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Stabilization of low-order cross-grid P_kQ_l mixed finite elements

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ABSTRACT

In this paper we analyze a low-order family of mixed finite element methods for the numerical solution of the Stokes problem and a second order elliptic problem, in two space dimensions. In these schemes, the pressure is interpolated on a mesh of rectangular elements, while the velocity is approximated on a triangular mesh obtained by dividing each rectangle into four triangles by its diagonals. For the lowest order P_1Q_0, a global spurious pressure mode is shown to exist and so this element, as P_1Q_1 case analyzed in Armentano and Blasco (2010) , is unstable. However, following the ideas given in Bochev et al. (2006) , a simple stabilization procedure can be applied, when we approximate the solution of the Stokes problem, such that the new P_1Q_0 and P_1Q_1 methods are unconditionally stable, and achieve optimal accuracy with respect to solution regularity with simple and straightforward implementations. Moreover, we analyze the application of our P_1Q_1 element to the mixed formulation of the elliptic problem. In this case, by introducing the modified mixed weak form proposed in Brezzi et al. (1993) , optimal order of accuracy can be obtained with our stabilized P_1Q_1 elements. Numerical results are also presented, which confirm the existence of the spurious pressure mode for the P_1Q_0 element and the excellent stability and accuracy of the new stabilized methods.

1. Introduction

The approximation by mixed finite element methods of the Stokes problem has been widely developed. In some works the two independent variables, velocity and pressure, are approximated by using spaces of different order of approximation for each one [1–10]. On the other hand, some stabilized formulations, which consists in modifying the discrete problem by the addition of new terms which enhance its stability, are introduced in order to use the same order approximation spaces for the velocity and the pressure (see, for example, [8,9,11–19] and the references therein). In particular, standard C_0 finite element spaces of low polynomial orders remain a popular choice in many engineering applications because, besides their simplicity, they offer reasonable accuracy and uniform data structures when using equal order interpolation and so the develop of stabilization procedure is still a focus of the interest.

In [1] we introduce and analyze a new family of mixed finite element methods in which the pressure is interpolated on a mesh of rectangular elements and the velocity on a triangular mesh obtained by dividing each rectangle into four triangles by its diagonals. We denote by P_kQ_l the elements in which the velocity is interpolated in each triangle by polynomials of degree no greater than k and the pressure is interpolated in each rectangle by polynomials of degree in each variable no greater than l. In that work we proved the existence of a global spurious pressure mode for the P_1Q_1 element, and that the cross-grid P_2Q_1 element satisfies the inf–sup condition getting optimally convergent solutions.

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In the present work we analyze the lowest P_1Q_0 element, and we show the existence of a global spurious pressure mode, so that convergence of the pressure does not hold for this element. Then, following the ideas given by Bochev, Dohrmann and Gunzburger in [11], which we denote by [BDG] procedure, we present a stabilized finite element method for the Stokes problem to counteract the lack of stability for our P_1Q_0 and P_1Q_1 cross-grid elements. The goal of these stabilized methods is that, in contrast to other stabilization procedures, they are parameter free, always lead to symmetric linear systems and have simple and straightforward computational implementation.

On the other hand, in the mixed formulation of second order elliptic problems, the approximation of the two variables has to be done taking into account some particular compatibility conditions in order to avoid instabilities (see [2,20,21] and the references therein). Moreover, the stable finite element approximations could be different from this problem to the Stokes problem, for example, the mini-elements are stable for the Stokes problem but not for the elliptic (see, for example, [2,22]). In this work we also analyze the application of our P_1Q_1 element to the mixed formulation of the elliptic problem. It is well known that in Raviart–Thomas spaces [23] (one of most used approaches for the elliptic problem), typically stable velocity approximations are continuous only in the normal direction and the most frequent approach RT_0P_0 has also discontinuous pressure [21–23]. In [24] the authors propose different approaches to stabilize the P_1P_1 elements, since the same strategy applied to the Stokes problem cannot be applied directly to the elliptic. In this paper we use the modified mixed weak form for the elliptic problem introduced in [20], which allows us to employ finite element spaces similar to those built for the Stokes problem. In particular, we can successfully apply our stabilized P_1Q_1 cross-grid elements to this modified mixed problem in order to obtain optimal equal-order continuous approximations. It is important to point out that the ideas present in this paper, could be extended to other interesting problems (see, for example, [24–28] and the references therein), with the purpose to obtain optimal, continuous and economical approximations.

Some numerical results are also presented which confirm the presence of the spurious pressure mode for the P_1Q_0 element and the successful stabilization procedure for our P_1Q_0 and P_1Q_1 cross-grid elements for the Stokes problem. Moreover, we show a numerical example for the application of our stabilized P_1Q_1 element to the modified mixed problem associated to the elliptic, which shows the good performance of our approximation method. Although in the stabilized procedure provided here we consider only rectangular elements, the methods we have developed can also be applied to meshes of general quadrilateral elements.

The rest of the paper is organized as follows. In Section 2 we state the Stokes problem, introduce the P_kQ_l mixed finite element approximations and we prove the instability of P_1Q_0, the lowest order case. In Section 3 we present the stabilization procedure for the cross-grid elements P_1Q_0 and P_1Q_1. In Section 4 we show the modified mixed formulation for second order elliptic equations and the approximation properties utilizing the stabilization method introduced in Section 3. Finally, in Section 5 we present some numerical examples which show the good performance of the stabilization procedure.

2. Cross-grid P_kQ_l finite element approximation of the Stokes problem

In this section we recall the Stokes problem and the family of cross-grid P_kQ_l mixed finite element methods introduced in [1] for its numerical approximation.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and polygonal domain, the classical Stokes problem is given by: Find the fluid velocity $\mathbf{u}$ and the pressure p such that

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega\,, \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega\,, \\ \mathbf{u} = 0 & \text{on } \Gamma := \partial\Omega\,, \end{cases} \tag{2.1}$$

where $\mathbf{f} \in (H^{-1}(\Omega))^2$ (the dual space of $(H_0^1(\Omega))^2$) is a given body force per unit mass and $\mu > 0$ is the kinematic viscosity, which we assume constant.

Let $V := (H_0^1(\Omega))^2$ and $Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$. The weak form of (2.1) is given by: Find $\mathbf{u} \in V$ and $p \in Q$ such that

$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \langle \mathbf{f},\mathbf{v}\rangle_{V'\times V} & \forall \mathbf{v} \in V\,, \\ b(\mathbf{u},q) = 0 & \forall q \in Q\,, \end{cases} \tag{2.2}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined on $V \times V$ and $V \times Q$, respectively, as

$$a(\mathbf{u},\mathbf{v}) = \mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} \qquad \mathbf{u},\mathbf{v} \in V\,,$$

$$b(\mathbf{v},q) = -\int_\Omega \nabla\cdot\mathbf{v}\, q \qquad \mathbf{v} \in V, q \in Q\,.$$

We denote by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$ the norms and seminorms in $H^m(D)$ or $(H^m(D))^2$ respectively and $(\cdot,\cdot)_D$ denotes the inner product in $L^2(D)$ or $(L^2(D))^2$ for any subdomain $D \subset \Omega$. The domain subscript is dropped for the case $D = \Omega$.

It is well known that the bilinear form $a(\cdot,\cdot)$ is coercive in V and there exists a constant $\beta > 0$ (see for instance [21]) such that for all $q \in Q$

$$\sup_{0\neq\mathbf{v}\in V} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_1} \geq \beta\|q\|_0.$$

According to the general theory of mixed problems [21,22] these conditions ensure that there exists a unique solution of problem (2.2).

Now, we consider a partition $\mathcal{C}_h$ of $\bar{\Omega}$ into rectangular elements K, which we assume to be regular, i.e., there exists a constant $\sigma > 0$, independent of the mesh size h, such that

$$h_K \leq \sigma \rho_K \quad \forall K \in \mathcal{C}_h,$$

where h_K denotes the diameter of K and ρ_K the diameter of the largest ball contained in K.

Then, we divide each rectangle K by its diagonals into four triangles and we call $\mathcal{T}_h$ the resulting mesh of triangular elements T.

Throughout the paper, we will denote by C a generic positive constant, not necessarily the same at each occurrence, which may depend on the mesh only through the parameter σ.

