

Generating degrees for graded projective resolutions

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We provide a framework connecting several well-known theories related to the linearity of graded modules over graded algebras. In the first part, we pay a particular attention to the tensor products of graded bimodules over graded algebras. Finally, we provide a tool to evaluate the possible degrees of a module appearing in a graded projective resolution once the generating degrees for the first term of some particular projective resolution are known.

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1. Introduction

Koszul algebras were introduced by Priddy in [13]. We will apply the notion of a Koszul algebra for algebras presented by quivers with relations. It can be stated as

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follows. Suppose that \mathbf{k} is a field, Q is a finite quiver, I is a homogeneous ideal of the path algebra $\mathbf{k}Q$ and $A = \mathbf{k}Q/I$. The algebra A is called *Koszul* if the maximal semisimple graded quotient A_0 of A has a graded A -projective resolution (P_\bullet, d_\bullet) such that for all $i \geq 0$, the A -module P_i is generated in degree i . Such a resolution is called a linear resolution and it is minimal whenever it exists, in the sense that $d_i(P_{i+1}) \subset P_i A_{>0}$ for all $i \geq 0$.

Green and Martínez-Villa proved in [10] that the quadratic algebra A is Koszul if and only if its Yoneda algebra, $E(A) = \bigoplus_{i \geq 0} \text{Ext}_A^i(A_0, A_0)$ is generated in degrees 0 and 1, which in turn is equivalent to the Yoneda algebra being isomorphic to the quadratic dual $A^!$ of A .

Koszulness has been generalized to various settings. Next, we describe some of these generalizations.

Berger introduced in [3] the notion of “nonquadratic Koszul algebra” for algebras of the form $A = T_{\mathbf{k}}V/I$, where V is a finite-dimensional \mathbf{k} -vector space and I is a two-sided ideal generated in degree s , for some $s \geq 2$. He required the trivial A -module \mathbf{k} to have a minimal graded projective resolution (P_\bullet, d_\bullet) such that each P_i is generated in degree $\frac{is}{2}$ for i even and $\frac{(i-1)s}{2} + 1$ for i odd.

The authors of [9] considered, under the name of s -Koszul algebras, non-necessarily quadratic Koszul algebras of the form $A = \mathbf{k}Q/I$, with Q a finite quiver and I an ideal generated by homogeneous elements of degree s , connecting this notion with the Yoneda algebra: the algebra A is s -Koszul if and only if $E(A)$ is generated in degrees 0, 1 and 2. Observe that 2-Koszul algebras are just Koszul algebras.

Later on, Green and Marcos generalized this notion defining δ -Koszul and δ -determined algebras. See [7] for details.

Moreover, Green and Marcos also introduced in [8] a family of algebras that they called 2- s -Koszul. They proved that these algebras also have the property that their Yoneda algebras are generated in degrees 0, 1, and 2.

The main objective of the current work is to place all these definitions in a unique framework. We next sketch how we will do this.

Let $A = \bigoplus_{i \geq 0} A_i$ be a graded \mathbf{k} -algebra generated in degrees 0 and 1, such that A_0 is a finite direct product of fields and A_1 is finite-dimensional. Given a graded A -module X , we consider a minimal graded projective resolution $(P_\bullet(X), d_\bullet(X))$ and we take into account in which degrees $P_i(X)$ is generated for each $i \geq 0$. We are specially interested in what we call \mathcal{S} -determined case.

In Sec. 2, we prove that, given graded \mathbf{k} -algebras A, B and C , a graded $A - B$ bimodule X and a graded $B - C$ bimodule Y , if X has a linear minimal $A - B$ -projective graded resolution, Y has a linear minimal $B - C$ -projective graded resolution, and $\text{Tor}_i^B(X, Y)$ vanishes for $i \geq 1$, then $X \otimes_B Y$ has a linear minimal $A - C$ -projective graded resolution. This is a particular case of Theorem 2 below. Note that this theorem shows that any graded bimodule over a Koszul algebra which is linear as a right module and flat as a left module is also linear as a bimodule and the tensor product with such a module gives a functor from the category of linear

graded modules to the category of linear graded modules. Moreover, it recovers and generalizes the fact that the tensor product of two Koszul algebras is Koszul. The same holds for the \mathcal{S} -determined (see Definition 1).

Section 3 is devoted to Gröbner bases. Loosely speaking, we show how one can use them to obtain generating degrees for the n th term of the minimal graded projective resolution of a module if one knows the generating degrees for terms of its projective presentation of a special form.

We fix a field \mathbf{k} . All algebras will be \mathbf{k} -algebras and all modules will be right A -modules unless otherwise stated. We will simply write \otimes for $\otimes_{\mathbf{k}}$ and \mathbb{N}_0 for the set of non-negative integer numbers.

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2. Tensor Products of \mathcal{S} -Determined Modules

In this section, we will prove Lemma 1, which is a graded version of the spectral sequences (2) and (3) from [4, p. 345].

Let A, B and C be \mathbf{k} -algebras. Let X be an A - B -bimodule, Y a B - C -bimodule and Z an A - C -bimodule. Given $a \in A$, we will denote left multiplication by a on X by $L_a \in \text{End}_B(X)$. We recall that for each $n \in \mathbb{N}$, $\text{Ext}_C^n(Y, Z)$ is an A - B bimodule with the structure given by

$$aTb := (\text{Ext}_C^n(L_b, Z) \circ \text{Ext}_C^n(Y, L_a))(T), \text{ for } T \in \text{Ext}_C^n(Y, Z), a \in A, b \in B.$$

Suppose now that A is a \mathbb{Z} -graded algebra and M is a graded A -module. Given $i \in \mathbb{Z}$, $M[i]$ will denote the i -shifted graded A -module with underlying A -module structure as before, whose grading is such that $M[i]_r = M_{i+r}$. For any graded A -module N and any $n \in \mathbb{N}$, we will denote by $\text{Hom}_{GrA}(M, N)$ the set of degree preserving A -module maps from M to N and by $\text{Ext}_{GrA}^n(M, N)$ the set of equivalence classes of exact sequences of graded A -modules with degree zero morphisms

$$0 \rightarrow N \xrightarrow{f_{n-1}} T_{n-1} \xrightarrow{f_{n-2}} \dots \xrightarrow{f_0} T_0 \xrightarrow{f_{-1}} M \rightarrow 0.$$

Let us consider as usual $\text{ext}_A^n(M, N) := \bigoplus_{i \in \mathbb{Z}} \text{Ext}_{GrA}^n(M, N[i])$, which is a subset of $\text{Ext}_A^n(M, N)$ in a natural way. Moreover, if M has an A -projective resolution with finitely generated modules, then both sets coincide.

