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Optimal distributed control problem for cubic nonlinear Schrödinger equation

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Abstract We consider an optimal internal control problem for the cubic nonlinear Schrödinger (NLS) equation on the line. We prove well-posedness of the problem and existence of an optimal control. In addition, we show first order optimality conditions. Also the paper includes the proof of a smoothing effect for the nonhomogeneous NLS, which is necessary to obtain the existence of an optimal control.

Keywords nonlinear Schrödinger equation · optimal control · optical fibers · noise immunity

Mathematics Subject Classification (2000) 35Q55 · 49J20 · 49K20

1 Introduction

1.1 The physical model

Ever since the first working fiber-optical data transmission system demonstrated by the German physicist Manfred Börner at Telefunken Research Labs in Ulm in 1965, the development of high-bit-rate transmission over optimal fibers has increased enormously its information-carrying capacity. However, there are limits on capacities imposed by various transmission impairments that distort and degrade the signal in a number of ways ([2], [19]). One common source of impairments in light-wave communication systems is the amplified spontaneous emission noise generated by the

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erbium-doped fiber amplifiers used to compensate loss in the fiber ([2], [19]). This additive noise perturbs the propagating pulses, producing amplitude, frequency, timing, and phase jitter, which can then lead to bit errors ([14], [15], [23]). The propagation of pulses in an optical fiber free of noise is governed by the nonlinear Schrödinger (NLS) equation ([16], [19], [25]) which in dimensionless units is

$$\partial_z u = i\partial_t^2 u + i|u|^2 u,\ z \in [0,\zeta],\ t \in \mathbb{R} \tag{1}$$

where z is the propagation distance, t is the retarded time (that is, the time in a reference frame that moves with the group velocity of the pulse), ζ is the length of the optical fiber and $u(z,t)$ is the slowly varying envelope of the electric field of an optical pulse in a fiber, all quantities are in dimensionless units.

Note that in other contexts such as acoustics waves in plasma, quantum mechanics, solid state physics, condensed matter physics, quantum chemistry, etc., NLS equation has the name of the variables exchanged.

In [26] the authors point out that Monte-Carlo simulations are not adequate to determine the effect of the noise on a system because of the small error rates. Therefore they propose a technique that concentrates Monte-Carlo simulation on those configurations that are most likely to lead to transmission errors. To do so, they use the analytical knowledge about the behavior of the system that comes from soliton perturbation theory and linearize the NLS equation around the soliton solution.

The goal of this paper is to consider a completely different approach to analyze the effect of the noise on the optical fiber transmission within the framework of optimal control. By studying the immunity noise level (see section 1.2), we would be able to find an upper bound for the error rate assuming that the distribution of the noise is known.

Following [12] and [29], we consider the evolution of the optical field by the non-homogeneous NLS equation

$$\partial_z u = i\partial_t^2 u + i|u|^2 u + g,\ z \in [0,\zeta],\ t \in \mathbb{R}.$$

where the term g describes the amplified spontaneous emission noise generation. Usually, noise is represented as circularly symmetric complex Gaussian noise with autocorrelation function $\langle g(z,t)\bar{g}(z',t')\rangle = \gamma^2\delta(z-z',t-t')$, where γ^2 is a parameter describing the noise power, $\langle\ \rangle$ denotes an ensemble average and δ denotes a delta function. Note that the autocorrelation function above implies infinite noise bandwidth. Assuming that any physical system (or any numerical computation) necessarily has a finite noise bandwidth, finite noise energy is considered (see [24], [26]). Consequently we will study the additive noise $g \in L^2([0,\zeta],L^2(\mathbb{R}))$.

It is known that for solitons in the absence of noise, the pulse shape remains fixed, but in the presence of noise the pulse shape can be degraded. We will consider that a pulse is degraded if it satisfies a certain restriction when it arrives at the end of the line. More precisely, in this simplified model of signal transmission, we will consider a prescribed (finite) set of possible sent pulses and their corresponding received ones. In this context, we analyze the anomalous transmissions where the error could not be detected, consisting with the reception of an admissible pulse not corresponding to the sent one.

In this framework, we consider g as a control and we will search for a control that minimizes a given cost functional having two terms involving the L^2 norm of the control and also an alternative term minimizing the distance to a given target.

In this paper we prove the existence and first order necessary conditions for a control satisfying the restriction at the end of the line that minimizes the given functional.

In the case where we consider only the first term of this objective functional (see the next subsection), minimizing the L^2 norm of the control, we will have proved the existence and first order necessary conditions of a control with minimum L^2 norm among all the controls that produces signal degradation. In this way we would know that if the noise acting on the line has less L^2 norm than the minimum noise, then it would not produce signal degradation.

We will now describe the mathematical setting.

1.2 The mathematical model

We consider the following model of data transmission: given a set of pulses $\{u_0, v_0\}$, the received pulses at the end of the line without noise are the pulses $\{u_\zeta, v_\zeta\}$. The received pulses $\{u_\zeta, v_\zeta\}$ are the evaluation of u, the solutions of usual NLS equation (1) with initial data $u(0,t) = u_0(t)$ and $u(0,t) = v_0(t)$ respectively, at $z = \zeta$. In order to differentiate each bit transmitted sequentially, a time window σ is used. Thus, let us consider that the pulse sent is u_0 if it is verified

$$\int_{\mathbb{R}} |u(\zeta,t) - u_\zeta(t)|^2 \sigma^2(t)dt \leq \eta,$$

(similarly for v_0) where u is the solution of

$$\partial_z u(z,t) = i\partial_t^2 u(z,t) + i|u(z,t)|^2 u(z,t) + g(z,t), \tag{2a}$$

$$u(0,t) = u_0(t), \tag{2b}$$

and $g \in L^2([0,\zeta], L^2(\mathbb{R}))$ represents the noise on the line. The η level must be chosen in order to distinguish between two pulses received without noise. For this, we choose

$$\eta < \frac{1}{2} \int_{\mathbb{R}} |u_\zeta(t) - v_\zeta(t)|^2 \sigma^2(t)dt \tag{3}$$

and since $\{u_\zeta, v_\zeta\}$ are the received pulses free of noise, η only depends on $\{u_0, v_0\}$. An error occurs when, due to noise, the solution of (2) verifies

$$\int_{\mathbb{R}} |u(\zeta,t) - v_\zeta(t)|^2 \sigma^2(t)dt \leq \eta.$$

The minimum noise that verifies this condition represents the noise immunity of the line.

Note that even if the noise only changes the phase, $u(\zeta,t) \cong e^{i\theta} u_\zeta(t)$, the error measure is written as

$$\int_{\mathbb{R}} |u(\zeta,t) - u_\zeta(t)|^2 \sigma^2(t)dt \cong \int_{\mathbb{R}} 2(1-\cos\theta)|u_\zeta(t)|^2 \sigma^2(t)dt,$$

which shows that perturbations of the phase are also considered in this approach.

With the aim of identifying the sequentially sent pulses, the receiver measures the pulses in a given window of time. We will consider this window of time as a real function σ localized around the predicted arrival time. For instance, we can take

$$\sigma(t) = ke^{-\left(\frac{t-t_r}{\tau}\right)^2}$$

where t_r is the time of the arrival, τ the width of the window and k a normalizing constant. More generally, we consider σ a real smooth function such that $\sup_{t\in\mathbb{R}} |t\sigma(t)| < +\infty$.

For $u_0 \in L^2(\mathbb{R})$ and a complex value control $g \in L^2([0,\zeta], L^2(\mathbb{R}))$, in section 3 we will prove the well posedness of equation (2) in $C([0,\zeta], L^2(\mathbb{R}))$. We will call $u[g]$ the solution associated to the control g. We introduce the set of admissible controls

$$\mathscr{G}_{\text{ad}} = \{g \in L^2([0,\zeta], L^2(\mathbb{R})) : \|\sigma\left(u[g](\zeta) - v_\zeta\right)\|_{L^2}^2 \leq \eta\}$$

and say that g is an admissible control if $g \in \mathscr{G}_{\text{ad}}$. Note that inequality (3) implies $0 \notin \mathscr{G}_{\text{ad}}$ and, from continuous dependence of the solutions on the control, we have $g \notin \mathscr{G}_{\text{ad}}$ for low noise levels. Our objective is to determine the minimum noise level that could cause an error in the transmission.

In this article we will consider the variational problem

$$0 \leq \mathscr{J}_\star = \inf_{g\in\mathscr{G}_{\text{ad}}} \mathscr{J}(g) \tag{4}$$

where

$$\mathscr{J}(g) = \|g\|_{L^2([0,\zeta],L^2)}^2 + \kappa\|\sigma\left(u[g](\zeta) - v_\zeta\right)\|_{L^2(\mathbb{R})}^2,$$

with $\kappa \geq 0$. Since $g \notin \mathscr{G}_{\text{ad}}$ if $\|g\|_{L^2([0,\zeta],L^2)}^2 < \varepsilon$, we see that $\mathscr{J}_\star > 0$.

We could consider a finite set of sent pulses at the beginning of the line $\{u_{1,0}, \ldots, u_{n,0}\}$ and $\{u_{1,\zeta}, \ldots, u_{n,\zeta}\}$ the pulses received at the end of the line without noise. Let $u_j[g]$ be the solution of (2a) with initial data $u_j[g](0,t) = u_{j,0}(t)$, define $\mathscr{G}_{\text{ad}} = \bigcup_{1\leq j\neq k\leq n} \mathscr{G}_{\text{ad}}^{j,k}$, where

$$\mathscr{G}_{\text{ad}}^{j,k} = \{g \in L^2([0,\zeta], L^2(\mathbb{R})) : \|\sigma\left(u_j[g](\zeta) - u_{k,\zeta}\right)\|_{L^2}^2 \leq \eta\}.$$

Then, $0 \notin \mathscr{G}_{\text{ad}}$ provided that

$$\eta < \frac{1}{2} \min_{1\leq j<k\leq n} \int_{\mathbb{R}} |u_{j,\zeta}(t) - u_{k,\zeta}(t)|^2 \sigma^2(t)\,dt,$$

and then $\mathscr{J}_\star > 0$, given that $\mathscr{J}_\star = \inf_{g\in\mathscr{G}_{\text{ad}}} \mathscr{J}(g) = \min_{1\leq j\neq k\leq n} \inf_{g\in\mathscr{G}_{\text{ad}}^{j,k}} \mathscr{J}(g)$. Therefore, it is sufficient to study the case initially considered.

