CONTINUOUS TIME RANDOM WALKS AND THE CAUCHY PROBLEM FOR THE HEAT EQUATION

By

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Abstract. We deal with anomalous diffusions induced by continuous time random walks - CTRW in $\mathbb{R}^n$. A particle moves in $\mathbb{R}^n$ in such a way that the probability density function $u(\cdot, t)$ of finding it in region Ω of $\mathbb{R}^n$ is given by $\int_\Omega u(x,t)dx$. The dynamics of the diffusion is provided by a space time probability density $J(x,t)$ compactly supported in $\{t \geq 0\}$. For t large enough, u satisfies the equation

$$u(x,t) = [(J - \delta) * u](x,t),$$

where δ is the Dirac delta in space-time. We give a sense to a Cauchy type problem for a given initial density distribution f. We use Banach fixed point method to solve it and prove that under parabolic rescaling of J, the equation tends weakly to the heat equation and that for particular kernels J, the solutions tend to the corresponding temperatures when the scaling parameter approaches 0.

1 Introduction and statement of the results

We are concerned with a probabilistic description of the motion of a particle in the space $\mathbb{R}^n$. As is usual, we write

$$\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n \text{ and } t \geq 0\}.$$

Sometimes we also consider the whole space-time

$$\mathbb{R}^{n+1} = \{(x,t) : x \in \mathbb{R}^n \text{ and } t \in \mathbb{R}\}.$$

The x variable is thought as a space variable, while t represents time.

For fixed t, we denote by $u(x,t)$ the probability density of the position of the particle at time t. Precisely, for a given Borel set E in $\mathbb{R}^n$, the quantity $\mathcal{P}(t,E) = \int_E u(x,t)dx$ measures the probability of finding the particle in E at time t.

The general problem is to find $u(x,t)$ when the dynamics of the system is known and some initial state is given.

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Regarding the dynamics of the system, we deal with anomalous diffusions, more precisely, with continuous time random walks (CTRW). For a comprehensive introduction to the subject, we refer to [6]. A CTRW in $\mathbb{R}^n$ is provided by a space-time probability density function and kernel $J(x,t)$, defined on $\mathbb{R}^{n+1}$. In this model, the particle has a probability density function $u(x,t)$ of arrival at position $x \in \mathbb{R}^n$ at time $t > 0$ which depends on the events of arrival at any $y \in \mathbb{R}^n$ (sometimes only on the events of arrival at any y in some neighborhood of x) at any previous time $s < t$. This dependence is given precisely by the convolution in $\mathbb{R}^{n+1}$ of J with u itself. In other words, for $t \geq 0$ and $x \in \mathbb{R}^n$,

$$u(x,t) = (J * u)(x,t) = \iint_{\mathbb{R}^{n+1}} J(x-y, t-s)u(y,s)\,dy\,ds. \tag{1.1}$$

The physical condition of the dependence of the current position of the particle only on the past ($s < t$) gives us the first natural condition on J:

(J1) $\operatorname{supp} J \subset \mathbb{R}^{n+1}_+$.

On the other hand, since J is a density in $\mathbb{R}^{n+1}$, we must have
(J2) $J \geq 0$, and
(J3) $J \in L^1(\mathbb{R}^{n+1})$ and $\iint_{\mathbb{R}^{n+1}} J(x,t)\,dx\,dt = 1$.

Following the notation in [6], we call the density function defined in $\mathbb{R}^n$ by

$$\lambda(x) = \int_{\mathbb{R}} J(x,t)\,dt$$

the **jump length probability function**. Notice that from *(J1)*, we have $\lambda(x) = \int_{\mathbb{R}^+} J(x,t)\,dt$. On the other hand, the **waiting time probability function** is given by

$$\tau(t) = \int_{\mathbb{R}^n} J(x,t)\,dx.$$

Regarding the initial condition, let us first assume that the particle is localized at the origin of $\mathbb{R}^n$ for $t < 0$. In other words, $u(x,t) = \delta_0(x)$ for $t < 0$. Hence, since $u(x,t)$ for $t \geq 0$ needs to satisfy (1.1), from *(J1)* we must have that

$$\begin{aligned} u(x,0) &= \iint_{\mathbb{R}^{n+1}} J(x-y,-s)u(y,s)dy\,ds \\ &= \int_{\mathbb{R}^-}\left(\int_{\mathbb{R}^n} J(x-y,-s)u(y,s)dy\right) ds \\ &= \int_{\mathbb{R}^-}\left(\int_{\mathbb{R}^n} J(x-y,-s)\delta_0(y)dy\right) ds \\ &= \int_{\mathbb{R}^-} J(x,-s)ds \\ &= \lambda(x). \end{aligned}$$

In other words, the deterministic situation *the particle is at the origin for* $t < 0$ produces *immediately* at time $t = 0$ a random situation modeled precisely by the jump length probability function $\lambda(x)$ associated to the density J.

More generally, if the position at time $t < 0$ of the particle distributes as indicates the density $f(x)$, then $u(x, 0) = (\lambda * f)(x)$. In this framework, the basic initial problem we are interested in, takes the following form. Given $J(x, t)$ and $f(x)$, find $u(x, t)$ for $(x, t) \in \mathbb{R}^{n+1}_+$ such that

$$(P) \qquad \begin{cases} u(x,t) = (J * \overline{u})(x,t), & x \in \mathbb{R}^n,\ t \geq 0; \\ \overline{u}(x,t) = \begin{cases} f(x), & t < 0; \\ u(x,t), & t \geq 0. \end{cases} \end{cases}$$

Sometimes, to emphasize the data J and f in (P), we write $P(J, f)$ for the problem P and $u(J, f)$ for its solution.

Let us observe that the expected initial condition is attained since, taking $t = 0$ in the first equation in (P), we get

$$u(x, 0) = (J * \overline{u})(x, 0) = \iint J(x - y, -s) f(y)\, dy\, ds = (\lambda * f)(x).$$

We consider wide families of kernels J, but there is one, the parabolic mean value kernels, which plays a more significant role for our subsequent analysis. We denote by $\mathscr{H}$ (for heat) these special occurrences of J. Let us introduce the most known of these kernels $\mathscr{H}$; see [7] or [5]. Denote by $\mathcal{W}(x, t)$ the Weierstrass kernel for $t > 0$ and $x \in \mathbb{R}^n$. Precisely, $\mathcal{W}(x, t) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$. Set

$$E = \{(x, t) \in \mathbb{R}^{n+1}_+ : \mathcal{W}(x, t) \geq 1\} \quad \text{and} \quad \mathcal{H}(x, t) = \frac{1}{4} \mathcal{X}_E(x, t) \frac{|x|^2}{t^2}.$$

It is easy to check that $\mathcal{H}$ satisfies properties *(J1)*, *(J2)* and *(J3)* stated above. Moreover,

(J4) $\mathcal{H}$ has compact support in $\mathbb{R}^{n+1}$;

(J5) $\mathcal{H}$ is radial as a function of $x \in \mathbb{R}^n$ for each t.