Using the standard notation $\mathcal{P}_k$ for the space of polynomials of degree not greater than k and Q_l for the space of polynomials of the form $q(x, y) = \sum_j \alpha_j p_j(x) q_j(y)$ with p_j and q_j polynomials of degree less than or equal to l, the cross-grid $P_k Q_l$ mixed finite element spaces for the approximation of the velocity and the pressure are defined, respectively, as follows:

$$V_h^k = \{\mathbf{v} \in V : \mathbf{v}_{|T} \in (\mathcal{P}_k)^2, \ \forall T \in \mathcal{T}_h\}, \tag{2.3}$$

$$Q_h^l = \{q \in Q : q_{|K} \in Q_l, \ \forall K \in \mathcal{C}_h\}. \tag{2.4}$$

The velocity and pressure nodes for the P_1Q_0 element are shown in Fig. 1 and for the P_1Q_1 element in Fig. 2.

The standard Galerkin approximation of (2.2) is given by: Find $(\mathbf{u}_h, p_h) \in V_h^k \times Q_h^l$ such that:

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle_{V' \times V} & \forall \mathbf{v}_h \in V_h^k, \\ b(\mathbf{u}_h, q_h) = 0 & \forall q_h \in Q_h^l. \end{cases} \tag{2.5}$$

In order to have a stable and convergent approximation, the discrete spaces V_h^k and Q_h^l have to satisfy the well-known LBB condition, i.e., there should exist a constant $\tilde{\beta} > 0$, independent of h, such that for the pair (k, l) we have that

$$\sup_{0 \neq \mathbf{v}_h \in V_h^k} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_1} \geq \tilde{\beta} \|q_h\|_0 \quad \forall q_h \in Q_h^l. \tag{2.6}$$

Then, if (2.6) holds the theory of mixed finite element methods [21,22] states that problem (2.5) has a unique solution which is stable and optimally convergent, i.e., there exists a positive constant C such that:

$$\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0 \leq C\{\inf_{\mathbf{v} \in V_h^k} \|\mathbf{u} - \mathbf{v}\|_1 + \inf_{q \in Q_h^l} \|p - q\|_0\}. \tag{2.7}$$

The cases of interest are, of course, those for which $l \leq k$. For $l = k$, we have an approximation of the same order for both variables, although not an equal approximation. For $l = k - 1$ the orders of the interpolation errors in the error estimate (2.7) are balanced.

2.1. *The cross-grid P_1Q_0 element*

The first thing that we have to observe when we choose a mixed finite element is the relation between $\dim V_h^k$ and $\dim Q_h^l$, in fact a necessary condition is that:

$$\dim V_h^k \geq \dim Q_h^l.$$

Indeed, consider a partition $\mathcal{C}_h$, and the corresponding triangulation $\mathcal{T}_h$, of $\bar{\Omega}$ and let us denote by NT the number of triangles, NR the number of rectangles, υ_T the number of internal vertices of the triangles, υ_R the number of internal vertices of the rectangles, and υ_B the number of boundary vertices (which is the same for triangles and rectangles).

We shall thus have $\dim V_h^1 = 2\upsilon_T$ (since the velocities vanish on the boundary) and $\dim Q_h^0 = NR - 1$ because of the zero mean value of the pressures. We observe that $\upsilon_T = \upsilon_R + NR$. Therefore, $\dim V_h^1 = 2\upsilon_T = 2\upsilon_R + 2NR \geq 2NR \geq \dim Q_h^0$ and this element has the chance of being stable. However, our indicator of the instability is the fact that, as the following lemma shows, the kernel of the operator $B_h : Q_h^0 \to (V_h^1)'$ is nontrivial. Thus, this element does not satisfy the inf–sup condition (2.6).

Lemma 2.1. *Let $\Omega = (0, 1) \times (0, 1)$ and let $\mathcal{C}_h$ be a uniform mesh consisting of $N \times N$ rectangles. Let us consider the $P_K Q_l$ mixed finite element approximation with $k = 1$ and $l = 0$. Then, there exists a global spurious pressure mode $q_h^* \in Q_h^0 \setminus \{0\}$ such that*

$$\int_\Omega q_h^* \nabla \cdot \mathbf{v}_h = 0 \qquad \forall \mathbf{v}_h \in V_h^1.$$

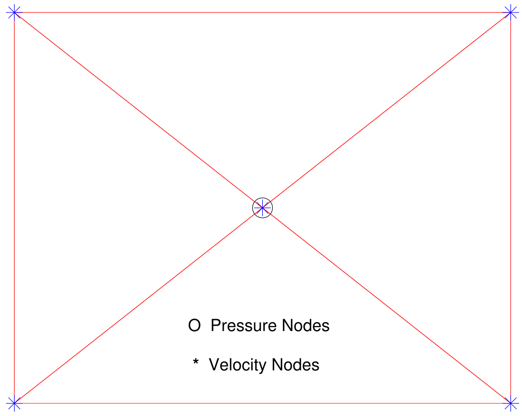

Fig. 1. Velocity and pressure nodes of the cross-grid P_1Q_0 mixed finite element.

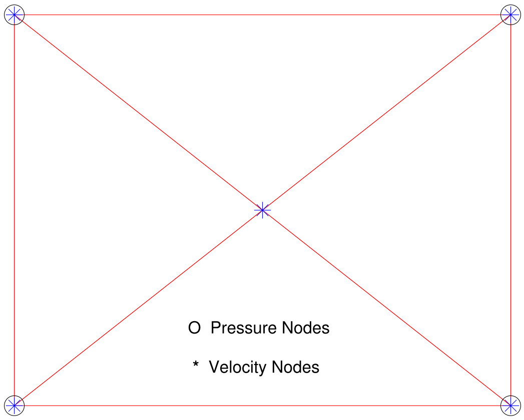

Fig. 2. Velocity and pressure nodes of the cross-grid P_1Q_1 mixed finite element.

Proof. Let $K_{i,j} = [(i-1)h, ih] \times [(j-1)h, jh]$ be the rectangles of the uniform mesh $\mathcal{C}_h$, with $h = 1/N$, $1 \le i, j \le N$ and let $n_{i,j} = (ih, jh)$, $0 \le i, j \le N$ be the nodes of the mesh $\mathcal{C}_h$. Let q_h be any function in Q_0, we denote by $q_{hK_{i,j}}$ the constant value of q_h in each rectangle $K_{i,j}$. Then,

$$\int_\Omega q_h \nabla \cdot \mathbf{v}_h = \sum_{K_{i,j}\in\mathcal{C}_h} \int_{K_{i,j}} q_h \nabla \cdot \mathbf{v}_h = \sum_{K_{i,j}\in\mathcal{C}_h} q_{hK_{i,j}} \int_{K_{i,j}} \nabla \cdot \mathbf{v}_h$$
$$= \sum_{K_{i,j}\in\mathcal{C}_h} q_{hK_{i,j}} \sum_{T\subset K_{i,j}} \int_T \nabla \cdot \mathbf{v}_h.$$

Since $\mathbf{v}_h = (v_1, v_2)$ is linear in each $T \in \mathcal{T}_h$ we get that

$$\int_\Omega q_h \nabla \cdot \mathbf{v}_h = \sum_{K_{i,j}\in\mathcal{C}_h} q_{hK_{i,j}} \sum_{T\subset K_{i,j}} \int_T \nabla \cdot \mathbf{v}_h = \sum_{K_{i,j}\in\mathcal{C}_h} q_{hK_{i,j}} \frac{h^2}{4} \sum_{T\subset K_{i,j}} \nabla \cdot \mathbf{v}_h|_T$$
$$= \sum_{K_{i,j}\in\mathcal{C}_h} q_{hK_{i,j}} \frac{h}{2} \{v_{1i,j-1} - v_{1i-1,j-1} + v_{1i,j} - v_{1i-1,j} - v_{2i,j-1} - v_{2i-1,j-1} + v_{2i,j} + v_{2i-1,j}\}$$
$$= \frac{h}{2} \sum_{i,j} v_{1i-1,j} \{-q_{hK_{i,j}} - q_{hK_{i,j+1}} + q_{hK_{i-1,j+1}} + q_{hK_{i-1,j}}\}$$
$$+ \frac{h}{2} \sum_{i,j} v_{2i-1,j} \{q_{hK_{i,j}} - q_{hK_{i,j+1}} - q_{hK_{i-1,j+1}} + q_{hK_{i-1,j}}\}.$$

We can define a global $q_h^* \in Q_h^0$ such that:

$$q^*_{hK_{i,j}} = \begin{cases} 1 & \text{if } i+j \text{ is even} \\ -1 & \text{if } i+j \text{ is odd} \end{cases}$$

which conduce to a *checkerboard pattern* and to conclude that

$$\int_\Omega q_h^* \nabla \cdot \mathbf{v}_h = 0 \qquad \forall \mathbf{v}_h \in V_h^1. \quad \square$$

3. The stabilization method

In this section we present the stabilization method for the cross-grid P_1Q_0 and P_1Q_1 mixed finite element. Our stabilization procedure follows the [BDG] procedure introduced in [11].