There is a large amount of literature on controllability for the internal or bilinear control problem of Schrödinger equations, for instance see the surveys [22] and [30] and the references therein. However, optimal control problems for Schrödinger equations have received recent attention in the last years. In [7], the authors prove necessary conditions for an optimal bilinear control problem for a Schrödinger equation with a Hartree type nonlinearity. In [5], an optimal bilinear control problem for a linear Schrödinger equation with a Coulombian potential is studied. In [20] existence of an optimal control and necessary optimality conditions are derived for an abstract bilinear optimal control problem for a linear Schrödinger equation. In [18] the authors study an optimal bilinear control problem of Gross-Pitaevskii equations. In this setting the existence of an optimal control relies strongly on the fact that the energy space is compactly embedded in $L^2(\mathbb{R})$. As well for the bilinear case, in [13] the optimal control problem of a nonlinear Schrödinger equations is studied. In this article, the authors recover some kind of compactness of a minimizing sequence using previous results. In [3] existence an necessary conditions are proved for an optimal bilinear control problem for a nonlinear Schrödinger equation with Dirichlet conditions in a interval. In [4] the authors derived necessary and sufficient conditions for an abstract bilinear optimal control problem for a linear Schrödinger equation. As it can be seen all of the previous works concern with bilinear optimal control. As far as we know there are no previous results for a distributed optimal control for a nonlinear Schrödinger equation as we present in this article.

On the other side, in most of the articles regarding optimal control for a Schrödinger equation the objective functional consists of two terms, one describing the cost it takes to obtain the desired outcome through the control process and the other being the desired physical quantity (observable) to be minimized. In the present work, we follow this idea and consider two terms, the first measuring the level of noise and the second one (which could be omitted with $\kappa = 0$) related with the distance between the pulse with noise at the end of the line and a desired signal.

In the present article, we prove existence of an optimal distributed control of a nonlinear Schrödinger equation in the whole line with state constraints, as the limit of a minimizing sequence.

In a future work, we hope to implement, from these results, a numerical method that allows us to calculate the value of $\mathscr{J}_\star$ in a specific problem.

1.3 Organization of the paper

The rest of the work is organized as follows. Section 2 is devoted to preliminary results which will be used to prove the well posedness and the compactness necessary for the existence of a minimizer. In section 3 we prove the well posedness of the non-homogeneous NLS and the Fréchet differentiability of the unique solution of the state equation with respect to the control, required for the derivation of the first order necessary conditions. In section 4 we begin by proving a regularizing effect of the solution of the NLS which is essential to prove the compactness. Finally, in section 5 we prove our two main results: existence of a minimizer (Theorem 3) and first order necessary conditions for an optimal control (Theorem 5) which we enunciate here

Theorem 3 *Let $g_n \in \mathscr{G}_{ad}$ be a minimizing sequence. Consider $g_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$ and $u_\star \in \mathscr{X}_\zeta \cap H^1([0,\zeta],H^{-2}(\mathbb{R}))$ given by Proposition 12. Then $u_\star = u[g_\star]$, $g_\star \in \mathscr{G}_{ad}$ and $\mathscr{J}(g_\star) = \mathscr{J}_\star$.*

Theorem 5 *Let $g_\star$ be an optimal solution of problem* (4) *and $u_\star = u[g_\star]$ its associated state. Then, there exists $\alpha \geq 0$ such that $g_\star$ and $u_\star$ satisfy the following equations*

$$\begin{aligned}
&\partial_z u_\star = i\partial_t^2 u_\star + i|u_\star|^2 u_\star + g_\star \\
&u_\star(0) = u_0 \\
&\partial_z g_\star = i\partial_t^2 g_\star + 2i|u_\star|^2 g_\star - iu_\star^2 \bar{g_\star} \\
&g_\star(\zeta) = -\left(\kappa + \frac{1}{2}\alpha\right)\sigma^2(u_*(\zeta) - v_\zeta) \\
&\|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2 \leq \eta \\
&\alpha\left(\eta - \|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2\right) = 0.
\end{aligned}$$

2 Preliminaries and notation

As usual, we call $L^2(\mathbb{R})$ the real Hilbert space of complex valued square-integrable function on $\mathbb{R}$, with the inner product

$$(g,h)_{L^2} = \mathrm{Re}\int_{\mathbb{R}} \overline{g}(t)h(t)dt = \mathrm{Re}\int_{\mathbb{R}} \overline{\hat{g}}(\xi)\hat{h}(\xi)d\xi,$$

where $\hat{g}$ is the Fourier transform of g. We define the Sobolev spaces $H^s(\mathbb{R})$ as the distributions $g \in \mathscr{S}'(\mathbb{R})$ verifying $h = (\mathbf{1} - \partial_t^2)^{s/2} g \in L^2(\mathbb{R})$, where h is defined by $\hat{h}(\xi) = \left(1 + |\xi|^2\right)^{s/2} \hat{g}(\xi)$. For any $s \in \mathbb{R}$, $H^s(\mathbb{R})$ is a real Hilbert space with the inner product

$$(g,h)_{H^s} = \mathrm{Re}\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s \overline{\hat{g}}(\xi)\hat{h}(\xi)d\xi.$$

It is known that $H^s(\mathbb{R}) = \{g \in \mathscr{S}'(\mathbb{R}) : g^{(k)} \in L^2(\mathbb{R}), 0 \leq k \leq s\}$ if $s \in \mathbb{N}$. We can identify $H^{-s}(\mathbb{R})$ with the dual space of $H^s(\mathbb{R})$ by the duality product

$$\begin{aligned}
\langle g,h\rangle_{H^{-s},H^s} &= ((\mathbf{1} - \partial_t^2)^{-s/2} g, (\mathbf{1} - \partial_t^2)^{s/2} h)_{L^2} \\
&= \mathrm{Re}\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^{-s/2} \overline{\hat{g}}(\xi) \left(1 + |\xi|^2\right)^{s/2} \hat{h}(\xi)d\xi = \mathrm{Re}\int_{\mathbb{R}} \overline{\hat{g}}(\xi)\hat{h}(\xi)d\xi.
\end{aligned}$$

Let X be a Banach space and $I \subset \mathbb{R}$ an interval, for $1 \leq p < \infty$ we define the Banach space $L^p(I,X)$ as the completion of $C_c(I,X)$ with the norm

$$\|f\|_{L^p(I,X)} = \left(\int_I \|f(z)\|_X^p dz\right)^{1/p}.$$

Note that if $X = L^p(\mathbb{R})$, then $L^p(I, L^p(\mathbb{R})) \equiv L^p(I \times \mathbb{R})$.

Given a weight function $w \in C(\mathbb{R})$, $w(t) > 0$, we denote by $L_w^2(\mathbb{R})$ the Hilbert space of square-integrable functions with respect to the measure $\nu(dt) = w(t)dt$.

In the next results we prove some compact embeddings which together with the regularizing properties given in section 4 will provide us the compactness necessary to prove the existence of a minimizer.

Although the following results are mostly known, we provide the proofs for completeness.

Lemma 1 *If w is a weight function such that $w(t) \overset{|t|\to\infty}{\longrightarrow} +\infty$, then $H^{1/2}(\mathbb{R})\cap L^2_w(\mathbb{R})$ is compactly embedded in $L^2(\mathbb{R})$.*

Proof Let $Y \subset H^{1/2}(\mathbb{R})\cap L^2_w(\mathbb{R})$ be a bounded set, for any $\varepsilon > 0$ there exists $\tau > 0$ such that $w(t)\varepsilon > 1$ for $|t| > \tau$, then

$$\int_{|t|>\tau} |u(t)|^2 dt \le \varepsilon \int_{|t|>\tau} |u(t)|^2 w(t) dt \le \varepsilon \|u\|^2_{L^2_w} \le C\varepsilon.$$

Given $h \in \mathbb{R}$, we define $u_h(t) = u(t-h)$, from Parseval's identity we get

$$\|u_h - u\|^2_{L^2} = \int_{\mathbb{R}} |e^{-ih\xi} - 1|^2 |\widehat{u}(\xi)|^2 d\xi,$$

using $|e^{-ih\xi} - 1| \le \min\{2, |h\xi|\} \le 2^{1/2}|h|^{1/2}|\xi|^{1/2}$ we obtain

$$\|u_h - u\|^2_{L^2} \le C|h| \int_{\mathbb{R}} |\xi||\widehat{u}(\xi)|^2 d\xi \le C|h|\|u\|^2_{H^{1/2}(\mathbb{R})},$$

thus $\|u_h - u\|_{L^2} \le C|h|^{1/2}$, for all $u \in Y$ and $h \in \mathbb{R}$. Therefore, Y is relatively compact in $L^2(\mathbb{R})$ (see [1] theorem 2.32.)

Corollary 1 $L^2([0,\zeta], H^{1/2}(\mathbb{R})\cap L^2_w(\mathbb{R})) \cap W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$ *is compactly embedded in $L^2([0,\zeta], L^2(\mathbb{R}))$.*

Proof Since $H^{1/2}(\mathbb{R})\cap L^2_w(\mathbb{R}) \overset{c}{\hookrightarrow} L^2(\mathbb{R}) \hookrightarrow H^{-2}(\mathbb{R})$, the result follows from Aubin–Lions–Simon Lemma.

If $L^2_1(\mathbb{R}) = L^2_{w_1}(\mathbb{R})$, with $w_1(t) = (1+t^2)$, it holds $L^2_1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ and

$$\|u\|^2_{L^2_1} = \|u\|^2_{L^2} + \|t\,u\|^2_{L^2}.$$

Proposition 1 *The space $L^2_1(\mathbb{R})$ is compactly embedded in $H^{-2}(\mathbb{R})$.*

Proof Let $\{\phi_n\}_{n\in\mathbb{N}} \subset L^2_1(\mathbb{R})$ be a bounded sequence. We will prove that there exists a subsequence convergent in $H^{-2}(\mathbb{R})$. Since $\mathbf{1} - \partial_t^2 : H^k(\mathbb{R}) \to H^{k-2}(\mathbb{R})$ is an isomorphism, we have that $(\mathbf{1} - \partial_t^2)^{-1}\phi_n$ is bounded in $H^2(\mathbb{R})$. We shall see that $(\mathbf{1} - \partial_t^2)^{-1}(\phi_n)$ is bounded in $L^2_1(\mathbb{R})$. Therefore the result will follow from Lemma 1. Let $\psi_n = (\mathbf{1} - \partial_t^2)^{-1}\phi_n$, then $\psi_n = h * \phi_n$, where $h(t) = \frac{1}{2} e^{-|t|}$. We can write

$$\begin{aligned} t\,\psi_n(t) &= \frac{1}{2}\int_{\mathbb{R}} t\, e^{-|t-t'|}\phi_n(t')dt' \\ &= \frac{1}{2}\int_{\mathbb{R}} (t-t')e^{-|t-t'|}\phi_n(t')dt' + \frac{1}{2}\int_{\mathbb{R}} t'\, e^{-|t-t'|}\phi_n(t')dt', \end{aligned}$$

but $h, t\,h \in L^1(\mathbb{R})$ and $\phi_n \in L^2_1(\mathbb{R})$, thus $t\,\psi_n \in L^2(\mathbb{R})$ and

$$\|\psi_n\|^2_{L^2_1} = \|\psi_n\|^2_{L^2} + \|t\,\psi_n\|^2_{L^2} \leq C\|\phi_n\|^2_{L^2_1} \leq CM.$$

Corollary 2 *The space $C([0,\zeta], L^2_1(\mathbb{R})) \cap H^1([0,\zeta], H^{-2}(\mathbb{R}))$ is compactly embedded in $C([0,\zeta], H^{-2}(\mathbb{R}))$.*

Proof Using $L^2_1(\mathbb{R}) \overset{c}{\hookrightarrow} H^{-2}(\mathbb{R})$, the result follows from Arzelà–Ascoli theorem.

