}(X)$ s.t. $\mathbf{S}'' = \bigcup_{i=1}^{n-1} \{(\mathcal{A}_i, \mathcal{A}_{i+1})\}$, with $(\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbb{S}$ is the support set of Σ, and $\mathsf{trg}(\alpha) = \mathcal{A}_1$. By Definition 24, $(S_{n-1}^-, \mathsf{src}(X)) \in \mathbb{R}_{AF}$ and $(S_{n-1}^-, X) \in \mathbb{R}_{AF}$. Hence, analogously to case $(b.i)$, it would be the case that for every $S_i \in \mathbf{S}''$ it holds that $S_i, S_i^- \in \mathbf{S}$ and $S_i^+ \notin \mathbf{S}$. Thus, $\mathbf{S}$ would not be D-conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.

□

Lemma 5. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF. If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is a complete extension of Δ, then $\mathsf{Map}(\mathbf{S})$ is a D-conflict-free set of Δ_{AF}.*

Proof. Suppose $\mathbf{S}$ is a complete extension of Δ and $\mathsf{Map}(\mathbf{S})$ is not D-conflict-free. Then, by Definition 2, it must be the case that $\exists X, Y \in \mathsf{Map}(\mathbf{S})$ s.t. $(X, Y) \in \mathbb{R}_{AF}$. Then, by Definition 24 this leads to one of the following cases: (a) $X \in \mathbb{R}$ and $Y = \mathsf{trg}(X)$; (b) $X = \mathsf{src}(\alpha)$ and $Y = \alpha^-$, with $\alpha \in \mathbb{S}$; (c) $X = \alpha^-$ and $Y = \mathsf{trg}(\alpha)$, with $\alpha \in \mathbb{S}$; (d) $X = \alpha^-$ and $Y = \beta^+$, with $\alpha, \beta \in \mathbb{S}$ and $\mathsf{trg}(\alpha) = \beta$; (e) $X = \alpha^-$ and $Y = \beta^-$, with $\alpha, \beta \in \mathbb{S}$ and $\mathsf{trg}(\alpha) = \beta$; (f) $X \in \mathbb{R}$ and $Y = \alpha^+$, with $\alpha \in \mathbb{S}$ and $\mathsf{trg}(X) = \alpha$; (g) $X \in \mathbb{R}$ and $Y = \alpha^-$, with $\alpha \in \mathbb{S}$ and $\mathsf{trg}(X) = \alpha$; (h) $X, Y \in \mathbb{R}$ and $\mathsf{trg}(X) = \mathsf{src}(Y)$; (i) $X = \alpha^-$ and $Y = \beta^+$, with $\alpha, \beta \in \mathbb{S}$ and $\mathsf{trg}(\alpha) = \mathsf{src}(\beta)$; (j) $X \in \mathbb{R}$ and $Y = \alpha^+$, with $\alpha \in \mathbb{S}$ and $\mathsf{trg}(X) = \mathsf{src}(\alpha)$; or (k) $X = \alpha^-$ and $Y \in \mathbb{R}$, with $\alpha \in \mathbb{S}$ and $\mathsf{trg}(\alpha) = \mathsf{src}(Y)$.

(a) If $X \in \mathbb{R}$ and $Y = \mathsf{trg}(X)$, then, by Definition 28, $X, Y \in \mathbf{S}$ and, by Definition 12, X $\mathsf{d\text{-}def}$ Y. Therefore, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a complete extension of Δ.

(b) If $Y = \alpha^- \in \mathsf{Map}(\mathbf{S})$, with $\alpha \in \mathbb{S}$, then, by Definition 28, it must be the case that $X = \mathsf{src}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Contradiction.

(c) If $X = \alpha^- \in \mathsf{Map}(\mathbf{S})$, with $\alpha \in \mathbb{S}$, then, by Definition 28, it must be the case that $Y = \mathsf{trg}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Contradiction.

(d) If $X = \alpha^- \in \mathsf{Map}(\mathbf{S})$, then, by Definition 28, it must be the case that $\mathsf{trg}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Moreover, if $\mathsf{trg}(\alpha) = \beta$, with $\beta \in \$$, it must be the case that $Y = \beta^+ \notin \mathsf{Map}(\mathbf{S})$. Contradiction.

(e) Analogous to case (d).

(f) If $X \in \mathbb{R}$ and $Y = \alpha^+$, with $\alpha \in \$$, then, by Definition 28, $X, \alpha \in \mathbf{S}$. Then, if $\mathsf{trg}(X) = \alpha$, by Definition 12, X $\mathsf{d\text{-}def}$ α. Therefore, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a complete extension of Δ.

(g) Analogous to case (f).

(h) $If X, Y \in \mathbb{R}$, then, by Definition 28, $X, Y \in \mathbf{S}$. If $\mathsf{trg}(X) = \mathsf{src}(Y)$, by Definition 13, X $\mathsf{i\text{-}def}$ Y. Therefore, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a complete extension of Δ.

(i) If $X = \alpha^- \in \mathsf{Map}(\mathbf{S})$, with $\alpha \in \$$, then, by Definition 28, it must be the case that $\mathsf{trg}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Moreover, if $\mathsf{trg}(\alpha) = \mathsf{src}(\beta)$, with $\beta \in \$$, it must be the case that $Y = \beta^+ \notin \mathsf{Map}(\mathbf{S})$. Contradiction

(j) If $Y = \alpha^+ \in \mathsf{Map}(\mathbf{S})$, with $\alpha \in \$$, then, by Definition 28, $\mathsf{src}(\alpha) \in \mathbf{S}$. Also, if $X \in \mathbb{R}$, by Definition 28, $X \in \mathbf{S}$. Then, if $\mathsf{trg}(X) = \mathsf{src}(\alpha)$, by Definition 12, X $\mathsf{d\text{-}def}$ $\mathsf{src}(\alpha)$. As a result, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a complete extension of Δ.

(k) If $X = \alpha^- \in \mathsf{Map}(\mathbf{S})$, with $\alpha \in \$$, then, by Definition 28, $\exists \gamma \in \mathbf{S}$, $\exists \mathbf{S'} \subseteq \mathbf{S}$ s.t. γ $\mathsf{u\text{-}def}$ $\mathsf{src}(\alpha)$ or γ $\mathsf{c\text{-}def}$ $\mathsf{src}(\alpha)$ given $\mathbf{S'}$. Also, if $Y \in \mathbb{R}$, by Definition 28, it holds that $Y \in \mathbf{S}$. Hence, if $\mathsf{trg}(\alpha) = \mathsf{src}(Y)$, by Definition 17, γ $\mathsf{ei\text{-}def}$ Y given $\{\alpha\}$ or γ $\mathsf{ei\text{-}def}$ Y given $\mathbf{S'} \cup \{\alpha\}$. As a result, $\mathbf{S}$ would not be conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a complete extension of Δ.

□

Lemma 6. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-complete extension of Δ_{AF} and $X \in \mathbb{A}_{AF}$ is D-acceptable w.r.t. $\mathbf{S}$, then $\mathsf{D\text{-}IMap}(X)$ is acceptable w.r.t. $\mathsf{D\text{-}Map}(\mathbf{S})$.

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \$$ is a complete extension of Δ and $X \in \mathbb{A} \cup \mathbb{R} \cup \$$ is acceptable w.r.t. $\mathbf{S}$, then X is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$.

Proof.

A) Suppose that X is D-acceptable w.r.t. $\mathbf{S}$. To prove that $\mathsf{D\text{-}IMap}(X)$ is acceptable w.r.t. $\mathsf{D\text{-}Map}(\mathbf{S})$ we have to prove: (1) if $\exists \alpha \in \mathbb{R}$ s.t. α $\mathsf{d\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$, then $\exists \beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists \mathbf{S'} \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. β $\mathsf{u\text{-}def}$ α or β $\mathsf{c\text{-}def}$ α given $\mathbf{S'}$; (2) if $\exists \alpha \in \mathbb{R}$ s.t. α $\mathsf{i\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$, then $\exists \beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists \mathbf{S'} \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. β $\mathsf{u\text{-}def}$ α or β $\mathsf{c\text{-}def}$ α given $\mathbf{S'}$; (3) if $\exists \alpha \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \$$ s.t. α $\mathsf{e\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$ given $\mathbf{T}$, then $\exists \beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists \mathbf{S'} \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. β $\mathsf{u\text{-}def}$ α, β $\mathsf{c\text{-}def}$ α given $\mathbf{S'}$, β $\mathsf{u\text{-}def}$ γ or β $\mathsf{c\text{-}def}$ γ given $\mathbf{S'}$, where $\gamma \in \mathbf{T}$; and (4) if $\exists \alpha \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \$$ s.t. α $\mathsf{ei\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$ given $\mathbf{T}$, then $\exists \beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists \mathbf{S'} \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. β $\mathsf{u\text{-}def}$ α, β $\mathsf{c\text{-}def}$ α given $\mathbf{S'}$, β $\mathsf{u\text{-}def}$ γ or β $\mathsf{c\text{-}def}$ γ given $\mathbf{S'}$, where $\gamma \in \mathbf{T}$.

(1) If α $\mathsf{d\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$, then, by Definition 12, $\mathsf{D\text{-}IMap}(X) \in \mathbb{A} \cup \mathbb{R} \cup \$$. If $\mathsf{D\text{-}IMap}(X) \in \mathbb{A} \cup \mathbb{R}$, then, by Definition 25, $X = \mathsf{D\text{-}IMap}(X)$ and thus, by Definition 24, $(\alpha, X) \in \mathbb{R}_{AF}$. If $\mathsf{D\text{-}IMap}(X) \in \$$, then, by Definition 25, either $X = \mathsf{D\text{-}IMap}(X)$, $X = \delta^+$ or $X = \delta^-$, where $\delta \in \$$. In the first case, by Definition 24, $(\alpha, X) \in \mathbb{R}_{AF}$. In the second and third cases, by Definition 24, respectively, $(\alpha, \delta^+) \in \mathbb{R}_{AF}$ and $(\alpha, \delta^-) \in \mathbb{R}_{AF}$. In all cases, since by hypothesis X is D-acceptable w.r.t. $\mathbf{S}$, it must be the case that $\exists Z \in \mathbf{S}$ s.t. $(Z, \alpha) \in \mathbb{R}_{AF}$. Hence, by Definition 24, since $\alpha \in \mathbb{R}$, it must be the case that either: (a) $Z \in \mathbb{R}$ and $\mathsf{trg}(Z) = \alpha$; (b) $Z \in \mathbb{R}$ and $\mathsf{trg}(Z) = \mathsf{src}(\alpha)$; ($c$) $Z = S_1^-$, with $S_1 \in \$$, and $\mathsf{trg}(S_1) = \alpha$; or ($d$) $Z = S_1^-$, with $S_1 \in \$$, and $\mathsf{trg}(S_1) = \mathsf{src}(\alpha)$.