First, we observe that for both pairs P_1Q_0 and P_1Q_1 some weak inf–sup bounds hold. Indeed, we have the following lemmas.

Lemma 3.1. *Let V_h^1 and Q_h^1 be the spaces defined above. Then, there exist positive constants C_1 and C_2 such that*

$$\sup_{0\neq \mathbf{v}\in V_h^1} \frac{\int_\Omega q_h \nabla \cdot \mathbf{v}_h}{\|\mathbf{v}_h\|_1} \geq C_1\|q_h\|_0 - C_2 h\|\nabla q_h\|_0 \quad \forall q_h \in Q_h^1.$$

Proof. see Lemma 2.1 in [11]. $\square$

Lemma 3.2. *Let V_h^1 and Q_h^0 be the spaces defined above. Then, there exist positive constants C_1 and C_2 such that*

$$\sup_{0\neq \mathbf{v}\in V_h^1} \frac{\int_\Omega q_h \nabla \cdot \mathbf{v}_h}{\|\mathbf{v}_h\|_1} \geq C_1\|q_h\|_0 - C_2 h^{\frac{1}{2}}\Big(\sum_{\ell\subset \mathcal{C}_h} \int_\ell [q_h]^2\Big)^{\frac{1}{2}} \quad \forall q_h \in Q_h^0,$$

where $[q_h]$ denotes the jump of q_h.

Proof. Since $Q_h^0 \subset L_0^2(\Omega)$ for any $q_h \in Q_h^0$ there exists $\mathbf{w} \in (H_0^1(\Omega))^2$ and a constant C_d such that

$$\frac{\int_\Omega \operatorname{div}\mathbf{w}\, q_h}{\|\mathbf{w}\|_1} \geq C_d\|q_h\|_0. \tag{3.8}$$

Let $I_h : V \to V_h$ be the Clément interpolation operator (see [29]). Since the mesh is regular, this operator satisfies that for $T \in \mathcal{T}_h$ and $\ell \subset \mathcal{T}_h$,

$$\begin{aligned} &\|\mathbf{w} - I_h\mathbf{w}\|_{m,T} \leq C h_T^{r-m}\|\mathbf{w}\|_{r,\omega_T}, \quad r = 0, 1, \quad m = 0, \ldots, r \\ &\|\mathbf{w} - I_h\mathbf{w}\|_{0,\ell} \leq C|\ell|^{1/2}\|\mathbf{w}\|_{1,\omega_\ell}, \end{aligned} \tag{3.9}$$

with

$$\omega_T := \bigcup_{\mathcal{N}(T)\cap\mathcal{N}(T')\neq\emptyset} T', \qquad \omega_\ell := \bigcup_{\mathcal{N}(\ell)\cap\mathcal{N}(T')\neq\emptyset} T',$$

where $\mathcal{N}(\ell)$ and $\mathcal{N}(T)$ denote the set of vertices of ℓ and T respectively. We observe that, in particular, $\|\mathbf{w} - I_h\mathbf{w}\|_m \leq Ch^{1-m}\|\mathbf{w}\|_1$, and therefore $\|I_h\mathbf{w}\|_1 \leq C\|\mathbf{w}\|_1$. Then, by using these estimations and (3.8) we get

$$\begin{aligned} \frac{|\int_\Omega \operatorname{div} I_h(\mathbf{w})\, q_h|}{\|I_h(\mathbf{w})\|_1} &\geq C\frac{|\int_\Omega \operatorname{div} I_h(\mathbf{w})\, q_h|}{\|\mathbf{w}\|_1} \\ &\geq C\left(\frac{|\int_\Omega \operatorname{div}\mathbf{w}\, q_h|}{\|\mathbf{w}\|_1} - \frac{|\int_\Omega \operatorname{div}(I_h(\mathbf{w}) - \mathbf{w})\, q_h|}{\|\mathbf{w}\|_1}\right) \\ &\geq C_1\|q_h\|_0 - C\frac{|\int_\Omega \operatorname{div}(I_h(\mathbf{w}) - \mathbf{w})\, q_h|}{\|\mathbf{w}\|_1}. \end{aligned}$$

Now, for the second term, integrating by parts and taking into account that, for each $K \in \mathcal{C}_h$, the constant value of q_h is the same for the four triangles contained in K, and that the functions in V_h^1 are continuous in the triangulations $\mathcal{T}_h$ we can get, by using (3.9), that

$$\int_{\Omega} \operatorname{div}(I_h(\mathbf{w}) - \mathbf{w})\, q_h = \sum_{K\in\mathcal{C}_h}\sum_{T\subset K}\int_T \operatorname{div}(I_h(\mathbf{w}) - \mathbf{w})\, q_h = \sum_{K\in\mathcal{C}_h}\sum_{T\subset K}\int_{\partial T}(I_h(\mathbf{w}) - \mathbf{w})\cdot\mathbf{n}\, q_h$$
$$= \sum_{K\in\mathcal{C}_h}\int_{\partial K}(I_h(\mathbf{w}) - \mathbf{w})\cdot\mathbf{n}\, q_h = \sum_{\ell\subset\mathcal{C}_h}\int_{\ell}(I_h(\mathbf{w}) - \mathbf{w})\cdot\mathbf{n}\,[q_h]$$
$$\le Ch^{1/2}\|\mathbf{w}\|_1 \sum_{\ell\subset\mathcal{C}_h}\Big(\int_{\ell}[q_h]^2\Big)^{\frac{1}{2}},$$

and we conclude the proof. □

These two lemmas show the inf–sup deficiency of the unstable pairs P_1Q_1 and P_1Q_0 respectively. In order to get an stabilized approach we introduce the following operators:

Let be $q \in L^2(\Omega)$, for each $K \in \mathcal{C}_h$ we denote by χ_K the characteristic of K. Then, the operator $\Pi_0 : L^2(\Omega) \to Q_h^0$ is defined such that

$$\int_{\Omega} \Pi_0(q)\,\chi_K = \int_{\Omega} q\,\chi_K, \quad \forall K \in \mathcal{C}_h.$$

Thus, for any $K \in \mathcal{C}_h$, we have $\Pi_0(q)|_K = \frac{1}{|K|}\int_K q$.

On the other hand, for the stabilization of the pair P_1Q_0, we consider for each vertex v of K the local operator I_v defined in page 110 of [30], i.e., let $\omega_v := \bigcup\{K \in \mathcal{C}_h :\ v \text{ is a vertex of } K\}$ we define $I_v : Q \to Q_0(\omega_v)$ as

$$\int_{\omega_v} I_v(p)\, q = \int_{\omega_v} p\, q \quad \text{for any } q \in Q_0(\omega_v).$$

Let $\{n_i\}_{1\le i\le N}$ be the set of the nodes of $\mathcal{C}_h$ and let $\{\beta_i\}_{1\le i\le N}$ be the corresponding nodal bases. Then, the operator $\Pi_1 : L^2(\Omega) \to Q_h^1$ is defined as:

$$\Pi_1(q) = \sum_{i=1}^{N} I_{n_i}(q)\beta_i.$$

From the interpolation theory (see, [29–31]) we can affirm that, these operators Π_0 and Π_1 satisfy

$$\|\Pi_l q\|_0 \le C\|q\|_0 \quad \text{and} \quad \|q - \Pi_l q\|_0 \le Ch\|q\|_1, \quad l = 0, 1.$$

Moreover, we also have the following result

Lemma 3.3. *There exists a positive constant C such that*

$$Ch\|\nabla q_h\|_0 \le \|q_h - \Pi_0 q_h\|_0 \quad \forall q_h \in Q_1.$$

$$Ch^{\frac{1}{2}}[q_h] \le \|q_h - \Pi_1 q_h\|_0 \quad \forall q_h \in Q_0.$$

Proof. see Lemma 2.3 of [11]. □

Therefore, combining this result, Lemmas 3.1 and 3.2 we can infer that

Corollary 3.1. *Let V_h^1 and Q_h^l, $l = 0, 1$ be the spaces defined above. Then, there exist positive constants C_1 and C_2 such that*

$$\sup_{0\ne\mathbf{v}\in V_h^1} \frac{\int_{\Omega} q_h \nabla\cdot\mathbf{v}_h}{\|\mathbf{v}_h\|_1} \ge C_1\|q_h\|_0 - C_2\|q_h - \Pi_l q_h\|_0 \quad \forall q_h \in Q_h^l, \quad l = 0, 1.$$

Now, let be $C : V \times Q \to \mathbb{R}$ the bilinear form defined as:

$$C((\mathbf{u}, p), (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u}) - G(p, q),$$

where $G(p, q) = \int_{\Omega}(p - \Pi_k(p))(q - \Pi_k(q))$ with $k = 1$ for the P_1Q_0 pair and $k = 0$ for P_1Q_1.