3 Well posedness

Although the well posedness of the homogeneous NLS has been widely studied (see [9], section 4.6), in this section we study thoroughly the non-homogeneous problem (2a)-(2b) aiming to obtain estimates of the solution and its derivatives. In particular, we get an equation for the Fréchet derivative of the solution with respect to the control variable (Proposition 6), which we use to obtain first order necessary conditions for the optimal control. The proof of the next results are similar to the ones of cubic NLS equation, using Strichartz estimates, which involves the spaces $L^q(\mathbb{R}, L^p(\mathbb{R}))$ for certain pairs of admissible exponents (p,q), i.e.

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p},$$

where $1 \leq p \leq \infty$. Note that $(6,6)$ and $(2,\infty)$ are pairs of admissible exponents.

Let $S(z)$ be the unitary group generated by $i\partial_t^2$. We recall the following classical estimates needed for well posedness (see [9]).

Proposition 2 *Let $I \subseteq \mathbb{R}$ be an interval. For any (p,q) pair of admissible exponent, there exists $C_p > 0$ such that for any $u_0 \in L^2(\mathbb{R})$ it holds that $S(z)u_0 \in L^q(I, L^p(\mathbb{R}))$ and*

$$\|S(z)u_0\|_{L^q(I,L^p(\mathbb{R}))} \leq C_p \|u_0\|_{L^2(\mathbb{R})}.$$

Proposition 3 *Let $I \subseteq \mathbb{R}$ be an interval and (p,q), (r',γ') two pairs of admissible exponents. Then there exists $C_{p,r} > 0$ such that for $g \in L^\gamma(I, L^r(\mathbb{R}))$ it holds $v \in L^q(I, L^p(\mathbb{R}))$ and*

$$\|v\|_{L^q(I,L^p(\mathbb{R}))} \leq C_{p,r} \|g\|_{L^\gamma(I,L^r(\mathbb{R}))},$$

where

$$v(z) = \int_0^z S(z-z')g(z')dz'$$

and r, γ are the conjugate exponents of r', γ' respectively.

In what follows we deduce classical estimates for the cubic nonlinearity. If $u \in L^6([0,\zeta],L^6(\mathbb{R}))$, then $|u|^2u \in L^2([0,\zeta],L^2(\mathbb{R}))$ and

$$\||u|^2u\|_{L^2([0,\zeta],L^2)} = \|u\|^3_{L^6([0,\zeta],L^6)}. \tag{5}$$

Using that $|u|^2u - |\tilde{u}|^2\tilde{u} = (|u|^2+|\tilde{u}|^2)(u-\tilde{u}) + u\tilde{u}\,\overline{(u-\tilde{u})}$ and Hölder's inequality, we obtain

$$\begin{aligned}\||u|^2u - |\tilde{u}|^2\tilde{u}\|_{L^2([0,\zeta],L^2)} \leq C\left(\|u\|^2_{L^6([0,\zeta],L^6)} + \|\tilde{u}\|^2_{L^6([0,\zeta],L^6)}\right)\\ \times \|u-\tilde{u}\|_{L^6([0,\zeta],L^6)}.\end{aligned} \tag{6}$$

and for any compact $K \subset \mathbb{R}$

$$\begin{aligned}\||u|^2u - |\tilde{u}|^2\tilde{u}\|_{L^1([0,\zeta]\times K)} \leq C\left(\|u\|^2_{L^4([0,\zeta]\times K)} + \|\tilde{u}\|^2_{L^4([0,\zeta]\times K)}\right)\\ \times \|u-\tilde{u}\|_{L^2([0,\zeta]\times K)}.\end{aligned} \tag{7}$$

Let $I \subseteq \mathbb{R}$ be an interval, consider the Banach space $\mathscr{X}_I = C(I,L^2(\mathbb{R})) \cap L^6(I,L^6(\mathbb{R}))$ with the norm

$$\|u\|_{\mathscr{X}_{\mathsf{z}}} = \|u\|_{C(I,L^2)} + \|u\|_{L^6(I,L^6)}.$$

Next, we prove local existence for a mild solution of (2) in the space $\mathscr{X}_{\mathsf{z}} = \mathscr{X}_{[0,\mathsf{z}]}$.

Theorem 1 *Let $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2(\mathbb{R}))$, if*

$$r = \max\left\{\|u_0\|_{L^2}, \|g\|_{L^1([0,\zeta],L^2)}\right\},$$

then there exists $\mathsf{z} = \mathsf{z}(r) \in (0,\zeta]$ and $u \in \mathscr{X}_{\mathsf{z}}$ solution of the integral equation

$$u(z) = S(z)u_0 + \int_0^z S(z-z')(i|u(z')|^2u(z') + g(z'))dz'. \tag{8}$$

There exists a constant $C > 0$ such that $\|u\|_{\mathscr{X}_{\mathsf{z}}} \leq Cr$ and u depends continuously on u_0 and g. Furthermore, there exists $L = L(r) > 0$ such that if $\tilde{u}_0 \in L^2(\mathbb{R})$ and $\tilde{g} \in L^1([0,\zeta],L^2(\mathbb{R}))$ are close to u_0 and g respectively, then the solution $\tilde{u}$ is defined on $[0,\mathsf{z}]$ and satisfies

$$\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} \leq L\left(\|u_0-\tilde{u}_0\|_{L^2} + \|g-\tilde{g}\|_{L^1([0,\zeta],L^2)}\right). \tag{9}$$

Proof Let $v(z) = S(z)u_0$ and $w(z) = \int_0^z S(z-z')g(z')dz'$, then from Propositions 2 and 3, it is obtained that

$$\|v+w\|_{\mathscr{X}_{\mathsf{z}}} \leq C\left(\|u_0\|_{L^2} + \|g\|_{L^1([0,\zeta],L^2)}\right) = R \leq 2Cr$$

for all $\mathsf{z} \in [0,\zeta]$. Consider the map $\Gamma : \mathscr{X}_{\mathsf{z}} \to \mathscr{X}_{\mathsf{z}}$ defined by

$$(\Gamma u)(z) = v(z) + w(z) + i\int_0^z S(z-z')|u(z')|^2u(z')dz',$$

and take $B_R(v+w)$ the ball in $\mathscr{X}_{\mathsf{z}}$ of radius R centered at $v+w$. If $u \in B_R(v+w)$, it holds $|u|^2u \in L^2([0,\zeta],L^2(\mathbb{R})) \subset L^1([0,\zeta],L^2(\mathbb{R}))$ and from Proposition 3 we have

$$\begin{aligned}\|\Gamma u-(v+w)\|_{L^q([0,\mathsf{z}],L^p(\mathbb{R}))} &\leq C\||u|^2u\|_{L^1([0,\mathsf{z}],L^2)} \\ &\leq C\mathsf{z}^{1/2}\||u|^2u\|_{L^2([0,\mathsf{z}],L^2)} = C\mathsf{z}^{1/2}\|u\|^3_{L^6([0,\mathsf{z}],L^6)} \\ &\leq C\mathsf{z}^{1/2}(\|v+w\|_{\mathscr{X}_{\mathsf{z}}}+R)^3 \leq 8C\mathsf{z}^{1/2}R^3.\end{aligned}$$

If $\mathsf{z} < \delta/R^4$, where $8C\delta^{1/2} < 1$, $\Gamma u \in B_R(v+w)$. Moreover, if $u,\tilde{u} \in B_R(v+w)$, we have

$$(\Gamma u)(z)-(\Gamma\tilde{u})(z) = i\int_0^z S(z-z')(|u(z')|^2u(z')-|\tilde{u}(z')|^2\tilde{u}(z'))dz',$$

form Strichartz estimates and (6), we obtain

$$\begin{aligned}\|\Gamma u-\Gamma\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} &\leq C\||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L^1([0,\mathsf{z}],L^2)} \\ &\leq C\mathsf{z}^{1/2}\||u|^2u-|\tilde{u}|^2\tilde{u}\|_{L^2([0,\mathsf{z}],L^2)} \leq CR^2\mathsf{z}^{1/2}\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}}.\end{aligned}$$

Thus, $\|\Gamma u-\Gamma\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} \leq \gamma\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}}$ with $0 \leq \gamma < 1$, and then there exists a unique fixed point of Γ in $B_R(v+w)$ solution of (8), satisfying

$$\|u\|_{\mathscr{X}_{\mathsf{z}}} \leq \|u-v-w\|_{\mathscr{X}_{\mathsf{z}}}+\|v+w\|_{\mathscr{X}_{\mathsf{z}}} \leq 2R \leq \tilde{C}r.$$

Let $\tilde{\mathsf{z}} \in [0,\zeta]$ and $\tilde{u} \in \mathscr{X}_{\tilde{\mathsf{z}}}$ be another solution of (8), Then there exists $0 < \mathsf{z}' \leq \min\{\mathsf{z},\tilde{\mathsf{z}}\}$, such that $\|\tilde{u}-v-w\|_{\mathscr{X}_{\mathsf{z}'}} < R$, and therefore $u(z) = \tilde{u}(z)$ for $0 \leq z \leq \mathsf{z}'$. Consider

$$\mathsf{z}_1 = \sup\{0 \leq z \leq \min\{\mathsf{z},\tilde{\mathsf{z}}\} : u(z') = \tilde{u}(z'), 0 \leq z' \leq z\}.$$

If $\mathsf{z}_1 < \min\{\mathsf{z},\tilde{\mathsf{z}}\}$, we can define $u^{(1)}(z) = u(z+\mathsf{z}_1)$ and $\tilde{u}^{(1)}(z) = \tilde{u}(z+\mathsf{z}_1)$, solutions of (2) defined on $[0,\min\{\mathsf{z},\tilde{\mathsf{z}}\}-\mathsf{z}_1]$. Since $u^{(1)}(0) = \tilde{u}^{(1)}(0)$, arguing as before, there would exist $\delta > 0$ such that $u^{(1)}(z) = \tilde{u}^{(1)}(z)$, for $0 \leq z < \delta$, contradicting that z_1 was the supreme. Therefore $u(z) = \tilde{u}(z)$, for all $z \in [0,\min\{\mathsf{z},\tilde{\mathsf{z}}\}]$.

Finally, let $\tilde{u}_0 \in L^2(\mathbb{R})$ and $\tilde{g} \in L^1([0,\zeta],L^2(\mathbb{R}))$ be near to u_0 and g respectively, and let $\tilde{\psi}$, $\tilde{v}$, $\tilde{\Gamma}$ be the functions and operator associated to $\tilde{u}_0$ and $\tilde{g}$. Then $\tilde{\Gamma}$ is a contraction and therefore there exists $\tilde{u} \in \mathscr{X}_{\mathsf{z}}$ a unique fixed point of $\tilde{\Gamma}$. It holds

$$\begin{aligned}\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} &= \|\Gamma u-\tilde{\Gamma}\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} \leq \|\Gamma u-\Gamma\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}}+\|\Gamma\tilde{u}-\tilde{\Gamma}\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} \\ &\leq \gamma\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}}+C(\|u_0-\tilde{u}_0\|_{L^2}+\|g-\tilde{g}\|_{L^1([0,\zeta],L^2)})\end{aligned}$$

and thus

$$\|u-\tilde{u}\|_{\mathscr{X}_{\mathsf{z}}} \leq \frac{C}{1-\gamma}(\|u_0-\tilde{u}_0\|_{L^2}+\|g-\tilde{g}\|_{L^1([0,\zeta],L^2)}),$$

proving the continuous dependence of the solution with respect to u_0 and g.