(1.a) and (1.b) If $Z \in \mathbb{R}$, then, by Definition 25, $\mathsf{D\text{-}IMap}(Z) = Z$ and thus, by Definition 26, $Z \in \mathsf{D\text{-}Map}(\mathbf{S})$. In case (1.$a$), since $\mathsf{trg}(Z) = \alpha$, by Definition 12 it holds that Z $\mathsf{d\text{-}def}$ α. Thus, by Definition 14, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = Z$) s.t. β $\mathsf{u\text{-}def}$ α. In case (1.b), since $\mathsf{trg}(Z) = \mathsf{src}(\alpha)$, by Definition 13 it holds that Z $\mathsf{i\text{-}def}$ α. Thus, by Definition 14, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = Z$) s.t. β $\mathsf{u\text{-}def}$ α.

(1.c) and (1.d) If $Z = S_1^-$, with $S_1 \in \mathbb{S}$, by Definition 25, $\mathsf{D\text{-}IMap}(Z) = S_1$ and thus, by Definition 26, it holds that $S_1 \in \mathsf{D\text{-}Map}(\mathbf{S})$. Also, by Definition 24, $(\mathsf{src}(S_1), S_1^-) \in \mathbb{R}_{AF}$. Then, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists W \in S$ s.t. $(W, \mathsf{src}(S_1)) \in \mathbb{R}_{AF}$. Therefore, by Definition 24 we have the following cases: (1.$c.i$) $W \in \mathbb{R}$ and $\mathsf{trg}(W) = \mathsf{src}(S_1)$; (1.$c.ii$) $W = S_2^-$, with $S_2 \in \mathbb{S}$, and $\mathsf{trg}(S_2) = \mathsf{src}(S_1)$; (1.$d.i$) $W \in \mathbb{R}$ and $\mathsf{trg}(W) = \mathsf{src}(S_1)$; or (1.$d.ii$) $W = S_2^-$, with $S_2 \in \mathbb{S}$, and $\mathsf{trg}(S_2) = \mathsf{src}(S_1)$.

(1.$c.i$) If $W \in \mathbb{R}$, then, by Definition 25, $\mathsf{D\text{-}IMap}(W) = W$ and thus, by Definition 26 it holds that $W \in \mathsf{D\text{-}Map}(\mathbf{S})$. Since $W, S_1 \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\mathsf{trg}(W) = \mathsf{src}(S_1)$ and $\mathsf{trg}(S_1) = \alpha$, by Definition 16, W $\mathsf{e\text{-}def}$ α given $\{S_1\}$. Therefore, by Definition 18, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists\mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = W$ and $\mathbf{S}' = \{S_1\}$) s.t. β $\mathsf{c\text{-}def}$ α given $\mathbf{S}'$.

(1.$c.ii$) If $W = S_2^-$, with $S_2 \in \mathbb{S}$, and $\mathsf{trg}(S_2) = \mathsf{src}(S_1)$, then, by Definition 25, $\mathsf{D\text{-}IMap}(W) = S_2$ and thus, by Definition 26, it holds that $S_2 \in \mathsf{D\text{-}Map}(\mathbf{S})$. By Definition 24, $(\mathsf{src}(S_2), S_2^-) \in \mathbb{R}_{AF}$. Then, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists H \in \mathbf{S}$ s.t. $(H, \mathsf{src}(S_2)) \in \mathbb{R}_{AF}$. This situation is similar to case (1.c), leading to cases analogous to (1.$c.i$) and (1.$c.ii$); hence, leading to the consideration of a chain of support ending with S_1, where each one of these supports belongs to $\mathsf{D\text{-}Map}(\mathbf{S})$. Moreover, by Definition 11, this chain of support is finite. Let S_n be the first support on the chain of the form $\mathsf{src}(S_n) \overset{S_n}{\Longrightarrow} \mathsf{src}(S_{n-1}) \overset{S_{n-1}}{\Longrightarrow} \ldots \mathsf{src}(S_2) \overset{S_2}{\Longrightarrow} \mathsf{src}(S_1) \overset{S_1}{\Longrightarrow} \alpha$. Following the same reasoning as in case (1.c), $\exists V = S_n^- \in \mathbf{S}$ s.t. $\mathsf{D\text{-}IMap}(V) = S_n$ and $(\mathsf{src}(S_n), S_n^-) \in \mathbb{R}_{AF}$. Hence, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists G \in \mathbf{S}$ s.t. $(G, \mathsf{src}(S_n)) \in \mathbb{R}_{AF}$. Given that S_n is the first link on the chain of support, by Definition 24 it must be the case that $G \in \mathbb{R}$ and $\mathsf{trg}(G) = \mathsf{src}(S_n)$. Also, by Definition 25, $\mathsf{D\text{-}IMap}(G) = G$ and thus, by Definition 26 it holds that $G \in \mathsf{D\text{-}Map}(\mathbf{S})$. Therefore, since $G \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\{S_n, \ldots, S_1\} \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$, $\mathsf{trg}(G) = \mathsf{src}(S_n)$, $\mathsf{trg}(S_i) = \mathsf{src}(S_{i-1})$ (with $2 \leq i \leq n$) and $\mathsf{trg}(S_1) = \alpha$, by Definition 16, G $\mathsf{e\text{-}def}$ α given $\{S_n, \ldots, S_1\}$. As a result, by Definition 18, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists\mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = G$ and $\mathbf{S}' = \{S_n, \ldots, S_1\}$) s.t. β $\mathsf{c\text{-}def}$ α given $\mathbf{S}'$.

(1.$d.i$) If $W \in \mathbb{R}$, then, by Definition 25, $\mathsf{D\text{-}IMap}(W) = W$ and thus, by Definition 26 it holds that $W \in \mathsf{D\text{-}Map}(\mathbf{S})$. Since $W, S_1 \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\mathsf{trg}(W) = \mathsf{src}(S_1)$ and $\mathsf{trg}(S_1) = \mathsf{src}(\alpha)$, by Definition 17, W $\mathsf{ei\text{-}def}$ α given $\{S_1\}$. Therefore, by Definition 18, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists\mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = W$ and $\mathbf{S}' = \{S_1\}$) s.t. β $\mathsf{c\text{-}def}$ α given $\mathbf{S}'$.

(1.$d.ii$) The proof in this case is analogous to case (1.$c.ii$), where the only difference is that $\mathsf{trg}(S_1) = \mathsf{src}(\alpha)$. Therefore, by Definition 17, G $\mathsf{ei\text{-}def}$ α given $\{S_n, \ldots, S_1\}$. As a result, by Definition 18, $\exists\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\exists\mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ (with $\beta = G$ and $\mathbf{S}' = \{S_n, \ldots, S_1\}$) s.t. β $\mathsf{c\text{-}def}$ α given $\mathbf{S}'$.

(2) If α $\mathsf{i\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$, then, by Definition 13, $\mathsf{D\text{-}IMap}(X) \in \mathbb{R}$ and thus, by Definition 25, it holds that $\mathsf{D\text{-}IMap}(X) = X$. Furthermore, by Definition 24, $(\alpha, X) \in \mathbb{R}_{AF}$. Hence, since by hypothesis X is D-acceptable w.r.t. $\mathbf{S}$, it must be the case that $\exists Z \in \mathbf{S}$ s.t. $(Z, \alpha) \in \mathbb{R}_{AF}$. The rest of the proof is now analogous to case (1).

(3) If α $\mathsf{e\text{-}def}$ $\mathsf{D\text{-}IMap}(X)$ given $\mathbf{T}$, then, by Definition 16, $\mathsf{D\text{-}IMap}(X) \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$. If $\mathsf{D\text{-}IMap}(X) \in \mathbb{A} \cup \mathbb{R}$, by Definition 25 it holds that $X = \mathsf{D\text{-}IMap}(X)$. Also, by Definition 24, $\exists S_1 \in \mathbf{T}$ s.t. $(S_1^-, X) \in \mathbb{R}_{AF}$. If $\mathsf{D\text{-}IMap}(X) \in \mathbb{S}$, by Definition 25 it holds that either $X = \mathsf{D\text{-}IMap}(X)$, $X = \delta^+$ or $X = \delta^-$, where $\delta \in \mathbb{S}$. In the first case, by Definition 24, $\exists S_1 \in \mathbf{T}$ s.t. $(S_1^-, X) \in \mathbb{R}_{AF}$. In the second and third cases, by Definition 24, $\exists S_1 \in \mathbf{T}$ s.t., respectively, $(S_1^-, \delta^+) \in \mathbb{R}_{AF}$

and $(S_1^-, \delta^-) \in \mathbb{R}_{AF}$. In all cases, since by hypothesis X is D-acceptable w.r.t. $\mathbf{S}$, it must be the case that $\exists Z \in \mathbf{S}$ s.t. $(Z, S_1^-) \in \mathbb{R}_{AF}$. Therefore, by Definition 24, we have to consider the following cases: $(3.a)$ $Z = \mathsf{src}(S_1)$; $(3.b)$ $Z \in \mathbb{R}$ and $\mathsf{trg}(Z) = S_1$; or $(3.c)$ $Z = Y^-$, with $Y \in \mathbb{S}$, and $\mathsf{trg}(Y) = S_1$.

$(3.a)$ If $Z = \mathsf{src}(S_1)$, then we have that either $(3.a.i)$ S_1 is the only support in $\mathbf{T}$; or $(3.a.ii)$ $\exists S_2 \in \mathbf{T}$ s.t. $S_1 \neq S_2$ and $\mathsf{trg}(S_2) = \mathsf{src}(S_1)$.

$(3.a.i)$ In this case, by Definition 24, $(\alpha, Z) \in \mathbb{R}_{AF}$. Hence, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, \alpha) \in \mathbb{R}_{AF}$. The rest of the proof is now analogous to case (1).

$(3.a.ii)$ In this case, by Definition 24, $(S_2^-, \mathsf{src}(S_1)) \in \mathbb{R}_{AF}$. Hence, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, S_2^-) \in \mathbb{R}_{AF}$. The rest of the proof follows by considering cases analogous to $(3.a)$, $(3.b)$ and $(3.c)$.

