OBLIQUE PROJECTIONS AND SAMPLING PROBLEMS

GUSTAVO CORACH AND JUAN I. GIRIBET

Abstract. In this work, the *consistent sampling* requirement of signals is studied. We establish how this notion is related with certain set of projectors which are selfadjoint with respect to a semi-inner product. We extend previous results and present some new problems related with sampling theory.

1. Introduction

In signal processing language, *sampling* is an operation which converts a continuous time (or space) signal (i.e., function) into a discrete time one. This is a previous step which allows the analysis of a signal in the computer. The classical sampling scheme is based on the Whittaker-Kotelnikov-Shannon theorem. Recall that the Paley-Wiener space of band-limited functions is the space $\mathcal{PW}$ of all $f \in L^2(\mathbb{R})$ which can be written as $f(t) = \int_{-\pi}^{\pi} g(\omega)e^{i\omega t}d\omega$, $t \in \mathbb{R}$ for some $g \in L^2(\mathbb{R})$. The Whittaker-Kotelnikov-Shannon theorem establishes that it is possible to reconstruct any signal $f \in \mathcal{PW}$ from its values at the integers $\{f(n)\}_{n\in\mathbb{Z}}$. More precisely, the series $\sum_{n\in\mathbb{Z}} f(n)\frac{sin(\pi(t-n))}{\pi(t-n)}$ converges uniformly to $f(t)$. If a signal $f \in L^2(\mathbb{R})$ does not belong to the Paley-Wiener space, a common strategy in signal processing applications is to apply a low pass filter (i.e., the operator mapping $f \mapsto \left\langle f, \frac{\sin(\pi(.-t))}{\pi(.-t)} \right\rangle$) to the signal f, obtaining a new one g. Although the signal recovered from the samples $\{g(n)\}_{n\in\mathbb{Z}}$ in general does not coincide with the original signal f, it turns out that it is always a good approximation of it. In fact, the recovered signal is the orthogonal projection of the original signal in $\mathcal{PW}$. For a detailed exposition of these facts see [17]. An usual way of representing the samples of a signal f is the following: given vectors $\{v_n\}_n \in \mathbb{N}$ which spans a closed subspace $\mathcal{S}$ *(sampling subspace)*, these samples are given by $\{f_n\}_{n\in\mathbb{N}} = \{\langle f, v_n \rangle\}_{n\in\mathbb{N}}$ [17]. On the other hand, given samples $\{f_n\}_{n\in\mathbb{N}}$, the reconstructed signal $\hat{f}$ is given by $\hat{f} = \sum_{n\in\mathbb{N}} f_n w_n$, where $\{w_n\}_{n\in\mathbb{N}}$ spans a closed subspace $\mathcal{R}$ *(reconstruction subspace)*.

In the classical sampling scheme the reconstruction and the sampling subspaces are assumed to be the same. However, in many applications, this not the case, and then it is not always possible to recover the best approximation of the original signal. Thus, different sampling techniques must be used. In [18], M. Unser and A. Aldroubi introduced the idea of consistent sampling. Here, the reconstructed

1991 *Mathematics Subject Classification.* Primary 94A20; Secondary 47A58.

Key words and phrases. Consistent sampling, Oblique projections.

This research was partially supported by CONICET, ANPCYT, PUNQ 0530/07, UBACyT.

signal $\hat{f}$ is not generally the best approximation of the original signal, but f and $\hat{f}$ are forced to have the same samples.

Originally, this idea has been studied in shift invariant spaces. Later, in [7, 8], the consistent sampling has been studied in abstract Hilbert spaces by Y. Eldar for finite dimensional spaces and by Y. Eldar and T. Werther for infinite dimensional spaces, respectively. An underlying assumption in these works is that the Hilbert space $\mathcal{H}$ of signals can be decomposed as $\mathcal{H} = \mathcal{R} \dot{+} \mathcal{S}^{\perp}$ (where $\dot{+}$ is the direct sum). Under this assumption, it is shown that the consistent sampling requirement is related with the (unique) projection P with range $R(P) = \mathcal{R}$ and nullspace $N(P) = \mathcal{S}^{\perp}$. More precisely, the unique signal $\hat{f} \in \mathcal{R}$ that satisfies $\left\langle \hat{f}, v_n \right\rangle = \left\langle \hat{f}, v_n \right\rangle$, for every $n \in \mathbb{N}$, is given by $\hat{f} = Pf$. However, in some signal applications the hypothesis $\mathcal{R} \cap \mathcal{S}^{\perp} = \{0\}$ is not satisfied. A. Hirabayashi and M. Unser, in [9], proved that in this case there are infinite projections P such that $\hat{f} = Pf$ satisfies the consistent sampling requirement.

The main goal of this paper is to give an interpretation of the consistent sampling in terms of the notion of compatibility between a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ and a positive semidefinite operator A acting on $\mathcal{H}$. This notion, defined in [4] and developed later in [1, 2, 3, 5], has a completely different origin. In [15], Z. Pasternak-Winiarski studied, for a fixed subspace $\mathcal{S}$, the analiticity of the map $A \to P_{A,\mathcal{S}}$ which associates to each positive invertible operator A the orthogonal projection onto $\mathcal{S}$ under the (equivalent) inner product $\langle f, g \rangle_A = \langle Af, g \rangle$, for $f, g \in \mathcal{H}$. These perturbations of the inner product occur quite frequently, the reader is referred to [12, 13, 14] for many applications. The notion of compatibility appears when A is allowed to be any positive semidefinite operator, not necessarily invertible (and even, a selfadjoint bounded linear operator). More precisely, A and $\mathcal{S}$ are said to be *compatible* if there exists a (bounded linear) projection Q with range $\mathcal{S}$ which satisfies $AQ = Q^*A$ (i.e., Q is Hermitian with respect to the semi-inner product $\langle\ ,\ \rangle_A$). Unlike for invertible A's, it may happen that there is no such Q or that there is an infinite number of them. However, there exists an angle condition between $\mathcal{S}^{\perp}$ and $\overline{A(\mathcal{S})}$ which determines the existence of such projections.

As far as we know, with the exception of [8], the consistent sampling requirement has not been studied in the context of perturbed inner spaces. In [8], the assumption $\mathcal{R} \cap \mathcal{S}^{\perp} = \{0\}$ forces the perturbations of the inner product to be defined by positive invertible operators, i.e., the consistent sampling idea is studied in an equivalent Hilbert space. But, as we show in this work, the consistent sampling requirement in semi-inner product spaces allows a simpler and more general way for studying this problem.