The outstanding fact regarding $\mathcal{H}$ is given by its role in the mean value formula for temperatures. If $v(x, t)$ is a solution of the heat equation $\frac{\partial v}{\partial t} = \triangle v$ in a domain Ω in $\mathbb{R}^{n+1}$, then, for $(x, t) \in \Omega$ and r small enough, we have

$$v(x, t) = \iint \mathcal{H}_r(x - y, t - s) v(y, s) dy\, ds,$$

where $\mathcal{H}_r$ denotes the parabolic r-mollifier of $\mathcal{H}$. Precisely

$$\mathcal{H}_r(x, t) = \frac{1}{r^{n+2}} \mathcal{H}\left(\frac{x}{r}, \frac{t}{r^2}\right) = \frac{1}{r^n} \mathcal{X}_{E(r)}(x, t) \frac{|x|^2}{t^2},$$

with $E(r) = \{(x,t) \in \mathbb{R}^{n+1}_+ : \mathcal{W}(x,t) \geq r^{-n}\}$. Figure 1 depicts the support $E(r)$ of $\mathcal{H}_r$.

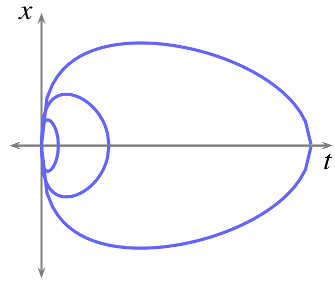

Figure 1. Sets $E(r)$ for $n = 1$ and $r = 1/2, 1/4, 1/8$.

In the sequel, for any kernel $J(x,t)$ and any $r > 0$, we denote by $J_r(x,t)$ the parabolic approximation to the identity given by $J_r(x,t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$. Moreover, the notation $v_r(x,t)$ or even $f_r(x)$ for functions depending on space-time or, only on the space variable always the same meaning. Precisely,

$$v_r(x,t) = r^{-n-2} v(r^{-1}x, r^{-2}t) \quad \text{and} \quad f_r(x) = r^{-n-2} f(r^{-1}x).$$

The results of this paper are in the spirit of those in [4] and [3]. Instead of dealing with generalization of boundary conditions, we are concerned with diffusion problems in the whole space $\mathbb{R}^n$, and the initial condition is generalized.

Let us state the main results of this paper. The first one is the weak convergence to the heat equation.

Theorem 1. Assume that $J(x,t)$ satisfies *(J2)*, *(J3)*, *(J4)* and *(J5)*. Then, for each φ in the Schwartz class of $\mathbb{R}^{n+1}$,

$$\lim_{r\to 0} \frac{1}{r^2}\left(J_r - \delta\right) * \varphi = \mu \frac{\partial \varphi}{\partial t} + \nu \triangle \varphi,$$

uniformly on $\mathbb{R}^{n+1}$, where $\mu = -\iint tJ(x,t)dx\,dt$ and $\nu = \frac{1}{2n}\iint |x|^2 J(x,t)dx\,dt$.

The second result concerns the existence of solutions for problem (P). For a given Lipschitz function $f \in \mathcal{C}^{0,\gamma}(\mathbb{R}^n)$ of order γ, we denote by $[f]_\gamma$ the corresponding seminorm of f. In the next statement, $\mathcal{C}$ denotes the space of continuous functions.

Theorem 2. Assume that $J(x,t)$ satisfies *(J1)*, *(J2)*, *(J3)*, and *(J4)*. Set $\alpha = \sup\{\beta : \iint_{s\leq\beta} J(y,s)dyds < 1\}$. Let $f \in L^\infty(\mathbb{R}^n)$ be given. Then there exists a unique solution $u(x,t)$ of (P) in the space $(\mathcal{C} \cap L^\infty)(\mathbb{R}^{n+1}_+)$. If $f \in (L^1 \cap L^\infty)(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} f(x)dx$ for every $t \geq 0$. In particular, if f is a density function, so is $u(\cdot,t)$ for every $t \geq 0$. Moreover, if f belongs to $(\mathcal{C}^{0,\gamma} \cap L^\infty)(\mathbb{R}^n)$, then

$$|u(x,t) - f(x)| \leq C[f]_\gamma \tag{1.2}$$

for $(x,t) \in \mathbb{R}^n \times [0,\alpha]$ and some C which does not depend on f.

The next result, which is interesting in and of itself, contains a maximum principle which is used in the proof of Theorem 4. Precisely, the supremum of the probability density function in the future of $\alpha = \sup\{\beta : \iint_{s\leq\beta} J(y,s)dy\,ds < 1\}$ coincides with its supremum in $\mathbb{R}^n \times [0,\alpha]$.

Theorem 3. Let J be a kernel satisfying *(J1)*, *(J2)*, *(J3)*, and *(J4)*. Let $w(x,t)$ be a bounded function defined in $\mathbb{R}^{n+1}_+$ such that

$$w(x,t) = \iint J(x-y,t-s)w(y,s)dy\,ds \tag{1.3}$$

for $(x,t) \in \mathbb{R}^n \times [\alpha,+\infty)$. Then

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} |w(x,t)| = \sup_{(x,t)\in\mathbb{R}^n\times[0,\alpha]} |w(x,t)|\,.$$

Let us proceed to state the fourth result of the paper.

Theorem 4. For each $H \in \mathscr{H}$, there exists $C > 0$ such that, for every $r > 0$ and every $f \in (\mathcal{C}^{0,\gamma} \cap L^\infty)(\mathbb{R}^n)$,

$$\|u(H_r,f) - u\|_{L^\infty(\mathbb{R}^{n+1}_+)} \leq C[f]_\gamma r^\gamma,$$

where u is the temperature in $\mathbb{R}^{n+1}_+$ given by $u(x,t) = (\mathcal{W}(\cdot,t) * f)(x)$.

Let us finally remark that in [2] the authors prove the Hölder regularity for solutions of the master equation associated to CTRW's.

In Section 2, we prove the weak convergence of parabolic rescalings to a heat equation. In Section 3, we show existence of solution for the Cauchy nonlocal problem. Section 4 is devoted to proving the maximum principle contained in Theorem 3. Finally, Section 5 deals with convergence of solutions of rescalings of (P) to temperatures.