$(3.b)$ If $Z \in \mathbb{R}$, then, by Definition 25, D-IMap$(Z) = Z$ and thus, by Definition 26 it holds that $Z \in$ D-Map$(\mathbf{S})$. Therefore, since $\mathsf{trg}(Z) = S_1$, by Definition 12, Z d-def S_1. As a result, by Definition 14, $\exists \beta \in$ D-Map$(\mathbf{S})$ (with $\beta = Z$) s.t. β u-def γ, with $\gamma \in \mathbf{T}$ and $\gamma = S_1$.

$(3.c)$ If $Z = Y^-$, then, by Definition 25, D-IMap$(Z) = Y$ and thus, by Definition 26 it holds that $Y \in$ D-Map$(\mathbf{S})$. By Definition 24, $(\mathsf{src}(Y), Y^-) \in \mathbb{R}_{AF}$. Then, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, \mathsf{src}(Y)) \in \mathbb{R}_{AF}$. The rest of the proof is now analogous to case $(1.c)$.

(4) If α ei-def D-IMap(X) given $\mathbf{T}$, then, by Definition 17, D-IMap$(X) \in \mathbb{R}$. Thus, by Definition 25, D-IMap$(X) = X$ and, by Definition 26 it holds that $X \in$ D-Map$(\mathbf{S})$. By Definition 24, $\exists S_1 \in \mathbf{T}$ s.t. $(S_1^-, X) \in \mathbb{R}_{AF}$. Therefore, since by hypothesis X is D-acceptable w.r.t. $\mathbf{S}$, it must be the case that $\exists Z \in \mathbf{S}$ s.t. $(Z, S_1^-) \in \mathbb{R}_{AF}$. The rest of the proof is now analogous to case (3).

Finally, by (1)-(4), if X is D-acceptable w.r.t. $\mathbf{S}$, then D-IMap(X) is acceptable w.r.t. D-Map$(\mathbf{S})$.

$B)$ If X is acceptable w.r.t. $\mathbf{S}$, then by Definition 22 and Lemma 1 it holds that $X \in \mathbf{S}$. Suppose by contradiction that X is acceptable w.r.t. $\mathbf{S}$ and X is not D-acceptable w.r.t. Map$(\mathbf{S})$. Then, by Definition 2, it must be the case that $\exists Y \in \mathbb{A}_{AF}$ s.t. $(Y, X) \in \mathbb{R}_{AF}$ and $\nexists Z \in$ Map$(\mathbf{S})$ s.t. $(Z, Y) \in \mathbb{R}_{AF}$. If $(Y, X) \in \mathbb{R}_{AF}$, since $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$, by Definition 24 it must be the case that either: (a) $Y \in \mathbb{R}$ and $X = \mathsf{trg}(Y)$; (b) $Y = \alpha^-$ and $X = \mathsf{trg}(\alpha)$, with $\alpha \in \mathbb{S}$; (c) $X, Y \in \mathbb{R}$ and $\mathsf{trg}(Y) = \mathsf{src}(X)$; or (d) $Y = \alpha^-$ and $X \in \mathbb{R}$, with $\alpha \in \mathbb{S}$ and $\mathsf{trg}(\alpha) = \mathsf{src}(X)$.

(a) If $Y \in \mathbb{R}$ and $X = \mathsf{trg}(Y)$, then by Definition 12, Y d-def X. Therefore, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it must be the case that either: $(a.i)$ $\exists \alpha \in \mathbf{S}$ s.t. α u-def Y; or $(a.ii)$ $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α c-def Y given $\mathbf{S}'$.

$(a.i)$ If α u-def Y then by Definition 14 either α d-def Y or α i-def Y. In both cases, by Definition 24, it holds that $(\alpha, Y) \in \mathbb{R}_{AF}$. Moreover, by Definition 28, $\alpha \in$ Map$(\mathbf{S})$. As a result, $\exists Z \in$ Map$(\mathbf{S})$ (with $Z = \alpha$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$. Contradiction.

$(a.ii)$ If α c-def Y given $\mathbf{S}'$, then by Definition 18 either α e-def Y given $\mathbf{S}'$ or α ei-def Y given $\mathbf{S}'$. In both cases, by Definitions 16 and 17, there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots \mathcal{A}_n]$ s.t. $\mathsf{trg}(\alpha) = \mathcal{A}_1$ and $\mathbf{S}'$ is the support set of Σ. Moreover, if α e-def Y given $\mathbf{S}'$, $\mathcal{A}_n = Y$; otherwise, if α ei-def Y given $\mathbf{S}'$, $\mathcal{A}_n = \mathsf{src}(Y)$. Then, by Definition 24, in both cases, $(\alpha, \mathcal{A}_1) \in \mathbb{R}_{AF}$ and $\forall S_i = (\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbf{S}'$ $(1 \leq i \leq n-1)$ it holds that $(\mathcal{A}_i, S_i^-) \in \mathbb{R}_{AF}$, $(S_i^-, S_{i+1}^+) \in \mathbb{R}_{AF}$ and $(S_i^-, \mathcal{A}_{i+1}) \in \mathbb{R}_{AF}$. Also, since $\alpha \in \mathbf{S}$, by Definition 28 it holds that $\alpha \in$ Map$(\mathbf{S})$. Furthermore, for every $S_i \in \mathbf{S}' \subseteq \mathbf{S}$ it holds that $S_i \in$ Map$(\mathbf{S})$. Then, since by hypothesis $\mathbf{S}$ is a complete extension of Δ and $\alpha \in$ Map$(\mathbf{S})$, $\mathcal{A}_1 \notin$ Map$(\mathbf{S})$. As a result, $S_1^- \in$ Map$(\mathbf{S})$ and $S_1^+ \notin$ Map$(\mathbf{S})$. In addition, since for every $S_i \in \mathbf{S}'$ it holds that $\mathsf{trg}(S_i) = \mathsf{src}(S_{i+1})$, by extension we have that $\mathcal{A}_j \notin$ Map$(\mathbf{S})$ $(2 \leq j \leq n)$, $S_k^+ \notin$ Map$(\mathbf{S})$ and $S_k^- \in$ Map$(\mathbf{S})$ $(2 \leq k \leq n-1)$; in particular, $S_{n-1}^- \in$ Map$(\mathbf{S})$. Finally, if α e-def Y given $\mathbf{S}'$, by Definition 24 it holds that $(S_{n-1}^-, Y) \in \mathbb{R}_{AF}$; otherwise, if α ei-def Y given $\mathbf{S}'$, by Definition 24 it also holds that $(S_{n-1}^-, Y) \in \mathbb{R}_{AF}$ (because the AFN associated with Δ is such that S_{n-1}^- attacks $\mathsf{src}(Y)$ and $\mathsf{src}(Y)$ supports Y). Therefore, $\exists Z \in$ Map$(\mathbf{S})$ (with $Z = S_{n-1}^-$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$. Contradiction.

(b) If $Y = \alpha^-$, $X = \mathsf{trg}(\alpha)$ with $\alpha \in \mathbb{S}$, and X is not D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$, then it must be the case that $\mathsf{src}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Therefore, by Definition 28, it would be the case that $\mathsf{src}(\alpha) \notin \mathbf{S}$. Hence, since $X \in \mathbf{S}$ and by hypothesis $\mathbf{S}$ is a complete extension of Δ, it must be the case that $\alpha \notin \mathbf{S}$. If $\mathsf{src}(\alpha) \notin \mathbf{S}$, by Lemma 1, it must be the case that $\mathsf{src}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$. Thus, by Definition 20, $\exists \gamma \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \mathbb{S}$ s.t. γ u-def $\mathsf{src}(\alpha)$ or γ c-def $\mathsf{src}(\alpha)$ given $\mathbf{T}$, and $\nexists \beta \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$ s.t. β u-def γ, β u-def δ, β c-def γ given $\mathbf{S}'$ or β c-def δ given $\mathbf{S}'$, where $\delta \in \mathbf{T}$. In particular, given such $\gamma \in \mathbb{R}$ and $\mathbf{T} \in \mathbb{S}$, by Definition 18 it would be the case that γ c-def X given $\{\alpha\}$ or γ c-def X given $\mathbf{T} \cup \{\alpha\}$. Therefore, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it must be the case that either: (b.i) $\exists \beta \in \mathbf{S}$ s.t. β u-def α; or (b.ii) $\exists \beta \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. β c-def α given $\mathbf{S}'$.

(b.i) If β u-def α, then by Definition 14 it must be the case that β d-def α and thus, $\mathsf{trg}(\beta) = \alpha$. Moreover, by Definition 24, $(\beta, \alpha) \in \mathbb{R}_{AF}$ and $(\beta, \alpha^-) \in \mathbb{R}_{AF}$. By Definition 28 we have $\beta \in \mathsf{Map}(\mathbf{S})$. As a result, $\exists Z \in \mathsf{Map}(\mathbf{S})$ (with $Z = \beta$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$. Contradiction.

(b.ii) If β c-def α given $\mathbf{S}'$, then by Definition 18 it must be the case that β e-def α given $\mathbf{S}'$. Then, by Definition 16, there exists a support sequence $\Sigma = [\mathcal{A}_1, \dots \mathcal{A}_n = \alpha]$ for α s.t. $\mathsf{trg}(\beta) = \mathcal{A}_1$ and $\mathbf{S}'$ is the support set of Σ. By Definition 24, $(\beta, \mathcal{A}_1) \in \mathbb{R}_{AF}$ and $\forall S_i = (\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbf{S}'$ $(1 \leq i \leq n-1)$ it holds that $(S_i^-, S_{i+1}^+) \in \mathbb{R}_{AF}$, $(\mathcal{A}_i, S_i^-) \in \mathbb{R}_{AF}$ and $(S_{n-1}^-, \alpha) \in \mathbb{R}_{AF}$. Also, by Definition 28 it holds that $\beta \in \mathsf{Map}(\mathbf{S})$ and thus, $\forall S_i \in \mathbf{S}'$ it holds that $S_i^- \in \mathsf{Map}(\mathbf{S})$; in particular, $S_{n-1}^- \in \mathsf{Map}(\mathbf{S})$. By Definition 24, $(S_{n-1}^-, \alpha^-) \in \mathbb{R}_{AF}$. As a result, $\exists Z \in \mathsf{Map}(\mathbf{S})$ (with $Z = S_{n-1}^-$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$. Contradiction.