Following, we obtain an estimate of the L^2 norm of the solution of (8), which allows us to prove the global existence.

Proposition 4 *Given $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2(\mathbb{R}))$, let $u \in \mathscr{X}_{\mathsf{z}}$ be the solution of* (8) *given by Theorem 1, then u satisfies*

$$\|u\|_{C([0,\mathsf{z}],L^2)} \leq \|u_0\|_{L^2} + 2\|g\|_{L^1([0,\zeta],L^2)}. \tag{10}$$

Proof Let $u_0 \in H^2(\mathbb{R})$ and $g \in C([0,\zeta],H^2(\mathbb{R}))$, it is easy to see that the solution of (8) verifies $u \in C([0,\mathsf{z}],H^2(\mathbb{R})) \cap C^1([0,\mathsf{z}],L^2(\mathbb{R}))$. Then

$$\frac{d}{dz}\|u\|_{L^2}^2 = 2\left(u, i\partial_t^2 u + i|u|^2 u + g\right)_{L^2} = 2\,(u,g)_{L^2} \leq 2\|u\|_{L^2}\|g\|_{L^2}, \tag{11}$$

integrating in $[0,z]$ for $0 \leq z \leq \mathsf{z}$, we obtain

$$\begin{aligned}\|u(z)\|_{L^2}^2 &\leq \|u_0\|_{L^2}^2 + 2\|u\|_{C([0,z],L^2)}\int_0^z \|g(z')\|_{L^2}dz' \\ &\leq \|u_0\|_{L^2}^2 + 2\|u\|_{C([0,\mathsf{z}],L^2)}\|g\|_{L^1([0,\zeta],L^2)}.\end{aligned}$$

Then $m = \|u\|_{C([0,\mathsf{z}],L^2)}$ satisfies

$$m^2 \leq \|u_0\|_{L^2}^2 + 2m\|g\|_{L^1(0,\zeta,L^2)},$$

from where we get that m satisfies (10).

Since $H^2(\mathbb{R})$ and $C([0,\zeta],H^2(\mathbb{R}))$ are dense in $L^2(\mathbb{R})$ and $L^1([0,\zeta],L^2(\mathbb{R}))$ respectively, from the continuous dependence, we extend the estimation for $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2(\mathbb{R}))$.

Next, we prove the global existence.

Theorem 2 *Given $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2(\mathbb{R}))$, there exists a unique $u \in \mathscr{X}_\zeta$ solution of* (8)*, which satisfies*

$$\|u\|_{\mathscr{X}_\zeta} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1([0,\zeta],L^2)}\right). \tag{12}$$

Furthermore, $u \in W^{1,1}([0,\zeta],H^{-2}(\mathbb{R}))$,

$$\|u\|_{W^{1,1}([0,\zeta],H^{-2})} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1([0,\zeta],L^2)}\right) \tag{13}$$

and the equation (2a) *is satisfied (or posed) in H^{-2} for almost all $z \in [0,\zeta]$.*

Proof Given $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2(\mathbb{R}))$, from Theorem 1, there exists $u \in \mathscr{X}_{\mathsf{z}}$ a local solution of (8) with $\mathsf{z} \in (0,\zeta]$. Let $\zeta^* \leq \zeta$ be the maximal time of existence of solution u. From inequality (10), we have that

$$\|u(z)\|_{L^2} \leq \|u_0\|_{L^2} + 2\|g\|_{L^1([0,\zeta],L^2)} \quad \text{for all } z \in [0,\zeta^*).$$

Let $r = \|u_0\|_{L^2} + 2\|g\|_{L^1([0,\zeta],L^2)}$ and $\mathsf{z}' \in (0,\zeta]$ be the minimum time of existence of the solution of (8) given by Theorem 1, with initial data u_1 such that $\|u_1\|_{L^2(\mathbb{R})} \leq r$. If $\zeta^* < \zeta$, taking $\zeta_1 \in (\zeta^* - \mathsf{z}', \zeta^*)$ and initial data $u_1 = u(\zeta_1)$, we would have an extension of the solution u to the interval $[0, \zeta_1 + \mathsf{z}']$, with $\zeta_1 + \mathsf{z}' > \zeta^*$, which contradicts the maximality of ζ^*. Therefore $\zeta^* = \zeta$ and $\|u\|_{C([0,\zeta],L^2)} \leq r$.

Let $n = [\zeta/\mathrm{z}'] + 1$ and $z_j = j\zeta/n$ with $j = 0, \dots, n$, then $z_j - z_{j-1} < \mathrm{z}'$. Since $u_j = u(z_j)$ satisfies that $\|u_j\|_{L^2(\mathbb{R})} \leq r$, we have that $\|u\|_{L^6([z_{j-1},z_j],L^6)} \leq Cr$ and therefore

$$\|u\|^6_{L^6([0,\zeta],L^6)} = \sum_{j=1}^{n} \|u\|^6_{L^6([z_{j-1},z_j],L^6)} \leq nCr^6 \leq \zeta Cr^6/\mathrm{z}(r),$$

proving (12).

Now, considering the operator $A = \partial_t^2 : D(A) \to X$, $D(A) = H^2(\mathbb{R})$, $X = L^2(\mathbb{R})$, using remark 1.6.1 (i) from [9] for $f = |u|^2 u - ig$, we obtain that the solution u of the integral equation (8) is in $W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$ and satisfies the equation (2a) for almost all $z \in [0,\zeta]$. Finally, since

$$\|\partial_t^2 u\|_{C([0,\zeta],H^{-2})} + \||u|^2 u\|_{L^2([0,\zeta],L^2)} \leq C\left(\|u\|_{\mathscr{X}_\zeta} + \|u\|^3_{\mathscr{X}_\zeta}\right),$$

from (12) and equation (2a) we obtain the estimation (13).

Given $u_0 \in L^2(\mathbb{R})$, for any $g \in L^1([0,\zeta], L^2(\mathbb{R}))$, we define $u[g] \in \mathscr{X}_\zeta$ as the solution of (8). We will prove $g \mapsto u[g]$ is a Fréchet differentiable map from $L^1([0,\zeta], L^2(\mathbb{R}))$ to $\mathscr{X}_\zeta$.

We begin with a lemma that will provide the global existence of a family of linear Schrödinger equations.

Lemma 2 *Let $B : \mathscr{X}_\zeta \to L^2([0,\zeta], L^2(\mathbb{R}))$ be a bounded linear operator. Assume that there exists $C > 0$ such that for $0 \leq a < b \leq \zeta$*

$$\|By\|_{L^2([a,b],L^2)} \leq C\|y\|_{\mathscr{X}_{[a,b]}}. \tag{14}$$

Let $h \in L^1([0,\zeta], L^2(\mathbb{R}))$. Then, for $y_0 \in L^2(\mathbb{R})$ there exists $y \in \mathscr{X}_\zeta$ solution of the linear integral equation

$$y(z) = S(z)y_0 + \int_0^z S(z-z')\left(By(z') + h(z')\right)dz', \tag{15}$$

such that $\|y\|_{\mathscr{X}_\zeta} \leq C\left(\|y_0\|_{L^2} + \|h\|_{L^1([0,\zeta],L^2)}\right)$. Moreover, $y \in W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$ and satisfies the differential equation

$$\begin{aligned} \partial_z y &= i\partial_t^2 y + By + h, \\ y(0) &= y_0. \end{aligned}$$

Proof We begin by proving a local existence, that is we will prove that there exists $\delta > 0$ such that for any $a \in [0,\zeta)$ and $y_a \in L^2(\mathbb{R})$, there exists a unique mild solution $y \in \mathscr{X}_{[a,a+\delta]}$ of the linear integral equation

$$y(z) = S(z-a)y_a + \int_a^z S(z-z')\left(By(z') + h(z')\right)dz'.$$

In order to do this, we define $\Gamma : \mathscr{X}_{[a,a+\delta]} \to \mathscr{X}_{[a,a+\delta]}$ given by

$$\Gamma(y)(z) = S(z-a)y_a + \int_a^z S(z-z')\left(By(z') + h(z')\right)dz'.$$

Since B is bounded, from Cauchy Schwarz, we deduce that if $y \in \mathscr{X}_{[a,a+\delta]}$, then

$$\|By\|_{L^1([a,a+\delta],L^2)} \leq \delta^{1/2}\|By\|_{L^2([a,a+\delta],L^2)} \leq C\delta^{1/2}\|y\|_{\mathscr{X}_{[a,a+\delta]}} \tag{16}$$

and therefore, from Strichartz estimates we have that $\Gamma(y) \in \mathscr{X}_{[a,a+\delta]}$. Moreover, since B is a linear operator, from (16)

$$\|\Gamma(y) - \Gamma(\tilde{y})\|_{\mathscr{X}_{[a,a+\delta]}} \leq C\delta^{1/2}\|y - \tilde{y}\|_{\mathscr{X}_{[a,a+\delta]}}.$$

Choosing $C\delta^{1/2} < 1/2$, using a fixed point argument, we can prove the local existence and continuous dependence in $\mathscr{X}_{[a,a+\delta]}$ for δ small depending only on the Strichartz constants. Moreover, since

$$\|y\|_{\mathscr{X}_{[a,a+\delta]}} \leq C\left(\|y_a\|_{L^2} + \delta^{1/2}\|y\|_{\mathscr{X}_{[a,a+\delta]}} + \|h\|_{L^1([0,\zeta],L^2)}\right)$$

we obtain that

$$\|y\|_{\mathscr{X}_{[a,a+\delta]}} \leq 2C\left(\|y_a\|_{L^2} + \|h\|_{L^1([0,\zeta],L^2)}\right). \tag{17}$$

Finally, let $n = [\zeta/\delta] + 1$ and $z_j = j\zeta/n$ with $j = 0,\dots,n$, then $z_j - z_{j-1} < \delta$ and $y_{z_j} = y(z_j)$ for $j \geq 1$. From (17) we obtain that

$$\|y(z_1)\|_{L^2} \leq \|y\|_{\mathscr{X}_{[0,z_1]}} \leq 2C\left(\|y_0\|_{L^2} + \|h\|_{L^1([0,\zeta],L^2)}\right),$$

from where inductively we deduce that

$$\|y\|_{\mathscr{X}_{[z_j,z_{j+1}]}} \leq C_j\left(\|y_0\|_{L^2} + \|h\|_{L^1([0,\zeta],L^2)}\right).$$

Using that, there exists a constant $C > 0$ such that

$$\|y\|_{\mathscr{X}_\zeta} \leq C\sum_{j=0}^{n-1}\|y\|_{\mathscr{X}_{[z_j,z_{j+1}]}}$$

we have that $\|y\|_{\mathscr{X}_\zeta} \leq C\left(\|y_0\|_{L^2} + \|h\|_{L^1([0,\zeta],L^2)}\right)$.