Suppose now that A, B and C are \mathbb{Z} -graded algebras, and that the bimodules X, Y and Z are graded. For each $n \geq 0$, the A - B -bimodule structure on $\text{Ext}_C^n(Y, Z)$ induces a graded A - B -bimodule structure on $\text{ext}_C^n(Y, Z)$ whose i th component is $\text{Ext}_{GrC}^n(Y, Z[i])$. Note also that $\text{ext}_C^n(Y, Z[i]) \cong \text{ext}_C^n(Y, Z)[i]$ as graded A - B -bimodule, moreover for any $n \geq 0$, $\text{Tor}_n^B(X, Y)$ is a graded A - C -bimodule in a natural way.

The main tool of this section is the following lemma.

Lemma 1. *Let A, B and C be \mathbb{Z} -graded algebras, X a graded A - B -bimodule, Y a graded B - C -bimodule, and Z a graded A - C -bimodule. There are two first*

quadrant cohomological spectral sequences with second pages

$$E_2^{i,j} = \text{Ext}_{Gr(A^{op} \otimes B)}^i(X, \text{ext}_C^j(Y, Z)) \quad \text{and} \quad \tilde{E}_2^{i,j} = \text{Ext}_{Gr(A^{op} \otimes C)}^i(\text{Tor}_j^B(X, Y), Z)$$

that converge to the same graded space.

Proof. Let

$$\dots \xrightarrow{d_n(X)} P_n(X) \xrightarrow{d_{n-1}(X)} \dots \xrightarrow{d_0(X)} P_0(X) \xrightarrow{\mu_X} X$$

be a graded $A - B$ -projective resolution of X and

$$(Z \xrightarrow{\iota_Z} I^0(Z) \xrightarrow{d^0(Z)} \dots \xrightarrow{d^{n-1}(Z)} I^n(Z) \xrightarrow{d^n(Z)} \dots$$

be a graded $A - C$ -injective resolution of Z . Consider two bicomplexes whose (i, j) -components are respectively

$$\text{Hom}_{Gr(A^{op} \otimes B)}(P_i(X), \text{hom}_C(Y, I^j(Z)))$$

and

$$\text{Hom}_{Gr(A^{op} \otimes C)}(P_j(X) \otimes_B Y, I^i(Z)).$$

Since there is an isomorphism of complexes

$$\begin{aligned} F_\bullet &= \text{Hom}_{Gr(A^{op} \otimes C)}(P_\bullet(X) \otimes_B Y, I^\bullet(Z)) \\ &\cong \text{Hom}_{Gr(A^{op} \otimes B)}(P_\bullet(X), \text{hom}_C(Y, I^\bullet(Z))), \end{aligned}$$

the respective total complexes are isomorphic. Here, as usually, for two complexes of graded modules $(U_\bullet, d_{U,\bullet})$ and $(V^\bullet, d^{V,\bullet})$ over the algebra D , we denote by $\text{Hom}_{GrD}(U_\bullet, V^\bullet)$ the complex with $(\text{Hom}_{GrD}(U_\bullet, V^\bullet))_n = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{GrD}(U_{i-n}, V^{-i})$ and differential d_\bullet defined by the equality $d_n(f) = d^{V,-i} f + (-1)^n f d_{U,i-n}$ for $f \in \text{Hom}_{GrD}(U_{i-n}, V^{-i})$.

The first two pages of the spectral sequence E corresponding to the first bicomplex are

$$\begin{aligned} E_1^{i,j} &= \text{Hom}_{Gr(A^{op} \otimes B)}(P_i(X), \text{ext}_C^j(Y, Z)) \quad \text{and} \\ E_2^{i,j} &= \text{Ext}_{Gr(A^{op} \otimes B)}^i(X, \text{ext}_C^j(Y, Z)), \end{aligned}$$

while the first two pages of the spectral sequence \tilde{E} corresponding to the second bicomplex are

$$\begin{aligned} \tilde{E}_1^{i,j} &= \text{Hom}_{Gr(A^{op} \otimes C)}(\text{Tor}_j^B(X, Y), I^i(Z)) \quad \text{and} \\ \tilde{E}_2^{i,j} &= \text{Ext}_{Gr(A^{op} \otimes B)}^i(\text{Tor}_j^B(X, Y), Z). \end{aligned}$$

Since both spectral sequences converge to the homology of F_\bullet , the lemma is proved. \square

From now on, any \mathbb{Z} -graded algebra A is assumed to be non-negatively graded, that is $A = \bigoplus_{i \geq 0} A_i$, where A_0 is isomorphic to a finite product of copies of \mathbf{k} as

an algebra, $\dim_{\mathbf{k}} A_1 < \infty$, and A is generated as an algebra by $A_0 \oplus A_1$. This is equivalent to say that $A \cong (\mathbf{k}Q)/I$, where Q is a finite quiver and I is an ideal generated by homogeneous elements of degree greater or equal 2.

Definition 1. Let $\mathcal{S} = (\mathcal{S}_i)_{i \geq 0}$ be a collection of subsets $\mathcal{S}_i \subset \mathbb{Z}$. A graded A -module X is called \mathcal{S} -determined if it has a graded projective resolution $P_{\bullet}(X)$ such that $P_i(X)$ is generated as A -module by elements of degrees belonging to \mathcal{S}_i , i.e. $P_i(X) = \langle \bigoplus_{j \in \mathcal{S}_i} P_i(X)_j \rangle_A$ for all $i \geq 0$. We say that X is \mathcal{S} -determined up to degree r if the condition on $P_i(X)$ holds for $0 \leq i \leq r$.

If the set $\mathcal{S}_i = \{i\}$, i.e. each $P_i(X)$ is generated in degree i , then we say that the resolution is linear.

Equivalently, a graded A -module X is \mathcal{S} -determined if and only if for any $i \geq 0$ and any graded A -module Y with support not intersecting \mathcal{S}_i — that is, $\bigoplus_{j \in \mathcal{S}_i} Y_j = 0$ — the space $\text{Ext}_{GrA}^i(X, Y)$ is zero. Analogously, the graded A -module X is \mathcal{S} -determined up to degree r if and only if the last mentioned condition holds for $0 \leq i \leq r$.

The notion of an \mathcal{S} -determined module provides a general framework for some well-known situations. We will now exhibit some well-known examples of \mathcal{S} -determined modules.