Hence, the stabilized problem associated to (2.5) can be written as: Find $(\mathbf{u}_h, p_h) \in V_h^k \times Q_h^l$, with $k = 1, l = 0$ for the P_1Q_0 pair and $k = 1, l = 1$ for P_1Q_1 element, such that

$$C((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = \langle \mathbf{f}, \mathbf{v}_h\rangle_{V'\times V}. \tag{3.10}$$

Thus, we have the following theorem which states the stabilization of our P_1Q_1 and P_1Q_0 cross-grid elements.

Theorem 3.1. *Let (V_h^k, S_h^l) be the spaces with $k=1, l=0$ or $k=1, l=1$. Then, there exists a positive constant C, independent of h, such that*

$$\sup_{(\mathbf{v}_h,q_h)\in V_h\times S_h} \frac{C((\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h))}{\|\mathbf{v}_h\|_1+\|q_h\|_0} \geq C\left(\|\mathbf{u}_h\|_1+\|p_h\|_0\right) \quad \forall(\mathbf{u}_h,p_h)\in V_h^k\times Q_h^l.$$

Proof. It is the same proof of Theorem 4.1 in [11]. □

Now, from the classical mixed finite approximations theory (see, for example, [21]), we are in condition to conclude that:

Corollary 3.2. *Assume that $(\mathbf{u},p)\in V\cap H^2(\Omega)\times Q\cap H^1(\Omega)$ solves the Stokes problem (2.2) and that $(\mathbf{u}_h,p_h)$ is the solution of the stabilized mixed problem (3.10) with our P_1Q_1 or P_1Q_0 cross-grid elements. Then,*

$$\|\mathbf{u}-\mathbf{u}_h\|_1+\|p-p_h\|_0\leq Ch(\|\mathbf{u}\|_2+\|p\|_1).$$

4. Cross-grid P_1Q_1 finite element approximation for second order elliptic equations

In this section we consider one of the simplest second order elliptic problems and its mixed formulation, and we propose a modification of the problem in order to guarantee the convergence with optimal order for the stabilized P_1Q_1 cross-grid elements.

The elliptic problem under consideration is: Find p such that

$$\begin{cases} -\mathrm{div}(\nabla p)=f & \text{in } \Omega\,, \\ \dfrac{\partial p}{\partial \mathbf{n}}=0 & \text{on } \Gamma:=\partial\Omega. \end{cases} \tag{4.11}$$

In many applications the variable of interest is $\mathbf{u}=-\nabla p$, and so a mixed finite element method seems to be appropriate in order to approximate $\mathbf{u}$ and p simultaneously. Indeed, the mixed formulation of (4.11) is: Find $\mathbf{u}\in H_0(\mathrm{div},\Omega)$ and $p\in L_0^2(\Omega)$ such that

$$\begin{cases} (\mathbf{u},\mathbf{v})+b(\mathbf{v},p)=0 & \forall\mathbf{v}\in H_0(\mathrm{div},\Omega)\,, \\ -b(\mathbf{u},q)=(f,q) & \forall q\in L_0^2(\Omega)\,, \end{cases} \tag{4.12}$$

where $H_0(\mathrm{div},\Omega)=\{\mathbf{v}\in(L^2(\Omega))^2 : \mathrm{div}(\mathbf{v})\in L^2(\Omega),\ \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma\}$ with the usual norm:

$$\|\mathbf{v}\|_H^2:=\|\mathbf{v}\|_0^2+\|\mathrm{div}(\mathbf{v})\|_0^2.$$

This problem involves a different differential operator of the Stokes problem, and it is not surprising that stable finite element approximations to the Stokes problem could not be appropriate for the elliptic. The main difference is that, while in Stokes the family of elements has only to satisfy the inf–sup conditions, the elliptic problem under consideration has to fulfill two conditions. In fact, assuming that we have finite element spaces $V_h\subset H_0(\mathrm{div},\Omega)$ and $Q_h\subset L_0^2(\Omega)$ the mixed discrete problem is: Find $\mathbf{u}_h\in V_h$ and $p_h\in Q_h$ such that

$$\begin{cases} (\mathbf{u}_h,\mathbf{v}_h)+b(\mathbf{v}_h,p_h)=0 & \forall\mathbf{v}_h\in V_h, \\ -b(\mathbf{u}_h,q_h)=(f,q_h) & \forall q_h\in Q_h. \end{cases} \tag{4.13}$$

It is well known that in order to guarantee the convergence we need that the discrete spaces satisfy the following two conditions:

(1) $\exists\alpha>0 : \|\mathbf{v}_h\|_0^2\geq\alpha\|\mathbf{v}_h\|_H^2, \quad \forall\mathbf{v}_h$ such that: $(\mathrm{div}\,\mathbf{v}_h,q_h)=0, \quad \forall q_h\in Q_h.$
(2) The LBB conditions, i.e., there exists $\hat{\beta}>0$ such that

$$\sup_{0\neq\mathbf{v}_h\in V_h}\frac{(\mathrm{div}\,\mathbf{v}_h,q_h)}{\|\mathbf{v}_h\|_H}\geq\hat{\beta}\|q_h\|_0 \qquad \forall q_h\in Q_h.$$

Since for any function $\mathbf{v}\in(H^1(\Omega))^2$ we have that $\|\mathbf{v}\|_1\geq\|\mathbf{v}\|_H$, it is clear that, if the family of finite elements satisfies the inf–sup condition (2.6) these also satisfy the LBB condition (2) but not necessary the condition (1), unless $\mathrm{div}(V_h)=Q_h$. Now, we introduce the modification given in [20] in order to guarantee the convergence, with optimal order, when our cross-grid stabilized finite element P_1Q_1 is applied. Hence, since $\nabla(\mathrm{div}\,\mathbf{u})=\nabla f$, we have the alternative variational formulation of (4.12): Find $\mathbf{u}\in H_0(\mathrm{div},\Omega)$ and $p\in L_0^2(\Omega)$ such that

$$\begin{cases} (\mathbf{u},\mathbf{v})+(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})+b(\mathbf{v},p)=(f,\mathrm{div}\,\mathbf{v}) & \forall\mathbf{v}\in H_0(\mathrm{div},\Omega), \\ -b(\mathbf{u},q)=(f,q) & \forall q\in L_0^2(\Omega). \end{cases} \tag{4.14}$$

Now, it is clear that this is a well-posed problem and the corresponding discrete approximation: Find $\mathbf{u}_h \in V_h$ and $p_h \in Q_h$ such that

$$\begin{cases} (\mathbf{u}_h, \mathbf{v}_h) + (\text{div}\mathbf{u}_h, \text{div}\mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (f, \text{div}\mathbf{v}_h) & \forall \mathbf{v}_h \in V_h, \\ -b(\mathbf{u}_h, q_h) = (f, q_h) & \forall q_h \in Q_h, \end{cases} \tag{4.15}$$

is well-posed too if the spaces V_h and S_h satisfy the LBB condition. Now, we consider the spaces V_h^1 and S_h^1, defined in (2.3) and (2.4) respectively, with $V = H_0(\text{div}, \Omega)$ and $Q = L_0^2(\Omega)$. Then, we define the bilinear form $D : V_h^1 \times Q_h^1 \to \mathbb{R}$ as:

$$D((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = (\mathbf{u}_h, \mathbf{v}_h) + (\text{div}\,\mathbf{u}_h, \text{div}\mathbf{v}_h) + b(\mathbf{v}_h, p_h) + b(\mathbf{u}_h, q_h) - G(p_h, q_h),$$

the new stabilized problem is: Find $(\mathbf{u}_h, p_h) \in V_h^1 \times Q_h^1$ such that

$$D((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)) = (f, \text{div}\mathbf{v}_h) - (f, q_h), \quad \forall (\mathbf{v}_h, q_h) \in V_h^1 \times Q_h^1. \tag{4.16}$$