2 Some space time nonlocal parabolic operators and their weak limit. Proof of Theorem 1

Since $\iint J(y,s)dyds = 1$, applying Taylor's formula, we get, for $0 < r < 1$,

$$\begin{aligned}
&\iint J_r(x-y,t-s)\varphi(y,s)dyds - \varphi(x,t)\\
&\quad = \iint J_r(x-y,t-s)(\varphi(y,s)-\varphi(x,t))dy\,ds\\
&\quad = \iint J_r(x-y,t-s)\left[\sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}(x,t)(y_i-x_i)+\frac{\partial\varphi}{\partial t}(x,t)(s-t)\right.\\
&\qquad\quad \left.+\frac{1}{2}(y-x,s-t)D^2\varphi(x,t)(y-x,s-t)^t + R(y-x,s-t)\right]dy\,ds,
\end{aligned}$$

where D^2 denotes the Hessian matrix of the second derivatives of φ with respect to x and t, and $|R(x,t)| = O(|x|^2+t^2)^{3/2}$.

The last integral in the above identities can be written as the sums of the following seven terms:

$$\begin{aligned}
I &= \sum_{i=1}^{n}\frac{\partial\varphi}{\partial x_i}(x,t)\left(\iint (y_i-x_i)J_r(x-y,t-s)dy\,ds\right),\\
II &= \frac{\partial\varphi}{\partial t}(x,t)\left(\iint (s-t)J_r(x-y,t-s)dy\,ds\right),\\
III &= \sum_{ij=1,i\neq j}^{n}\frac{\partial^2\varphi}{\partial x_i\partial x_j}(x,t)\left(\frac{1}{2}\iint (y_i-x_i)(y_j-x_j)J_r(x-y,t-s)dy\,ds\right),\\
IV &= \sum_{i=1}^{n}\frac{\partial^2\varphi}{\partial x_i^2}(x,t)\left(\frac{1}{2}\iint (y_i-x_i)^2J_r(x-y,t-s)dy\,ds\right),\\
V &= \sum_{i=1}^{n}\frac{\partial^2\varphi}{\partial x_i\partial t}(x,t)\left(\frac{1}{2}\iint (y_i-x_i)(s-t)J_r(x-y,t-s)dy\,ds\right),\\
VI &= \frac{\partial^2\varphi}{\partial t^2}(x,t)\left(\frac{1}{2}\iint (s-t)^2J_r(x-y,t-s)dy\,ds\right),
\end{aligned}$$

and

$$VII = \iint J_r(x-y,t-s)R(y-x,s-t)dy\,ds.$$

Since, for t fixed, J is radial as a function of x, we see that I, III and V vanish.

For the other four integrals, we perform the parabolic change of variables $(z, \zeta) = (\frac{x-y}{r}, \frac{t-s}{r^2})$ to obtain

$$\begin{aligned} II &= \frac{\partial \varphi}{\partial t}(x,t) r^2 \left(- \iint \zeta J(z,\zeta) dz\, d\zeta \right), \\ IV &= \sum_{i=1}^{n} \frac{\partial^2 \varphi}{\partial x_i^2}(x,t) r^2 \left(\frac{1}{2} \iint z_i^2 J(z,\zeta) dz\, d\zeta \right), \\ VI &= \frac{\partial^2 \varphi}{\partial t^2}(x,t) r^4 \left(\frac{1}{2} \iint \zeta^2 J(z,\zeta) dz\, d\zeta \right), \\ VII &= \iint J(z,\zeta) R(rz, r^2\zeta) dz\, d\zeta. \end{aligned}$$

Finally, since, as a function of r, VI and VII are of order at least r^3 close to 0, we see that

$$\lim_{r\to 0} \frac{1}{r^2} \left[(J_r - \delta) * \varphi \right](x,t) = \lim_{r\to 0} \left(\frac{II}{r^2} + \frac{IV}{r^2} \right) = \mu \frac{\partial \varphi}{\partial t}(x,t) + \nu \triangle \varphi(x,t),$$

where μ and ν are defined as in the statement of Theorem 1. That convergence is uniform in $\mathbb{R}^{n+1}$ follows from the fact that φ is a Schwartz function, and so VI and VII converge to 0 uniformly.

Lemma 5. If $J = \mathcal{H}$, then $\mu = -\nu$; and the limit equation in Theorem 1 is the heat equation multiplied by a constant.

Proof. All we need to show is that

$$\iint \mathcal{H}(y,s) s\, dy\, ds = \frac{1}{2n} \iint \mathcal{H}(y,s) |y|^2\, dy\, ds. \tag{2.1}$$

Let us compute both of these integrals in terms of the Euler gamma function and the area surface of the unit ball of $\mathbb{R}^n$, S^{n-1}. On one hand,