For instance, it is easy to characterize, in infinite dimensional spaces, the set of (possibly infinite) oblique projections related with the consistent sampling idea. It is important to remark that, although some of the results given in [9] can be directly extended to infinite dimensional spaces, by means of Moore-Penrose pseudo inverses, for some others results this is not possible. In fact, there is an angle condition that must be fulfilled. Here the notion of compatibility plays an important role because, as we show below, this notion is closely related with the notion of angle condition between subspaces. If $\mathcal{S}^\perp \cap \mathcal{R} \neq \{0\}$; by this reason there are infinite oblique projections related with the consistent sampling requirement, in [9] criteria for selecting one among these projections have been considered. These criteria are motivated by signal processing applications. Let $\mathcal{C}$ be the set of all projections that satisfies the consistent sampling requirement. In [9], given a subspace $\mathcal{M} \subseteq \mathcal{R}$ all projections $Q \in \mathcal{C}$ such that $\mathcal{M} \subseteq R(Q)$ are characterized. In this paper, we study a more general problem: given a subspace $\mathcal{M} \subseteq \mathcal{R}$, we give necessary and sufficient conditions for the existence of a projection $Q_0 \in \mathcal{C}$ such that $\|f - Q_0 f\| \leq \|f - Qf\|$ for every $f \in \mathcal{M}$ and every $Q \in \mathcal{P}$, and, if those conditions hold, we characterize the set of all these Q_0. As a corollary we study the case $Q_0 f = f$, for every $f \in \mathcal{M}$. Also motivated by signal processing applications, we characterize those projectors in $\mathcal{C}$ which minimize the so called *aliasing norm* [7, 10, 11].

The paper is organized as follows: Section 2 contains the preliminaries. In Section 3 we present some variational problems in the set $\mathcal{P}(A, \mathcal{S})$. These problems are related with two sampling problems presented in Section 5. In Section 4, we present the notion of *consistent sampling* and the link with the compatibility. We characterize the set of operators satisfying a consistent sampling requirement and characterize some operators with particular properties. In Section 5, we study when it is possible to impose to the consistent sampling requirement the additional property of recovering the best approximation of a signal, for certain set of signals. Finally, in Section 6 a problem related with the sampling of perturbed signals is presented.

2. Oblique projections and compatibility

In this section, we present a survey of useful results concerning the existence of projections which are orthogonal, in some sense, with respect to a fixed positive semidefinite operator. We start with some notation.

Along this work $\mathcal{H}$ denotes a (complex, separable) Hilbert space with inner product $\langle \ , \ \rangle$. Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $L(\mathcal{H}, \mathcal{K})$ denotes the space of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ and $L(\mathcal{H}) = L(\mathcal{H}, \mathcal{H})$. If $T \in L(\mathcal{H})$ then T^* denotes the adjoint operator of T, $R(T)$ stands for the range of T and $N(T)$ for its nullspace. If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{T}$ is a closed subspace of $\mathcal{K}$, then $L(\mathcal{S}, \mathcal{T})$ will be identified with the subspace of $L(\mathcal{H}, \mathcal{K})$ consisting of all $T \in L(\mathcal{H}, \mathcal{K})$ such that $R(T) \subseteq \mathcal{T}$ and $\mathcal{S}^\perp \subseteq N(T)$.

Let $GL(\mathcal{H})$ denote the group of invertible operators of $L(\mathcal{H})$, $L(\mathcal{H})^+$ the cone of (semidefinite) positive operators of $L(\mathcal{H})$, $GL(\mathcal{H})^+ = L(\mathcal{H})^+ \cap GL(\mathcal{H})$ and $\mathcal{Q}$ the set of projections of $L(\mathcal{H})$, i.e., $\mathcal{Q} = \{Q \in L(\mathcal{H}) \ : \ Q^2 = Q\}$.

If $\mathcal{S}$ and $\mathcal{T}$ are two (closed) subspaces of $\mathcal{H}$, denote by $\mathcal{S} \dotplus \mathcal{T}$ the direct sum of $\mathcal{S}$ and $\mathcal{T}$, $\mathcal{S} \oplus \mathcal{T}$ the (direct) orthogonal sum of them and $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp$. If $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T}$, the (oblique) projection $P_{\mathcal{S}//\mathcal{T}}$ onto $\mathcal{S}$ along $\mathcal{T}$ is the projection uniquely determined by $R(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{S}$ and $N(P_{\mathcal{S}//\mathcal{T}}) = \mathcal{T}$. In particular, $P_{\mathcal{S}} = P_{\mathcal{S}//\mathcal{S}^{\perp}}$ is the orthogonal projection onto $\mathcal{S}$.

Given two subspaces $\mathcal{S}, \mathcal{T}$, the cosine of the *Friedrichs angle* $\theta(\mathcal{S}, \mathcal{T}) \in [0, \pi/2]$ between them is defined by

$$c(\mathcal{S}, \mathcal{T}) = sup\{|\langle f, g \rangle| : f \in \mathcal{S} \ominus \mathcal{T}, \|f\| < 1, g \in \mathcal{T} \ominus \mathcal{S}, \|g\| < 1\}.$$

Remark 2.1. *The following conditions are equivalent:*

(1) $c(\mathcal{S}, \mathcal{T}) < 1$*;*
(2) $\mathcal{S} + \mathcal{T}$ *is closed;*
(3) $\mathcal{S}^{\perp} + \mathcal{T}^{\perp}$ *is closed;*
(4) $c(\mathcal{S}^{\perp}, \mathcal{T}^{\perp}) < 1$*.*

The *Dixmier angle* between $\mathcal{S}$ and $\mathcal{T}$ is the angle in $[0, \pi/2]$ whose cosine is defined by

$$c_0(\mathcal{S}, \mathcal{T}) = sup\{|\langle f, g \rangle| : f \in \mathcal{S}, \|f\| < 1, g \in \mathcal{T}, \|g\| < 1\}.$$

Observe that, in general $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$ and if $\mathcal{S} \cap \mathcal{T} = \{0\}$ then the equality holds. Notice that $c_0(\mathcal{S}, \mathcal{T}) < 1 \iff \mathcal{S} \cap \mathcal{T} = \{0\}$ and $\mathcal{S} + \mathcal{T}$ is closed. Observe that, by its definition, $c_0(\mathcal{S}, \mathcal{T})$ is monotone in each variable: if $\mathcal{S} \subseteq \mathcal{S}'$ and $\mathcal{T} \subseteq \mathcal{T}'$ then $c_0(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}', \mathcal{T}')$. However, this is not true, in general, for Friedrichs cosine.

In [18], Unser and Aldroubi introduced a notion of *largest angle* or *uniform cosine angle* between two subspaces $\mathcal{S}$ and $\mathcal{T}$: this is the angle in $[0, \pi/2]$ whose cosine is

$$R(\mathcal{S}, \mathcal{T}) = inf\{\|P_{\mathcal{T}} s\|, s \in \mathcal{S}, \|s\| = 1\}.$$

This notion, widely used in signal processing literature, is related with the Dixmier angle. In fact, $R(\mathcal{S}, \mathcal{T}) = \sqrt{1 - {c_0}^2(\mathcal{S}, \mathcal{T}^{\perp})}$, and all relevant properties of R can be easily deduced from those of c_0.

Given $A \in L(\mathcal{H})^+$ consider $\langle f, g \rangle_A = \langle Ax, y \rangle$, for every $f, g \in \mathcal{H}$. Then $\langle\, , \,\rangle_A$ defines a semi-inner product on $\mathcal{H}$. There is also a seminorm associated to $\langle\, , \,\rangle_A$, namely $\|f\|_A = \langle Af, f \rangle^{1/2}$ for every $f \in \mathcal{H}$.

An operator $T \in L(\mathcal{H})$ is A-*selfadjoint* if $\langle Tf, g \rangle_A = \langle f, Tg \rangle_A$, for every $f, g \in \mathcal{H}$. The following lemma characterizes the A-selfadjoint projections.