(c) If $X, Y \in \mathbb{R}$ and $\mathsf{trg}(Y) = \mathsf{src}(X)$, by Definition 17 it holds that Y i-def X. Therefore, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it must be the case that either: (c.i) $\exists \alpha \in \mathbf{S}$ s.t. α u-def Y; or (c.ii) $\exists \alpha \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α c-def Y given $\mathbf{S}'$. The proofs in these cases are analogous to cases (a.i) and (a.ii).

(d) If $Y = \alpha^-$ and $\mathsf{trg}(\alpha) = \mathsf{src}(X)$, with $\alpha \in \mathbb{S}$ and $X \in \mathbb{R}$, and X is not D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$, then it must be the case that $\mathsf{src}(\alpha) \notin \mathsf{Map}(\mathbf{S})$. Thus, by Definition 28, $\mathsf{src}(\alpha) \notin \mathbf{S}$. In addition, since $X \in \mathbf{S}$, by Lemma 1 it holds that $\mathsf{src}(X) = \mathsf{trg}(\alpha) \in \mathbf{S}$. As a result, since $\mathsf{trg}(\alpha) \in \mathbf{S}$, $\mathsf{src}(\alpha) \notin \mathbf{S}$ and, by hypothesis $\mathbf{S}$ is a complete extension of Δ, it must be the case that $\alpha \notin \mathbf{S}$. Moreover, if $\mathsf{src}(\alpha) \notin \mathbf{S}$, by Lemma 1, it must be the case that $\mathsf{src}(\alpha)$ is not acceptable w.r.t. $\mathbf{S}$. Thus, by Definition 20, $\exists \gamma \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \mathbb{S}$ s.t. γ u-def $\mathsf{src}(\alpha)$ or γ c-def α given $\mathbf{T}$ and $\nexists \beta \in \mathbf{S}$, $\nexists \mathbf{S}' \subseteq \mathbf{S}$ s.t. β u-def γ, β u-def δ, β c-def γ given $\mathbf{S}'$ or β c-def δ given $\mathbf{S}'$, where $\delta \in \mathbf{T}$. In particular, given such $\gamma \in \mathbb{R}$ and $\mathbf{T} \in \mathbb{S}$, by Definition 18 it would be the case that γ c-def X given $\{\alpha\}$ or γ c-def X given $\mathbf{T} \cup \{\alpha\}$. Therefore, since by hypothesis X is acceptable w.r.t. $\mathbf{S}$, it must be the case that either (d.i) $\exists \beta \in \mathbf{S}$ s.t. β u-def α or (d.ii) $\exists \beta \in \mathbf{S}$, $\exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. β c-def α given $\mathbf{S}'$. The proofs in these cases are analogous to cases (b.i) and (b.ii).

□

Lemma 7. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF, $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF, $X \in \mathbb{S}$ and $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ a complete extension of Δ.*

A) If X and $\mathsf{src}(X)$ are acceptable w.r.t. $\mathbf{S}$, then X^+ is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$.

B) If X is acceptable w.r.t. $\mathbf{S}$ and $\exists \alpha \in \mathbb{R}$, $\exists \mathbf{T} \subseteq \mathbb{S}$ s.t. α is acceptable w.r.t. $\mathbf{S}$, $\forall S_i \in \mathbf{T}$ it holds that S_i is acceptable w.r.t. $\mathbf{S}$, and α u-def $\mathsf{src}(X)$ or α c-def $\mathsf{src}(X)$ given $\mathbf{T}$, then X^- is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$.

Proof.

A) If X and $\mathsf{src}(X)$ are acceptable w.r.t. $\mathbf{S}$, by Lemma 6 it holds that X and $\mathsf{src}(X)$ are D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$. By Definition 24, if $\exists Y \in \mathbb{A}_{AF}$ s.t. $(Y, X^+) \in \mathbb{R}_{AF}$, it must be the case that $(Y, X) \in \mathbb{R}_{AF}$ or $(Y, \mathsf{src}(X)) \in \mathbb{R}_{AF}$. Then, since X and $\mathsf{src}(X)$ are D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$, it must be the case that $\exists Z \in \mathsf{Map}(\mathbf{S})$ s.t. $(Z, Y) \in \mathbb{R}_{AF}$. As a result, X^+ is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$.

B) To prove that X^- is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$, we need to prove that if $\exists Y \in \mathbb{A}_{AF}$ s.t. $(Y, X^-) \in \mathbb{R}_{AF}$, then $\exists Z \in \mathsf{Map}(\mathbf{S})$ s.t. $(Z, Y) \in \mathbb{R}_{AF}$. By Definition 24, if $\exists Y \in \mathbb{A}_{AF}$ s.t. $(Y, X^-) \in \mathbb{R}_{AF}$, then it must be the case that either: (a) $(Y, X) \in \mathbb{R}_{AF}$; or (b) $Y = \mathsf{src}(X)$.

   (a) If X is acceptable w.r.t. $\mathbf{S}$, by Lemma 6 it holds that X is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$. Thus, $\forall Y \in \mathbb{A}_{AF}$ s.t. $(Y, X) \in \mathbb{R}_{AF}$, $\exists Z \in \mathbb{A}_{AF}$ s.t. $(Z, Y) \in \mathbb{R}_{AF}$. As a result, it holds that X^- is D-acceptable w.r.t. $\mathsf{Map}(\mathbf{S})$.

   (b) If $Y = \mathsf{src}(X)$, we need to prove that $\exists Z \in \mathsf{Map}(\mathbf{S})$ s.t. $(Z, Y) \in \mathbb{R}_{AF}$. Let us now consider the following cases: $(b.i)$ α $\mathsf{u\text{-}def}$ $\mathsf{src}(X)$; or $(b.ii)$ α $\mathsf{c\text{-}def}$ $\mathsf{src}(X)$ given $\mathbf{T}$.

      $(b.i)$ If α $\mathsf{u\text{-}def}$ $\mathsf{src}(X)$ then, by Definition 14, it must be the case that α $\mathsf{d\text{-}def}$ $\mathsf{src}(X)$ and thus, by Definition 24 it holds that $(\alpha, \mathsf{src}(X)) \in \mathbb{R}_{AF}$. Also, if α is acceptable w.r.t. $\mathbf{S}$, then by Definition 22 and Lemma 1 it holds that $\alpha \in \mathbf{S}$. Therefore, by Definition 28, $\alpha \in \mathsf{Map}(\mathbf{S})$. As a result, $\exists Z \in \mathsf{Map}(\mathbf{S})$ (with $Z = \alpha$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$.

      $(b.ii)$ If α $\mathsf{c\text{-}def}$ $\mathsf{src}(X)$ given $\mathbf{T}$, then, by Definition 18 it must be the case that α $\mathsf{e\text{-}def}$ $\mathsf{src}(X)$ given $\mathbf{T}$ and thus, there exists a support sequence $\Sigma = [\mathcal{A}_1, \ldots, \mathcal{A}_n = \mathsf{src}(X)]$ for $\mathsf{src}(X)$ s.t. $\mathsf{trg}(\alpha) = \mathcal{A}_1$ and $\mathbf{T}$ is the support set of Σ. By Definition 24, $(\alpha, A_1) \in \mathbb{R}_{AF}$ and $\forall S_i = (\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbf{T}$ $(1 \leq i \leq n{-}1)$ it holds that $(S_i^-, S_{i+1}^+) \in \mathbb{R}_{AF}$ and $(\mathcal{A}_i, S_i^-) \in \mathbb{R}_{AF}$. In addition, by Definition 24, $(S_{n-1}^-, \mathsf{src}(X)) \in \mathbb{R}_{AF}$. Since by hypothesis α is acceptable w.r.t. $\mathbf{S}$, by Definition 22 and Lemma 1 it holds that $\alpha \in \mathbf{S}$. Therefore, by Definition 28, $\alpha \in \mathsf{Map}(\mathbf{S})$. Similarly, every $S_i \in \mathbf{T}$ is acceptable w.r.t. $\mathbf{S}$ and thus, $S_i \in \mathsf{Map}(\mathbf{S})$; in particular, $S_{n-1} \in \mathsf{Map}(\mathbf{S})$. Then, by Definition 28, it holds that $\forall S_i \in \mathbf{T}$: $S_i^- \in \mathsf{Map}(\mathbf{S})$; in particular, $S_{n-1}^- \in \mathsf{Map}(\mathbf{S})$. As a result, $\exists Z \in \mathsf{Map}(\mathbf{S})$ (with $Z = S_{n-1}^-$) s.t. $(Z, Y) \in \mathbb{R}_{AF}$.

□

Lemma 8. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) *If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-complete extension of Δ_{AF}, then $\mathsf{D\text{-}Map}(\mathbf{S})$ is admissible.*

B) *If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is a complete extension of Δ, then $\mathsf{Map}(\mathbf{S})$ is D-admissible.*

Proof.

A) It follows directly from Definition 21 and Lemmas 4 and 6.

B) It follows directly from Definition 2 and Lemmas 5, 6 and 7.

□

Theorem 1. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) *If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-complete extension of Δ_{AF}, then $\mathsf{D\text{-}Map}(\mathbf{S})$ is a complete extension of Δ.*

B) *If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is a complete extension of Δ, then $\mathsf{Map}(\mathbf{S})$ is a D-complete extension of Δ_{AF}.*

Proof.

A) It follows directly from Definition 22 and Lemmas 6 and 8.

B) It follows directly from Definition 3 and Lemmas 6, 7 and 8.

□

Theorem 2. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) *If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-preferred extension of Δ_{AF}, then $\mathsf{D\text{-}Map}(\mathbf{S})$ is a preferred extension of Δ.*

B) *If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is a preferred extension of Δ, then $\mathsf{Map}(\mathbf{S})$ is a D-preferred extension of Δ_{AF}.*

Proof.

A) It follows directly from Definition 22, [30, Theorem 25] and Lemma 8.

B) It follows directly from Definition 3 and Lemmas 2 and 8.

□

Theorem 3. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \$ \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) *If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-stable extension of Δ_{AF}, then* D-Map($\mathbf{S}$) *is a stable extension of* Δ.

B) *If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \$$ is a stable extension of Δ, then* Map($\mathbf{S}$) *is a D-stable extension of* Δ_{AF}.

Proof.