$$
\begin{aligned}
\iint \mathcal{H}(y,s)s\,dy\,ds &= \frac{1}{4}\iint \mathcal{X}_{E(1)}(-y,-s)\frac{|y|^2}{s^2}s\,dy\,ds = -\frac{1}{4}\iint_{E(1)} \frac{|y|^2}{s}\,dy\,ds\\
&= \frac{1}{4}\int_{-\frac{1}{4\pi}}^{0}\int_{B\left(0,(2ns\ln(4\pi(-s)))^{\frac{1}{2}}\right)} \frac{|y|^2}{-s}\,dy\,ds\\
&= \frac{1}{4}\int_{-\frac{1}{4\pi}}^{0}\frac{1}{-s}\int_{0}^{(2ns\ln(4\pi(-s)))^{\frac{1}{2}}} \rho^{n+1}\int_{S^{n-1}} d\sigma\,d\rho\,ds\\
&= \frac{\sigma(S^{n-1})}{4(n+2)}\int_{-\frac{1}{4\pi}}^{0}\frac{1}{-s}\,(2ns\ln(4\pi(-s)))^{\frac{n+2}{2}}\,ds\\
&= \frac{\sigma(S^{n-1})}{4(n+2)}\int_{0}^{1}\frac{1}{t}\left(\frac{n}{2\pi}t(-\ln(t))\right)^{\frac{n+2}{2}}\,dt\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}}{4(n+2)2^{\frac{n+2}{2}}\pi^{\frac{n+2}{2}}}\int_{0}^{\infty} e^{-\theta(\frac{n+2}{2})}\theta^{\frac{n+2}{2}}\,d\theta\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}}{4(n+2)2^{\frac{n+2}{2}}\pi^{\frac{n+2}{2}}}\frac{2}{(n+2)}\frac{2^{\frac{n+2}{2}}}{(n+2)^{\frac{n+2}{2}}}\int_{0}^{\infty} e^{-\zeta}\zeta^{\frac{n+2}{2}}\,d\zeta\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}}{2(n+2)^{\frac{n+6}{2}}\pi^{\frac{n+2}{2}}}\Gamma\left(\frac{n+4}{2}\right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{1}{2n}\iint \mathcal{H}(y,s)|y|^2\,dy\,ds &= \frac{1}{8n}\iint \mathcal{X}_{E(1)}(-y,-s)\frac{|y|^2}{s^2}|y|^2\,dy\,ds = \frac{1}{8n}\iint_{E(1)} \frac{|y|^4}{s^2}\,dy\,ds\\
&= \frac{1}{8n}\int_{-\frac{1}{4\pi}}^{0}\int_{B\left(0,(2ns\ln(4\pi(-s)))^{\frac{1}{2}}\right)} \frac{|y|^4}{s^2}\,dy\,ds\\
&= \frac{1}{8n}\int_{-\frac{1}{4\pi}}^{0}\frac{1}{s^2}\int_{0}^{(2ns\ln(4\pi(-s)))^{\frac{1}{2}}} \rho^{n+3}\int_{S^{n-1}} d\sigma\,d\rho\,ds\\
&= \frac{\sigma(S^{n-1})}{8n(n+4)}\int_{-\frac{1}{4\pi}}^{0}\frac{1}{s^2}\,(2ns\ln(4\pi(-s)))^{\frac{n+4}{2}}\,ds\\
&= \frac{\sigma(S^{n-1})}{8(n+4)}\frac{n^{\frac{n+4}{2}}4\pi}{2^{\frac{n+4}{2}}\pi^{\frac{n+4}{2}}}\int_{0}^{1}\frac{1}{t^2}\,(t(-\ln(t)))^{\frac{n+2}{2}}\,dt\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}4\pi}{8(n+4)2^{\frac{n+4}{2}}\pi^{\frac{n+4}{2}}}\int_{0}^{\infty} e^{-\theta(\frac{n+2}{2})}\theta^{\frac{n+4}{2}}\,d\theta\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}4\pi}{8(n+4)2^{\frac{n+4}{2}}\pi^{\frac{n+4}{2}}}\frac{2}{(n+2)}\frac{2^{\frac{n+4}{2}}}{(n+2)^{\frac{n+4}{2}}}\int_{0}^{\infty} e^{-\zeta}\zeta^{\frac{n+4}{2}}\,d\zeta\\
&= \frac{\sigma(S^{n-1})n^{\frac{n+2}{2}}}{(n+4)(n+2)^{\frac{n+6}{2}}\pi^{\frac{n+2}{2}}}\Gamma\left(\frac{n+6}{2}\right).
\end{aligned}
$$

Now, since $\Gamma(z+1) = z\Gamma(z)$, we have that $\frac{1}{n+4}\Gamma\left(\frac{n+6}{2}\right) = \frac{1}{2}\Gamma\left(\frac{n+4}{2}\right)$. □

3 Existence of solutions for (*P*). Proof of Theorem 2

Let $J(x,t)$ be a kernel defined in space time $\mathbb{R}^{n+1}$ satisfying *(J1)–(J4)*. Let $f \in L^\infty(\mathbb{R}^n)$. Following the ideas in [4], [3], and [1], we solve (*P*) by iterated application of the Banach fixed point theorem. From *(J3)* and *(J4)*, we have that $\alpha = \sup\{\beta : \iint_{s<\beta} J(x,s)dx\,ds < 1\}$ is positive and finite. For the first step in the use of the fixed point theorem, we consider the Banach space $\mathscr{B}_1 = (\mathcal{C} \cap L^\infty)(\mathbb{R}^n \times [0,\alpha/2])$ with the L^∞ norm.

As in the statement of (*P*), set

$$\overline{v}(x,t) = \begin{cases} f(x), & t<0; \\ v(x,t), & t \in [0,\alpha/2], \end{cases}$$

where $v \in \mathscr{B}_1$. Since $\overline{v}$ is bounded on $\mathbb{R}^n \times (-\infty, \alpha/2]$ and $J \in L^1(\mathbb{R}^{n+1})$, the integral

$$g(x,t) := \iint_{\mathbb{R}^n \times (-\infty,\alpha/2]} J(x-y,t-s)\overline{v}(y,s)dy\,ds$$

is absolutely convergent for $(x,t) \in \mathbb{R}^n \times [0,\alpha/2]$. Let us prove that, as a function of $(x,t) \in \mathbb{R}^n \times [0,\alpha/2]$, the function g belongs to $\mathscr{B}_1$. From the definition of g, we see that

$$|g(x,t)| \leq \left(\iint J dy\,ds\right) \|\overline{v}\|_\infty \leq \sup\{\|f\|_\infty, \|v\|_\infty\}.$$

Let us check the continuity of g. For $h \in \mathbb{R}^n$ and $k \in \mathbb{R}$ such that $(x+h,t+k) \in (-\infty,\alpha/2]$, we have

$$\begin{aligned} &|g(x+h,t+k) - g(x,t)| \\ &\qquad \leq \iint |J(x+h-y,t+k-s) - J(x-y,t-s)|\,|\overline{v}(y,s)|\,dy\,ds \\ &\qquad \leq \omega_1(\sqrt{|h|^2+k^2})\,\|\overline{v}\|_\infty, \end{aligned}$$

where ω_1 is the modulus of continuity in L^1 of J. Hence, for $v \in \mathscr{B}_1$, we also have that $g \in \mathscr{B}_1$ when restricted to the strip $\mathbb{R}^n \times [0,\alpha/2]$.

Define the mapping $T_1 : \mathscr{B}_1 \to \mathscr{B}_1$ by $T_1 v = g$. Let us now prove that T_1 is a contractive mapping in $\mathscr{B}_1$. Let $v, w \in \mathscr{B}_1$. Let $(x,t) \in \mathbb{R}^n \times [0,\alpha/2]$. Then, with

$$\overline{w}(x,t) = \begin{cases} f(x), & t<0; \\ w(x,t), & t \in [0,\alpha/2], \end{cases}$$

we have

$$\begin{aligned} T_1 v(x,t) - T_1 w(x,t) &= \iint_{s\leq\alpha/2} J(x-y,t-s)(\overline{v}(y,s) - \overline{w}(y,s))dy\,ds \\ &= \iint_{0<s\leq\alpha/2} J(x-y,t-s)(v(y,s) - w(y,s))dy\,ds. \end{aligned}$$