- Consider a function $\delta : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}$, and define $\mathcal{S}_i = \{\delta(i)\}$ for all $i \geq 0$, the \mathcal{S} -determined modules are called δ -determined modules. If A_0 is a δ -determined module over A , then the graded algebra A is called δ -determined.
- With the same notations, if moreover the Ext algebra, $E(A)$, of A is finitely generated, then A is called δ -Koszul, see [7]. In particular, if δ is the identity, then δ -determined modules are called linear modules and δ -Koszul algebras are exactly Koszul algebras [13].
- Also, given $s \in \mathbb{N}$, let us define $\chi_s : \mathbb{N}_0 \rightarrow \mathbb{Z}$ by

$$\chi_s(i) = \begin{cases} \frac{is}{2} & \text{if } i \text{ is even,} \\ \frac{(i-1)s}{2} + 1 & \text{if } i \text{ is odd.} \end{cases}$$

The χ_s -linear modules are called χ_s -determined modules and χ_s -Koszul algebras are s -Koszul algebras, see [3]. Denoting by \mathcal{S}_i the set $\{j \mid j \leq \chi_s(i)\}$, \mathcal{S} -determined modules correspond to 2- s -linear modules. If moreover A_0 is a 2- s -determined module over A , then the graded algebra A is called 2- s -determined, see [8].

Using minimal graded projective resolutions, it is not difficult to see that the A -module A_0 is \mathcal{S} -determined if and only if A is an \mathcal{S} -determined module over $A^{op} \otimes A$. This fact follows, for example, from [14, Theorem 2].

Given two collections $\mathcal{S} = (\mathcal{S}_i)_{i \geq 0}$ and $\mathcal{R} = (\mathcal{R}_i)_{i \geq 0}$ of subsets of \mathbb{Z} we define the collection $\mathcal{S} \otimes \mathcal{R} = ((\mathcal{S} \otimes \mathcal{R})_i)_{i \geq 0}$ by

$$(\mathcal{S} \otimes \mathcal{R})_i = \bigcup_{\substack{j+k=i \\ j, k \geq 0}} \{n+m \mid n \in \mathcal{S}_j, m \in \mathcal{R}_k\}.$$

Lemma 1 allows us to prove the following theorem, which generalizes some well known results about Koszul algebras and Koszul modules concerning tensor products.

Theorem 2. *Let $\mathcal{S} = (\mathcal{S}_i)_{i \geq 0}$ and $\mathcal{R} = (\mathcal{R}_i)_{i \geq 0}$ be two collections of subsets of \mathbb{Z} . Let A, B and C be \mathbb{Z} -graded algebras, and finally let X be a graded $A - B$ -bimodule which is \mathcal{S} -determined as bimodule and Y be a graded $B - C$ -bimodule which is \mathcal{R} -determined as C -module. If $\text{Tor}_i^B(X, Y) = 0$ for $1 \leq i \leq r - 1$, then $X \otimes_B Y$ is an $\mathcal{S} \otimes \mathcal{R}$ -determined until r th degree $A - C$ -bimodule. In particular, if $\text{Tor}_i^B(X, Y) = 0$ for all $i \geq 1$, then $X \otimes_B Y$ is an $\mathcal{S} \otimes \mathcal{R}$ -determined $A - C$ -bimodule.*

Proof. Let us fix n and r such that $0 \leq n \leq r$. For any graded $A - C$ -bimodule Z such that $\bigoplus_{m \in (\mathcal{S} \otimes \mathcal{R})_n} Z_m = 0$, we will prove that $\text{Ext}_{Gr(A^{op} \otimes C)}^n(X \otimes_B Y, Z) = 0$. By Lemma 1, there are spectral sequences

$$E_{i,j}^2 = \text{Ext}_{Gr(A^{op} \otimes B)}^i(X, \text{ext}_C^j(Y, Z)) \quad \text{and} \quad \tilde{E}_{i,j}^2 = \text{Ext}_{Gr(A^{op} \otimes C)}^i(\text{Tor}_j^B(X, Y), Z)$$

that converge to the same graded space T_\bullet . It follows easily from the condition on $\text{Tor}_*^B(X, Y)$ that $T_n = \text{Ext}_{Gr(A^{op} \otimes C)}^n(X \otimes_B Y, Z)$ if $n < r$ and that $T_r = \text{Ext}_{Gr(A^{op} \otimes C)}^r(X \otimes_B Y, Z) \oplus V$ for some $V \subset \text{Hom}_{Gr(A^{op} \otimes C)}(\text{Tor}_r^B(X, Y), Z)$.

Thus, it is enough to prove that $E_{i,j}^2 = 0$ for all integers $i, j \geq 0$ such that $i + j = n$. Let us fix such i and j . If $k \in \mathcal{S}_i$, then for any $l \in \mathcal{R}_j$ it is clear that $k + l \in (\mathcal{S} \otimes \mathcal{R})_n$ and so $Z[k]_l = Z_{k+l} = 0$. Since Y is an \mathcal{R} -determined C -module, we know that $\text{ext}_C^j(Y, Z)_k = \text{Ext}_{Gr C}^j(Y, Z[k]) = 0$ for any $k \in \mathcal{S}_i$; from this, since X is an \mathcal{S} -determined $A^{op} \otimes B$ -module, $E_{i,j}^2 = \text{Ext}_{Gr(A^{op} \otimes B)}^i(X, \text{ext}_C^j(Y, Z)) = 0$. We have proven that for any $0 \leq n \leq r$ and any graded $A - C$ -bimodule Z such that $\bigoplus_{m \in (\mathcal{S} \otimes \mathcal{R})_n} Z_m = 0$ one has $\text{Ext}_{Gr(A^{op} \otimes C)}^n(X \otimes_B Y, Z) = 0$. Consequently, $X \otimes_B Y$ is an $\mathcal{S} \otimes \mathcal{R}$ -determined until r th degree $A - C$ -bimodule. \square

Example 1. Let A be the \mathbf{k} -algebra with generators x and y subject to the relations $xy = yx = 0$ and $x^3 = y^3$. Let $X = A/\langle x \rangle$ and $Y = A/\langle y \rangle$. Note that X is a graded A -module and Y is a graded A -bimodule in a natural way. We will show that the conclusion of Theorem 2 about the tensor product $X \otimes_A Y$ of \mathbf{k} - A -bimodule X and A - A -bimodule Y does not hold. One can easily see that there are short exact sequences

$$Y[1] \hookrightarrow A \twoheadrightarrow X$$

and

$$X[1] \hookrightarrow A \twoheadrightarrow Y$$

of graded right A -modules. It follows from these two short exact sequences that X and Y are linear as right A -modules. On the other hand, $X \otimes_A Y \cong A/\langle x, y \rangle$ is the unique simple A -module whose minimal projective resolution P_\bullet is not linear at P_2 . This example shows that Theorem 2 is not valid without the vanishing condition on $\text{Tor}_*^B(X, Y)$.