Using the same argument as in Theorem 2, we get that $y \in W^{1,1}([0,\zeta],H^{-2}(\mathbb{R}))$ and satisfies the differential equation.

Following we prove some previous results that will be used to prove the Fréchet differentiability of $u[g]$ in Proposition 6 and the continuous dependence of the solutions of the state equation (8) given by Theorem 2.

Corollary 3 *Let $B_j : \mathscr{X}_\zeta \to L^2([0,\zeta],L^2(\mathbb{R}))$ for $j = 1,2$ be two bounded linear operators verifying* (14) *and $y_1, y_2 \in \mathscr{X}_\zeta$ the solutions of* (15) *given by Lemma 2. Then, it holds*

$$\|y_1 - y_2\|_{\mathscr{X}_\zeta} \leq C\|(B_1 - B_2)y_j\|_{L^1([0,\zeta],L^2)}.$$

Proof Let $w = y_1 - y_2$ and $h = (B_1 - B_2)y_1$, we can write

$$w(z) = \int_0^z S(z-z')\left(B_2\, w(z') + h(z')\right) dz',$$

using Lemma 2 we obtain the result.

Lemma 3 *Let $u_1, u_2 \in \mathscr{X}_\zeta$ and $B_j : \mathscr{X}_\zeta \to L^2([0,\zeta], L^2(\mathbb{R}))$ the operators defined by*

$$B_j\, y = 2i\mathrm{Re}(\overline{u}_j\, y)u_j + i|u_j|^2 y$$

for $j = 1,2$. Then B_j satisfies (14) *and*

$$\|(B_1 - B_2)y\|_{L^2([0,\zeta],L^2)} \le C(\|u_1\|_{\mathscr{X}_\zeta} + \|u_2\|_{\mathscr{X}_\zeta})\|u_1 - u_2\|_{\mathscr{X}_\zeta}\|y\|_{\mathscr{X}_\zeta} \tag{18}$$

Proof Let $y \in \mathscr{X}_\zeta$, then for all $0 \le a < b \le \zeta$

$$\begin{aligned}\|B_j y\|_{L^2([a,b],L^2)} &= \|2i\mathrm{Re}(\overline{u}_j\, y)u_j + i|u_j|^2 y\|_{L^2([a,b],L^2)} \\ &\le C\|u_j\|^2_{L^6([0,\zeta],L^6)}\|y\|_{L^6([a,b],L^6)},\end{aligned}$$

therefore inequality (14) is satisfied. Inequality (18) follows analogously.

Proposition 5 *Let $u_0, \tilde{u}_0 \in L^2(\mathbb{R})$, $g, \tilde{g} \in L^1([0,\zeta], L^2)$ and $u, \tilde{u} \in \mathscr{X}_\zeta$ the solutions given by Theorem 2. There exists $C = C(\zeta, u_0, \tilde{u}_0, g, \tilde{g}) > 0$ such that*

$$\|\tilde{u} - u\|_{\mathscr{X}_\zeta} \le C\left(\|\tilde{u}_0 - u_0\|_{L^2} + \|\tilde{g} - g\|_{L^1([0,\zeta],L^2)}\right).$$

Proof Let $\delta u_0 = \tilde{u}_0 - u_0$, $\delta g = \tilde{g} - g$ and $\delta u = \tilde{u} - u$, it is verified

$$\delta u(z) = S(z)\delta u_0 + \int_0^z S(z-z')(B\delta u(z') + \delta g(z'))dz',$$

where $By = i(|\tilde{u}|^2 + |u|^2)y + iu\,\tilde{u}\,\bar{y}$. From Lemma 2, using inequality (12) we have the result.

Proposition 6 *Let $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta], L^2)$, then $u[\cdot]$ is Fréchet differentiable and $y = D_g u[g](\delta g) \in \mathscr{X}_\zeta$ is the solution of the linear integral equation*

$$y(z) = \int_0^z S(z-z')(2i\mathrm{Re}(\overline{u}[g]\, y)u[g] + i|u[g]|^2 y + \delta g)(z')dz'. \tag{19}$$

Moreover, $y \in W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$ and satisfies the differential equation

$$\begin{aligned}&\partial_z y = i\partial_t^2 y + 2i\mathrm{Re}(\overline{u}[g]\, y)u[g] + i|u[g]|^2 y + \delta g, \\ &y(0) = 0.\end{aligned}$$

Proof We begin by proving the existence of the solution of equation (19). Given $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta],L^2)$, let $u = u[g] \in \mathscr{X}_\zeta$. We consider the linear operator $B : \mathscr{X}_\zeta \to L^2([0,\zeta],L^2(\mathbb{R}))$ given by

$$By = 2i\mathrm{Re}(\overline{u}y)u + i|u|^2 y.$$

From Lemma 3, B satisfies (14). Since $\delta g \in L^1([0,\zeta],L^2(\mathbb{R}))$, from Lemma 2 there exists $y \in \mathscr{X}_\zeta$ solution of the linear non homogeneous equation (19).

Let $u = u[g]$, $\tilde{u} = u[g+\delta g]$ and $\delta u = \tilde{u} - u$, using that

$$|\tilde{u}|^2\tilde{u} - |u|^2 u = 2\mathrm{Re}(\overline{u}\delta u)u + |u|^2\delta u + |\delta u|^2 u + 2\mathrm{Re}(\overline{u}\delta u)\delta u + |\delta u|^2\delta u,$$

we have that

$$\begin{aligned}\delta u(z) &= \int_0^z S(z-z')(i|\tilde{u}|^2\tilde{u} - i|u|^2 u + \delta g)(z')dz' \\ &= \int_0^z S(z-z')\left(2i\mathrm{Re}(\overline{u}\delta u)u + i|u|^2\delta u + \delta g + \rho[g,\delta g]\right)(z')dz',\end{aligned}$$

where $\rho[g,\delta g] = (i|\delta u|^2 u + i2\mathrm{Re}(\overline{u}\,\delta u)\delta u + i|\delta u|^2\delta u)$. Then

$$(\delta u - y)(z) = \int_0^z S(z-z')(i2\mathrm{Re}(\overline{u}(\delta u - y))u + i|u|^2(\delta u - y) + \rho[g,\delta g])(z')dz'.$$

Since $\rho[g,\delta g] \in L^2([0,\zeta],L^2(\mathbb{R}))$ and $\|\rho[g,\delta g]\|_{L^2([0,\zeta],L^2(\mathbb{R}))} \le C\|\delta u\|^2_{\mathscr{X}_\zeta}$, Lemma 2 implies

$$\|\delta u - y\|_{\mathscr{X}_z} \le C\|\delta u\|^2_{\mathscr{X}_\zeta}. \tag{20}$$

Finally, from (9) we have that $\|\delta u\|_{\mathscr{X}_\zeta} \le C\|\delta g\|_{L^1([0,\zeta],L^2)}$ proving that u is Fréchet differentiable.

4 Regularizing properties

During the past several years there have been a number of papers concerning local smoothness properties of linear and nonlinear Schrödinger equations (see [17], [21] and references in therein). Adapting the ideas of [27] for Benjamin-Ono equation, in [28] and [11] local regularizing properties of Schrödinger equation are proved. In this section, we consider a similar problem with the non homogeneous term g. We will use the smoothness of solutions to obtain compactness.

Let H be the Hilbert transform given by $\widehat{(H\varphi)}(\xi) = -i\,\mathrm{sign}(\xi)\widehat{u}(\xi)$. If we set the operators $P_\pm = \frac{1}{2}(1 \pm iH)$ we have that $P_+ + P_- = 1$, $P_+ - P_- = iH$, and using that $H^2 = -1$, we obtain $HP_\pm = \mp iP_\pm$ and $[H,\partial_t] = [P_\pm,\partial_t] = 0$. We will prove the following

Proposition 7 *Let $u \in \mathscr{X}_\zeta$ be the solution of equation* (2)*, for any $\omega \in \mathscr{S}(\mathbb{R})$, it is verified $\omega u \in L^2([0,\zeta],H^{1/2})$ and*

$$\|\omega u\|^2_{L^2([0,\zeta],H^{1/2})} \le C(\omega,\zeta,\|u_0\|_{L^2},\|g\|_{L^1([0,\zeta],L^2)}) \tag{21}$$

Proof Let Ω be a primitive function of $|\omega|^2$, then $\Omega \in L^\infty(\mathbb{R})$. We will show the result for $u_0 \in H^2(\mathbb{R})$, $g \in C([0,\zeta],H^2(\mathbb{R}))$, and therefore $u \in C([0,\zeta],H^2(\mathbb{R})) \cap C^1([0,\zeta],L^2(\mathbb{R}))$, for any $z \in [0,\zeta]$ we have

$$\frac{1}{2}\frac{d}{dz}(\Omega P_\pm u(z), P_\pm u(z))_{L^2} = \big(i\Omega P_\pm \partial_t^2 u(z), P_\pm u(z)\big)_{L^2} + \big(i\Omega P_\pm |u(z)|^2 u(z), P_\pm u(z)\big)_{L^2} + (i\Omega P_\pm g(z), P_\pm u(z))_{L^2} = I_1^\pm(z) + I_2^\pm(z) + I_3^\pm(z).$$

We drop the z dependence in the rest of the proof for readability. We begin by analyzing the first term, integrating by parts we derive

$$I_1^\pm = -(i|\omega|^2 P_\pm \partial_t u, P_\pm u)_{L^2} - (i\Omega P_\pm \partial_t u, P_\pm \partial_t u)_{L^2}.$$

The integrand in the second term is pure imaginary and thus its real part is zero. Since $iP_\pm = \mp HP_\pm$, commutating ω and H we obtain

$$\begin{aligned} I_1^\pm &= \pm(|\omega|^2 H\partial_t P_\pm u, P_\pm u)_{L^2} \\ &= \pm(\overline{\omega} H\omega\partial_t P_\pm u, P_\pm u)_{L^2} \mp (\overline{\omega}[H,\omega]\partial_t P_\pm u, P_\pm u)_{L^2}, \end{aligned}$$

from the product rule, we have

$$\begin{aligned} I_1^\pm = &\pm(\overline{\omega} H\partial_t(\omega P_\pm u), P_\pm u)_{L^2} \mp (\overline{\omega} H(\partial_t\omega)P_\pm u, P_\pm u)_{L^2} \\ &\mp (\overline{\omega}[H,\omega]\partial_t P_\pm u, P_\pm u)_{L^2}. \end{aligned}$$

Being $\omega \in W^{1,\infty}(\mathbb{R})$ and $P_\pm, H, [H,\omega]\partial_t$ bounded operators in $L^2(\mathbb{R})$ (see [6]), we can estimate

$$|(\overline{\omega} H(\partial_t\omega)P_\pm u, P_\pm u)_{L^2}| + |(\overline{\omega}[H,\omega]\partial_t P_\pm u, P_\pm u)_{L^2}| \le C\|\omega\|_{W^{1,\infty}}^2 \|u\|_{L^2}^2.$$