Hence

$$\|T_1 v - T_1 w\|_\infty \leq \left(\sup_{(x,t)\in\mathbb{R}^n\times[0,\alpha/2]} \iint_{0<s\leq\alpha/2} J(x-y,t-s)dyds \right) \|v-w\|_\infty$$

Now *(J1)* and the definition of α give

$$\begin{aligned} \iint_{0<s\leq\alpha/2} J(x-y,t-s)dyds &= \iint_{t-\alpha/2<\sigma\leq t} J(z,\sigma)dz\,d\sigma \\ &= \iint_{0<\sigma\leq t} J(z,\sigma)dz\,d\sigma \\ &\leq \iint_{0<\sigma\leq\alpha/2} J(z,\sigma)dz\,d\sigma =: \tau < 1, \end{aligned}$$

so that $\|T_1 v - T_1 w\|_\infty \leq \tau\,\|v-w\|_\infty$. Hence T_1 is a contractive mapping of $\mathscr{B}_1$. Thus there exists a unique fixed point $u_1 \in \mathscr{B}_1$ of T_1. In other words,

$$u_1(x,t) = \iint J(x-y,t-s)\overline{u_1}(y,s)dy\,ds \tag{3.1}$$

for $x \in \mathbb{R}^n$ and $0 \leq t \leq \alpha/2$.

Let us check that

$$\int_{\mathbb{R}^n} u_1(x,t)dx = \int_{\mathbb{R}^n} f(x)dx$$

for every $0 \leq t \leq \alpha/2$ when $f \in L^1(\mathbb{R}^n)$. Since u_1 can be realized as the limit of the sequence of iterations of T_1 applied to any function $v \in \mathscr{B}_1$, we may take $v(x,t) = f(x)$ as the starting point. In doing so, we see that the integral in the variable x of $\left|T_1^m f(x,t)\right|$ does not exceed $\int |f|\,dx$. In fact, from *(J3)*, we see that

$$\begin{aligned} \int |T_1 f(x,t)|\,dx &= \int \left| \iint J(x-y,t-s)f(y)dy\,ds \right| dx \\ &\leq \int \left(\iint J(x-y,t-s)dx\,ds \right) |f(y)|\,dy = \int |f|\,dy. \end{aligned}$$

Hence, inductively, assuming $\int \left|T_1^m f(x,t)\right| dx \le \int |f|\, dx$, we have

$$\begin{aligned}
\int \left|T_1^{m+1} f(x,t)\right| dx &= \int \left|T_1(T_1^m f)(x,t)\right| dx \\
&= \int \left| \iint J(x-y,t-s)\overline{T_1^m f}(y,s) dy\, ds \right| dx \\
&= \int \left| \iint J(y,t-s)\overline{T_1^m f}(x-y,s) dy\, ds \right| dx \\
&= \int \left| \iint_{s<0} J(y,t-s) f(x-y) dy\, ds \right. \\
&\quad \left. + \iint_{s>0} J(y,t-s) T_1^m f(x-y,s) dy\, ds \right| dx \\
&\le \iint_{s<0} J(y,t-s) \left| \int f(x-y) dx \right| dy\, ds \\
&\quad + \iint_{s>0} J(y,t-s) \left| \int T_1^m f(x-y,s) dx \right| dy\, ds \\
&\le \int |f|\, dx.
\end{aligned}$$

By the same arguments, we can conclude that $\int T_1^{m+1} f(x,t) dx = \int f(x) dx$ for $0 \le t \le \alpha/2$. The result then follows since, for $f \in L^1 \cap L^\infty$, we have that $T_1^m f \to u_1$ also in $\mathcal{C}([0,\alpha/2], L^1(\mathbb{R}^n))$. In fact, if

$$|||T_1^{m+1} f - T_1^m f||| \le \tau^m |||T_1^1 f - f||| \tag{3.2}$$

where $|||v||| = \sup_{t\in[0,\alpha/2]} \|v(\cdot,t)\|_{L^1(\mathbb{R}^n)}$, then $T_1^m f$ is also a Cauchy sequence in $\mathcal{C}([0,\alpha/2], L^1(\mathbb{R}^n))$. Since $T_1^m f$ converges uniformly to u_1 we get the desired preservation of the integral. It remains to prove (3.2).

Let us first check that $T_1^m f$ is continuous as a function of $t \in [0,\alpha/2]$ with values in $L^1(\mathbb{R}^n)$ for each m. For $t, t+h \in [0,\alpha/2]$,

$$\begin{aligned}
\int |T_1^m f(x,t) - T_1^m f(x,t+h)| dx &= \int \left| \iint J(x-y,t-s)\overline{T_1^{m-1} f}(y,s) d\, y ds \right. \\
&\quad \left. - \iint J(x-y,t+h-s)\overline{T_1^{m-1} f}(y,s) dy\, ds \right| dx \\
&= \int \left| \iint \langle J(z,t-s) - J(z,t+h-s) \rangle \overline{T_1^{m-1} f}(x-z,s) dz\, ds \right| dx \\
&\le \int \int |J(z,t-s) - J(z,t+h-s)| \left(\int \left| \overline{T_1^{m-1} f}(x-z,s) \right| dx \right) dz\, ds \\
&\le \int |f(x)|\, dx \int \int |J(z,t-s) - J(z,t+h-s)|\, dz\, ds,
\end{aligned}$$

which tends to 0 as $h \to 0$ because $J \in L^1(\mathbb{R}^{n+1})$.

Similar calculations show that $T_1^m f$ is a Cauchy sequence in the $|||\cdot|||$. In fact, for $t \in [0, \alpha/2]$,

$$\begin{aligned}
&\int |T_1^{m+1} f(x,t) - T_1^m f(x,t)| dx \\
&\quad = \int \left| \iint J(x-y, t-s) \left(\overline{T_1^m f}(y,s) - \overline{T_1^{m-1} f}(y,s) \right) dy\, ds \right| dx \\
&\quad = \int \left| \int \int_{0 \le s \le t} J(x-y, t-s) \left(T_1^m f(y,s) - T_1^{m-1} f(y,s) \right) dy\, ds \right| dx \\
&\quad \le \int \int_{0 \le s \le t} J(z, t-s) \left(\int \left| T_1^m f(x-z, s) - T_1^{m-1} f(x-z, s) \right| dx \right) dz\, ds \\
&\quad = \int \int_{0 \le s \le t} J(z, t-s) \left(\int \left| T_1^m f(x, s) - T_1^{m-1} f(x, s) \right| dx \right) dz\, ds \\
&\quad \le \left(\sup_{s \in [0, \alpha/2]} \int \left| T_1^m f(x,s) - T_1^{m-1} f(x,s) \right| dx \right) \iint_{0 \le s \le \alpha/2} J(z, t-s) dz\, ds \\
&\quad = \tau |||T_1^m f - T_1^{m-1} f|||.
\end{aligned}$$

Hence

$$|||T_1^{m+1} f - T_1^m f||| \le \tau |||T_1^m f - T_1^{m-1} f|||.$$

By iteration, we obtain (3.2).