Corollary 2. *Let X be a graded $A - B$ -bimodule. If A_0 is an \mathcal{S} -determined right A -module and X is an \mathcal{R} -determined B -module, then X is an $\mathcal{S} \otimes \mathcal{R}$ -determined $A - B$ -bimodule. In particular, if A is 2- s -determined and X is a 2- s -determined B -module, then X is a 2- s -determined $A - B$ -bimodule. When $s = 2$ we obtain that if A is Koszul and X is a linear B -module, then X is a linear $A - B$ -bimodule.*

Proof. Since A_0 is an \mathcal{S} -determined right A -module, A is an \mathcal{S} -determined module over $A^{op} \otimes A$. Since A is flat as a right A -module, the result follows from Theorem 2, since $A \otimes_A X \cong X$. \square

It was proved in [2, 10] that if A and B are Koszul algebras, then $A \otimes B$ is a Koszul algebra too. In the next corollary, we give a very short and easy proof of a generalization of this fact. Note that this generalization follows from [12, Chap. 3, Proposition 1.1.] for algebras A, B such that $A_0 = B_0 = \mathbf{k}$.

Corollary 3. *If X is an \mathcal{S} -determined A -module and Y is an \mathcal{R} -determined B -module, then $X \otimes Y$ is an $\mathcal{S} \otimes \mathcal{R}$ -determined module over $A \otimes B$. In particular, if A and B are 2- s -determined, then $A \otimes B$ is 2- s -determined. It also follows that if A and B are Koszul, then $A \otimes B$ is Koszul.*

Proof. It follows from Theorem 2 since any \mathbf{k} -module is flat. The second part follows from the fact that $(A \otimes B)_0 = A_0 \otimes B_0$. \square

It is interesting to mention also the following special case of Theorem 2.

Corollary 4. *Let X be a graded $A - B$ -bimodule that is flat as a left A -module. If X is 2- s -determined as B -module, then the functor $- \otimes_A X : \text{Mod}A \rightarrow \text{Mod}B$ induces a functor from the category of 2- s -determined A -modules to the category of 2- s -determined B -modules. In particular, if X is a linear B -module, then $- \otimes_A X$ induces a functor from the category of linear A -modules to the category of linear B -modules.*

3. Using Gröbner Bases

In this section, we will use Gröbner bases techniques to study graded projective resolutions of a graded module X over an algebra $A = \mathbf{k}Q/I$, where Q is a finite quiver and I is a homogeneous ideal contained in $(\mathbf{k}Q_{>0})^2$. Our aim is to estimate the degrees of the modules appearing in the minimal projective resolution of X using Gröbner basis of I and a particular graded projective presentation of X .

We fix a set of paths $S \subset Q_{\geq 2}$. Next, we introduce some notation. Given two paths p and q in Q , we write $p|q$ if there are paths u and v in Q — possibly of length 0 — such that $q = upv$. If $q = pv$ (respectively, $q = up$) we say that p divides q on the left (respectively, right) and we write $p|_lq$ (respectively, $p|_rq$). We say that S is *reduced* if for any $q \in S$, there is no $p \in S$, $p \neq q$ such that $p|q$. Let us write $\text{len}(q) = n$ if $q \in Q_n$.

We next define, for $n \in \mathbb{N}$, the notion of n -*overlap* for S . These elements will provide a minimal set of generators for each projective module of the minimal projective resolution of A_0 as A -module. Note that n -overlaps are called n -chains by Anick in [1] and n -ambiguities by Chouhy and Solotar in [5].

Given a quiver Q we also denote by Q the set of paths in Q , the context makes it clear what we mean.

Definition 3. Let Q be a quiver and $S \subset Q_{\geq 2}$ is a set of paths. We say that $p \in Q$ is an S -*path* if there exists $s \in S$ such that $s|_rp$. We denote by Q_S the set of S -paths in Q . Suppose that $p = qu$ for some $u, q \in Q$. We say that q S -*vanishes* p if there is no $s \in S$ dividing u . We say that q *almost S -vanishes* p if q does not S -vanish p and, for any presentation $u = u_1u_2$ with $u_2 \in Q_{>0}$, q S -vanishes qu_1 . We write $q|_Sp$ if q S -vanishes p and $q|_S^ap$ if q almost S -vanishes p (note that the relations $|_S$ and $|_S^a$ are not transitive). If $q|_S^ap$, then we automatically have $p \in Q_S$.

We next define the set of n -*overlaps* $O_n(S) \subset Q$ and the set of n -*quasioverlaps* $QO_n(S) \subset Q \times Q_{>0}$ inductively on n .

- For $n = 0$, we define $O_0(S) = Q_1$ and $QO_0(S) = \{(w, v) \mid w \in Q_0, v \in Q_{>0}, vw = v\}$.
- For $n = 1$, we define $O_1(S) = S$ and $QO_1(S) = \{(w, v) \mid w, v \in Q_{>0}, vw \in S\}$.
- For $n > 1$, we define

$$O_n(S) = \{w \mid \exists w_1 \in O_{n-1}(S), w_2 \in O_{n-2}(S) \text{ such that } w_1|_Sw, w_2|_S^aw\}$$

and

$$QO_n(S) = \{(w, v) \mid \exists (w_1, v) \in QO_{n-1}(S), (w_2, v) \in QO_{n-2}(S) \text{ such that } w_1|_Sw, w_2|_S^aw\}.$$

Lemma 5. *Suppose that S is reduced and $n \geq 1$. If $w \in O_n(S)$, then there is a unique w' such that $w'|_lw$ and $w' \in O_{n-1}(S)$. If $(w, v) \in QO_n(S)$, then there is a unique w' such that $w'|_lw$ and $(w', v) \in QO_{n-1}(S)$.*

Proof. We will only prove the assertion about n -overlaps since the proof for n -quasioverlaps is similar. The existence of w' is a direct consequence of the definition of an n -overlap. Thus we only need to prove the uniqueness. We proceed by induction on n . For $n = 1$, the assertion is obvious. For $n = 2$, the assertion follows directly from the fact that S is reduced. Suppose now that $n > 2$ and we have already proven the statement for $n - 1$ and $n - 2$. Suppose that there are two different paths $w', w'' \in O_{n-1}(S)$ such that $w'|_lw$ and $w''|_lw$. Without loss of generality, we may assume that $w'' = w'u$ for some $u \in Q_{>0}$. By the definition of $O_{n-1}(S)$,

there exist $w'_1 \in O_{n-2}(S)$ and $w'_2 \in O_{n-3}(S)$ such that $w'_1|_S w'$ and $w'_2|_S w'$. The inductive hypothesis assures that w'_1 and w'_2 are unique elements of $O_{n-2}(S)$ and $O_{n-3}(S)$, respectively such that $w'_2|_l w'_1|_l w''$. Then $w'_2|_S w''$. Since $u \in Q_{>0}$, we have $w'_2|_S w'$. Since this is a contradiction, the proof is complete. \square

Given $w \in O_n(S)$, and i such that $0 \leq i \leq n$, $\rho_i(w)$ will denote the unique element of $O_i(S)$ that divides w on the left. Analogously, for $(w, v) \in QO_n(S)$, $0 \leq i \leq n$, $\rho_i^v(w)$ will denote the unique element such that $(\rho_i^v(w), v) \in QO_i(S)$ and $\rho_i^v(w)|_l w$. The next two lemmas give alternative definitions for n -overlaps and n -quasioverlaps.