Since $H\partial_t = D$, where $\widehat{Du}(\xi) = |\xi|\hat{u}(\xi)$, we deduce

$$\begin{aligned} (\overline{\omega} H\partial_t(\omega P_\pm u), P_\pm u)_{L^2} &= (H\partial_t(\omega P_\pm u), \omega P_\pm u)_{L^2} \\ &= (D(\omega P_\pm u), \omega P_\pm u)_{L^2} = \|D^{1/2}\omega P_\pm u\|_{L^2}^2. \end{aligned}$$

We conclude that $I_1^\pm = \pm\|D^{1/2}\omega P_\pm u\|_{L^2}^2 + J_1^\pm$, with $|J_1^\pm| \le C\|\omega\|_{W^{1,\infty}}^2 \|u\|_{L^2}^2$. Using Cauchy–Schwarz inequality we have that

$$\begin{aligned} |I_2^\pm| &\le \|\Omega\|_{L^\infty} \|P_\pm |u|^2 u\|_{L^2} \|P_\pm u\|_{L^2} \le \|\Omega\|_{L^\infty} \|u\|_{L^6}^3 \|u\|_{L^2} \\ |I_3^\pm| &= |(i\Omega P_\pm g, P_\pm u)_{L^2}| \le \|\Omega\|_{L^\infty} \|g\|_{L^2} \|u\|_{L^2}. \end{aligned}$$

Then

$$\sum_{j=1}^3 I_j^\pm = \pm\|D^{1/2}\omega P_\pm u\|_{L^2}^2 + J_1^\pm + I_2^\pm + I_3^\pm.$$

If $Q = (\Omega P_+ u, P_+ u)_{L^2} - (\Omega P_- u, P_- u)_{L^2}$, it holds that $|Q| \le C\|u\|_{L^2}^2$ and

$$\frac{1}{2}\frac{d}{dz}Q = \|D^{1/2}(\omega P_+ u)\|_{L^2}^2 + \|D^{1/2}(\omega P_- u)\|_{L^2}^2 + K, \tag{22}$$

where $K = J_1^+ - J_1^- + I_2^+ - I_2^- + I_3^+ - I_3^-$. Integrating (22) in $[0,\zeta]$, we obtain

$$\frac{1}{2}(Q(\zeta) - Q(0)) = \int_0^\zeta \|D^{1/2}(\omega P_+ u)\|_{L^2}^2 + \|D^{1/2}(\omega P_- u)\|_{L^2}^2 dz + \int_0^\zeta K dz.$$

Being $|K| \le C(\Omega)(\|u\|_{L^2}\|g\|_{L^2} + \|u\|_{L^2}^2 + \|u\|_{L^6}^6)$, from (12) we have

$$\left| \int_0^\zeta K dz \right| \le C(\omega, \zeta, \|u_0\|_{L^2}, \|g\|_{L^1([0,\zeta],L^2)}),$$

and using $\|D^{1/2}(\omega u)\|_{L^2} \le \|D^{1/2}(\omega P_+ u)\|_{L^2} + \|D^{1/2}(\omega P_- u)\|_{L^2}$, we get

$$\|D^{1/2}(\omega u)\|_{L^2([0,\zeta],L^2)}^2 \le C(\omega, \zeta, \|u_0\|_{L^2}, \|g\|_{L^1([0,\zeta],L^2)})$$

what proves the desired result. Since the right hand side of the last estimate depends only on $\|u_0\|_{L^2}$ and $\|g\|_{L^1([0,\zeta],L^2)}$, using a continuous dependence argument, we obtain the general result.

Remark 1 In Proposition 7, we have used only the fact that $\omega \in L^2(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. For instance, if $\omega(t) = (1+t^2)^{-\alpha}$ with $\alpha > 1/4$, estimate (21) holds.

In order to study the existence of minimizer, we will consider a minimizing sequence which will be bounded in $L^2([0,\zeta], L^2(\mathbb{R}))$. Then we will need to prove that this sequence and the sequence of associated controls converge. Consider $\{g_n\}_{n\in\mathbb{N}}$ a minimizing sequence and $u_n = u[g_n]$ the corresponding solutions given by Theorem 2. We will prove the existence of a subsequence of $\{g_n\}_{n\in\mathbb{N}}$, such that the associated solutions are convergent in different senses.

Proposition 8 *Assume $\{g_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^1([0,\zeta],L^2(\mathbb{R}))$, then there exists a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ and $u_\star \in \mathscr{X}_\zeta$ such that the associated solutions $u_{n_j} = u[g_{n_j}]$ converge weakly to $u_\star$ in $L^2([0,\zeta],L^2(\mathbb{R}))$ and also in $L^6([0,\zeta],L^6(\mathbb{R}))$.*

Proof From Theorem 2, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{X}_\zeta$. Being $L^2([0,\zeta],L^2(\mathbb{R}))$ and $L^6([0,\zeta],L^6(\mathbb{R}))$ reflexive Banach spaces, the result is a consequence of Banach–Alaoglu Theorem.

Under the same assumptions of Proposition 8 we can now prove convergence in $L^2([0,\zeta],L^2_{\text{loc}}(\mathbb{R}))$.

Proposition 9 *Assume $\{g_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^1([0,\zeta],L^2(\mathbb{R}))$, then there exists a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ and $u_\star \in L^2([0,\zeta],L^2(\mathbb{R})) \cap L^6([0,\zeta],L^6(\mathbb{R}))$ such that the associated solutions $u_{n_j} = u[g_{n_j}]$ converge to $u_\star$ in $L^2([0,\zeta],L^2_{loc}(\mathbb{R}))$, that is*

$$\lim_{j\to\infty} \int_0^\zeta \int_{-\tau}^\tau |u_{n_j}(z,t) - u_\star(z,t)|^2 dt dz = 0, \tag{23}$$

for all $\tau > 0$.

Proof Let $\{g_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^1([0,\zeta],L^2(\mathbb{R}))$ and $\{u_n\}_{n\in\mathbb{N}}$ the sequence of associated solutions in $\mathscr{X}_\zeta$ given by Theorem 2, which is also bounded in $\mathscr{X}_\zeta$. From Proposition 8, without loss of generality we can assume that u_n converge weakly to $u_\star$. Let $\omega \in C_c^\infty(\mathbb{R})$ be such that $0 \le \omega \le 1$, $\omega \equiv 1$ in the interval $[-1,1]$ and $\operatorname{supp}(\omega) \subset (-2,2)$. We define $\omega_k(t) = \omega(t/k)$, then we have that $\omega_k u_n = u_n$ if $|t| < k$ and $\operatorname{supp}(\omega_k u_n) \subset (-2k,2k)$. For each $k \in \mathbb{N}$, the sequence $\{\omega_k u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2([0,\zeta],L^2_1(\mathbb{R}))$, from Proposition 7 we have that is bounded in the space $L^2([0,\zeta],H^{1/2}(\mathbb{R})\cap L^2_1(\mathbb{R}))$ and from Theorem 2 is also bounded in the space $W^{1,1}([0,\zeta],H^{-2}(\mathbb{R}))$. Then, from Corollary 1, there exists a subsequence $\{u_{1,n}\}_{n\in\mathbb{N}}$ and $v_1 \in L^2([0,\zeta],L^2(\mathbb{R})$ such that $\{\omega_1 u_{1,n}\}_{n\in\mathbb{N}}$ converges to v_1 in $L^2([0,\zeta],L^2(\mathbb{R}))$. Let $\varphi \in C([0,\zeta]\times\mathbb{R})$ such that $\operatorname{supp}(\varphi) \subset [0,\zeta]\times[-1,1]$, since $\omega_1 u_{1,n} = u_{1,n}$ in $[-1,1]$, we have

$$\begin{aligned}\int_0^\zeta \int_{\mathbb{R}} u_\star(z,t)\varphi(z,t)dtdz &= \lim_{n\to\infty}\int_0^\zeta \int_{-1}^1 u_{1,n}(z,t)\varphi(z,t)dtdz\\ &= \lim_{n\to\infty}\int_0^\zeta \int_{-1}^1 \omega_1\, u_{1,n}(z,t)\varphi(z,t)dtdz\\ &= \int_0^\zeta \int_{-1}^1 v_1(z,t)\varphi(z,t)dtdz.\end{aligned}$$

Therefore, $u_{1,n}\big|_{[0,\zeta]\times[-1,1]} \to v_1\big|_{[0,\zeta]\times[-1,1]} = u_\star\big|_{[0,\zeta]\times[-1,1]}$ in $L^2([0,\zeta]\times[-1,1])$. Applying an inductive argument, from the sequence $\{u_{k-1,n}\}_{n\in\mathbb{N}}$, we can construct a subsequence $\{u_{k,n}\}_{n\in\mathbb{N}}$ such that

$$u_{k,n}\big|_{[0,\zeta]\times[-k,k]} \to u_\star\big|_{[0,\zeta]\times[-k,k]}, \text{ in } L^2([0,\zeta]\times[-k,k]).$$

Taking the diagonal sequence $\{u_{n,n}\}_{n\in\mathbb{N}}$, we get $u_{n,n}\big|_{[0,\zeta]\times[-k,k]} \to u_\star\big|_{[0,\zeta]\times[-k,k]}$ for all $k \in \mathbb{N}$.

Under the same conditions of the last two propositions, we are now in position to prove that the cubic nonlinearity converges weakly in $L^2([0,\zeta],L^2(\mathbb{R}))$.

Proposition 10 *Assume $\{g_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^1([0,\zeta],L^2(\mathbb{R}))$, then there exists a subsequence $\{g_{n_j}\}_{j\in\mathbb{N}}$ and $u_\star \in L^2([0,\zeta],L^2(\mathbb{R}))\cap L^6([0,\zeta],L^6(\mathbb{R}))$ such that the associated solutions $u_{n_j} = u[g_{n_j}]$ verify that $|u_{n_j}|^2 u_{n_j} \rightharpoonup |u_\star|^2 u_\star$ in $L^2([0,\zeta],L^2(\mathbb{R}))$.*

Proof From Theorem 2, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathscr{X}_\zeta$, then from (5) we have that $|u_n|^2 u_n \in L^2([0,\zeta],L^2(\mathbb{R}))$ is bounded and therefore converges weakly to some function $\psi \in L^2([0,\zeta],L^2(\mathbb{R}))$. Since $L^2([0,\zeta]\times\mathbb{R}) \hookrightarrow \mathscr{D}'((0,\zeta)\times\mathbb{R})$, we have that $|u_n|^2 u_n$ converges to ψ in $\mathscr{D}'((0,\zeta)\times\mathbb{R})$. Using (7) and Hölder inequality, we obtain

$$\begin{aligned}\||u_n|^2u_n - |u_\star|^2u_\star\|_{L^1([0,\zeta]\times[-\tau,\tau])} \le\ & C(\zeta,\|u_n\|_{\mathscr{X}_\zeta},\|u_\star\|_{\mathscr{X}_\zeta})\\ & \times \|u_n - u_\star\|_{L^2([0,\zeta]\times[-\tau,\tau])}.\end{aligned}$$

Being $\{u_n\}_{n\in\mathbb{N}}$ bounded in $\mathscr{X}_\zeta$, from (23) it holds that $|u_n|^2 u_n \xrightarrow{\mathscr{D}'} |u_\star|^2 u_\star$, and therefore $\psi = |u_\star|^2 u_\star$.