A) Suppose by contradiction that $\mathbf{S}$ is a D-stable extension of Δ_{AF} and D-Map($\mathbf{S}$) is not a stable extension of Δ. By [30, Lemma 15 and Theorem 25], $\mathbf{S}$ is a D-complete extension of Δ_{AF}. Thus, by Theorem 1, D-Map($\mathbf{S}$) is a complete extension of Δ. Hence, by Definition 22, D-Map($\mathbf{S}$) is conflict-free. As a result, it should be the case that $\exists X \in \mathbb{A}_{AF}$ s.t. $X \notin \mathbf{S}$ and $\exists Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$ and D-IMap(X) $\notin$ D-Map($\mathbf{S}$), but $\nexists \alpha \in$ D-Map($\mathbf{S}$), $\nexists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. α u-def D-IMap(X) or α c-def D-IMap(X) given $\mathbf{S}'$. Now we have to consider the following cases: (1) $X \in \mathbb{R}$; (2) $X \in \mathbb{A}$; (3) $X = \delta^+$, with $\delta \in \$$; (4) $X = \delta^-$, with $\delta \in \$$; or (5) $X \in \$$.

(1) If $X \in \mathbb{R}$, then by Definition 25 it holds that D-IMap(X) $= X$. In addition, given $Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$, by Definition 24 it must be the case that either: (1.*a*) $Z \in \mathbb{R}_{AF}$ and $X =$ trg(Z); (1.*b*) $Z = Y^-$, with $Y \in \$$, and trg(Y) $= X$; (1.*c*) $Z \in \mathbb{R}$ and trg(Z) $=$ src(X); or (1.*d*) $Z = Y^-$, with $Y \in \$$, and trg(Y) $=$ src(X).

(1.*a*) If $Z \in \mathbb{R}$ and trg(Z) $= X$ then, by Definition 12 it holds that Z d-def X. By Definition 25, D-IMap(Z) $= Z$ and D-IMap(X) $= X$. Then, by Definition 26, it holds that $Z \in$ D-Map($\mathbf{S}$). Thus, $\exists \alpha \in$ D-Map($\mathbf{S}$) (with $\alpha = Z$) s.t. α u-def D-IMap(X). Contradiction.

(1.*b*) This case is analogous to case (*A*.1.*c*) in Lemma 6, leading to the fact that $\exists \beta \in$ D-Map($\mathbf{S}$), $\exists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. β c-def D-IMap(X) given $\mathbf{S}'$. Thus, $\exists \alpha \in$ D-Map($\mathbf{S}$) (with $\alpha = \beta$), $\exists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. α c-def D-IMap(X) given $\mathbf{S}'$. Contradiction.

(1.*c*) If $Z \in \mathbb{R}$ and trg(Z) $=$ src(X), then, by Definition 13, it holds that Z i-def X. Also, by Definition 25, D-IMap(Z) $= Z$ and, by Definition 26, $Z \in$ D-Map($\mathbf{S}$). Thus, $\exists \alpha \in$ D-Map($\mathbf{S}$) (with $\alpha = Z$) s.t. α u-def D-IMap(X). Contradiction.

(1.*d*) The proof in this case is analogous to case (*A*.1.*d*) in Lemma 6, leading to the fact that $\exists \beta \in$ D-Map($\mathbf{S}$), $\exists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. β c-def D-IMap(X) given $\mathbf{S}'$. Thus, $\exists \alpha \in$ D-Map($\mathbf{S}$) (with $\alpha = \beta$), $\exists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. α c-def D-IMap(X) given $\mathbf{S}'$. Contradiction.

(2) If $X \in \mathbb{A}$, then, by Definition 25, it holds that D-IMap(X) $= X$. In addition, given $Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$, by Definition 24 it must be the case that either: (2.*a*) $Z \in \mathbb{R}$ and $X =$ trg(Z); or (2.*b*) $Z = Y^-$, with $Y \in \$$, and trg(Y) $= X$.

(2.*a*) The proof in this case is analogous to case (1.*a*).

(2.*b*) The proof in this case is analogous to case (1.*b*).

(3) If $X = \delta^+$, with $\delta \in \$$, then, by Definition 25, it holds that D-IMap(X) $= \delta$. Also, if D-IMap(X) $\notin$ D-Map($\mathbf{S}$), by Definition 26 it must be the case that $\delta^- \notin \mathbf{S}$. In addition, given $Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$, by Definition 24 it must be the case that either: (3.*a*) $Z = Y^-$, with $Y \in \$$, and trg(Y) $=$ D-IMap(X); (3.*b*) $Z \in \mathbb{R}$ and trg(Z) $=$ D-IMap(X); (3.*c*) $Z \in \mathbb{R}$ and trg(Z) $=$ src(D-IMap(X)); or (3.*d*) $Z = Y^-$, with $Y \in \$$, and trg(Y) $=$ src(D-IMap(X)).

(3.*a*) By Definition 25, D-IMap(Z) $= Y$ and thus, by Definition 26, $Y \in$ D-Map($\mathbf{S}$). Also, by Definition 24, it holds that (src(Y), Z) $\in \mathbb{R}_{AF}$. Then, since by hypothesis $\mathbf{S}$ is a D-stable (and thus, a D-complete) extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. (W, src(Y)) $\in \mathbb{R}_{AF}$. Hence, by Definition 24 we have the following cases: (3.*a*.*i*) $W \in \mathbb{R}$ and trg(W) $=$ src(Y); or (3.*a*.*ii*) $W = \varepsilon^-$, with $\varepsilon \in \$$, and trg(ε) $=$ src(Y).

(3.*a*.*i*) In this case, by Definitions 25 and 26, it holds that $W \in$ D-Map($\mathbf{S}$). Then, by Definition 16, W e-def D-IMap(X) given $\{Y\}$. Therefore, $\exists \alpha \in$ D-Map($\mathbf{S}$), $\exists \mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) (with $\alpha = W$ and $\mathbf{S}' = \{Y\}$) s.t. α c-def D-IMap(X) given $\mathbf{S}'$. Contradiction.

(3.$a.ii$) The proof in this case is analogous to case (A.1.$c.ii$) in Lemma 6, leading to the fact that $\exists\beta \in$ D-Map($\mathbf{S}$), $\exists\mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. β c-def D-IMap(X) given $\mathbf{S}'$. Therefore, $\exists\alpha \in$ D-Map($\mathbf{S}$) (with $\alpha = \beta$), $\exists\mathbf{S}' \subseteq$ D-Map($\mathbf{S}$) s.t. α c-def D-IMap(X) given $\mathbf{S}'$. Contradiction.

(3.b) The proof in this case is analogous to case (1.a).

(3.c) If $\delta^- \notin \mathbf{S}$, then, since by hypothesis $\mathbf{S}$ is a D-stable extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, \delta^-) \in \mathbb{R}_{AF}$. Then, by Definition 24 we have the following cases: (3.$c.i$) $W =$ src(D-IMap(X)); (3.$c.ii$) $W = \varepsilon^-$, with $\varepsilon \in \mathbb{S}$, and trg($\varepsilon$) = D-IMap($X$); or (3.$c.iii$) $W \in \mathbb{R}$ and trg(W) = D-IMap(X).

(3.$c.i$) Suppose $W =$ src(D-IMap(X)). By Definition 24 it holds that (Z, src(D-IMap(X))) $\in \mathbb{R}_{AF}$. Then, it would be the case that $\exists Z, W \in \mathbf{S}$ s.t. $(Z, W) \in \mathbb{R}_{AF}$ and thus, $\mathbf{S}$ would not be D-conflict-free, contradicting the hypothesis that $\mathbf{S}$ is a D-stable extension of Δ_{AF}.

(3.$c.ii$) The proof in this case is analogous to case (1.b).

(3.$c.iii$) The proof in this case is analogous to case (1.a).

(3.d) If $d^- \notin \mathbf{S}$ then, since by hypothesis $\mathbf{S}$ is a D-stable extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, \delta^-) \in \mathbb{R}_{AF}$. Then, by Definition 24 we have the following cases: (3.$d.i$) $W =$ src(D-IMap(X)); (3.$d.ii$) $W = \varepsilon^-$, with $\varepsilon \in \mathbb{S}$, and trg($\varepsilon$) = D-IMap($X$); or (3.$d.iii$) $W \in \mathbb{R}$ and trg(W) = D-IMap(X).

(3.$d.i$) The proof in this case is analogous to case (3.$c.i$).

(3.$d.ii$) The proof in this case is analogous to case (3.$c.ii$).

(3.$d.iii$) The proof in this case is analogous to case (3.$c.iii$).

(4) If $X = \delta^-$, with $\delta \in \mathbb{S}$, then, by Definition 25, it holds that D-IMap(X) = δ. Also, if D-IMap(X) $\notin$ D-Map($\mathbf{S}$), it must be the case that $\delta^+ \notin \mathbf{S}$. In addition, given $Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$, it must be the case that either: (4.a) $Z =$ src(D-IMap(X)); (4.b) $Z = Y^-$, with $Y \in \mathbb{S}$, and trg(Y) = D-IMap(X); or (4.c) $Z \in \mathbb{R}$ and trg(Z) = D-IMap(X).

(4.a) If $\delta^+ \notin \mathbf{S}$, then, since by hypothesis $\mathbf{S}$ is a D-stable extension of Δ_{AF}, it must be the case that $\exists W \in \mathbf{S}$ s.t. $(W, \delta^+) \in \mathbb{R}_{AF}$. Hence, by Definition 24, we have to consider the following cases: (4.$a.i$) $W = \varepsilon^-$, with $\varepsilon \in \mathbb{S}$, and trg($\varepsilon$) = D-IMap($X$); (4.$a.ii$) $W \in \mathbb{R}$ and trg(W) = D-IMap(X); (4.$a.iii$) $W \in \mathbb{R}$ and trg(W) = src(D-IMap(X)); or (4.$a.iv$) $W = \varepsilon^-$, with $\varepsilon \in \mathbb{S}$, and trg($\varepsilon$) = src(D-IMap($X$)).

(4.$a.i$) The proof in this case is analogous to case (3.a).

(4.$a.ii$) The proof in this case is analogous to case (3.b).

(4.$a.iii$) The proof in this case is analogous to case (3.c).

(4.$a.iv$) The proof in this case is analogous to case (3.d).

(4.b) The proof in this case is analogous to case (3.a).

(4.c) The proof in this case is analogous to case (3.b).