Observe that since $u_1(x,t)$ can be obtained as the iteration of T_1 starting with any function $v \in \mathscr{B}_1$, we can take, in particular, the constant function $v = (s(f) - i(f))/2$, where $s(f) = \sup f$ and $i(f) = \inf f$. Then $\overline{v} = v \mathcal{X}_{\{0 \le t \le \alpha/2\}} + f \mathcal{X}_{\{t<0\}}$, so that $i(f) \le \overline{v} \le s(f)$ everywhere. From *(J2)* and *(J3)*, we also have $i(f) \le T_1 v \le s(f)$ on $\mathbb{R}^n \times [0, \alpha/2]$. The same argument shows that for every iteration $T_1^k v$ of $T_1 v$, we have $i(f) \le T_1^k v \le s(f)$. Since u_1 is the uniform limit of $T_1^k v$, we get $i(f) \le u_1(x,t) \le s(f)$ on the strip $\mathbb{R}^n \times [0, \alpha/2]$. So far we have existence and mass preservation for $t \in [0, \alpha/2]$.

Now proceed inductively by covering $\mathbb{R}^+$ with intervals of the type $[(i-1)\alpha/2, i\alpha/2]$. The first step, $i = 1$, is precisely the one described above. Assume that $u_i \in \mathscr{B}_i = (\mathcal{C} \cap L^\infty)(\mathbb{R}^n \times [(i-1)\alpha/2, i\alpha/2])$ for each $i = 1, \ldots, j$ have been built in such a way that

$$u_i(x,t) = \iint J(x-y, t-s) \overline{u_i}(y,s) dy\, ds$$

with

$$\overline{u_i}(x,t) = \begin{cases} \overline{u_{i-1}}(x,t), & t < (i-1)\alpha/2; \\ u_i(x,t), & (i-1)\alpha/2 \le t \le i\alpha/2. \end{cases}$$

Moreover, $\int_{\mathbb{R}^n} u_i(x,t)dx = \int_{\mathbb{R}^n} f(x)dx$ for $(i-1)\alpha/2 \le t \le i\alpha/2$,

$$i(f) \le u_i(x,t) \le s(f) \tag{3.3}$$

for every $(x,t) \in \mathbb{R}^n \times [(i-1)\alpha/2, i\alpha/2]$, and $u_i(x,(i-1)\alpha/2) = u_{i-1}(x,(i-1)\alpha/2)$ for every x.

Define $\mathscr{B}_{j+1}$ as the space $(\mathcal{C} \cap L^\infty)(\mathbb{R}^n \times [j\alpha/2,(j+1)\alpha/2])$ with the complete metric induced by the L^∞ norm. For $v \in \mathscr{B}_{j+1}$, define

$$T_{j+1}v(x,t) = \iint J(x-y,t-s)\overline{v}(y,s)dy\,ds$$

with

$$\begin{cases} \overline{v}(x,t) = \overline{u_j}(x,t), & t < j\alpha/2; \\ v(x,t), & j\alpha/2 \le t \le (j+1)\alpha/2. \end{cases}$$

As in the case of $i = 1$, it easy to check that with $(x,t) \in \mathbb{R}^n \times [j\alpha/2,(j+1)\alpha/2]$, $T_{j+1}v \in \mathscr{B}_{j+1}$. Hence $T_{j+1} : \mathscr{B}_{j+1} \to \mathscr{B}_{j+1}$. It is also easy to prove that T_{j+1} is contractive on $\mathscr{B}_{j+1}$ with the same rate of contraction τ obtained when $i = 1$.

Also, by the same argument as in the case $i = 1$, with $\int \overline{u_j}(x,t)dx = \int f(x)dx$ when $t \le j\alpha/2$, we have $\int_{\mathbb{R}^n} u_{j+1}(x,t)dx = \int_{\mathbb{R}} f(x)dx$ for $t \in [j\alpha/2,(j+1)\alpha/2]$. To check that $u_{j+1}(x,j\alpha/2) = u_j(x,j\alpha/2)$, we need only observe that for $j\alpha/2 \le t \le (j+1)\alpha/2$, the fixed point equation is given by

$$u_{j+1}(x,t) = \iint J(x-y,t-s)\overline{u_{j+1}}(y,s)dy\,ds.$$

For $t = j\alpha/2$, property *(J1)* shows that the above integral involves only values of s which are bounded above by $j\alpha/2$. For those values, $\overline{u_{j+1}}(y,s) = \overline{u_j}(y,s)$, so that

$$u_{j+1}(x,j\alpha/2) = \iint J(x-y,j\alpha/2-s)\overline{u_j}(y,s)dy\,ds = u_j(x,j\alpha/2),$$

as desired.

Property (3.3) for $i = j+1$ can be proved following the same argument used in the case $i = 1$. Notice that the function $u(x,t)$ defined on $\mathbb{R}^{n+1}_+$ by $u(x,t) = u_{j(t)}(x,t)$ with $j(t)$ the only positive integer for which $(j(t)-1)\alpha/2 \le t < j(t)\alpha/2$ is continuous and bounded. Moreover, $i(f) \le u(x,t) \le s(f)$ for every $(x,t) \in \mathbb{R}^{n+1}_+$.

The above remarks prove that $u \in \mathscr{B} = (\mathcal{C} \cap L^\infty)(\mathbb{R}^{n+1}_+)$ and solves (P).

To prove the uniqueness of the solution u, we argue as follows. If u and $\widetilde{u}$ are solutions, their restrictions on the strip $\mathbb{R}^n \times [0,\alpha/2]$ coincide. Since the fixed point of T_1 is unique and, being a solution of (P) in $\mathbb{R}^n \times [0,\alpha/2]$ is equivalent to being a fixed point for T_1, we see that $u \equiv \widetilde{u}$ on $\mathbb{R}^n \times [0,\alpha/2]$. For the next time interval $[\alpha/2, \alpha]$, the restriction of both u and $\widetilde{u}$ to this interval are fixed points of the *same* operator T_2. Again the uniqueness given by the Banach fixed point guarantees $u \equiv \widetilde{u}$ on $\mathbb{R}^n \times [\alpha/2, \alpha]$. Proceeding inductively, we get that $u \equiv \widetilde{u}$ everywhere.