5 Variational problem

We go back to consider the variational problem

$$0 \leq \mathscr{J}_\star = \inf_{g \in \mathscr{G}_{\text{ad}}} \mathscr{J}(g) \tag{24}$$

with

$$\mathscr{J}(g) = \|g\|^2_{L^2([0,\zeta],L^2)} + \kappa \|\sigma(u[g](\zeta) - v_\zeta)\|^2_{L^2} \tag{25}$$

for $\kappa \geq 0$, and $\mathscr{G}_{\text{ad}}$ is the space of controls $g \in L^2([0,\zeta],L^2(\mathbb{R}))$ such that the solution $u[g] \in \mathscr{X}_\zeta$ of equation (8) satisfies

$$\int_{\mathbb{R}} \sigma^2 |u[g](\zeta,t) - v_\zeta(t)|^2 dt \leq \eta. \tag{26}$$

It is clear that since the control g is not localized, we can reach any target we want. For instance, consider $\tilde{u}_0, \tilde{v}_\zeta \in H^2(\mathbb{R})$ and $\tilde{u} \in C^1([0,\zeta],H^2(\mathbb{R}))$ such that $\tilde{u}(0) = \tilde{u}_0$, $\tilde{u}(\zeta) = \tilde{v}_\zeta$. If we define $g \in C([0,\zeta],L^2(\mathbb{R}))$ as $g = \partial_z \tilde{u} - i(\partial_t^2 \tilde{u} + |\tilde{u}|^2 \tilde{u})$, then obviously $\tilde{u}$ is the solution of (2a) with initial data $\tilde{u}_0$ and control g. Given $\varepsilon > 0$, from the continuous dependence, there exists $\delta > 0$ such that if $\|u_0 - \tilde{u}_0\|_{L^2} < \delta$, it holds that the solution u of (2) verifies that $\|u - \tilde{u}\|_{\mathscr{X}_\zeta} < \varepsilon$. Then, $\|u(\zeta) - v_\zeta\|_{L^2} \leq \|u - \tilde{u}\|_{\mathscr{X}_\zeta} + \|\tilde{v}_\zeta - v_\zeta\|_{L^2} < 2\varepsilon$, provided $\|\tilde{v}_\zeta - v_\zeta\|_{L^2} < \varepsilon$. Therefore, the admissible set of controls $\mathscr{G}_{\text{ad}}$ is not empty.

Since it holds the continuous inclusion $L^2([0,\zeta],L^2(\mathbb{R})) \hookrightarrow L^1([0,\zeta],L^2(\mathbb{R}))$ with

$$\|g\|_{L^1([0,\zeta],L^2)} \leq \zeta^{1/2} \|g\|_{L^2([0,\zeta],L^2)},$$

from now on we will consider $g \in L^2([0,\zeta],L^2(\mathbb{R}))$ which is a Hilbert space and all previous results are valid. In this case, the solution u of equation (8), given by Theorem 1, satisfies $u \in H^1([0,\zeta],H^{-2}(\mathbb{R}))$ and

$$\|\partial_z u\|_{L^2([0,\zeta],H^{-2})} \leq C(\zeta, \|u_0\|_{L^2}, \|g\|_{L^2([0,\zeta],L^2)}).$$

Proposition 11 *Let $g_n \in \mathscr{G}_{ad}$ be a minimizing sequence of the variational problem, then the function $u_\star \in L^2([0,\zeta],L^2(\mathbb{R})) \cap L^6([0,\zeta],L^6(\mathbb{R}))$ given by Proposition 8 satisfies $u_\star \in H^1([0,\zeta],H^{-2}(\mathbb{R}))$.*

Proof Since $\{g_n\}_{n\in\mathbb{N}}$ is bounded in $L^2([0,\zeta],L^2(\mathbb{R}))$, from inequality (13) we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1([0,\zeta],H^{-2}(\mathbb{R}))$. Therefore there exists a subsequence, that we will keep on calling u_n, weakly convergent. Recall that $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{X}_\zeta$. Let $\theta \in C_c^1([0,\zeta],H^2(\mathbb{R}))$, then

$$\int_0^\zeta \langle \partial_z u_n, \theta \rangle_{H^{-2},H^2}\, dz = -\int_0^\zeta \langle u_n, \partial_z \theta \rangle_{H^{-2},H^2}\, dz = -\int_0^\zeta (u_n, \partial_z \theta)_{L^2}\, dz,$$

passing to the limit we get

$$\lim_{n\to\infty} \int_0^\zeta \langle \partial_z u_n, \theta \rangle_{H^{-2},H^2}\, dz = -\int_0^\zeta (u_\star, \partial_z \theta)_{L^2}\, dz = -\int_0^\zeta \langle u_\star, \partial_z \theta \rangle_{H^{-2},H^2}\, dz.$$

Therefore $u_\star \in H^1([0,\zeta],H^{-2}(\mathbb{R}))$ and $\partial_z u_n \rightharpoonup \partial_z u_\star$ in $L^2([0,\zeta],H^{-2}(\mathbb{R}))$.

Next, we will prove that the function $u_\star \in C([0,\zeta],L^2(\mathbb{R}))$ and that there exists a control $g_\star$ associated to $u_\star$.

Proposition 12 *Let $g_n \in \mathscr{G}_{ad}$ be a minimizing sequence of the variational problem. Consider the function $u_\star \in L^2([0,\zeta],L^2(\mathbb{R})) \cap L^6([0,\zeta],L^6(\mathbb{R})) \cap H^1([0,\zeta],H^{-2}(\mathbb{R}))$, given by Proposition 8 and Proposition 11. Then there exists $g_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$ such that*

$$\partial_z u_\star = i\partial_t^2 u_\star + i|u_\star|^2 u_\star + g_\star, \tag{27}$$

and $u_\star \in C([0,\zeta],L^2(\mathbb{R}))$.

Proof Since $\{g_n\}_{n\in\mathbb{N}}$ is bounded in $L^2([0,\zeta],L^2(\mathbb{R}))$, there exists a subsequence of $\{g_n\}_{n\in\mathbb{N}}$, which we call $\{g_n\}_{n\in\mathbb{N}}$, and $g_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$ such that $g_n \rightharpoonup g_\star$. From Proposition 8, we have that u_n converges weakly to $u_\star$ in $L^2([0,\zeta],L^2(\mathbb{R}))$. Furthermore, from Proposition 10, $|u_n|^2 u_n \rightharpoonup |u_\star|^2 u_\star$ in $L^2([0,\zeta],L^2(\mathbb{R}))$. From Proposition 11, we obtain that $\partial_z u_n \rightharpoonup \partial_z u_\star$ in $L^2([0,\zeta],H^{-2}(\mathbb{R}))$. Since it is verified that $0 = \partial_z u_n - i\partial_t^2 u_n - i|u_n|^2 u_n - g_n$ in $L^2([0,\zeta],H^{-2}(\mathbb{R}))$, passing to the limit we deduce

$$0 = \partial_z u_\star - i\partial_t^2 u_\star - i|u_\star|^2 u_\star - g_\star.$$

Since $u_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$, there exists $z_0 \in [0,\zeta]$ such that $u_\star(z_0) \in L^2(\mathbb{R})$, using Proposition 4.1.9 from [10] (with $X = H^{-2}(\mathbb{R})$, $D(A) = L^2(\mathbb{R})$ and $f = i|u_\star|^2 u_\star + g_\star$) we obtain that $u_\star$ verifies

$$u_\star(z) = S(z-z_0)u_\star(z_0) + \int_{z_0}^{z} S(z-z')(i|u_\star(z')|^2 u_\star(z') + g_\star(z'))dz'.$$

Since $f \in L^2([0,\zeta],L^2(\mathbb{R}))$, using Lemma 4.1.5 from [10] (with $X = L^2(\mathbb{R})$) we obtain that $u_\star \in C([0,\zeta],L^2(\mathbb{R}))$.

Proposition 13 *Let $\omega \in W^{2,\infty}(\mathbb{R})$ be such that $\sup_{t\in\mathbb{R}} |t\omega(t)| < \infty$, $\{g_n\}_{n\in\mathbb{N}}$ a bounded sequence in $L^2([0,\zeta],L^2(\mathbb{R}))$ and $\{u_n\}_{n\in\mathbb{N}}$ the sequence given by Proposition 9 that converges to $u_\star$ in $L^2([0,\zeta],L^2_{loc}(\mathbb{R}))$, Then, there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ such that ωu_n converges to $\omega u_\star$ in $C([0,\zeta],H^{-2}(\mathbb{R}))$.*

Proof Being $\{u_n\}_{n\in\mathbb{N}}$ a bounded sequence in $C([0,\zeta],L^2(\mathbb{R}))$, $\{\omega u_n\}_{n\in\mathbb{N}}$ is bounded in $C([0,\zeta],L^2_1(\mathbb{R}))$ and in $H^1([0,\zeta],H^{-2}(\mathbb{R}))$. Then, from Corollary 2 there exist a subsequence and a function $\psi \in C([0,\zeta],H^{-2}(\mathbb{R}))$ such that ωu_n converges to ψ in $C([0,\zeta],H^{-2}(\mathbb{R}))$. In particular, $\omega u_n \rightharpoonup \psi$ in $L^2([0,\zeta],H^{-2}(\mathbb{R}))$. Since $\omega u_n \rightharpoonup \omega u_\star$ in $L^2([0,\zeta],L^2(\mathbb{R}))$, we obtain $\psi = \omega u_\star$.