(5) If $X \in \mathbb{S}$, then, by Definition 25, it holds that D-IMap(X) = X. Then, if $X \notin \mathbf{S}$, given $Z \in \mathbf{S}$ s.t. $(Z, X) \in \mathbb{R}_{AF}$, by Definition 24, it must be the case that either: (5.a) $Z \in \mathbb{R}$ and trg(Z) = X; or (5.b) $Z = Y^-$, with $Y \in \mathbb{S}$, and trg(Y) = X.

(5.a) The proof in this case is analogous to case (1.a).

(5.b) The proof in this case is analogous to case (1.b).

B) If $\mathbf{S}$ is a stable extension of Δ, then, by Corollary 1, $\mathbf{S}$ is a complete extension of Δ. Moreover, by Theorem 1, Map($\mathbf{S}$) is a D-complete extension of Δ_{AF}. Thus, to prove tat Map($\mathbf{S}$) is a D-stable extension of Δ_{AF} we need to show that $\forall X \notin$ Map($\mathbf{S}$) : $\exists Y \in$ Map($\mathbf{S}$) s.t. $(Y, X) \in \mathbb{R}_{AF}$. Now we have to consider the following cases: (a) $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$; ($b$) $X = \delta^+$, with $\delta \in \mathbb{S}$; or (c) $X = \delta^-$, with $\delta \in \mathbb{S}$.

(a) If $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ and $X \notin$ Map($\mathbf{S}$), then, by Definition 28, it holds that $X \notin \mathbf{S}$. Therefore, since by hypothesis $\mathbf{S}$ is a stable extension of Δ, it must be the case that either: ($a.i$) $\exists\alpha \in \mathbf{S}$ s.t. α u-def X; or ($a.ii$) $\exists\alpha \in \mathbf{S}, \exists\mathbf{S}' \subseteq \mathbf{S}$ s.t. α c-def X given $\mathbf{S}'$.

(a.i) If α $\mathsf{u\text{-}def}$ X, then, by Definition 24, $(\alpha, X) \in \mathbb{R}_{AF}$. Moreover, by Definition 28 it holds that $\alpha \in \mathsf{Map}(\mathbf{S})$. Thus, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = \alpha$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(a.ii) If α $\mathsf{c\text{-}def}$ X given $\mathbf{S}'$, then, by Definition 18 either α $\mathsf{e\text{-}def}$ X given $\mathbf{S}'$ or α $\mathsf{ei\text{-}def}$ X given $\mathbf{S}'$ (the latter being only possible if $X \in \mathbb{R}$). In both cases, by Definitions 16 and 17, there exists a support sequence $\Sigma = [\mathcal{A}_1, \dots \mathcal{A}_n]$ s.t. $\mathsf{trg}(\alpha) = \mathcal{A}_1$ and $\mathbf{S}'$ is the support set of Σ. Moreover, if α $\mathsf{e\text{-}def}$ X given $\mathbf{S}'$, $\mathcal{A}_n = X$; otherwise, if α $\mathsf{ei\text{-}def}$ X given $\mathbf{S}'$, $\mathcal{A}_n = \mathsf{src}(X)$. Then, by Definition 24, in both cases, $(\alpha, \mathcal{A}_1) \in \mathbb{R}_{AF}$ and $\forall S_i = (\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathbf{S}'$ $(1 \leq i \leq n-1)$ it holds that $(\mathcal{A}_i, S_i^-) \in \mathbb{R}_{AF}$, $(S_i^-, S_{i+1}^+) \in \mathbb{R}_{AF}$ and $(S_i^-, \mathcal{A}_{i+1}) \in \mathbb{R}_{AF}$. Also, since $\alpha \in \mathbf{S}$, by Definition 28 it holds that $\alpha \in \mathsf{Map}(\mathbf{S})$. Furthermore, for every $S_i \in \mathbf{S}' \subseteq \mathbf{S}$ it holds that $S_i \in \mathsf{Map}(\mathbf{S})$. Then, since by hypothesis $\mathbf{S}$ is a stable (also, complete) extension of Δ and $\alpha \in \mathsf{Map}(\mathbf{S})$, $\mathcal{A}_1 \notin \mathsf{Map}(\mathbf{S})$. As a result, $S_1^- \in \mathsf{Map}(\mathbf{S})$ and $S_1^+ \notin \mathsf{Map}(\mathbf{S})$. In addition, since for every $S_i \in \mathbf{S}'$ it holds that $\mathsf{trg}(S_i) = \mathsf{src}(S_{i+1})$, by extension we have that $\mathcal{A}_j \notin \mathsf{Map}(\mathbf{S})$ $(2 \leq j \leq n)$, $S_k^+ \notin \mathsf{Map}(\mathbf{S})$ and $S_k^- \in \mathsf{Map}(\mathbf{S})$ $(2 \leq k \leq n-1)$; in particular, $S_{n-1}^- \in \mathsf{Map}(\mathbf{S})$. Finally, if α $\mathsf{e\text{-}def}$ X given $\mathbf{S}'$, by Definition 24 it holds that $(S_{n-1}^-, X) \in \mathbb{R}_{AF}$; otherwise, if α $\mathsf{ei\text{-}def}$ X given $\mathbf{S}'$, by Definition 24 it also holds that $(S_{n-1}^-, X) \in \mathbb{R}_{AF}$ (because the AFN associated with Δ is such that S_{n-1}^- attacks $\mathsf{src}(X)$ and $\mathsf{src}(X)$ supports X). Therefore, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = S_{n-1}^-$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(b) If $X = \delta^+$, with $\delta \in \$$, and $X \notin \mathsf{Map}(\mathbf{S})$ then, by Definition 28, it could be the case that either: (b.i) $\delta \notin \mathsf{Map}(\mathbf{S})$; (b.ii) $\delta \in \mathsf{Map}(\mathbf{S})$ and $\delta^- \notin \mathsf{Map}(\mathbf{S})$; or (b.iii) $\delta, \delta^- \in \mathsf{Map}(\mathbf{S})$.

(b.i) If $\delta \notin \mathsf{Map}(\mathbf{S})$, then, by Definition 28, $\delta \notin \mathbf{S}$. Hence, since by hypothesis $\mathbf{S}$ is a stable extension of Δ, it must be the case that: (b.i.I) $\exists \alpha \in \mathbf{S}$ s.t. α $\mathsf{u\text{-}def}$ δ; or (b.i.II) $\exists \alpha \in \mathbf{S}, \exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α $\mathsf{c\text{-}def}$ δ given $\mathbf{S}'$.

(b.i.I) In this case, by Definition 14, it must be the case that α $\mathsf{d\text{-}def}$ δ and $\mathsf{trg}(\alpha) = \delta$. Hence, by Definition 24, $(\alpha, \delta) \in \mathbb{R}_{AF}$, $(\alpha, \delta^+) \in \mathbb{R}_{AF}$ and $(\alpha, \delta^-) \in \mathbb{R}_{AF}$. Moreover, by Definition 28, $\alpha \in \mathsf{Map}(\mathbf{S})$. As a result, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = \alpha$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(b.i.II) In this case, by Definitions 18 and 16, there exists a support sequence $\Sigma = [\mathcal{A}_1, \dots, \mathcal{A}_n]$ s.t. $\mathbf{S}' = \bigcup_{i=1}^{n-1} \{S_i\}$, with $S_i = (\mathcal{A}_i, \mathcal{A}_{i+1})$, is the support set of Σ. Then, analogously to case (a.ii), $(S_{n-1}^-, \delta) \in \mathbb{R}_{AF}$ and $S_{n-1}^- \in \mathsf{Map}(\mathbf{S})$. Furthermore, by Definition 24, $(S_{n-1}^-, \delta^+) \in \mathbb{R}_{AF}$ and $(S_{n-1}^-, \delta^-) \in \mathbb{R}_{AF}$. As a result, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = S_{n-1}^-$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(b.ii) If $\delta \in \mathsf{Map}(\mathbf{S})$ and $\delta^- \notin \mathsf{Map}(\mathbf{S})$, then, by Definition 28, it must be the case that $\mathsf{src}(\delta) \notin \mathsf{Map}(\mathbf{S})$, $\mathsf{trg}(\delta) \notin \mathsf{Map}(\mathbf{S})$ and $\nexists \alpha \in \mathbf{S}, \nexists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α $\mathsf{u\text{-}def}$ $\mathsf{src}(\delta)$ or α $\mathsf{c\text{-}def}$ $\mathsf{src}(\delta)$ given $\mathbf{S}'$. Moreover, by Definition 28, it would be the case that $\mathsf{src}(\delta) \notin \mathbf{S}$, contradicting the hypothesis that $\mathbf{S}$ is a stable extension of Δ. As a result, given a stable extension $\mathbf{S}$ of Δ, it can never be the case that $\delta \in \mathsf{Map}(\mathbf{S})$ (with $\delta \in \$$) and $\delta^+, \delta^- \notin \mathsf{Map}(\mathbf{S})$.

(b.iii) If $\delta, \delta^- \in \mathsf{Map}(\mathbf{S})$, then, by Definition 28, it must be the case that either: (b.iii.I) $\exists \alpha \in \mathbf{S}$ s.t. α $\mathsf{u\text{-}def}$ $\mathsf{src}(\delta)$; or (b.iii.II) $\exists \alpha \in \mathbf{S}, \exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α $\mathsf{c\text{-}def}$ $\mathsf{src}(\delta)$ given $\mathbf{S}'$.

(b.iii.I) If α $\mathsf{u\text{-}def}$ $\mathsf{src}(\delta)$, then, by Definitions 14 and 12, it must be the case that α $\mathsf{d\text{-}def}$ $\mathsf{src}(\delta)$. Then, by Definition 24, it holds that $(\alpha, \mathsf{src}(\delta)) \in \mathbb{R}_{AF}$ and $(\alpha, \delta^+) \in \mathbb{R}_{AF}$. Moreover, by Definition 28, $\alpha \in \mathsf{Map}(\mathbf{S})$. As a result, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = \alpha$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(b.iii.II) If α $\mathsf{c\text{-}def}$ $\mathsf{src}(\delta)$ given $\mathbf{S}'$, then, by Definitions 18 and 16, it must be the case that α $\mathsf{e\text{-}def}$ $\mathsf{src}(\delta)$ given $\mathbf{S}'$ and there exists a support sequence $\Sigma = [\mathcal{A}_1, \dots, \mathcal{A}_n = \mathsf{src}(\delta)]$ for $\mathsf{src}(\delta)$ s.t. $\mathbf{S}' = \bigcup_{i=1}^{n-1} \{S_i\}$, with $S_i = (\mathcal{A}_i, \mathcal{A}_{i+1})$, is the support set of Σ. Then, analogously to case (a.ii), $S_{n-1}^- \in \mathsf{Map}(\mathbf{S})$. Furthermore, by Definition 24, $(S_{n-1}^-, \mathsf{src}(\delta)) \in \mathbb{R}_{AF}$ and $(S_{n-1}^-, \delta^+) \in \mathbb{R}_{AF}$. As a result, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = S_{n-1}^-$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

(c) If $X = \delta^-$, with $\delta \in \$$, and $X \notin \mathsf{Map}(\mathbf{S})$, then, by Definition 28, it could be the case that either: (c.i) $\delta \notin \mathsf{Map}(\mathbf{S})$; (c.ii) $\delta \in \mathsf{Map}(\mathbf{S})$ and $\delta^+ \notin \mathsf{Map}(\mathbf{S})$; or (c.iii) $\delta, \delta^+ \in \mathsf{Map}(\mathbf{S})$.