First, we observe that

Lemma 4.1. *There exist positive constants C_1 and C_2 such that*

$$\sup_{0 \neq \mathbf{v}_h \in V_h^1} \frac{\int_\Omega \text{div}\,\mathbf{v}_h\, q_h}{\|\mathbf{v}_h\|_H} \geq C_1 \|q_h\|_0 - C_2 h \|\nabla q_h\|_0 \quad \forall q_h \in Q_h^1.$$

Proof. Although the proof follows similar arguments of the proof of Lemma 2.1 in [11], we include it for the sake of completeness. We know that given any $q \in L^2(\Omega)$ there exists a function $\mathbf{u} \in H_0^1(\Omega) \subset H_0(\text{div}, \Omega)$ and a constant C_d such that $\text{div}\,\mathbf{u} = q$ and $\|\mathbf{u}\|_1 \leq C_d \|q\|_0$ and thus, the following inf–sup holds

$$\sup_{0 \neq \mathbf{v} \in H_0(\text{div},\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_H} \geq \beta_d \|q\|_0.$$

Therefore, given $q_h \in Q_h^1$ we have that there exists a function $\mathbf{w} \in H_0^1(\Omega) \subset H_0(\text{div}, \Omega)$ such that

$$\frac{\int_\Omega \text{div}\,\mathbf{w}\, q_h}{\|\mathbf{w}\|_H} \geq \beta_d \|q_h\|_0. \tag{4.17}$$

Let $\mathbf{w}_h \in V_h$ be an interpolator of $\mathbf{w}$ such that

$$\|\mathbf{w} - \mathbf{w}_h\|_0 \leq Ch\|\mathbf{w}\|_1 \quad \text{and} \quad \|\mathbf{w}_h\|_1 \leq \|\mathbf{w}\|_1,$$

which can be normalized such that $\|\mathbf{w}_h\|_H = \|p_h\|_0$. Hence, integrating by parts and using that $(\mathbf{w} - \mathbf{w}_h) \cdot \mathbf{n} = 0$ on Γ, we obtain

$$\begin{aligned} \frac{\left|\int_\Omega \text{div}\,\mathbf{w}_h\, q_h\right|}{\|\mathbf{w}_h\|_H} &\geq \frac{\left|\int_\Omega \text{div}\,\mathbf{w}_h\, q_h\right|}{\|\mathbf{w}_h\|_1} \geq \frac{\left|\int_\Omega \text{div}\,\mathbf{w}_h\, q_h\right|}{\|\mathbf{w}\|_1} \geq \frac{\int_\Omega \text{div}\,\mathbf{w}\, q_h}{\|\mathbf{w}\|_1} - \frac{\left|\int_\Omega \text{div}\,(\mathbf{w} - \mathbf{w}_h)\, q_h\right|}{\|\mathbf{w}\|_1} \\ &\geq \beta_d \|q_h\|_0 - \frac{\left|\int_\Omega (\mathbf{w} - \mathbf{w}_h) \nabla q_h\right|}{\|\mathbf{w}\|_1} \geq \beta_d \|q_h\|_0 - Ch\|\nabla q_h\|_0, \end{aligned} \tag{4.18}$$

and the proof concludes. □

Therefore, we can infer the following result

Theorem 4.1. *There exists a positive constant C, independent of h, such that*

$$\sup_{(\mathbf{v}_h, q_h) \in V_h^1 \times Q_h^1} \frac{D((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{\|\mathbf{v}_h\|_H + \|q_h\|_0} \geq C(\|\mathbf{u}_h\|_H + \|p_h\|_0) \quad \forall (\mathbf{u}_h, p_h) \in V_h^1 \times Q_h^1.$$

Proof.

Given any $p_h \in Q_h^1$, we can consider a function $\mathbf{w}$ such that (4.17) holds, and let $\mathbf{w}_h$ be its corresponding interpolator satisfying (4.18). Hence, from Lemma 4.1, for any $\alpha \in \mathbb{R}$ we have that:

$$\begin{aligned} D((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha \mathbf{w}_h, -p_h)) &= \|\mathbf{u}_h\|_H^2 - \alpha(\mathbf{u}_h, \mathbf{w}_h)_H - \alpha b(\mathbf{w}_h, p_h) + G(p_h, p_h) \\ &\geq \|\mathbf{u}_h\|_H^2 + \|(I - \Pi_0)p_h\|_0^2 - \alpha \|\mathbf{u}_h\|_H \|\mathbf{w}_h\|_H + \alpha(C_1\|p_h\|_0 - C_2 h\|\nabla p_h\|_0)\|\mathbf{w}_h\|_H. \end{aligned}$$

On the other hand, since $\Pi_0(p_h)$ is constant on each element we have that

$$h^2 \|\nabla p_h\|_0^2 = \sum_{K \subset Q_h} h^2 \|\nabla(p_h - \Pi_0(p_h))\|_{0,K}^2 \leq C_I \sum_{K \subset Q_h} \|p_h - \Pi_0(p_h)\|_{0,K}^2 = C_I \|p_h - \Pi_0(p_h)\|_0^2,$$

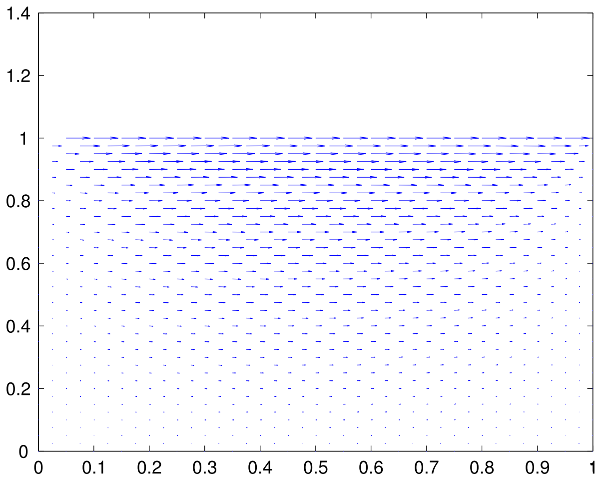

Fig. 3. Cavity flow, P_1Q_0 element: velocity vectors.

where in the last inequality we use the inverse estimate $\|\nabla q_h\|_0 \le C_I \frac{1}{h}\|q_h\|_0$. Thus, since $\|\mathbf{w}_h\|_H = \|p_h\|_0$ we have that

$$D((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha \mathbf{w}_h, -p_h)) \ge \|\mathbf{u}_h\|_H^2 + \|(I - \Pi_0)p_h\|_0^2 + \alpha C_1 \|p_h\|_0^2 \\ - \alpha \|\mathbf{u}_h\|_H \|p_h\|_0 - \alpha C \|(I - \Pi_0)p_h\|_0 \|p_h\|_0.$$

Hence, using the arithmetic–geometric inequality $ab \le \frac{1}{2\epsilon^2}a^2 + \frac{\epsilon}{2}b^2$ valid for all $\epsilon > 0$, we can write

$$D((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha \mathbf{w}_h, -p_h)) \ge (1 - K_1\alpha)\|\mathbf{u}_h\|_H^2 + (1 - K_2\alpha)\|(I - \Pi_0)p_h\|_0^2 + \alpha K_3 \|p_h\|_0^2,$$

for some positive constants K_1, K_2 and K_3. Then, we can choose an appropriate $\alpha > 0$ such that

$$D((\mathbf{u}_h, p_h), (\mathbf{u}_h - \alpha \mathbf{w}_h, -p_h)) \ge C(\|\mathbf{u}_h\|_H^2 + \|(I - \Pi_0)p_h\|_0^2 + \|p_h\|_0^2) \\ \ge C(\|\mathbf{u}_h\|_H + \|p_h\|_0)^2.$$

On the other hand, since $\|\mathbf{u}_h - \alpha \mathbf{w}_h\|_H \le \|\mathbf{u}_h\|_H + C\|\mathbf{w}_h\|_H = \|\mathbf{u}_h\|_H + C\|p_h\|_0$, taking $\mathbf{v}_h = \mathbf{u}_h - \alpha \mathbf{w}_h$ and $q_h = -p_h$, we get

$$\frac{D((\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h))}{\|\mathbf{v}_h\|_H + \|q_h\|_0} \ge C(\|\mathbf{u}_h\|_H + \|p_h\|_0),$$

and the result follows. □

Therefore, by similar arguments to those given above for the Stokes problem, we can infer that

Corollary 4.1. *Assume that* $(\mathbf{u}, p) \in V \cap H^2(\Omega) \times Q \cap H^1(\Omega)$ *solves the elliptic problem* (4.12) *and that* $(\mathbf{u}_h, p_h)$ *is the solution of the stabilized mixed problem* (4.16). *Then,*

$$\|\mathbf{u} - \mathbf{u}_h\|_H + \|p - p_h\|_0 \le Ch(\|\mathbf{u}\|_2 + \|p\|_1).$$

Remark 4.1. Let us notice that for cross-grid meshes our P_1Q_1 stabilized element has the same optimal order of convergence as the P_1P_1 stabilized element, with the advantage that our element requires one less pressure node in each rectangle.