Lemma 6. *Consider a reduced set of paths S and an integer n such that $n \geq 1$. Given a path $w \in Q$, $w \in O_n(S)$ if and only if it can be represented in the form $w = v_0 u_1 v_1 u_2 \cdots u_{n-1} v_{n-1} u_n v_n$, where $u_1, \dots, u_n \in Q$, $v_1, \dots, v_{n-1} \in Q_{>0}$, and $v_0, v_n \in Q_0$ are such that*

- (1) $v_{i-1} u_i v_i \in S$ for $1 \leq i \leq n$,
- (2) for all $1 \leq i \leq n-1$, there are no $u, v \in Q_{>0}$ such that $vv_i u \in S$, $u|_l u_{i+1} v_{i+1}$ and $v|_r u_i$,
- (3) $\text{len}(u_1) > 0$ if $n \leq 2$.

Proof. Suppose that $w \in O_n(S)$. Given i , $1 \leq i \leq n$, there exists a unique $w_i \in S$ such that $w_i|_r \rho_i(w)$. It follows from the definition of $O_i(S)$ that, for $2 \leq i \leq n$, there exist $v_{i-1}, u'_i \in Q_{>0}$ such that $\rho_i(w) = \rho_{i-1}(w) u'_i$ and $w_i = v_{i-1} u'_i$. We also define $u'_1 = \rho_1(w)$. Using again the definition of i -overlap, we get $v_i|_r u'_i$ for $1 \leq i \leq n-1$. It remains to define u_i from the equality $u'_i = u_i v_i$ for $1 \leq i \leq n-1$ and $u_n = u'_n$. It is clear that v_0 and v_n are simply the ending and the starting vertices of w .

Now, if $u_1, \dots, u_n \in Q$, $v_1, \dots, v_{n-1} \in Q_{>0}$, and $v_0, v_n \in Q_0$ satisfy all the required conditions, then the induction on $1 \leq i \leq n$ shows that $u_1 v_1 \cdots u_i v_i \in O_i(S)$. \square

Lemma 7. *Let S be a reduced set of paths and let n be an integer, $n \geq 1$. Given paths $w \in Q$ and $v_0 \in Q_{>0}$, the element $(w, v_0) \in QO_n(S)$ if and only if w can be represented in the form $w = u_1 v_1 u_2 \cdots u_{n-1} v_{n-1} u_n v_n$, where $u_1, \dots, u_n \in Q$, $v_1, \dots, v_{n-1} \in Q_{>0}$, and $v_n \in Q_0$ are such that*

- (1) $v_{i-1} u_i v_i \in S$ for $1 \leq i \leq n$,
- (2) for all $1 \leq i \leq n-1$, there are no $u, v \in Q_{>0}$ such that $vv_i u \in S$, $u|_l u_{i+1} v_{i+1}$ and $v|_r u_i$,
- (3) $\text{len}(u_1) > 0$ if $n = 1$.

Proof. The proof is analogous to the proof of Lemma 6 and so it is left to the reader. \square

Example 2. Let Q be the quiver with $Q_0 = \{e\}$ and $Q_1 = \{x, y\}$. Fix $S = \{x^2 y^3, x^3\}$. The element $(w_0, v_0) = (xxxyyy, xx)$ is a 3-quasioverlap with $\rho_2^{v_0}(w_0) =$

xxx and $\rho_1^{v_0}(w_0) = x$. The elements $u_1 = e, v_1 = u_2 = v_2 = x$, and $u_3 = yyy$ provide the partition of Lemma 7. At the same time, the element $w = v_0w_0 = xxxxyyyy$ is a 3-overlap with $\rho_2(w) = xxx$ and $\rho_1(w) = xxx \in S$. In this case, the paths $u_1 = x, v_1 = xx, u_2 = e, v_2 = x$, and $u_3 = xyyy$ provide the partition of Lemma 6. Note that, though v_0w_0 is a 3-overlap, we have $\rho_2(v_0w_0) \neq v_0\rho_2^{v_0}(w_0)$ and this fact causes differences in the partitions of v_0w_0 and (w_0, v_0) .

Example 3. Let us take Q as in the previous example and $S = \{x^3, xy^2\}$. Consider $(w_0, v_0) = (xxxxyy, xx)$. It is a 3-quasioverlap with $\rho_2^{v_0}(w_0) = xxx$ and $\rho_1^{v_0}(w_0) = x$. The elements $u_1 = e, v_1 = u_2 = v_2 = x$, and $u_3 = yy$ provide the partition of Lemma 7. At the same time, $v_0w_0 = xxxxyyy \notin O_3(S)$ while $w = xxxxyy$ is a 3-overlap with $\rho_2(w) = xxx$ and $\rho_1(w) = xxx \in S$. The paths $u_1 = x, v_1 = xx, u_2 = e, v_2 = x$, and $u_3 = yy$ provide the partition of Lemma 6. This example shows that it is possible that $(w_0, v_0) \in QO_n(S)$ while $v_0w_0 \notin O_n(S)$.

Let us introduce the following notation. Given $n \in \mathbb{N}$,

$$\max_{O_n}(S) = \sup\{\text{len}(w) \mid w \in O_n(S)\},$$

$$\min_{O_n}(S) = \inf\{\text{len}(w) \mid w \in O_n(S)\},$$

$$\max_{QO_n}(S) = \sup\{\text{len}(w) \mid \exists v \text{ such that } (w, v) \in QO_n(S)\},$$

$$\min_{QO_n}(S) = \inf\{\text{len}(w) \mid \exists v \text{ such that } (w, v) \in QO_n(S)\}.$$

By definition, we set $\min_{O_n}(S) = +\infty$ and $\max_{O_n}(S) = -\infty$ if $O_n(S)$ is empty and $\min_{QO_n}(S) = +\infty$ and $\max_{QO_n}(S) = -\infty$ if $QO_n(S)$ is empty. Note that under this convention, we have $\min_{O_n}(S) \geq n + 1$ and $\max_{O_n}(S) \leq \text{len}(S)n - n + 1$, where $\text{len}(S)$ denotes the maximal length of the paths in S .