Let us finally prove the estimate (1.2). First we show that (1.2) holds when $(x, t) \in \mathbb{R}^n \times [0, \alpha/2]$. This is because the function u in the first time interval $[0, \alpha/2]$ coincides with u_1 provided by the Banach fixed point theorem, and the rate of convergence can be estimated by the contraction constant τ. We already know that $\tau = \iint_{s\leq\alpha/2} J(y, s)dy\,ds < 1$. Denote by u_1^m the m-th iteration of T_1 applied to the initial guess $u_1^0 = f$. Then, since $\|u_1^{m+1} - u_1^m\|_\infty \leq \tau^m \|u_1^1 - u_1^0\|_\infty$, we see that

$$\|u_1^m - f\|_\infty \leq \left(\sum_{j=0}^{m} \tau^j\right) \left\|u_1^1 - f\right\|_\infty \leq \frac{1}{1-\tau} \left\|u_1^1 - f\right\|_\infty$$

for every $m = 1, 2, \ldots$

Let us now show that for $(x, t) \in \mathbb{R}^n \times [0, \alpha/2]$ there exists a constant $\widetilde{C}$ depending only on J such that $\|u_1^1 - f\|_\infty \leq \widetilde{C}[f]_\gamma$. In fact,

$$\begin{aligned} \left|u_1^1(x, t) - f(x)\right| &= |(T_1 f)(x, t) - f(x)| \\ &= \left|\iint J(x - y, t - s)(f(y) - f(x))dy\,ds\right| \\ &\leq [f]_\gamma \left(\iint J(x - y, t - s)\,|x - y|^\gamma\, dy\,ds\right). \end{aligned}$$

Hence for every $m = 1, 2, \ldots$,

$$\|u_1^m - f\|_{L^\infty(\mathbb{R}^n \times [0,\alpha/2])} \leq C[f]_\gamma,$$

where C depends only on J. The same is true for the uniform limit u_1 of the sequence u_1^m. In other words,

$$\|u_1 - f\|_{L^\infty(\mathbb{R}^n \times [0,\alpha/2])} \leq C[f]_\gamma. \tag{3.4}$$

Let us now see how to get the same type of estimate for the time interval $[\alpha/2, \alpha]$. From the construction of u, we have that on $\mathbb{R}^n \times [\alpha/2, \alpha]$, $u = u_2$ with

$$u_2(x, t) = \iint J(x - y, t - s)\overline{u_2}(y, s)dy\,ds, \quad \overline{u_2}(x, t) = \begin{cases} f(x), & t < 0; \\ u_1(x, t), & t \in [0, \alpha/2]; \\ u_2(x, t), & t \in [\alpha/2, \alpha]. \end{cases}$$

On $\mathbb{R}^n \times [\alpha/2, \alpha]$, the solution u_2 is the only fixed point for the operator T_2 and, since the limit u_2 of iterations u_2^m of $T_2 u_2^0 = u_2^1$ is independent of the starting point u_2^0, let us take again $u_2^0 = f$. Hence $\|u_2 - f\|_\infty \leq \frac{1}{1-\tau} \left\|u_2^1 - f\right\|_\infty$. Notice that, writing

$$\overline{f}(y, s) = f(y)\mathcal{X}_{s<0}(s) + u_1(y, s)\mathcal{X}_{[0,\alpha/2]}(s) + f(y)\mathcal{X}_{[\alpha/2,\alpha]}(s),$$

we have

$$u_2^1(x, t) = \iint J(x - y, t - s)\overline{f}(y, s)dy\, ds.$$

Let us finally check that the desired estimate holds for $\left\|u_2^1 - f\right\|_\infty$ in $\mathbb{R}^n \times [\alpha/2, \alpha]$. Take $(x, t) \in \mathbb{R}^n \times [\alpha/2, \alpha]$, then

$$\begin{aligned}
\left|u_2^1(x, t) - f(x)\right| &= |(T_2 f)(x, t) - f(x)| \\
&= \left|\iint J(x - y, t - s)\overline{f}(y, s)dyds - f(x)\right| \\
&\leq \iint_{s\leq 0} J(x - y, t - s)\, |f(y) - f(x)|\, dy\, ds \\
&\quad + \iint_{0<s<\alpha/2} J(x - y, t - s)\, |u_1(y, s) - f(y)|\, dy\, ds \\
&\quad + \iint_{s\leq\alpha} J(x - y, t - s)\, |f(y) - f(x)|\, dy\, ds.
\end{aligned}$$

The first and the third terms on the right hand side of the above inequality are bounded by the product of the Lipγ seminorm of f and a constant depending only on J. For the second term, we use (3.4), and we are done.

4 Maximum principle. Proof of Theorem 3

Recall that $\alpha = \sup\{\beta : \iint_{s\leq\beta} J(y, s)dy\, ds < 1\}$. Since the function $I(\beta) = \iint_{s\leq\beta} J dy\, ds$ is increasing and continuous as a function of β, α is also the infimum of those values of β for which $I(\beta) = 1$. Moreover, from definition of α, we have $0 < I(\alpha/2) < 1$.

Let $t_k = \alpha + (k - 1)\alpha/2$, $B_k = \mathbb{R}^n \times [0, t_k]$, $S_k = \sup_{B_k} |w|$ for $k = 1, 2, \ldots$; see Figure 2. Let us see that $S_k = S_{k-1}$. Let $(x, t) \in \mathbb{R}^n \times [t_{k-1}, t_k]$, hence

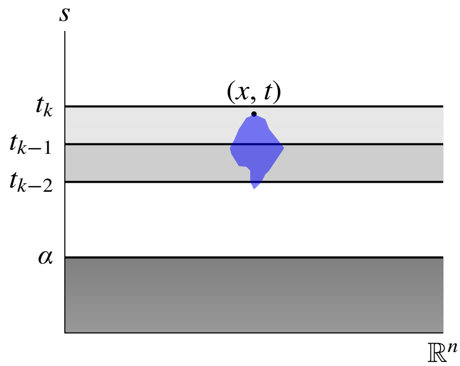

Figure 2. The relative position of supp J and the stripes B_k.

$$\begin{aligned}
|w(x,t)| &= \left|\iint J(x-y,t-s)w(y,s)dyds\right| \\
&= \left|\iint_{t_{k-1}\leq s\leq t} J(x-y,t-s)w(y,s)dy\,ds + \iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)w(y,s)dy\,ds\right| \\
&\leq S_k \iint_{t_{k-1}\leq s\leq t} J(x-y,t-s)dyds + S_{k-1}\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dy\,ds \\
&= S_k\left(1-\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dyds\right) + S_{k-1}\left(\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dy\,ds\right) \\
&= S_k-(S_k-S_{k-1})\left(\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dy\,ds\right).
\end{aligned}$$