Theorem 3 *Let $g_n \in \mathscr{G}_{ad}$ be a minimizing sequence. Consider $g_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$ and $u_\star \in \mathscr{X}_\zeta \cap H^1([0,\zeta],H^{-2}(\mathbb{R}))$ given by Proposition 12. Then $u_\star = u[g_\star]$, $g_\star \in \mathscr{G}_{ad}$ and $\mathscr{J}(g_\star) = \mathscr{J}_\star$.*

Proof From Proposition 12, there exists $g_\star \in L^2([0,\zeta],L^2(\mathbb{R}))$ satisfying equation (27) with $u_\star \in C([0,\zeta],L^2(\mathbb{R})) \cap H^1([0,\zeta],H^{-2}(\mathbb{R}))$. To see that $u_\star = u[g_\star]$ it remains to prove that $u_\star(0) = u_0$. Let $\omega > 0$ be such that hypothesis from Proposition 13 are satisfied, then $\omega u_0 = \omega u_n|_{z=0} \to \omega u_\star|_{z=0}$, therefore $u_\star|_{z=0} = u_0$, from where we obtain that $u_\star = u[g_\star]$. Using once more Proposition 13 we deduce that $\sigma u_n|_{z=\zeta} \to \sigma u_\star|_{z=\zeta}$ en $H^{-2}(\mathbb{R})$. From the inequality $\|\sigma(u_n(\zeta) - v_\zeta)\|_{L^2}^2 \le \eta$, we have that $\|\sigma u_n(\zeta)\|_{L^2}$ is bounded, then there exists a function $\psi \in L^2(\mathbb{R})$ such that $\sigma u_n(\zeta) \rightharpoonup \psi$ and therefore $\psi = \sigma u_\star(\zeta)$. Then, $\sigma(u_n(\zeta) - v_\zeta) \rightharpoonup \sigma(u_\star(\zeta) - v_\zeta)$ and $\|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2 \le \eta$. Thus, $g_\star \in \mathscr{G}_{\text{ad}}$. Finally, since $g_n \rightharpoonup g_\star$ and $\sigma(u_n(\zeta) - v_\zeta) \rightharpoonup \sigma(u_\star(\zeta) - v_\zeta)$ we obtain that

$$
\begin{aligned}
\mathscr{J}_\star &\le \kappa\|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2 + \|g_\star\|_{L^2([0,\zeta],L^2)}^2 \\
&\le \liminf_{n\to\infty} \kappa\|\sigma(u_n(\zeta) - v_\zeta)\|_{L^2}^2 + \|g_n\|_{L^2([0,\zeta],L^2)}^2 = \mathscr{J}_\star
\end{aligned}
$$

proving the optimality of $g_\star$.

Remark 2 From the continuity of $u[g]$ with respect to g, we have that for $\kappa = 0$, $\|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2 = \eta$.

Theorem 4 (Casas 1993 [8]) *Let $\mathscr{G}, Y$ be Banach spaces, $\mathscr{C} \subset Y$ a convex subset, $\mathscr{C}$ with nonempty interior. If $g_\star$ is a solution of the problem*

$$
\begin{cases}
\min \mathscr{J}(g), \\
g \in \mathscr{G}, \Lambda(g) \in \mathscr{C},
\end{cases}
$$

where $\mathscr{J} : \mathscr{G} \to \mathbb{R}$ and $\Lambda : \mathscr{G} \to Y$ are Gateaux differentiable function on $g_\star$. Then, there exist $\lambda \ge 0$ and $\mu_\star \in Y^*$ such that

$$\lambda + \|\mu_\star\|_{Y^*} > 0, \tag{28a}$$

$$\langle \mu_\star, \theta - \Lambda(g_\star)\rangle_{Y^*,Y} \le 0, \text{ for all } \theta \in \mathscr{C}, \tag{28b}$$

$$\langle \lambda \mathscr{J}'(g_\star) + (D\Lambda(g_\star))^*\mu_\star, g - g_\star\rangle_{\mathscr{G}^*,\mathscr{G}} \ge 0, \text{ for all } g \in \mathscr{G}. \tag{28c}$$

We define the operator $\Lambda(g) = \sigma(u[g](\zeta) - v_\zeta)$, where σ is a continuous function such that $\sup_{t\in\mathbb{R}} |t\sigma(t)| < +\infty$. In proposition 6 we have proved that the operator $u[g]$ is Fréchet differentiable which provides us the differentiability of $\Lambda(g)$. In the following proposition we will derive the adjoint operator of $D\Lambda(g)$ in order to apply the previous theorem.

Proposition 14 *Let $g \in L^2([0,\zeta],L^2(\mathbb{R}))$, $u = u[g] \in \mathscr{X}_\zeta$. Given $\mu_\zeta \in L^2(\mathbb{R})$, then $(D\Lambda(g))^*\mu_\zeta = \mu$, where $\mu \in \mathscr{X}_\zeta$ is the mild solution of*

$$\partial_z \mu = i\partial_t^2 \mu + 2i|u|^2\mu - iu^2\overline{\mu}, \tag{29a}$$

$$\mu(\zeta) = \sigma\mu_\zeta, \tag{29b}$$

Proof Considering $B : \mathscr{X}_\zeta \to L^2([0,\zeta], L^2(\mathbb{R}))$, the bounded linear operator given by $B(\mu) = 2i|u|^2\mu - iu^2\overline{\mu}$, and using the reversibility of the Schrödinger group S, from Lemma 2 we get that there exists a solution of the integral equation

$$\mu(z) = S(z-\zeta)\sigma\mu_\zeta - \int_z^\zeta S(z-z')\left(2i|u(z')|^2\mu(z') - iu^2(z')\overline{\mu}(z')\right)dz'$$

proving the existence of a mild solution of (29). Given $\delta g \in L^2([0,\zeta], L^2(\mathbb{R}))$, let $\theta = D_g[u](\delta g) \in \mathscr{X}_\zeta \cap W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$ be the solution of (19) and therefore $D\Lambda(g)\delta g = \sigma\theta(\zeta)$. Suppose $u \in C([0,\zeta], H^2(\mathbb{R}))$ and $\delta g \in C([0,\zeta], L^2(\mathbb{R}))$, using Theorem 4.8.1 and Remark 1.6.1 from [9], we have that $\theta \in C([0,\zeta], H^2(\mathbb{R})) \cap C^1([0,\zeta], L^2(\mathbb{R}))$. Therefore, it holds

$$\begin{aligned}
\left(\mu_\zeta, D\Lambda(g)\delta g\right)_{L^2} &= \left(\mu_\zeta, \sigma\theta(\zeta)\right)_{L^2} \\
&= \langle \sigma\mu_\zeta, \theta(\zeta)\rangle_{H^{-2},H^2} = \int_0^\zeta \frac{d}{dz}\langle \mu, \theta\rangle_{H^{-2},H^2} dz \\
&= \int_0^\zeta \langle \partial_z\mu, \theta\rangle_{H^{-2},H^2} + (\mu, \partial_z\theta)_{L^2}\, dz = \int_0^\zeta (\mu, \delta g)_{L^2}\, dz.
\end{aligned}$$

Using a density argument from Corollary 3 and Lemma 3, we obtain the latter equality for $u \in \mathscr{X}_\zeta$ and $\delta g \in L^2([0,\zeta], L^2(\mathbb{R}))$ and therefore $(D\Lambda(g))^*\mu_\zeta = \mu$.

Theorem 5 *Let $g_\star$ be an optimal solution of problem* (4) *and $u_\star = u[g_\star]$ its associated state. Then, there exists $\alpha \geq 0$ such that $g_\star$ and $u_\star$ satisfy the following equations*

$$\partial_z u_\star = i\partial_t^2 u_\star + i|u_\star|^2 u_\star + g_\star \tag{30}$$
$$u_\star(0) = u_0 \tag{31}$$
$$\partial_z g_\star = i\partial_t^2 g_\star + 2i|u_\star|^2 g_\star - iu_\star^2 \bar{g_\star} \tag{32}$$
$$g_\star(\zeta) = -\left(\kappa + \frac{1}{2}\alpha\right)\sigma^2(u_*(\zeta) - v_\zeta) \tag{33}$$
$$\|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2 \leq \eta \tag{34}$$
$$\alpha\left(\eta - \|\sigma(u_\star(\zeta) - v_\zeta)\|_{L^2}^2\right) = 0. \tag{35}$$

Proof Take $\mathscr{G} = \mathscr{G}^* = L^2([0,\zeta], L^2(\mathbb{R}))$, $Y = Y^* = L^2(\mathbb{R})$, the convex set $\mathscr{C} = \{\theta \in L^2(\mathbb{R}) : \|\theta\|_{L^2}^2 \leq \eta\}$, the functional $\mathscr{J}(g) = \kappa\|u[g](\zeta) - v_\zeta\|_{L^2_\sigma}^2 + \|g\|_{L^2([0,\zeta],L^2)}^2$ and the operator $\Lambda(g) = \sigma(u[g](\zeta) - v_\zeta)$. Then, from Theorem 4 there exists $\lambda \geq 0$ and $\mu_\star \in L^2(\mathbb{R})$, satisfying (28). That is,

$$\lambda + \|\mu_\star\|_{L^2} > 0, \tag{36a}$$
$$(\mu_\star, \theta - \Lambda(g_\star))_{L^2} \leq 0, \text{ for all } \theta \in \mathscr{C}, \tag{36b}$$
$$\left(\lambda\mathscr{J}'(g_\star) + (D\Lambda(g_\star))^*\mu_\star, g - g_\star\right)_{L^2([0,\zeta],L^2)} \geq 0, \text{ for all } g \in L^2([0,\zeta], L^2(\mathbb{R})). \tag{36c}$$

From (36c) we obtain $\lambda \mathscr{J}'(g_\star) + (D\Lambda(g_\star))^*\mu_\star = 0$. Assume $\lambda = 0$, then we would have $D\Lambda(g_\star)^*\mu_\star = 0$ and from Proposition 14 we would get $\mu_\star = 0$ contradicting (36a). Thus we can take $\lambda = 1$. Moreover, since

$$\mathscr{J}(g) = \|g\|^2_{L^2([0,\zeta],L^2)} + \kappa\|\Lambda(g)\|^2_{L^2}$$

we obtain

$$\begin{aligned}\mathscr{J}'(g_\star)(\delta g) &= (2g_\star, \delta g)_{L^2([0,\zeta],L^2)} + (2\kappa\Lambda(g_\star), D\Lambda(g_\star)(\delta g))_{L^2}\\ &= (2g_\star + (D\Lambda(g_\star))^*(2\kappa\Lambda(g_\star)), \delta g)_{L^2([0,\zeta],L^2)},\end{aligned}$$

and then

$$g_\star = -\frac{1}{2}(D\Lambda(g_\star))^*(2\kappa\Lambda(g_\star) + \mu_\star). \tag{37}$$

Since $\Lambda(g_*) \in \mathscr{C}$, from (36b) we get that $(\mu_\star, \Lambda(g_\star))_{L^2} = \max(\mu_\star, \theta)_{L^2}$ for all $\theta \in \mathscr{C}$ and therefore there exists $\alpha \geq 0$ such that $\mu_\star = \alpha\Lambda(g_\star)$ (for $\mu_\star \neq 0$, we have $\alpha > 0$ and for $\mu_\star = 0$ we take $\alpha = 0$). Also, if inequality (34) is strict ($\Lambda(g_\star)$ is in the interior of $\mathscr{C}$), from (36b) we have $\mu_\star = 0$ and therefore $\alpha = 0$, from where we obtain equation (35). Moreover, from (37) and Proposition 14, we obtain equations (32) and (33).

Remark 3 Note that for $\kappa = 0$, if we assume $\alpha = 0$, from (33) we would have $g_\star = 0$, which is not admissible from condition (3). Then, for $\kappa = 0$, we have $\alpha > 0$, therefore $\mu_\star \neq 0$ and $\|\sigma(u_\star(\zeta) - v_\zeta)\|^2_{L^2} = \eta$ (see remark (2)). Thus, from (33)

$$\frac{g_*(\zeta)}{\sigma} = -\frac{1}{2}\alpha\Lambda(g_*)$$

and therefore $\alpha = \frac{2}{\eta^{1/2}}\|g_\star(\zeta)/\sigma\|_{L^2}$.

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