Now, we are going to prove a theorem that allows to estimate the values of $\max_{O_n}(S)$ and $\min_{O_n}(S)$ using $\max_{QO_n}(S)$ and $\min_{QO_n}(S)$.

Theorem 8. *Given $n \geq 0$, for any reduced set S , we have*

- $\max_{QO_n}(S) \leq \max_{O_n}(S) - 1$,
- $\min_{QO_n}(S) \geq \min_{O_n}(S) - \text{len}(S) + 1$.

Proof. The result is obvious for $n = 0$. For $n \geq 1$, we are going to prove the following assertion. If (w, v) is an n -quasioverlap, then there is $v' \in Q_{>0}$ such that $v'|_r v$ and $v'w$ is an n -overlap. Since it follows easily from the definition of 1-overlap that $\text{len}(v) < \text{len}(S)$, we have

$$\text{len}(v'w) - \text{len}(S) + 1 \leq \text{len}(w) + \text{len}(v) - \text{len}(S) + 1 \leq \text{len}(w) \leq \text{len}(v'w) - 1$$

for such v' . Thus, after proving the existence of v' , we will be done.

More precisely, we will prove the following statement by induction on n . If $(w, v) \in QO_n(S)$, then there exists $v' \in Q_{>0}$ such that $v'|_r v, v'w \in O_n(S)$, $v'\rho_i^v(w)|_l \rho_i(v'w)$ for odd i , and $\rho_i(v'w)|_l v'\rho_i^v(w)$ for even i .

For $n = 1$, we define $v' = v$.

Let us now consider the case $n = 2$. If vw cannot be presented in the form $vw = u'su$ with $u, u' \in Q_{>0}$ and $s \in S$, then we can take $v' = v$ and obtain the 2-overlap wv satisfying all the required conditions. Suppose that it is possible to write $vw = u'su$ with $u, u' \in Q_{>0}$ and $s \in S$. Choose such a presentation with minimal $\text{len}(u)$. It follows from the definition of 2-quasioverlap that $\text{len}(u') < \text{len}(v)$, i.e. there is $v' \in Q_{>0}$ such that $v = u'v'$. It is easy to see that $v'w$ is a 2-overlap with $\rho_1(v'w) = s$ satisfying all the required conditions.

Let us now prove the inductive step. Suppose that the assertion above is true for all $(n - 1)$ -quasioverlaps. We will prove it for n by induction on $\text{len}(v)$. Since the assertion is obvious for any $(w, v) \in QO_n(S)$ with $\text{len}(v) = 1$, we may assume that, when we try to prove the assertion for some n -quasioverlap, we have already proved it for all n -quasioverlaps with length smaller than $\text{len}(v)$.

Let us consider $(w, v) \in QO_n(S)$ and denote by w' the path $\rho_{n-1}^v(w)$. Since $(w', v) \in QO_{n-1}(S)$, we can apply the induction hypothesis. Thus, there exists $\tilde{v} \in Q_{>0}$ such that $\tilde{v}|_r v, \tilde{v}w' \in O_{n-1}(S)$, $\tilde{v}\rho_i^v(w')|_l \rho_i(\tilde{v}w')$ for odd i , and $\rho_i(\tilde{v}w')|_l \tilde{v}\rho_i^v(w')$ for even i . Note also that $\rho_i^v(w') = \rho_i^v(w)$ for $0 \leq i \leq n - 1$. Let us consider three cases:

Case 1: $\rho_{n-2}(\tilde{v}w')|_S^a \tilde{v}w$. In this case, we can simply define $v' = \tilde{v}$. It is easy to see that $v'w$ is an n -overlap with $\rho_{n-1}(v'w) = v'w'$ satisfying all the required conditions.

Case 2: $\rho_{n-2}(\tilde{v}w')|_S \tilde{v}w$. In this case, $\tilde{v}\rho_{n-2}^v(w)|_l \rho_{n-2}(\tilde{v}w')$, $\tilde{v}\rho_{n-2}^v(w) \neq \rho_{n-2}(\tilde{v}w')$, and, hence, $2 \nmid n$.

Suppose that i is odd and $\rho_i(\tilde{v}w')|_S \tilde{v}\rho_{i+2}^v(w)$. Since

$$\begin{aligned} &\rho_i(\tilde{v}w')|_l \rho_{i+1}(\tilde{v}w'), \quad \rho_{i+1}(\tilde{v}w')|_l \tilde{v}\rho_{i+1}^v(w'), \\ &\tilde{v}\rho_{i+1}^v(w')|_l \tilde{v}\rho_{i+2}^v(w), \quad \text{and} \quad \rho_i(\tilde{v}w') \neq \rho_{i+1}(\tilde{v}w'), \end{aligned}$$

we know that $\rho_i(\tilde{v}w') \neq \tilde{v}\rho_{i+2}^v(w)$.

Also, $\tilde{v}\rho_i^v(w)|_S^a \tilde{v}\rho_{i+2}^v(w)$ implies $\tilde{v}\rho_i^v(w)|_S \rho_i(\tilde{v}w')$ and $\tilde{v}\rho_i^v(w) \neq \rho_i(\tilde{v}w')$. Thus, it follows from $\rho_{i-2}(\tilde{v}w')|_S^a \rho_i(\tilde{v}w')$ that $\rho_{i-2}(\tilde{v}w')|_S \tilde{v}\rho_i^v(w)$.

Now, the descending induction on i gives us $\rho_i(\tilde{v}w')|_S \tilde{v}\rho_{i+2}^v(w)$ and $\tilde{v}\rho_i^v(w)|_S \rho_i(\tilde{v}w')$ for all odd i such that $1 \leq i \leq n - 2$. As before, we have $\tilde{v}\rho_1^v(w) \neq \rho_1(\tilde{v}w')$. Consequently, $\tilde{v} \neq v$ and there is $v_0 \in Q_{>0}$ such that $v = v_0\tilde{v}$. Then $(\tilde{v}w, v_0) \in QO_{n-1}(S)$ and $\text{len}(v_0) < \text{len}(v)$. Thus, we have v'_0 such that $v'_0\tilde{v}w \in O_n(S)$ satisfies all required conditions. It is easy to check that we can take $v' = v'_0\tilde{v}$ in this case.

Case 3: There is a presentation $\tilde{v}w = \rho_{n-2}(\tilde{v}w')u'su$ with $u \in Q_{>0}$, $u' \in Q$ and $s \in S$. Let us choose such a presentation with minimal $\text{len}(u')$. Since $\rho_{n-2}(\tilde{v}w')|_l \tilde{v}\rho_{n-2}^v(w)$ and $\rho_{n-2}(\tilde{v}w') \neq \tilde{v}\rho_{n-2}^v(w)$, n must be even.