(c.i) The proof in this case is the same as case (b.i)

(c.ii) The proof in this case is analogous to case (b.ii)

($c.iii$) If $\delta^+ \in \mathsf{Map}(\mathbf{S})$, then, by Definition 28, it must be the case that $\mathsf{src}(\delta) \in \mathbf{S}$ and thus, $\mathsf{src}(\delta) \in \mathsf{Map}(\mathbf{S})$. Also, by Definition 24, $(\mathsf{src}(\delta), \delta^-) \in \mathbb{R}_{AF}$. As a result, $\exists Y \in \mathsf{Map}(\mathbf{S})$ (with $Y = \mathsf{src}(\delta)$) s.t. $(Y, X) \in \mathbb{R}_{AF}$.

□

Theorem 4. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is the D-grounded extension of Δ_{AF}, then $\mathsf{D\text{-}Map}(\mathbf{S})$ is the grounded extension of Δ.

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is the grounded extension of Δ, then $\mathsf{Map}(\mathbf{S})$ is the D-grounded extension of Δ_{AF}.

Proof.

A) It follows directly from Definitions 3 and 22, and Theorem 1, since the grounded extension of Δ (respectively, of Δ_{AF}) corresponds to its smallest (w.r.t. $\subseteq$) complete (respectively, D-complete) extension.

B) It follows directly from Definitions 22 and 3, and Theorem 1, since the grounded extension of Δ_{AF} (respectively, of Δ) corresponds to its smallest (w.r.t. $\subseteq$) D-complete (respectively, complete) extension.

□

Lemma 9. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF and $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF.*

A) If $\mathbf{S} \subseteq \mathbb{A}_{AF}$ is a D-complete extension of Δ_{AF}, then $\mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S})) = \mathbf{S}$.

B) If $\mathbf{S} \subseteq \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$ is a complete extension of Δ, then $\mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S})) = \mathbf{S}$.

Proof.

A) We need to prove the following: (1) $\mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S})) \subseteq \mathbf{S}$; and (2) $\mathbf{S} \subseteq \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S}))$.

(1) We need to prove that $\forall X \in \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S})) : X \in \mathbf{S}$. Let us now consider the following cases: ($A.1.a$) $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$; ($1.b$) $X = \beta^+$, with $\beta \in \mathbb{S}$; or ($1.c$) $X = \beta^-$, with $\beta \in \mathbb{S}$.

($1.a$) If $X \in \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S}))$ and $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$, then, by Definition 28, $X \in \mathsf{D\text{-}Map}(\mathbf{S})$. As a result, by Definitions 26 and 25, $X \in \mathbf{S}$.

($1.b$) If $X = \beta^+$, with $\beta \in \mathbb{S}$, then, by Definition 28, $\mathsf{src}(\beta) \in \mathsf{D\text{-}Map}(\mathbf{S})$, $\mathsf{trg}(\beta) \in \mathsf{D\text{-}Map}(\mathbf{S})$ and $\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$. By Definitions 26 and 25, $\mathsf{src}(\beta) \in \mathbf{S}$ and either: ($1.b.i$) $\beta, \beta^+ \in \mathbf{S}$ and $\beta^- \notin \mathbf{S}$; ($1.b.ii$) $\beta, \beta^- \in \mathbf{S}$ and $\beta^+ \notin \mathbf{S}$; or ($1.b.iii$) $\beta \in \mathbf{S}$ and $\beta^+, \beta^- \notin \mathbf{S}$.

($1.b.i$) If this is the case, then $X \in \mathbf{S}$.

($1.b.ii$) If $\beta^+ = X \notin \mathbf{S}$, then, by [30, Lemma 10], it would be the case that X is not D-acceptable w.r.t. $\mathbf{S}$ and $\exists \alpha \in \mathbb{A}_{AF}$ s.t. $(\alpha, X) \in \mathbb{R}_{AF}$ but $\nexists \gamma \in \mathbf{S}$ s.t. $(\gamma, \alpha) \in \mathbb{R}_{AF}$. By Definition 24, given such $\alpha \in \mathbb{R}_{AF}$ it must be the case that $(\alpha, \mathsf{src}(\beta)) \in \mathbb{R}_{AF}$ or $(\alpha, \beta) \in \mathbb{R}_{AF}$. Thus, $\mathsf{src}(\beta)$ or β would not be D-acceptable w.r.t. $\mathbf{S}$, contradicting they hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.

($1.b.iii$) The proof in this case is the same as in case ($1.b.ii$).

($1.c$) If $X = \beta^-$, with $\beta \in \mathbb{S}$, then, by Definition 28, $\beta \in \mathsf{Map}(\mathbf{S})$ and $\mathsf{src}(\beta), \mathsf{trg}(\beta) \notin \mathsf{Map}(\mathbf{S})$. By Definitions 26 and 25, it could be the case that either: ($1.c.i$) $\beta^+, \beta \in \mathbf{S}$ and $\beta^- \notin \mathbf{S}$; ($1.c.ii$) $\beta^-, \beta \in \mathbf{S}$ and $\beta^+ \notin \mathbf{S}$; or ($1.c.iii$) $\beta \in \mathbf{S}$ and $\beta^+, \beta^- \notin \mathbf{S}$.

($1.c.i$) If this is the case, then $X \in \mathbf{S}$.

($1.c.ii$) If $\mathsf{src}(\beta) \notin \mathsf{D\text{-}Map}(\mathbf{S})$, then, by Definition 26, $\mathsf{src}(\beta) \notin \mathbf{S}$. Then, by [30, Lemma 10], $\mathsf{src}(\beta)$ would not be D-acceptable w.r.t. $\mathbf{S}$ and $\exists \delta \in \mathbb{A}_{AF}$ s.t. $(\delta, \mathsf{src}(\beta)) \in \mathbb{R}_{AF}$ and $\nexists \gamma \in \mathbf{S}$ s.t. $(\gamma, \delta) \in \mathbb{R}_{AF}$. By Definition 24, given such $\delta \in \mathbb{A}_{AF}$, it must be the case that $(\delta, \beta^+) \in \mathbb{R}_{AF}$. As a result, β^+ would not be D-acceptable w.r.t. $\mathbf{S}$, contradicting the hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.

(1.*c.iii*) If $\beta^- = X \notin \mathbf{S}$, then, by [30, Lemma 10], X would not be D-acceptable w.r.t. $\mathbf{S}$ and thus, $\exists \delta \in \mathbb{A}_{AF}$ s.t. $(\delta, X) \in \mathbb{R}_{AF}$ but $\nexists \gamma \in \mathbf{S}$ s.t. $(\gamma, \delta) \in \mathbb{R}_{AF}$. By Definition 24, given such $\delta \in \mathbb{A}_{AF}$, it should be the case that $\delta = \mathsf{src}(\beta)$ or $(\delta, \beta) \in \mathbb{R}_{AF}$. Hence, since $\beta \in \mathbf{S}$ and by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, it must be the case that $\delta = \mathsf{src}(\beta)$. By Definition 28, it also holds that either: (1.*c.iii.I*) $\exists \alpha \in \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. α u-def $\mathsf{src}(\beta)$; or (1.*c.iii.II*) $\exists \alpha \in \mathsf{D\text{-}Map}(\mathbf{S}), \exists \mathbf{S}' \subseteq \mathsf{D\text{-}Map}(\mathbf{S})$ s.t. α c-def $\mathsf{src}(\beta)$ given $\mathbf{S}'$.

(1.*c.iii.I*) In this case, by Definition 24, $(\alpha, \mathsf{src}(\beta)) \in \mathbb{R}_{AF}$. Also, by Definitions 26 and 25, $\alpha \in \mathbf{S}$. As a result, $\exists \gamma \in \mathbf{S}$ (with $\gamma = \alpha$) s.t. $(\gamma, \delta) \in \mathbb{R}_{AF}$. Contradiction.

(1.*c.iii.II*) In this case, there exists a support sequence $\Sigma = [\mathcal{A}_1, \dots \mathcal{A}_n = \mathsf{src}(\beta)]$ with $\mathsf{trg}(\alpha) = \mathcal{A}_1$ s.t. $\mathbf{S}' = \bigcup_{i=1}^{n-1} \{S_i\}$, with $S_i = (\mathcal{A}_i, \mathcal{A}_{i+1})$, is the support set of Σ. By Definitions 26 and 25, $\alpha \in \mathbf{S}$. Also, since by hypothesis $\mathbf{S}$ is a D-complete extension of Δ_{AF}, $\forall S_i \in \mathbf{S}' : S_i^- \in \mathbf{S}$; in particular, $S_{n-1}^- \in \mathbf{S}$. Moreover, by Definition 24, $(S_{n-1}^-, \mathsf{src}(\beta)) \in \mathbb{R}_{AF}$. As a result, $\exists \gamma \in \mathbf{S}$ (with $\gamma = S_{n-1}^-$) s.t. $(\gamma, \delta) \in \mathbb{R}_{AF}$. Contradiction.

(2) We need to prove that $\forall X \in \mathbf{S} : X \in \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$. Let us now consider the following cases: (2.*a*) $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$; (2.*b*) $X = \beta^+$, with $\beta \in \mathbb{S}$; or (2.*c*) $X = \beta^-$, with $\beta \in \mathbb{S}$.

(2.*a*) If $X \in \mathbf{S}$ and $X \in \mathbb{A} \cup \mathbb{R} \cup \mathbb{S}$, then, by Definitions 26 and 25, $X \in \mathsf{D\text{-}Map}(\mathbf{S})$. Therefore, by Definition 28, $X \in \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S}))$.