Lemma 2.2. (**Krein**, [4]) *Let* $A \in L(\mathcal{H})^+$ *and* $Q \in \mathcal{Q}$*. Then the following conditions are equivalent:*

a. $AQ = Q^*A$*, i.e.,* Q *is* A*-selfadjoint,*
b. $N(Q) \subseteq A^{-1}(R(Q)^{\perp})$*.*

Definition 1. *If* $A \in L(\mathcal{H})^+$ *and* $\mathcal{S}$ *is a closed subspace of* $\mathcal{H}$*, the pair* $(A, \mathcal{S})$ *is said to be* compatible *if there exists a projection* $Q \in L(\mathcal{H})$ *such that* $R(Q) = \mathcal{S}$ *and* $AQ = Q^*A$*. The following result gives a list of equivalent conditions to the compatibility.*

Lemma 2.3. [4] *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$. The following conditions are equivalent:*

(1) *The pair $(A, \mathcal{S})$ is compatible,*
(2) *$\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$,*
(3) *$c_0(\mathcal{S}^\perp, \overline{A\mathcal{S}}) < 1$.*

The next theorem, due to R. G. Douglas [6] plays a relevant role in what follows.

Theorem 2.4. (**Douglas**, [6]) *Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent:*

(1) *$R(A) \subseteq R(B)$.*
(2) *There exists a positive number λ such that $AA^* \le \lambda BB^*$.*
(3) *There exists $D \in L(\mathcal{H})$ such that $A = BD$.*

Moreover, in this case there exists a unique solution D of the equation $AX = B$ such that $R(D) \subseteq N(A)^\perp$. D is called the reduced solution *of the equation $AX = B$. If $A^\dagger$ denotes the Moore-Penrose inverse of A, then $D = A^\dagger B$. It also satisfies that $\|D\|^2 = \inf\{\lambda : AA^* \le \lambda BB^*\}$.*

Corollary 2.5. [4] *Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and write $A \in L(\mathcal{H})^+$, in terms of the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$, as*

$$A = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \tag{1}$$

Then the pair $(A, \mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$.

Denote $\mathcal{P}(A, \mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S} \text{ and } AQ = Q^*A\}$, i.e., $\mathcal{P}(A, \mathcal{S})$ is the set of A-selfadjoint projections with range $\mathcal{S}$.

From now on, denote $\mathcal{N} = \mathcal{S} \cap N(A)$. If $\mathcal{N} = \{0\}$ then $\mathcal{P}(A, \mathcal{S})$ is a singleton (see Theorem 2.7).

Remark 2.6. *Using the decomposition given in equation* (1)*, we can write $\mathcal{P}(A, \mathcal{S}) = \{Q = \begin{bmatrix} I & x \\ 0 & 0 \end{bmatrix} : x \in L(\mathcal{S}^\perp, \mathcal{S})$ and $ax = b\}$.*

The set $\mathcal{P}(A, \mathcal{S})$ can be empty, a singleton (for example, if A is positive definite) or an infinite set. Indeed, this set is an affine manifold. The next theorem provides a parametrization of $\mathcal{P}(A, \mathcal{S})$, if A and $\mathcal{S}$ are compatible.

Let $P_{A,\mathcal{S}} = \begin{bmatrix} I & d \\ 0 & 0 \end{bmatrix}$, where d is the reduced solution of the equation $ax = b$.

Theorem 2.7. [4, 5] *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that $(A, \mathcal{S})$ is compatible. Then $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ is the projection onto $\mathcal{S}$ with nullspace $A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$. The set $\mathcal{P}(A, \mathcal{S})$ is an affine manifold and it can be parametrized as*

$$\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} + L(\mathcal{S}^\perp, \mathcal{N}),$$

or, in terms of the matrix decomposition given above,

$$\mathcal{P}(A,\mathcal{S}) = \begin{bmatrix} I & d+z \\ 0 & 0 \end{bmatrix},$$

with $R(z) \subseteq N(a)$*. Moreover* $P_{A,\mathcal{S}}$ *has minimal norm in* $\mathcal{P}(A,\mathcal{S})$*, but it is not the unique with this property, in general.*

If $f \in \mathcal{H}$ then $(I - P_{A,\mathcal{S}})f$ is the unique minimal norm element in the set

$$\{(I-Q)f : Q \in \mathcal{P}(A,\mathcal{S})\}. \tag{2}$$

In the following, we recall basic definitions and results related to frames of closed subspaces.

Definition 2. *Let* $\mathcal{S}$ *be a closed subspace of* $\mathcal{H}$*. The set* $\mathcal{V} = \{v_n\}_{n\in\mathbb{N}} \subseteq \mathcal{S}$ *is a* frame *for* $\mathcal{S}$ *if there exist numbers* $\gamma_1, \gamma_2 > 0$ *such that*

$$\gamma_1 \|f\|^2 \leq \sum_{n\in\mathbb{N}} |\langle f, v_n \rangle|^2 \leq \gamma_2 \|f\|^2, \text{ for every } f \in \mathcal{S}. \tag{3}$$

If the set $\mathcal{V} = \{v_n\}_{n\in\mathbb{N}}$ is also linearly independent then it is called a *Riesz basis* of $\mathcal{S}$.

Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and let $\mathcal{V} = \{v_n\}_{n\in\mathbb{N}}$ be a frame for $\mathcal{S}$. Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be the canonical orthonormal basis of ℓ^2. The unique operator $F \in L(\ell^2, \mathcal{H})$ such that $Fe_n = v_n$, for every $n \in \mathbb{N}$ is called the *synthesis operator* of $\mathcal{V}$. The adjoint operator $F^* \in L(\mathcal{H}, \ell^2)$ is called the *analysis operator* of $\mathcal{V}$, and it is given by $F^* f = \sum_{n\in\mathbb{N}} \langle f, v_n \rangle e_n$. Finally, the operator $T = FF^* \in L(\mathcal{H})$ is called the *frame operator* of $\mathcal{V}$ and, from equation (3), it satisfies: $\gamma_1 P_{\mathcal{S}} \leq P_{\mathcal{S}} T P_{\mathcal{S}} \leq \gamma_2 P_{\mathcal{S}}$ and then $1/\gamma_1 P_{\mathcal{S}} \leq P_{\mathcal{S}} T^\dagger P_{\mathcal{S}} \leq 1/\gamma_2 P_{\mathcal{S}}$, where $T^\dagger$ denotes the Moore-Penrose inverse of T.

3. Minimization problems in the set $\mathcal{P}(A,\mathcal{S})$

The purpose of this section is to study the following problem which, as will be shown below, is related with some signal processing applications.

As it was stated in Section 2, the reduced solution of $ax = b$ gives the element $P_{A,\mathcal{S}} \in \mathcal{P}(A,\mathcal{S})$. In [3], another interesting characterization of the element $P_{A,\mathcal{S}}$ is given in terms of the solution of a variational problem. If $\mathcal{S}$ is a closed subspace, it is easy to see that the orthogonal projection $P_{\mathcal{S}}$ is the unique solution of the problem

$$min_{Q\in\mathcal{Q}, R(Q)=\mathcal{S}} QQ^*.$$

In a similar way, $P_{\mathcal{S}^\perp}$ is the unique solution of

$$min_{Q\in\mathcal{Q}, N(Q)=\mathcal{S}} Q^*Q;$$

In [3] the following results are proven. Let $\mathcal{A} = \{Q \in \mathcal{Q} : R(Q) \subseteq \mathcal{M}^\perp, N(Q) = \mathcal{S}\}$.