5. Numerical results

5.1. The Stokes problem

We present in this section some numerical results obtained with the P_1Q_0 and the P_1Q_1 cross-grid mixed finite elements on two test cases of the Stokes problem.

The first example is the classical lid-driven cavity flow problem in $\Omega = (0, 1) \times (0, 1)$ with constant velocity $\mathbf{u} = (1, 0)$ in the top lid $\{y = 1\,,\ 0 < x < 1\}$ and homogeneous Dirichlet conditions in the rest. In this flow problem we consider $\mathbf{f} = 0$, and we took $\mu = 0.1$.

We solved this problem first with the P_1Q_0 and the P_1Q_1 mixed elements (which are unstable) and then with P_1Q_0 and P_1Q_1 stabilized. Both elements, with and without stabilization, produced correct velocity solutions, which are plotted in Figs. 3 and 4.

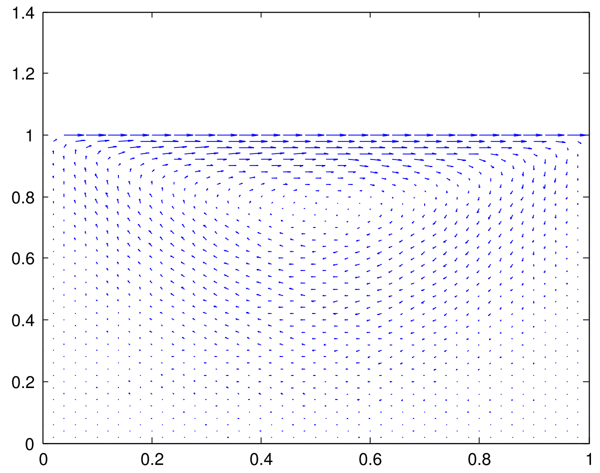

Fig. 4. Cavity flow, P_1Q_1 element: velocity vectors.

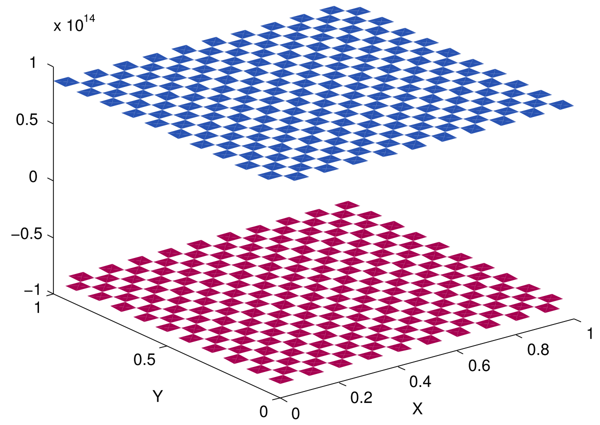

Fig. 5. Cavity flow, P_1Q_0 element: 3D view of the pressure.

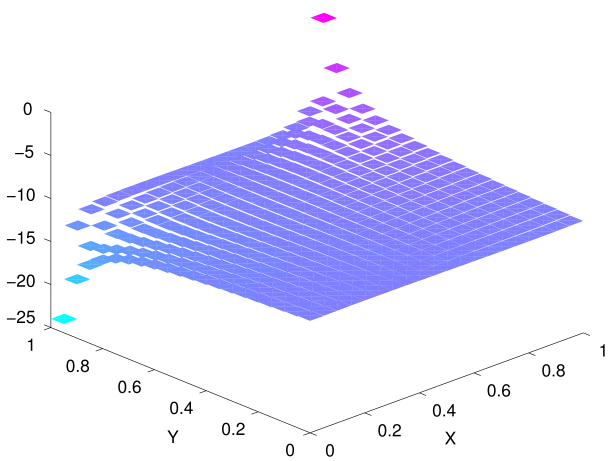

Fig. 6. Cavity flow, P_1Q_0 element with stabilization: 3D view of the pressure.

As we proved in Lemma 4.1 of [1], a clear nodal checkerboard mode phenomenon can be seen in the solution of the P_1Q_1 element (see Figures 7 and 9 in [1]). Likewise, Fig. 5 presents a three-dimensional view of the pressure solution in the P_1Q_0 case, which shows the nodal nature of the spurious pressure mode as we predicted by Lemma 2.1.

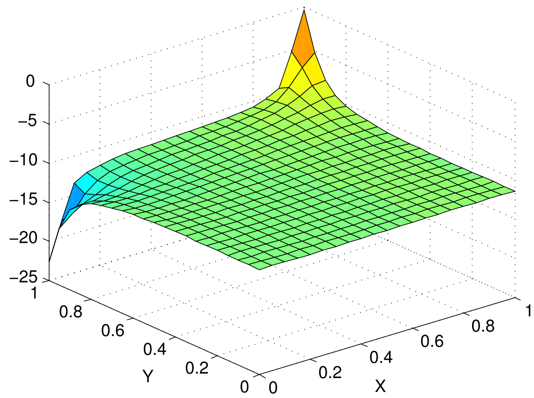

Fig. 7. Cavity flow, P_1Q_1 element with stabilization: 3D view of the pressure.

Table 1
Errors for the approximation of the solution $(\mathbf{u}, p)$ by P_1Q_0 and P_1Q_1 stabilized elements (second example).

N	P_1Q_0		P_1Q_1	
	$\|\mathbf{u}-\mathbf{u}_h\|_1$	$\|p-p_h\|_0$	$\|\mathbf{u}-\mathbf{u}_h\|_1$	$\|p-p_h\|_0$
10	1.6726	0.3001	1.6285	0.3878
15	1.0978	0.2012	1.0938	0.2162
20	0.8239	0.1801	0.8239	0.1298
25	0.6626	0.1514	0.6736	0.0788
30	0.5749	0.1021	0.5521	0.0541

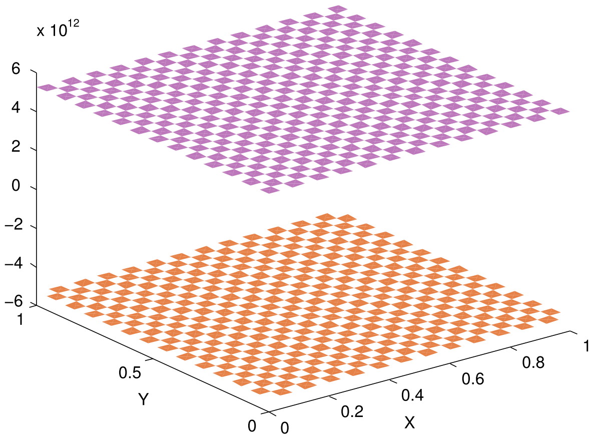

Fig. 8. P_1Q_0 element: 3D view of the pressure (second example).

Table 2
Errors in the approximation of the solution $(\mathbf{u}, p)$ by P_1Q_1 stabilized elements for the modified elliptic problem.

N	P_1Q_1	
	$\|\mathbf{u}-\mathbf{u}_h\|_1$	$\|p-p_h\|_0$
10	0.8446	0.5812
15	0.5628	0.3578
20	0.4216	0.2250
25	0.3373	0.1357
30	0.2810	0.0652

On the other hand, the pressure solutions obtained with our stabilization procedure, for the two elements P_1Q_0 and P_1Q_1, are shown in Figs. 6 and 7 respectively, where we can observe that stability has been reached.