It follows from the minimality of u' that $\rho_{n-2}(\tilde{v}w')|_S^a \rho_{n-2}(\tilde{v}w')u's$. Since $\rho_{n-2}(\tilde{v}w')|_S \tilde{v}w'$, we know that $\tilde{v}w'|_I \rho_{n-2}(\tilde{v}w')u's$ and, hence, $\rho_{n-2}(\tilde{v}w')u's|_S \tilde{v}w'$. Since $\tilde{v}\rho_{n-2}^v(w)|_S^a \tilde{v}w'$, we have also $\tilde{v}\rho_{n-2}^v(w)|_S \rho_{n-2}(\tilde{v}w')u's$.

For even $2 \leq i \leq n - 2$, we have $\tilde{v}\rho_{i-1}^v(w)|_I \rho_{i-1}(\tilde{v}w')|_I \tilde{v}\rho_i^v(w)$ and $\tilde{v}\rho_{n-1}^v(w)|_S \tilde{v}\rho_i^v(w)$. Thus, $\rho_i(\tilde{v}w')|_S \tilde{v}\rho_i^v(w)$ for such i . Suppose that $\rho_i(\tilde{v}w')|_S \tilde{v}\rho_i^v(w)$ and $\rho_i(\tilde{v}w') \neq \tilde{v}\rho_i^v(w)$ for some $2 \leq i \leq n - 2$. Since $\tilde{v}\rho_{i-2}^v(w)|_S^a \tilde{v}\rho_i^v(w)$, we have $\tilde{v}\rho_{i-2}^v(w)|_S \rho_i(\tilde{v}w')$. Moreover, $\rho_{i-2}(\tilde{v}w')|_S^a \rho_i(\tilde{v}w')$ implies $\rho_{i-2}(\tilde{v}w') \neq \tilde{v}\rho_{i-2}^v(w)$.

Thus, the descending induction on i gives us $\rho_i(\tilde{v}w')|_S \tilde{v}\rho_i^v(w)$ and $\tilde{v}\rho_{i-2}^v(w)|_S \rho_i(\tilde{v}w')$ for even i , $2 \leq i \leq n - 2$. In particular, we get $\tilde{v}|_S \rho_2(\tilde{v}w')$. On the other hand, there is $r \in S$ such that $r|_r \rho_2(\tilde{v}w')$ and $r \neq \rho_2(\tilde{v}w')$. As a consequence, $\rho_2(\tilde{v}w') = v'_0 r$ and $\tilde{v} = v'_0 v_0$ for some $v_0, v'_0 \in Q_{>0}$. In particular, $\text{len}(v_0) < \text{len}(v)$. It follows from our arguments that $(w, v_0) \in QO_n(S)$ with $v'_0 r_{n-1}^{v_0}(w) = \rho_{n-2}(\tilde{v}w')u's$. We thus have got v' such that $v'w \in O_n(S)$ satisfies all required conditions. \square

Corollary 9. For any reduced set of paths S ,

$$\max o_{n+m}(S) \leq \max o_n(S) + \max o_m(S) - 1$$

and

$$\min o_{n+m}(S) \geq \min o_n(S) + \min o_m(S) - \text{len}(S) + 1.$$

Proof. The proof follows from Theorem 8 and the fact that any $w \in O_{n+m}(S)$ can be represented in the form $w = w'w''$ with $w' \in O_n(S)$ and $w'' \in QO_m(S)$. \square

From now on, we fix an admissible order \geq on the set of paths in Q , [6]. More precisely, this means that there is a well order \geq such that for any paths $p, q, u, v \in Q$,

- if $p \geq q$, then $upv \geq uqv$ if the products are paths.
- $p \geq q$ if $q|p$.

Given a linear space V , its basis B , and $x \in V$, we call the sum $\sum_{i=1}^m \alpha_i b_i$ a *reduced expression* of x as a linear combination of the elements of B if $\alpha_i \in \mathbf{k}^*$, $b_i \in B$ for $1 \leq i \leq m$, $b_i \neq b_j$ for $1 \leq i < j \leq m$, and $x = \sum_{i=1}^m \alpha_i b_i$.

For $x \in \mathbf{k}Q$, $\text{tip}(x)$ is the maximal path of Q , with respect to the order \geq , appearing in the reduced expression of x as a linear combination of paths. A subset $G \subset I$ is called a *Gröbner basis* of I if for any $x \in I$, there exists $g \in G$ such that $\text{tip}(g) | \text{tip}(x)$.

From now on, we fix $A = \mathbf{k}Q/I$, where I is an ideal of $\mathbf{k}Q$ that has a finite Gröbner basis G .

We will use the notation of Green and Solberg in [11]. Let X be a graded A -module and let $P_0(X) \xrightarrow{\mu_X} X$ be its minimal graded projective cover. Suppose also that X is finitely presented. The projective module $P_0(X)$ can be presented in the form $P_0(X) = \bigoplus_{i \in T_0} f_i^0 A$, where T_0 is a finite set and, for any $i \in T_0$ there exist

$e_i \in Q_0$, $m_i \in \mathbb{Z}$, and an isomorphism of graded modules $f_i^0 A \cong e_i A[m_i]$ that sends f_i^0 to e_i .

Consider the graded space $V_X = \bigoplus_{i \in T_0} f_i^0 \mathbf{k}Q$. The set whose elements are of the form $f_i^0 p$, where $i \in T_0$ and p is a path ending in e_i , is a basis of V_X , we denote this set by B_X and introduce the following well order on it. We set $f_i^0 p \geq f_j^0 q$ if either $p > q$ or $p = q$ and $i \geq j$. We say that $f_i^0 p$ divides $f_j^0 q$ on the right if $i = j$ and $p|_r q$. For $x \in V_X$, we write $\text{tip}(x)$ for the maximal element of B_X , with respect to the order \geq , appearing in the reduced expression of x as a linear combination of elements of B_X . The set x_1, \dots, x_l of nonzero elements of V_X is called *right tip reduced* if $\text{tip}(x_i)$ does not divide $\text{tip}(x_j)$ on the right for any $1 \leq i, j \leq l$, $i \neq j$.