Proposition 3.1. *Let $\mathcal{S}$ and $\mathcal{M}$ be two closed subspaces of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{S} + \mathcal{M}^\perp$. Then $P_0 = P_{\mathcal{M}^\perp \ominus \mathcal{S}//\mathcal{S}}$ is the unique solution of*

$$min_{Q\in\mathcal{A}} Q^*Q.$$

Corollary 3.2. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ such that the pair $(A,\mathcal{S})$ is compatible. Then $P_{A,\mathcal{S}} \in \mathcal{P}(A,\mathcal{S})$ is the unique solution of*

$$min_{Q\in\mathcal{P}(A,\mathcal{S})}(I-Q)^*(I-Q)$$

Problem 3.3. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, such that the pair $(A,\mathcal{S})$ is compatible. Characterize the closed subspaces $\mathcal{M}$ for which there exists $Q_0 \in \mathcal{P}(A,\mathcal{S})$ such that*

$$\|Q_0 f\| \le \|Qf\|, \text{ for every } Q \in \mathcal{P}(A,\mathcal{S}), f \in \mathcal{M} \tag{4}$$

For each such $\mathcal{M}$, determine the set of all Q_0.

Observe that, if $\|Q_0 f\| \le \|Qf\|$, for every $f \in \mathcal{M}, Q \in \mathcal{P}(A,\mathcal{S})$ then $\langle P_\mathcal{M} Q_0^* Q_0 P_\mathcal{M} f, f\,\rangle \le \langle P_\mathcal{M} Q^* Q P_\mathcal{M} f, f\,\rangle$, for every $f \in \mathcal{H}$, i.e., $Q_0 \in \mathcal{P}(A,\mathcal{S})$ is a solution of

$$min_{Q\in\mathcal{P}(A,\mathcal{S})} P_\mathcal{M} Q^* Q P_\mathcal{M}.$$

The following Theorem gives necessary and sufficient conditions for the existence of solutions of Problem 3.3. Furthermore, it is shown that the solutions of Problem 3.3 can be related with the solution of certain operator equations.

Theorem 3.4. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, such that the pair $(A,\mathcal{S})$ is compatible. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, there exists $Q_0 \in \mathcal{P}(A,\mathcal{S})$ satisfying equation (4) if and only if $\mathcal{M}^\perp + \mathcal{N} \subseteq \mathcal{M}^\perp + \mathcal{S}^\perp$. Moreover, $Q_0 = P_{A,\mathcal{S}} + W_0 P_{\mathcal{S}^\perp}$, where $W_0^* \in L(\mathcal{H})$ is a solution of the equation,*

$$P_\mathcal{M} P_{\mathcal{S}^\perp} X = P_\mathcal{M} P_\mathcal{N}. \tag{5}$$

Proof. By Theorem 2.7, any $Q \in \mathcal{P}(A,\mathcal{S})$ can be written as $Q = P_{A,\mathcal{S}} + W P_{\mathcal{S}^\perp}$, with $R(W) \subseteq \mathcal{N} = N(A) \cap \mathcal{S} = A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S}$. Observe that, if $Q_0 \in \mathcal{P}(A,\mathcal{S})$ is decomposed as $Q_0 = P_{A,\mathcal{S}} + W_0 P_{\mathcal{S}^\perp}$, with $R(W_0) \subseteq \mathcal{N}$, then Q_0 is a solution of (4) if and only if $W_0 \in L(\mathcal{H},\mathcal{N})$ is such that,

$$\|P_{A,\mathcal{S}} P_\mathcal{M} f - W_0 P_{\mathcal{S}^\perp} P_\mathcal{M} f\| \le \|P_{A,\mathcal{S}} P_\mathcal{M} f - W P_{\mathcal{S}^\perp} P_\mathcal{M} f\|, \tag{6}$$

for every $f \in \mathcal{H}$ and for every bounded linear operator W with $R(W) \subseteq \mathcal{N}$.

Given $f \in \mathcal{H}$, observe that if $f \in N(P_{\mathcal{S}^\perp} P_\mathcal{M})$, then equation (6) holds for every $W \in L(\mathcal{H},\mathcal{N})$. By the other hand, if $f \notin N(P_{\mathcal{S}^\perp} P_\mathcal{M})$, given $\eta \in \mathcal{N}$, let $W \in L(\mathcal{H},\mathcal{N})$ such that $\eta = W P_{\mathcal{S}^\perp} P_\mathcal{M} f$. Then, $W_0 \in L(\mathcal{H},\mathcal{N})$ satisfies (6) if and only if

$$\|P_{A,\mathcal{S}} P_\mathcal{M} f - W_0 P_{\mathcal{S}^\perp} P_\mathcal{M} f\| \le \|P_{A,\mathcal{S}} P_\mathcal{M} f - \eta\|, \tag{7}$$

for every $\eta \in \mathcal{N}$. Then $W_0 P_{\mathcal{S}^\perp} P_\mathcal{M} f \in \mathcal{N}$ it is the best approximation of $P_{A,\mathcal{S}} P_\mathcal{M} f$ in the subspace $\mathcal{N}$, i.e., $W_0 P_{\mathcal{S}^\perp} P_\mathcal{M} f = P_\mathcal{N} P_{A,\mathcal{S}} P_\mathcal{M} f = P_\mathcal{N} P_\mathcal{M}$, since $P_\mathcal{N} P_{A,\mathcal{S}} = P_\mathcal{N}$. Thus, we look for those $W_0 \in L(\mathcal{H})$ such that,

$$P_\mathcal{M} P_{\mathcal{S}^\perp} W_0^* = P_\mathcal{M} P_\mathcal{N}.$$

By Theorem 2.4, the above equation has a solution if and only if $R(P_{\mathcal{M}}P_{\mathcal{N}}) \subseteq R(P_{\mathcal{M}}P_{\mathcal{S}^\perp})$ or, analogously if and only if $\mathcal{M}^\perp + \mathcal{N} \subseteq \mathcal{M}^\perp + \mathcal{S}^\perp$. □

Observe that the set of solutions of (5) is the affine manifold:

$$(P_{\mathcal{M}}P_{\mathcal{S}^\perp})^\dagger P_{\mathcal{M}}P_{\mathcal{N}} + L(\mathcal{H}, N(P_{\mathcal{M}}P_{\mathcal{S}^\perp}))$$

Examples. *The following are sufficient conditions for the existence of solutions of* (4).