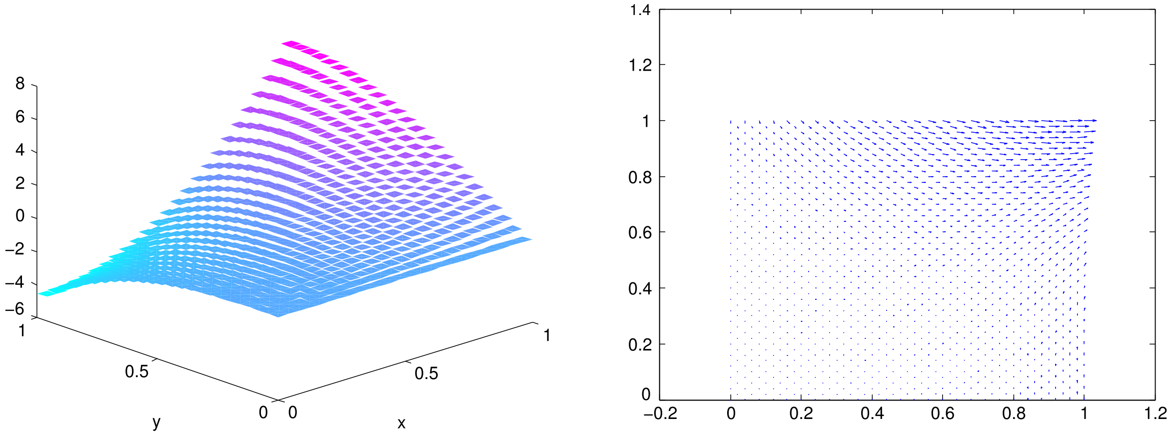

Fig. 9. P_1Q_0 element with stabilization: 3D view of the pressure (left)—velocity vectors (right) (second example).

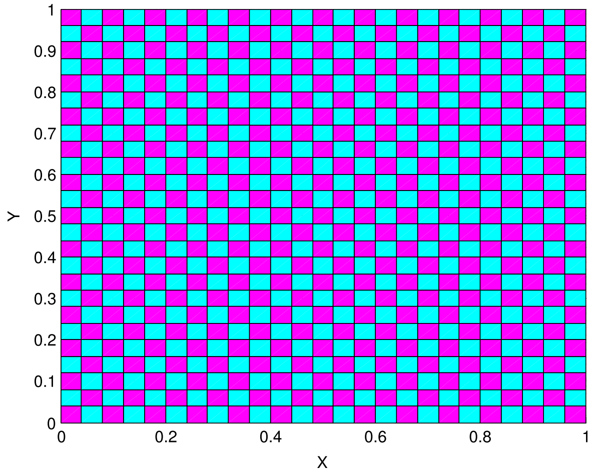

Fig. 10. P_1Q_1 element: 2D view of the pressure (second example).

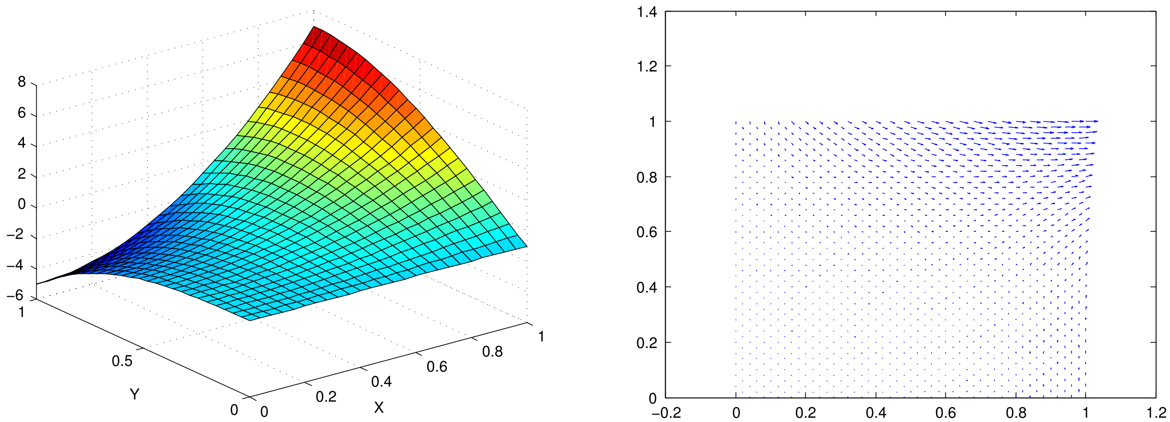

Fig. 11. P_1Q_1 element with stabilization: 3D view of the pressure (left)—velocity vectors (right) (second example).

The second example is given by taking $\Omega = (0, 1) \times (0, 1)$, $\mathbf{u} = (20xy^3, 5x^4 - 5y^4)$ and $p = 12x^2y - 4y^3 - 1$ and replacing them in the Stokes equations (2.1) in order to obtain the source $\mathbf{f}$. The purpose of this second example, in which we know the analytical solution, is, in particular, to estimate convergence rates. As an example, we show the figures corresponding

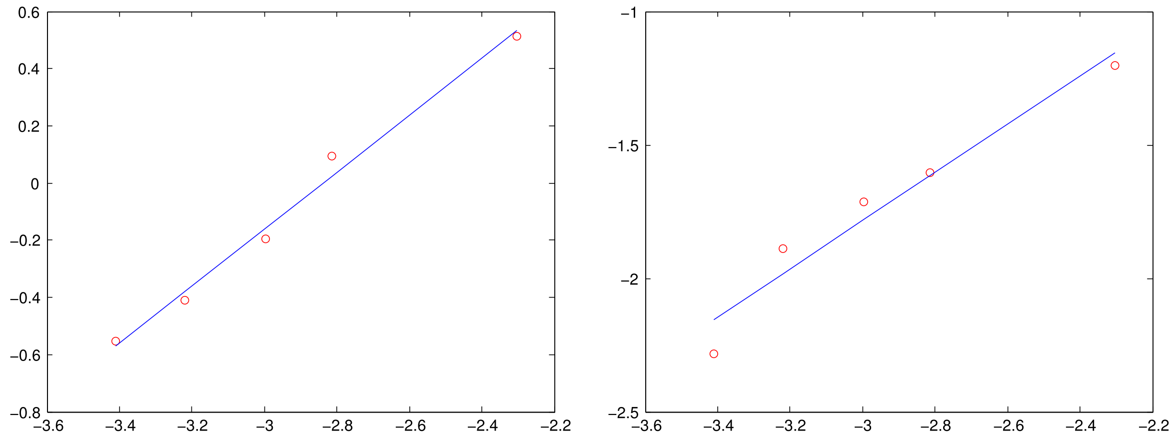

Fig. 12. P_1Q_0 element: $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ (right) and $log(\|p-p_h\|_0)$ (left) versus $log(h)$ (second example).

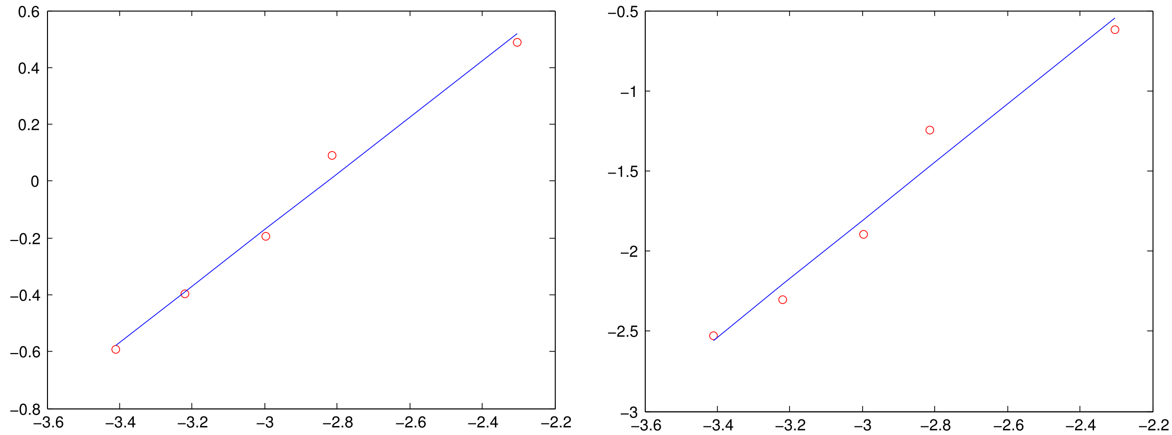

Fig. 13. P_1Q_1 element: $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ (right) and $log(\|p-p_h\|_0)$ (left) versus $log(h)$ (second example).