(2.*b*) If $X = \beta^+$, with $\beta \in \mathbb{S}$, then, by Definitions 26 and 25, $\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$. Let us now consider the following cases: (2.*b.i*) $\mathsf{src}(\beta) \in \mathbf{S}$; or (2.*b.ii*) $\mathsf{src}(\beta) \notin \mathbf{S}$.

(2.*b.i*) If $\mathsf{src}(\beta) \in \mathbf{S}$, then, by Definitions 26 and 25, $\mathsf{src}(\beta) \in \mathsf{D\text{-}Map}(\mathbf{S})$. Therefore, by Definition 28, $X \in \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S}))$.

(2.*b.ii*) If $\mathsf{src}(\beta) \notin \mathbf{S}$, then, by [30, Lemma 10], it must be the case that $\mathsf{src}(\beta)$ is not D-acceptable w.r.t. $\mathbf{S}$ and $\exists \alpha \in \mathbb{A}_{AF}$ s.t. $(\alpha, \mathsf{src}(\beta)) \in \mathbb{R}_{AF}$ but $\nexists \gamma \in \mathbf{S}$ s.t. $(\gamma, \alpha) \in \mathbb{R}_{AF}$. By Definition 24, given such $\alpha \in \mathbb{A}_{AF}$, it must be the case that $(\alpha, X) \in \mathbb{R}_{AF}$. Hence, X would not be D-acceptable w.r.t. $\mathbf{S}$, contradicting the hypothesis that $\mathbf{S}$ is a D-complete extension of Δ_{AF}.

(2.*c*) If $X = \beta^-$, with $\beta \in \mathbb{S}$, then, by Definitions 26 and 25, $\beta \in \mathsf{D\text{-}Map}(\mathbf{S})$. By Definition 24, $(\mathsf{src}(\beta), X) \in \mathbb{R}_{AF}$ and $(X, \mathsf{trg}(\beta)) \in \mathbb{R}_{AF}$. Then, it must be the case that $\mathsf{src}(\beta), \mathsf{trg}(\beta) \notin \mathbf{S}$. Moreover, by Definitions 26 and 25, $\mathsf{src}(\beta), \mathsf{trg}(\beta) \notin \mathsf{D\text{-}Map}(\mathbf{S})$. Therefore, by [30, Lemma 10], $\mathsf{src}(\beta)$ is not D-acceptable w.r.t. $\mathbf{S}$. By Definition 24, it must be the case that $\exists \alpha \in \mathbf{S}, \exists \mathbf{S}' \subseteq \mathbf{S}$ s.t. α u-def $\mathsf{src}(\beta)$ or α c-def $\mathsf{src}(\beta)$ given $\mathbf{S}'$. As a result, by Definition 28, $X \in \mathsf{Map}(\mathsf{D\text{-}Map}(\mathbf{S}))$.

B) We need to prove the following: (1) $\mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S})) \subseteq \mathbf{S}$, and (2) $\mathbf{S} \subseteq \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$.

(1) We need to prove that $\forall X \in \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S})) : X \in \mathbf{S}$. Let us consider the following cases: (1.*a*) $X \in \mathbb{A} \cup \mathbb{R}$; or (1.*b*) $X \in \mathbb{S}$.

(1.*a*) If $X \in \mathbb{A} \cup \mathbb{R}$ and $X \in \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$, then, by Definitions 26 and 25, $X \in \mathsf{Map}(\mathbf{S})$. Therefore, by Definition 28, $X \in \mathbf{S}$.

(1.*b*) If $X \in \mathbb{S}$ and $X \in \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$, then, by Definitions 26 and 25, it could be the case that $X \in \mathsf{Map}(\mathbf{S})$, $X^+ \in \mathsf{Map}(\mathbf{S})$ or $X^- \in \mathsf{Map}(\mathbf{S})$. In either case, by Definition 28, it holds that $X \in \mathbf{S}$.

(2) We need to prove that $\forall X \in \mathbf{S} : \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$. Given $X \in \mathbf{S}$, by Definition 28, $X \in \mathsf{Map}(\mathbf{S})$. Furthermore, by Definitions 26 and 25, $X \in \mathsf{D\text{-}Map}(\mathsf{Map}(\mathbf{S}))$.

□

Theorem 5. *Let $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ be an ASAF, $\Delta_{AF} = \langle \mathbb{A}_{AF}, \mathbb{R}_{AF} \rangle$ its associated AF and a semantics $\sigma \in \{$complete, preferred, stable, grounded$\}$. It holds that E is an extension of Δ under the σ semantics iff $\mathsf{Map}(\mathsf{E})$ is an extension of Δ_{AF} under the D-σ semantics. Equivalently, E' is an extension of Δ_{AF} under the D-σ semantics iff $\mathsf{D\text{-}Map}(\mathsf{E}')$ is an extension of Δ under the σ semantics .*

Proof. It follows directly from Theorems 1–4 and Lemma 9. □

Proposition 7. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \mathbb{S} \rangle$ *be an ASAF s.t.* $\forall \alpha \in (\mathbb{R} \cup \mathbb{S})$, $\mathsf{trg}(\alpha) \in \mathbb{A}$, $\Phi = \langle \mathbb{A}, \mathbb{R}, \mathbb{S}^+ \rangle$ *an AFN (where* $\mathbb{S}^+$ *is the transitive closure of* $\mathbb{S}$*) and a semantics* $\sigma \in \{$*complete, preferred, stable, grounded*$\}$*. It holds that:*

(1) *If* E_Δ *is an extension of* Δ *under the semantics* σ*, then* E_Φ *is an extension of* Φ *under the semantics* σ*, where* $\mathsf{E}_\Phi = \{\mathcal{A} \in \mathsf{E}_\Delta \mid \mathcal{A} \in \mathbb{A}\}$*.*

(2) *If* E_Φ *is an extension of* Φ *under the semantics* σ*, then there exists an extension* E_Δ *of* Δ *under the semantics* σ *s.t.* $\mathsf{E}_\Phi = \{\mathcal{A} \in \mathsf{E}_\Delta \mid \mathcal{A} \in \mathbb{A}\}$*.*

Proof. Let $\Phi_{AF} = \langle \mathbb{A}, \mathbb{R}_\Phi \rangle$ be the AF associated with Φ according to Definition 10. We prove the Lemma by showing that the defeats in Δ and the attacks in Φ_{AF} are equivalent in the sense that they lead to the same acceptability constraints on the arguments from Δ and Φ. Let us consider the following cases, which correspond to situations leading to defeats on arguments in Δ and attacks on arguments in Φ_{AF}:

- Let $\alpha = (\mathcal{A}, \mathcal{B}) \in \mathbb{R}$:

  Δ : By Definition 12, α $\mathsf{d\text{-}def}$ $\mathcal{B}$. Since by Definition 22 E_Δ is conflict-free, if $\alpha \in \mathsf{E}_\Delta$, then $\mathcal{B} \notin \mathsf{E}_\Delta$. Moreover, by Proposition 4, if $\alpha \in \mathsf{E}_\Delta$ then $\mathcal{A} \in \mathsf{E}_\Delta$. Hence, since by hypothesis all attacks and supports in Δ occur at the argument level, if $\mathcal{A} \in \mathsf{E}_\Delta$, then $\mathcal{B} \notin \mathsf{E}_\Delta$.

  Φ_{AF} : By Definition 10, $\alpha = (\mathcal{A}, \mathcal{B}) \in \mathbb{R}_\Phi$. Then, since by Definition 3 E_Φ is D-conflict-free, if $\mathcal{A} \in \mathsf{E}_\Phi$, then $\mathcal{B} \notin \mathsf{E}_\Phi$.

- Let $\alpha = (\mathcal{A}, \mathcal{B}) \in \mathbb{R}$ and a support sequence $\Sigma = [\mathcal{B}, \ldots, \mathcal{C}]$:

  Δ : By Definition 16, α $\mathsf{e\text{-}def}$ $\mathcal{C}$ given S, where S is the support set associated with Σ. Since by Definition 22 E_Δ is conflict-free, if $\alpha \in \mathsf{E}_\Delta$, then $\mathcal{C} \notin \mathsf{E}_\Delta$. Moreover, by Proposition 4, if $\alpha \in \mathsf{E}_\Delta$ then $\mathcal{A} \in \mathsf{E}_\Delta$. Hence, since by hypothesis all attacks and supports in Δ occur at the argument level, if $\mathcal{A} \in \mathsf{E}_\Delta$, then $\mathcal{C} \notin \mathsf{E}_\Delta$.

  Φ_{AF} : By Definition 8, the existence of the support sequence Σ implies that $(\mathcal{B}, \mathcal{C}) \in \mathbb{S}^+$. Hence, by Definition 9, there exists an extended attack from $\mathcal{A}$ to $\mathcal{C}$ in Φ and, by Definition 10, $(\mathcal{A}, \mathcal{C}) \in \mathbb{R}_\Phi$. As a result, since by Definition 3 E_Φ is D-conflict-free, if $\mathcal{A} \in \mathsf{E}_\Phi$, then $\mathcal{C} \notin \mathsf{E}_\Phi$.

As a result, every argument $\mathcal{A} \in \mathbb{A}$ belonging to an extension E_Δ of Δ under the semantics σ will also belong to the corresponding extension E_Φ of Φ under the same semantics and vice-versa. □

Proposition 8. *Let* $\Delta = \langle \mathbb{A}, \mathbb{R}, \emptyset \rangle$ *be an ASAF,* $\Gamma = \langle \mathbb{A}, \mathbb{R} \rangle$ *an AFRA and a semantics* $\sigma \in \{$*complete, preferred, stable, grounded*$\}$*. It holds that* E *is an extension of* Δ *under the* σ *semantics iff* E *is an extension of* Γ *under the* σ *semantics.*

Proof. Since the support relation of Δ is the empty set, by Definitions 14 and 18, the only defeats that may occur in Δ are unconditional defeats, that is, direct defeats or indirect defeats. By Definitions 12, 13, and 5, direct and indirect defeats of Δ and Γ coincide. In the absence of conditional defeats, the notion of conflict-freeness characterized in Definition 19 and the notion of acceptability characterized in Definition 20 are equivalent to the ones given in Definition 6. Moreover, by Definitions 21 and 6, the notions of admissibility in the ASAF and the AFRA coincide. As a result, by Definitions 22 and 7, the extensions of Δ and Γ under the σ semantics coincide. □

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