1. *If $\mathcal{N} = \{0\}$ easily follows that $\mathcal{M}^\perp + \mathcal{N} \subseteq \mathcal{M}^\perp + \mathcal{S}^\perp$. Recall that, in this case, $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$, so that* (4) *has solution for every $\mathcal{M}$.*
2. *If $\mathcal{S}$ and $\mathcal{M}$ are closed subspaces such that $c_0(\mathcal{S}, \mathcal{M}) < 1$, then $\mathcal{H} = \mathcal{S}^\perp + \mathcal{M}^\perp$. Thus $\mathcal{M}^\perp + \mathcal{N} \subseteq \mathcal{M}^\perp + \mathcal{S}^\perp$, and there exists a solution of equation* (4). *In this case, if X is a solution of* (5) *then $X \in (P_{\mathcal{M}}P_{\mathcal{S}^\perp})^\dagger P_{\mathcal{N}} + L(\mathcal{H}, \mathcal{S} + \mathcal{S}^\perp \cap \mathcal{M}^\perp)$, since $N((P_{\mathcal{M}}P_{\mathcal{S}^\perp})^\dagger) = \mathcal{M}^\perp$.*

Proposition 3.5. *Let $(A, \mathcal{S})$ be a compatible pair. Then there exists $Q_0 \in \mathcal{P}(A, \mathcal{S})$ such that $Q_0 P_{\mathcal{M}} = 0$ if and only if $\mathcal{M} \subseteq A^{-1}(\mathcal{S}^\perp)$ and $c_0(\mathcal{M}, \mathcal{S}) < 1$.*

Proof. Suppose that $\mathcal{M} \subseteq A^{-1}(\mathcal{S}^\perp)$ and $c_0(\mathcal{M}, \mathcal{S}) < 1$, it follows that $\mathcal{M} \cap \mathcal{S} = \{0\}$ and $\mathcal{M} \dotplus \mathcal{S}$ is closed. Let $\mathcal{T} = A^{-1}(\mathcal{S}^\perp) \ominus (\mathcal{S} \dotplus \mathcal{M})$, by Lemma 2.3 it follows that $\mathcal{H} = \mathcal{S} \dotplus \mathcal{T} \dotplus \mathcal{M}$. Define $Q_0 = P_{\mathcal{S}//\mathcal{M}+\mathcal{T}}$, by Lemma 2.2 it follows that $Q_0 \in \mathcal{P}(A, \mathcal{S})$. Furthermore, for every $f \in \mathcal{M}$, $Q_0 f = 0$.

Conversely, if there exists $Q_0 \in \mathcal{P}(A, \mathcal{S})$ such that $Q_0 P_{\mathcal{M}} = 0$, then $\mathcal{M} \subseteq N(Q_0) \subseteq A^{-1}(\mathcal{S}^\perp)$. Furthermore, since $c(N(Q_0), \mathcal{S}) = c_0(N(Q_0), \mathcal{S}) < 1$, it follows that the Dixmier cosine $c_0(N(Q_0), \mathcal{S}) < 1$, then $c_0(\mathcal{M}, \mathcal{S}) < 1$. □

The following result characterizes the set of projections $Q \in \mathcal{P}(A, \mathcal{S})$ that satisfies $QP_{\mathcal{M}} = 0$.

Proposition 3.6. *Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ be a closed subspace of $\mathcal{H}$, such that the pair $(A, \mathcal{S})$ is compatible. Let $\mathcal{M} \subseteq A^{-1}(\mathcal{S}^\perp)$ be a closed subspace of $\mathcal{H}$ such that $c_0(\mathcal{M}, \mathcal{S}) < 1$. Then, the set of projections $Q \in \mathcal{P}(A, \mathcal{S})$ satisfying $QP_{\mathcal{M}} = 0$, is the affine manifold $P_{\mathcal{S}//\mathcal{M}+\mathcal{T}} + L(\mathcal{S}^\perp \cap \mathcal{M}^\perp, \mathcal{N})$, where the (closed) subspace $\mathcal{T} \subseteq A^{-1}(\mathcal{S}^\perp)$ is any complement of $\mathcal{N} \dotplus \mathcal{M}$ in $A^{-1}(\mathcal{S}^\perp)$.*

Proof. Since $\mathcal{M} \subseteq A^{-1}(\mathcal{S}^\perp)$ satisfies $c_0(\mathcal{M}, \mathcal{S}) < 1$ it follows that $c_0(\mathcal{M}, \mathcal{N}) < 1$ then $\mathcal{M} \dotplus \mathcal{N} \subseteq A^{-1}(\mathcal{S}^\perp)$ is closed. Let $\mathcal{T}$ be a complementary subspace of $\mathcal{M} \dotplus \mathcal{N}$ in $A^{-1}(\mathcal{S}^\perp)$, and let $Q_0 = P_{\mathcal{S}//\mathcal{M}\dotplus\mathcal{T}}$. By Lemma 2.3, it follows that $Q_0 \in \mathcal{P}(A, \mathcal{S})$ and for every $f \in \mathcal{M}$, $Q_0 f = 0$. Let $T \in L(\mathcal{S}^\perp \cap \mathcal{M}^\perp, \mathcal{N})$, then, since $T \in L(\mathcal{S}^\perp, \mathcal{N})$, it follows that $Q = Q_0 + T \in \mathcal{P}(A, \mathcal{S})$. Furthermore, if $f \in \mathcal{M}$, it is easy to see that $Qf = 0$. Then every $Q \in Q_0 + L(\mathcal{S}^\perp \cap \mathcal{M}^\perp) \subseteq \mathcal{P}(A, \mathcal{S})$, satisfies $QP_{\mathcal{M}} = 0$.

Conversely, suppose that $Q, P \in \mathcal{P}(A, \mathcal{S})$ satisfy $QP_{\mathcal{M}} = 0$. Then, by Theorem 2.7, $Q - P \in L(\mathcal{S}^\perp, \mathcal{N})$. If $f \in \mathcal{M}$, it follows that $Qf = Pf = (Q - P)f = 0$, i.e., $\mathcal{M} + \mathcal{S} \subseteq N(Q - P)$. Then, $Q - P \in L(\mathcal{S}^\perp \cap \mathcal{M}^\perp, \mathcal{N})$. □

4. Consistent sampling

In this section we study a generalization of a sampling problem proposed by Unser and Aldroubi in [18] and generalized in [7, 8, 9]. More specifically, let f be a signal (a vector) of a suitable Hilbert space $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{R}$ be two (closed) subspace of $\mathcal{H}$, called respectively, the sampling and reconstruction subspaces. Given $\mathcal{V}_s = \{v_n\}_{n\in\mathbb{N}}$, a frame of $\mathcal{S}$, with synthesis operator H, $g = H^*f = \sum_{\mathbb{N}} \langle f, v_n \rangle e_n$ is called the samples of f. As an *inverse* procedure, we have the reconstruction process. Given $\mathcal{V}_r = \{w_n\}_{n\in\mathbb{N}}$, a frame of $\mathcal{R}$ with synthesis operator F, given the samples $g \in \ell^2$, the reconstructed signal associated to them, is $f = Fg$. In some signal applications we deal with the problem of, given the synthesis and the analysis operators, find a *filter* (a bounded linear operator) $X \in L(\ell^2)$ such that the signal reconstructed from the filtered samples i.e., the signal reconstructed from the samples $h = Xg$, have good (in some sense) approximation properties. Particularly, the *consistent sampling* condition imposes that, given the samples $g = H^*f$, the reconstructed signal $\tilde{f} = FXg = FXH^*f$ has the same samples as f, for every $f \in \mathcal{H}$ (i.e., $H^*f = H^*\tilde{f}$). From the point of view of its samples, the original signal and the reconstructed one are undistinguished.