Hence

$$\left(\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dyds\right)(S_k-S_{k-1}) \leq S_k-|w(x,t)|,$$

since

$$\begin{aligned}
\iint_{t-\alpha\leq s\leq t_{k-1}} J(x-y,t-s)dyds &= \iint_{t-t_{k-1}\leq s_1\leq\alpha} J(x-y,s_1)dy\,ds_1 \\
&\geq \iint_{\alpha/2\leq s_1\leq\alpha} J(x-y,s_1)dy\,ds_1 = 1-I(\alpha/2).
\end{aligned}$$

Then

$$(1-I(\alpha/2))(S_k-S_{k-1}) \leq S_k - \sup_{B_k\setminus B_{k-1}} |w|. \tag{4.1}$$

Of course, $S_k \geq S_{k-1}$, if $S_k > S_{k-1}$. Then $S_k = \sup_{B_k \setminus B_{k-1}} |w|$, and the right hand of (4.1) vanishes. Since $1 - I(\alpha/2)$ is positive, $S_k \leq S_{k-1}$, which is a contradiction.

5 Convergence of solutions. Proof of Theorem 4

Throughout this section, we assume that J in problem (P) is a fixed parabolic rescaling of a mean value kernel $H \in \mathscr{H}$. More precisely, for $r > 0$, we consider the CTRW with past density distribution $f(x)$ generated by the space-time probability density $H_r(x,t) = \frac{1}{r^{n+2}} H(\frac{x}{r}, \frac{t}{r^2})$, with $H(x,t)$ satisfying *(J1)–(J5)* and the mean value formula. The space probability density function for the localization of the moving particle at time t is the solution $u(H_r, f)(\cdot, t)$ of $P(H_r, f)$.

The result contained in Theorem 4 is a consequence of the two following lemmas and the maximum principle contained in Theorem 3.

Lemma 6. Let J be a kernel satisfying *(J1)–(J4)*. Set

$$\alpha = \sup \left\{ \beta : \iint_{s \leq \beta} J(y,s) dy\, ds < 1 \right\}.$$

Let $f \in (\mathcal{C}^{0,\gamma} \cap L^\infty)(\mathbb{R}^n)$, $0 < \gamma \leq 1$. Then there exists a constant $C > 0$ such that for every $r > 0$,

$$\|u(J_r, f) - f\|_{L^\infty(\mathbb{R}^n \times [0, \alpha r^2])} \leq C[f]_\gamma r^\gamma.$$

The next lemma contains a well-known result on the rate of convergence of temperatures when the initial condition belongs to a Lipschitz class.

Lemma 7. Let $f \in (\mathcal{C}^{0,\gamma} \cap L^\infty)(\mathbb{R}^n)$ and

$$u(x,t) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t} f(y) dy$$

be the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = \triangle u, & \mathbb{R}^{n+1}_+; \\ u(x,0) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

Then there exists a constant $C > 0$ such that

$$|u(x,t) - f(x)| \leq C[f]_\gamma t^{\gamma/2}$$

for every $(x,t) \in \mathbb{R}^{n+1}_+$.

Proof of Theorem 4. From Lemmas 6 and 7, we have

$$\sup_{(x,t) \in \mathbb{R}^n \times [0, \alpha r^2]} |u(H_r, f)(x,t) - u(x,t)| \leq C[f]_\gamma r^\gamma.$$

Now, since $\alpha = \sup\{\beta : \iint_{s\leq\beta} H(y,s)dyds < 1\}$, we also have

$$\alpha r^2 = \sup\left\{\beta : \iint_{s\leq\beta} H_r(y,s)dy\,ds < 1\right\}.$$

Hence, for $(x,t)\in\mathbb{R}^n\times(\alpha r^2,+\infty)$, the support of $H_r(x-y,t-s)$ as a function of (y,s) is contained in $\mathbb{R}^{n+1}_+$, o that for a temperature u defined on $\mathbb{R}^{n+1}_+$ and $t>\alpha r^2$, since $H\in\mathscr{H}$, the mean value formula holds, and

$$u(x,t) = \iint H_r(x-y,t-s)u(y,s)dy\,ds$$

for $x\in\mathbb{R}^n$ and $t>\alpha r^2$.

On the other hand, we also have that $u(H_r,f) = H_r * u(H_r,f)$ for $x\in\mathbb{R}^n$ and $t>\alpha r^2$, because $u(H_r,f)$ solves $P(H_r,f)$. Hence, applying Theorem 3 with H_r instead of J, αr^2 instead of α, and $u(H_r,f)-u$ instead of w, we get

$$\sup_{\mathbb{R}^{n+1}_+}|u(H_r,f)(x,t)-u(x,t)| \leq \sup_{\mathbb{R}^n\times[0,\alpha r^2]}|u(H_r,f)(x,t)-u(x,t)| \leq C[f]_\gamma r^\gamma,$$

as desired. □

Proof of Lemma 6. The result follows from (1.2), after parabolic rescaling. In fact, denote by $u(J,g)$ the solution in $\mathbb{R}^{n+1}_+$ of $P(J,g)$. Then

$$u(J_r,f) = \left[u\left(J,f_{\frac{1}{r}}\right)\right]_r,$$

for each $r>0$. Hence, for $x\in\mathbb{R}^n$ and $0\leq t/r^2\leq\alpha$, from (1.2) with $f(r\cdot)$ instead f, we have

$$\begin{aligned}|u(J_r,f)(x,t)-f(x)| &= \left|\left[u\left(J,f_{\frac{1}{r}}\right)\right]_r(x,t)-f(x)\right|\\ &= \left|u(J,f(r\cdot))\left(\tfrac{x}{r},\tfrac{t}{r^2}\right)-f\left(r\left(\tfrac{x}{r}\right)\right)\right|\\ &\leq C[f(r\cdot)]_\gamma = Cr^\gamma[f]_\gamma.\end{aligned}$$

□

Proof of Lemma 7. Since the Weierstrass kernel has integral equal to 1, for $(x,t)\in\mathbb{R}^{n+1}_+$,

$$\begin{aligned}|u(x,t)-f(x)| &= \left|\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}f(y)dy - f(x)\right|\\ &\leq \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}|f(y)-f(x)|\,dy\\ &\leq \frac{[f]_\gamma}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}|y-x|^\gamma\,dy = \frac{[f]_\gamma}{\pi^{n/2}}\int_{\mathbb{R}^n} e^{-|z|}|z|^\gamma\,dz\,t^{\frac{\gamma}{2}}.\end{aligned}$$

□

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