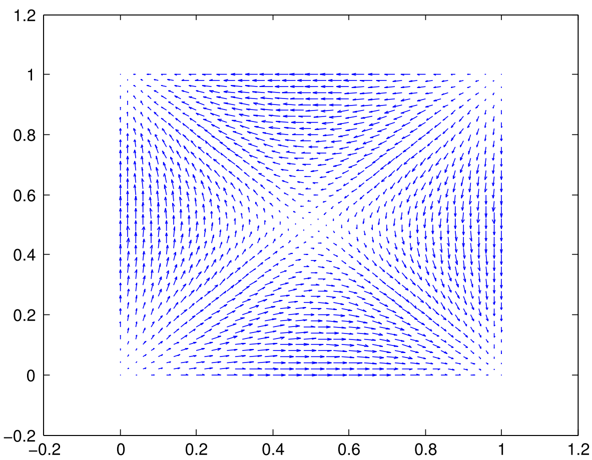

Fig. 14. P_1Q_1 element: velocity vectors for the elliptic problem.

to a uniform mesh of 25 × 25 rectangular elements for the pressure approximation. Again, a typical nodal checkerboard mode phenomenon can be seen in Fig. 8 when the cross-grid P_1Q_0 elements are used. The velocity vector and the pressure for the P_1Q_0 stabilized approximation are shown in Fig. 9. Analogous behavior can be observed for the P_1Q_1 approximation, see Figs. 10 and 11.

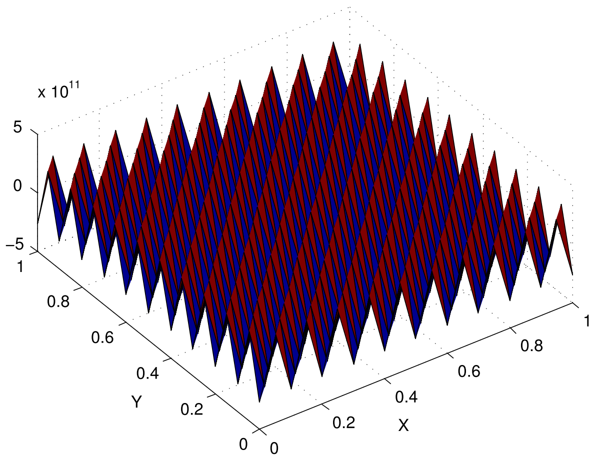

Fig. 15. P_1Q_1 element for the elliptic problem (4.13), 3D view of the pressure.

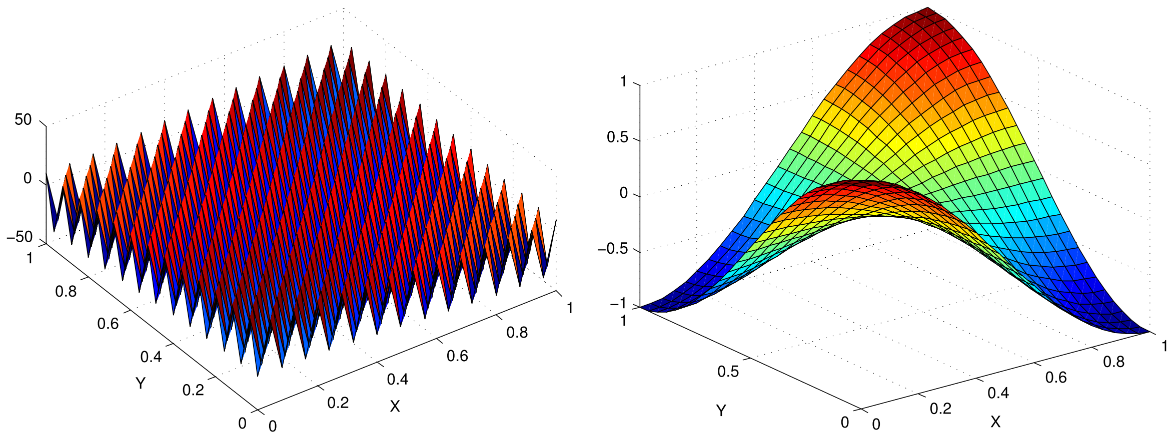

Fig. 16. P_1Q_1 element for the modified elliptic problem (4.14), 3D view of the pressure: without (left) and with (right) stabilization.

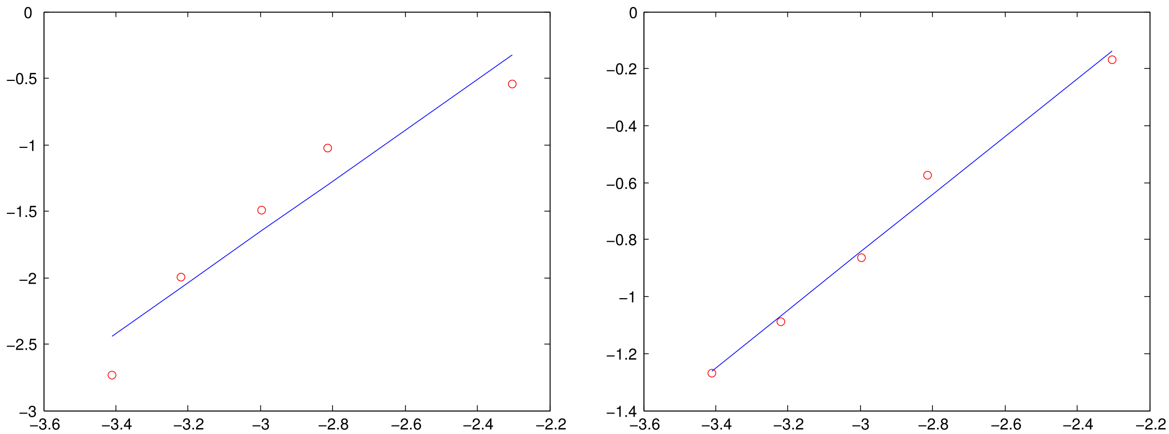

Fig. 17. P_1Q_1 element: $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ (right) and $log(\|p-p_h\|_0)$ (left) versus $log(h)$.

Moreover, with the objective of estimating the rates of convergence we solve this problem for $N = 10, 15, 20, 25, 30$, where N denotes the number of subdivisions of each boundary used in order to construct the quadrilateral mesh. The corresponding P_1Q_0 and P_1Q_1 errors, for the velocity in H^1 norm and the pressure in L^2 norm, are shown in Table 1. We observe that the error $\|\mathbf{u}-\mathbf{u}_h\|_1$ for the continuous P_1Q_1 and discontinuous P_1Q_0 pressure elements are almost the same.

Fig. 12(a) and (b) shows plots of $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ and $log(\|p-p_h\|_0)$ versus $log(h)$, where $h=\frac{1}{N}$, for the P_1Q_0 stabilized method. The numerical order, obtained by means of least-squares fitting, for the velocity error in H^1 norm is 0.99 and 0.91 for the L^2 error of the pressure. On the other hand, Fig. 13(a) and (b) shows plots of $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ and $log(\|p-p_h\|_0)$ versus $log(h)$, for the P_1Q_1 stabilized method. The numerical order for the velocity error in H^1 norm is in this case 0.99 and 1.89 for the L^2 error of the pressure.

5.2. The elliptic problem

In this section we show a numerical example of the application of our P_1Q_1 stabilized cross-grid mixed finite elements to the mixed problem (4.14).

In this example we take $\Omega=(0,1)\times(0,1)$ and $p(x,y)=\cos(\pi x)\cos(\pi y)$, i.e., $f=2\pi^2\cos(\pi x)\cos(\pi y)$, and solve the problems (4.13) and (4.15) using a uniform grid of 25×25 elements. The velocity, which is correct in all approximations under consideration, is shown in Fig. 14. The behavior of the pressure for the original problem (4.13) is shown in Fig. 15. On the other hand, the pressures obtained for the modified problem (4.15), without and with stabilization, are shown in Fig. 16 (left) and (right) respectively.

We also solve this problem for $N=10,15,20,25,30$. The corresponding errors in H^1 norm for the velocity and in L^2 norm the pressure, are shown in Table 2. Fig. 17(a) and (b) shows plots of $log(\|\mathbf{u}-\mathbf{u}_h\|_1)$ and $log(\|p-p_h\|_0)$ versus $log(h)$, where $h=\frac{1}{N}$, for the P_1Q_1 stabilized method. The numerical order, obtained by means of least-squares fitting, for the velocity error in H^1 norm is 1.01 and 1.91 for the L^2 error of the pressure.

Acknowledgments

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