Observe that, if the consistent sampling condition is satisfied then, for every $f \in \mathcal{H}$, $H^*f = H^*\tilde{f}$, so that

$$(FXH^*)^2 f = FXH^*(FXH^*f) = FXH^*\tilde{f} = FXH^*f,$$

it follows that FXH^* is a projection. Moreover, if $f \in N(FXH^*)$, then $FXH^*f = 0 = H^*FXH^*f = H^*\tilde{f} = H^*f$, thus $f \in N(H^*) = R(H)^\perp = \mathcal{S}^\perp$. Furthermore, since $\mathcal{S}^\perp \subseteq N(FXH^*)$, it follows that FXH^* is a projection with $N(FXH^*) = \mathcal{S}^\perp$. Based on this idea we give the following definition.

Definition 3. *Given $H, F \in L(\mathcal{K}, \mathcal{H})$, the operator $X \in L(\mathcal{K})$ satisfies the* consistent sampling requirement *for H* and *F if $X \in \mathcal{CS}(F, H) = \{X \in L(\mathcal{K}) : FXH^* \in \mathcal{Q}, N(FXH^*) = N(H^*)\}$.*

Since $R(FXH^*) \subseteq R(F)$, notice that, if there exists an operator $X \in L(\mathcal{K})$ satisfying the *consistent sampling requirement*, then $\mathcal{H} = R(F) + N(H^*)$. The converse is also true since, if $\mathcal{H} = R(F)+N(H^*)$ then $X = F^\dagger P_{R(F)\ominus N(H^*)//N(H^*)} H^{*\dagger} \in L(\mathcal{K})$ is well defined and it is easy to see that satisfies the consistent sampling requirement.

In [7, 8, 18] the consistent sampling requirement have been studied under the condition that $\mathcal{H} = N(H^*) \dotplus R(F)$. Recently, in [9], this notion has been studied (for finite dimensional spaces) without the assumption $R(F) \cap N(H^*) = \{0\}$, and it was shown that in this case, the set of projections of the type FXH^* with $N(FXH^*) = N(H^*)$ can be infinite. In this work we relate the consistent sampling condition in infinite dimensional spaces with the notion of compatibility introduced in [4] and developed in [3, 4, 5]. We show that the results given in [9] can be easily obtained using some characterizations of the set $\mathcal{P}(A, N(H^*))$, for an appropriate $A \in L(\mathcal{H})^+$.

The following set will be useful to characterize, in terms of the compatibility, the projections associated with the consistent sampling requirement. Given a closed range operator $H \in L(\mathcal{H},\mathcal{K})$, and a closed subspace $\mathcal{W}$ of $\mathcal{K}$ let

$$\mathcal{A}_H(\mathcal{W}) := \{A \in L(\mathcal{H})^+ : A^{-1}(R(H)) = \mathcal{W}\}$$

Observe that, if $\mathcal{R}$ is a closed subspace of $\mathcal{H}$ such that $\mathcal{H} = N(H^*) + \mathcal{R}$, then $P_{\mathcal{R}^\perp} \in \mathcal{A}_H(\mathcal{R})$, because $P_{\mathcal{R}^\perp}^{-1}(R(H)) = R(P_{\mathcal{R}^\perp}P_{N(H^*)})^\perp = N(P_{N(H^*)}P_{\mathcal{R}^\perp}) = \mathcal{R} \dotplus \mathcal{R}^\perp \cap R(H) = \mathcal{R}$. Also notice that, in this case, for any $A \in \mathcal{A}_H(\mathcal{R})$ the pair $(A, N(H^*))$ is compatible (see Lemma 2.3).

Lemma 4.1. *Consider operators $F, H \in L(\mathcal{K},\mathcal{H})$ with closed range such that $\mathcal{H} = R(F) + N(H^*)$ and $A \in \mathcal{A}_H(R(F))$. Then $X \in \mathcal{CS}(F,H)$ if and only if $FXH^* = I - Q$, for some $Q \in \mathcal{P}(A, N(H^*))$.*

Proof. Suppose that $X \in L(\mathcal{K})$ is such that $I - FXH^* \in \mathcal{P}(A, N(H^*))$. Then, $R(I - FXH^*) = N(FXH^*) = N(H^*)$, and therefore $X \in \mathcal{CS}(F,H)$.

Conversely, if $X \in \mathcal{CS}(F,H)$ then $E = FXH^*$ is a projection with $N(E) = N(H^*)$. Furthermore $N(I-E) = R(E) \subseteq R(F) = A^{-1}(R(H))$. Then $Q = I - E \in \mathcal{P}(A, N(H^*))$, since $R(H) = \overline{R(H)} = N(H^*)^\perp$ □

The above Lemma allows us to give a parameterization of the operators that satisfies a consistent sampling requirement. A similar result, for finite dimensional spaces, can be found in [9].

Theorem 4.2. *Let $F, H \in L(\mathcal{K},\mathcal{H})$ be closed range operators such that $\mathcal{H} = R(F) + N(H^*)$, $\mathcal{T} = R(F) \ominus N(H^*)$ and $\mathcal{R} = R(F^*) \cap F^{-1}(N(H^*))$. Then,*

$$\mathcal{CS}(F,H) = F^\dagger P_{\mathcal{T}//N(H^*)} H^{*\dagger} + L(R(H^*),\mathcal{R}) + L(\mathcal{K}, N(F)) + L(N(H),\mathcal{K})$$

Moreover, the operator $X_0 = F^\dagger P_{\mathcal{T}//N(H^*)} H^{*\dagger} \in \mathcal{CS}(F,H)$ satisfies that, given $f \in \mathcal{H}$, it holds $\|FX_0H^*f\| \leq \|FXH^*f\|$ for every $X \in \mathcal{CS}(F,H)$.

Proof. Let $A \in \mathcal{A}_H(R(F))$ so that $R(F) = A^{-1}(R(H))$, since $\mathcal{H} = R(F) + N(H^*)$ it follows that the pair $(A, N(H^*))$ is compatible. Suppose that $X \in L(\mathcal{K}) \in \mathcal{CS}(F,H)$, by Proposition 4.1, $I - FXH^* \in \mathcal{P}(A, N(H^*))$. Then, by Theorem 2.7, $I - FXH^* = P_{A,N(H^*)} + W$, for some $W \in L(R(H), N(H^*) \cap A^{-1}(R(H)))$ (i.e., for some $W \in L(R(H), N(H^*) \cap R(F))$). Recall that $P_{A,N(H^*)} = P_{N(H^*)//\mathcal{T}}$, where $\mathcal{T} = A^{-1}(R(H)) \ominus N(H^*) = R(F) \ominus N(H^*)$; then

$$P_{N(F)^\perp} X P_{N(H)^\perp} = F^\dagger F X H^* H^{*\dagger} = F^\dagger (I - P_{N(H^*)//\mathcal{T}}) H^{*\dagger} - F^\dagger W H^{*\dagger}.$$

Furthermore, since $R(H^{*\dagger}) = N(H^*)^\perp = R(H)$ and $N(H^{*\dagger})^\perp = R(H^*)$, it follows that

$$P_{N(F)^\perp} X P_{N(H)^\perp} = F^\dagger P_{\mathcal{T}//N(H^*)} H^{*\dagger} + F^\dagger \tilde{W},$$

with $\tilde{W} \in L(R(H^*), N(H^*) \cap R(F))$. Then,

$$X \in H^\dagger P_{\mathcal{T}//N(H^*)} F^{*\dagger} + L(\mathcal{K}, N(F)) + L(N(H),\mathcal{K}) + L(R(H^*),\mathcal{R}), \tag{8}$$

since $R(F^\dagger) = N(F)^\perp = R(F^*)$.