By [11, Proposition 5.1], there are finite sets T_1 and T'_1 , elements $h_i^1 \in V_X$ ($i \in T_1$), and elements $h_i^{1'} \in V_X$ ($i \in T'_1$) such that

- (1) any element h in the kernel of the composition $V_X \rightarrow P_0(X) \xrightarrow{\mu_X} X$ can be uniquely represented in the form $h = \sum_{i \in T_1} f_i + \sum_{i \in T'_1} f'_i$, where $f_i \in h_i^1 \mathbf{k}Q$ and $f'_i \in h_i^{1'} \mathbf{k}Q$,
- (2) for any element $h \in \{h_i^1\}_{i \in T_1} \cup \{h_i^{1'}\}_{i \in T'_1}$, there exists $e_h \in Q_0$ such that $he_h = h$,
- (3) $h_i^{1'} \in \bigoplus_{j \in T_1} h_j^0 I$ for any $i \in T'_1$,
- (4) the set $\{h_i^1\}_{i \in T_1} \cup \{h_i^{1'}\}_{i \in T'_1}$ is right tip reduced.

Moreover, it is clear that all the elements in the set $\{h_i^1\}_{i \in T_1} \cup \{h_i^{1'}\}_{i \in T'_1}$ can be chosen homogeneous. Let us define $\bar{P}_1(X) = \bigoplus_{i \in T_1} f_i^1 A$, where, for any $i \in T_1$ there exists an isomorphism of graded modules $f_i^1 A \cong e_{h_i^1} A[-\text{deg}(h_i^1)]$ that sends f_i^1 to $e_{h_i^1}$. Here $\text{deg}(h_i^1)$ denotes the degree of h_i^1 . Note that the order in the basis of V_X induces an order on the set $\{f_i^1\}_{i \in T_1}$. In this way, we obtain a graded projective presentation

$$\bar{P}_1(X) \xrightarrow{\bar{d}_0(X)} P_0(X) \xrightarrow{\mu_X} X$$

of X , where $\bar{d}_0(X)$ sends f_i^1 to the class of h_i^1 in $P_0(X)$ for any $i \in T_1$. Let also

$$\dots \xrightarrow{d_n(X)} P_n(X) \xrightarrow{d_{n-1}(X)} \dots \xrightarrow{d_0(X)} P_0(X) \xrightarrow{\mu_X} X$$

be the minimal graded A -projective resolution of X .

As before, given a graded A -module M , we will denote by M_j its j th homogeneous component. The next theorem will be crucial for us. It allows to estimate the degrees of the generators of the terms of the resolution constructed using the algorithm from [11].

Theorem 10. *Let us fix the notation as above. If k and l are the minimal and the maximal degrees of f_i^1 , $i \in T_1$, then, for any*

$$n \geq 1, \quad P_n(X) = \left(\bigoplus_{j=k+\min q_0 n(\text{tip}(G))}^{l+\max q_0 n(\text{tip}(G))} P_n(X)_j \right) A.$$

Proof. The algorithm described in [11, Sec. 3] gives a graded projective resolution

$$\dots \xrightarrow{\bar{d}_n(X)} \bar{P}_n(X) \xrightarrow{\bar{d}_{n-1}(X)} \dots \xrightarrow{\bar{d}_0(X)} P_0(X)(\xrightarrow{\mu_X} X),$$

of X . Due to this algorithm $\bar{P}_n(X)$ can be presented in the form $\bar{P}_n(X) = \bigoplus_{i \in T_n} f_i^n A$ and $\bar{d}_{n-1}(X)$ sends f_i^n to $h_i^n \in \bigoplus_{i \in T_{n-1}} f_i^{n-1} A$, where all these elements satisfy the following property. If $\text{tip}(h_i^n) = f_j^{n-1} p$, then $\text{tip}(h_j^{n-1}) p \in Q_S$ and if $q | \text{tip}(h_j^{n-1}) p$, $q \neq \text{tip}(h_j^{n-1}) p$, then q is not S -path.

For $i \in T_n$, let us introduce $p_i \in Q$ and $t(i) \in T_{n-1}$ by the equality $\text{tip}(h_i^n) = f_{t(i)}^{n-1} p_i$. It is possible to prove inductively on n that $(w, v) \in QO_n(S)$ for $w = p_{t^{n-2}(i)} \dots p_{t(i)} p_i$ and some $v \in Q_{>0}$ such that $v |_r \text{tip}(h_{t^{n-1}(i)}^1)$. Consequently, the required statement follows from the equality

$$\deg(f_i^n) = \deg(f_{t^{n-1}(i)}^1) + \sum_{m=0}^{n-2} \text{len}(p_{t^m(i)}) = \deg(f_{t^{n-1}(i)}^1) + \text{len}(w). \quad \square$$

In other words, for any $n \geq 1$, we are able to deduce the possible degrees of the generators of $P_n(X)$ from those of the generators of $\bar{P}_1(X)$ and the lengths of n -quasioverlaps for $\text{tip}(G)$.

We obtain two corollaries.

Corollary 11. *Let $A = \mathbf{k}Q/I$, X be a graded finitely presented A -module and let G be a homogeneous Gröbner basis for I such that $\text{tip}(G)$ is reduced. Let $P_\bullet(X)$ be a minimal graded A -projective resolution of the graded A -module X and $\bar{P}_1(X)$ be as above. If k and l are the minimal and the maximal degrees of f_i^1 , $i \in T_1$, then, for any*

$$n \geq 1, \quad P_n(X) = \left(\begin{array}{c} l + \max_{o_n}(\text{tip}(G)) - 1 \\ \bigoplus \\ j = k + \min_{o_n}(\text{tip}(G)) - \text{len}(\text{tip}(G)) + 1 \end{array} P_n(X)_j \right) A.$$

Proof. It follows directly from Theorems 8 and 10. □

Corollary 12. *Let $A = \mathbf{k}Q/I$ where I has a homogeneous Gröbner basis G such that $\text{len}(\text{tip}(G)) \leq s$ and $\max_{o_2}(\text{tip}(G)) \leq s + 1$. If additionally $\min_{o_1}(\text{tip}(G)) = s$, then the algebra A is s -Koszul.*

Proof. It follows from Corollaries 9 and 11 since $\bar{P}_1(A_0) = \bigoplus_{\alpha \in Q_1} e_\alpha A[-1]$, where e_α is the starting vertex of the arrow α . More precisely, we get by induction on $i \geq 3$ that

$$\max_{o_i}(\text{tip}(G)) \leq \max_{o_{i-2}}(\text{tip}(G)) + \max_{o_2}(\text{tip}(G)) - 1 \leq \chi_s(i - 2) + s = \chi_s(i).$$

If additionally $\min_{o_1}(\text{tip}(G)) = s$, then all the elements of $\text{tip}(G)$ have length s . Then we get that $P_2(A_0)$ is generated in degree $s + 1$ and that if the minimal generating degree for $P_i(A_0)$ is m , then the minimal generating degree for $P_{i+2}(A_0)$ is not less than $m + s$. Then we get by induction that $P_i(A_0)$ is generated in degree $\chi_s(i)$. □

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