Conversely, if X satisfies equation (8), then

$$FXH^* = P_{R(F)}P_{\mathcal{T}//N(H^*)}P_{R(H)} + FWH^*,$$

for some $W \in L(R(H^*), \mathcal{R})$.

Since $R(P_{\mathcal{T}//N(H^*)}) \subseteq R(F)$, $R(H)^\perp = N(P_{\mathcal{T}//N(H^*)})$ and $R(FW) \subseteq N(H^*) \cap R(F)$, then it holds $I - FXH^* = P_{N(H^*)//\mathcal{T}} + \tilde{W}$, for some $\tilde{W} \in L(N(H^*)^\perp, N(H^*) \cap R(F))$ i.e., $I - FXH^* \in \mathcal{P}(A, N(H^*))$, for any $A \in \mathcal{A}_H(R(F))$ and, in virtue of Proposition 4.1, $X \in L(\mathcal{K}) \in \mathcal{CS}(F, H)$.

Now, let $A \in \mathcal{A}_H(R(F))$ and $X \in F^\dagger P_{\mathcal{T}//N(H^*)}{H^*}^\dagger + L(\mathcal{K}, N(H)) + L(N(H), \mathcal{K}) + L(R(H^*), \mathcal{R})$, from the paragraph above it follows that $FXH^* = I - Q$ for some $Q \in \mathcal{P}(A, N(H^*))$. Observe that $FX_0H^* = I - P_{A,N(H^*)}$ and, by equation (2), it follows that $\|FX_0H^*f\| \leq \|FXH^*f\|$, for every $f \in \mathcal{H}$.

□

Observe that, if $N(F) = \{0\}$ and $N(H) = \{0\}$ then

$$\mathcal{CS}(F, H) = F^\dagger P_{R(F)\ominus N(H^*)//N(H^*)}{H^*}^\dagger + L(\mathcal{K}, F^{-1}(N(H^*))).$$

Remark 4.3. *Assume that, according to equation* (1), $A \in \mathcal{A}_H(\overline{R(F)})$ *has a decomposition* $A = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$. *Then* $X_0 = \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix}$, *where d is the Douglas solution of the equation $ax = b$.*

An important magnitude related with signal processing applications is the *aliasing norm* (see [11, 10, 7]): suppose that $T \in L(\mathcal{H})$ is the operator that assign to every signal $f \in \mathcal{H}$ the reconstructed signal $\hat{f} = Tf$. The *aliasing norm* is given by $\|TP_{R(T)^\perp}\|$. Given $X \in \mathcal{CS}(F, H)$, let a_X be the aliasing norm corresponding to FXH^*. Then, by Theorem 4.2, $X_0 = F^\dagger P_{R(F)\ominus N(H^*)//N(H^*)}{H^*}^\dagger$, satisfies $a_{X_0} \leq a_X$, for every $X \in \mathcal{CS}(F, H)$.

Suppose that $F, F' \in L(\mathcal{K}, \mathcal{H})$ and $H, H' \in L(\mathcal{K}, \mathcal{H})$ satisfy $R(F) = R(F')$ and $R(H) = R(H')$. Although, in general $\mathcal{CS}(F, H) \neq \mathcal{CS}(F', H')$, it is possible to establish a relation between these sets.

Lemma 4.4. *Let $F, H \in L(\mathcal{K}, \mathcal{H})$ be closed range operators such that $\mathcal{H} = R(F) + N(H^*)$ and let $A \in \mathcal{A}_H(R(F))$. Given $Q \in \mathcal{P}(A, N(H^*))$ there is a unique $X \in \mathcal{CS}(F, H) \cap L(R(H^*), R(F^*))$ such that $Q = I - FXH^*$.*

Proof. Given $Q \in \mathcal{P}(A, N(H^*))$, it is easy to see that $F^\dagger(I - Q){H^*}^\dagger \in \mathcal{CS}(F, H) \cap L(R(H^*), R(F^*))$. Suppose that $Q = I - FXH^* = I - FX'H^*$, with $X, X' \in \mathcal{CS}(F, H) \cap L(R(H^*), R(F^*))$ then $0 = F(X - X')H^* = F^\dagger F(X - X')H^*{H^*}^\dagger = P_{R(F^*)}(X - X')P_{R(H^*)} = X - X'$. □

Proposition 4.5. *Let $F, H \in L(\mathcal{K}, \mathcal{H})$ be closed range operators such that $\mathcal{H} = R(F) + N(H^*)$. Suppose that $H' \in L(\mathcal{K}, \mathcal{H})$, $F' \in L(\mathcal{K}, \mathcal{H})$ satisfy $R(F) = R(F')$ and $R(H) = R(H')$. Then, there is a bijection between $\mathcal{CS}(F, H) \cap L(R(H^*), R(F^*))$ and $\mathcal{CS}(F', H') \cap L(R(H'^*), R(F'^*))$.*

Proof. Let $\mathcal{S} = R(H)^\perp = R(H')^\perp$, $\mathcal{T} = R(F) = R(F')$ and $A \in \mathcal{A}_H(\mathcal{T})$. Given $X \in \mathcal{CS}(F,H) \cap L(R(H^*), R(F^*))$, by Proposition 4.1 $Q = I - FXH^* \in \mathcal{P}(A, \mathcal{S})$ and by Lemma 4.4, there exists a unique $X' \in \mathcal{CS}(F', H') \cap L(R(H'^*), R(F'^*))$ such that $Q = I - F'X'H'^*$. Let $\mathcal{F}(X') = F^\dagger F' X' H'^* H^{*\dagger}$, it follows that $\mathcal{F}$ is a bijection between $\mathcal{CS}(F,H) \cap L(R(H^*), R(F^*))$ and $\mathcal{CS}(F', H') \cap L(R(H'^*), R(F'^*))$, and $\mathcal{F}^{-1}(X) = F'^\dagger F X H^* H'^{*\dagger}$. ☐

5. Best approximation of signals with consistency requirement

As it was mentioned in the introduction, when we can not recover signals by its samples (i.e., $f \notin \mathcal{PW}$), a common strategy in signal processing is to apply a filter and thus recover the orthogonal projection of f in $\mathcal{PW}$.

Although in the consistent sampling requirement there is a projection Q involved, the situation is different. Given $F, H \in L(\mathcal{K}, \mathcal{H})$ such that $\mathcal{H} = R(F) + N(H^*)$, Proposition 4.1 establishes that, for a signal $f \in \mathcal{H}$, the reconstructed signal $\tilde{f} \in R(F)$ is given by $\tilde{f} = (I - Q)f$, for some $Q \in \mathcal{P}(A, N(H^*))$, with $A \in \mathcal{A}_H(R(F))$. Since, generally, Q is not selfadjoint, the reconstructed signal is not the best approximation of the signal $f \in \mathcal{H}$ in $R(F)$. When $R(F) \cap N(H^*) \neq \{0\}$, there is a certain degree of freedom in choosing the projection $Q \in \mathcal{P}(A, N(H^*))$. It is an interesting question if for certain signals belonging to a given (closed) subspace $\mathcal{M}$, it is possible to find a $Q_0 \in \mathcal{P}(A, N(H^*))$, such that the recovered signal $(I - Q_0)f$ be the best approximation of the signal f, over all possible reconstructed signals that satisfies the consistent sampling requirement. In other words, we are interested in finding $Q_0 \in \mathcal{P}(A, N(H^*))$ such that $\|f - (I - Q_0)f\| \leq \|f - (I - Q)f\|$, for every $f \in \mathcal{M}$ and for every $Q \in \mathcal{P}(A, N(H^*))$. The solution of this problem is based on Theorem 3.4.

Theorem 5.1. *Let $F, H \in L(\mathcal{K}, \mathcal{H})$ be two closed range operators such that $\mathcal{H} = R(F) + N(H^*)$, and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then, there exists $X_0 \in \mathcal{CS}(F,H)$ such that $\|f - FX_0H^*f\| \leq \|f - FXH^*f\|$, for every $X \in \mathcal{CS}(F,H)$ and for every $f \in \mathcal{M}$, if and only if $\mathcal{M}^\perp + N(H^*) \cap R(F) \subseteq \mathcal{M}^\perp + R(H)$. Moreover, $FX_0H^* = P_{R(F) \ominus N(H^*)//N(H^*)} + W_0 P_{R(H)}$, where $W_0^* \in L(\mathcal{H})$ is a solution of the equation $P_\mathcal{M} P_{R(H)} X = P_\mathcal{M} P_{N(H^*) \cap R(F)}$.*

Proof. It is a consequence of Theorem 3.4 and Proposition 4.1. ☐

In [9], the following problem is studied in finite dimensional spaces. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, find an operator $X \in L(\mathcal{K})$ satisfying the consistent sampling requirement and such that every signal $f \in \mathcal{M}$ be perfectly recovered. This problem can be restated as, finding $Q \in \mathcal{P}(A, N(F^*))$ (with $A \in A_F(R(H))$) such that $f - (I - Q)f = Qf = 0$, for every $f \in \mathcal{M}$. Notice that this problem is a particular case of Theorem 5.1 (see Proposition 3.5).

Corollary 5.2. *Let $F, H \in L(\mathcal{K}, \mathcal{H})$ be a closed range operator such that $\mathcal{H} = R(F) + N(H^*)$, and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then, there exists $X_0 \in \mathcal{CS}(F,H)$ such that every $f \in \mathcal{M}$ is perfectly recovered (i.e., $f - FX_0H^*f = 0$), if and only if $\mathcal{M} \subseteq R(F)$ and $c_0(\mathcal{M}, N(H^*)) < 1$.*

Proof. It is a consequence of Proposition 3.5 and Proposition 4.1. □

Let $\mathcal{M} \subseteq R(F)$ be a closed subspace such that $c_0(\mathcal{M}, N(H^*)) < 1$, and suppose that $\mathcal{T}$ is a closed subspace such that $\mathcal{M} \dot{+} \mathcal{T} \dot{+} N(H^*) = \mathcal{H}$. Let $A \in \mathcal{A}_H(\mathcal{T} \dot{+} \mathcal{M})$; observe that, $\mathcal{P}(A, N(H^*))$ is a singleton because $\mathcal{N} = N(H^*) \cap A^{-1}(R(H)) = \{0\}$ (see Theorem 2.7). Also notice that $Q \in \mathcal{P}(A, N(H^*))$ satisfies that every $f \in \mathcal{M}$ is perfectly recovered, since $N(Q) = A^{-1}(R(H)) = \mathcal{M} \dot{+} \mathcal{T}$.

Based on Proposition 3.6, given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, it is possible to characterize the operators $X \in \mathcal{CS}(F, H)$ that perfectly recover the signals $f \in \mathcal{M}$.

Proposition 5.3. *Consider closed range operators $F, H \in L(\mathcal{K}, \mathcal{H})$ such that $\mathcal{H} = R(F) + N(H^*)$, and a closed subspace $\mathcal{M} \subseteq R(F)$. Then, $X \in \mathcal{CS}(F, H)$ satisfies $f - FXH^*f = 0$ for every $f \in \mathcal{M}$ if and only if, given a complementary subspace $\mathcal{T}$ of $\mathcal{M} + N(H^*) \cap R(F)$ in $R(F)$ and $T \in L(R(H) \cap \mathcal{M}^\perp, N(H^*) \cap R(F))$,*

$$P_{N(F)^\perp} X P_{N(H)^\perp} = F^\dagger P_{\mathcal{M} \dot{+} \mathcal{T} // N(H^*)} H^{*\dagger} - F^\dagger T H^{*\dagger}.$$

Proof. Let $A \in \mathcal{A}_H(R(F))$, since $\mathcal{H} = R(F) + N(H^*)$ it follows that the pair $(A, N(H^*))$ is compatible. Suppose that $X \in L(\mathcal{K}) \in \mathcal{CS}(F, H)$. By Proposition 4.1, if holds $I - FXH^* \in \mathcal{P}(A, N(H^*))$, and by Proposition 3.6 we get $I - FXH^* = P_{N(H^*)//\mathcal{M} \dot{+} \mathcal{T}} + T$, where $\mathcal{T}$ is a closed subspace such that, $R(F) = \mathcal{T} \dot{+} \mathcal{M} \dot{+} N(H^*) \cap R(F)$ and $T \in L(R(H) \cap \mathcal{M}^\perp, R(H) \cap R(F))$, since $R(H)$ is a closed subspace. Then, $P_{N(F)^\perp} X P_{N(H)^\perp} = F^\dagger P_{\mathcal{M} \dot{+} \mathcal{T} // N(H^*)} H^{*\dagger} - F^\dagger T H^{*\dagger}$.

Conversely, let $X \in L(\mathcal{K})$ such that, $P_{N(F)^\perp} X P_{N(H)^\perp} = F^\dagger P_{\mathcal{M} \dot{+} \mathcal{T} // N(H^*)} H^{*\dagger} - F^\dagger T H^{*\dagger}$, with $T \in L(R(H) \cap \mathcal{M}^\perp, R(H) \cap R(F))$. Then, it is easy to see that $I - FXH^* = P_{\mathcal{M} \dot{+} \mathcal{T} // N(H^*)} + T$, i.e., $I - FXH^* \in \mathcal{P}(A, N(H^*))$, for any $A \in \mathcal{A}_H(R(F))$. Then, by Proposition 4.1, $X \in L(\mathcal{K}) \in \mathcal{CS}(F, H)$. Furthermore, if $f \in \mathcal{M}$, $(I - FXH^*)f = P_{\mathcal{M} \dot{+} \mathcal{T} // N(H^*)} f = f$, i.e., every signal $f \in \mathcal{M}$ is perfectly recovered.

□

6. Consistent sampling of perturbed signals

Given a signal $f \in L^2(\mathbb{R})$, perturbed by a stochastic process δf (see below for a proper definition), in this section, we are interested in studying the existence of a consistent sampling requirement which is unbiased for a certain family of signals, and the influence of the perturbation δf on the reconstructed signal is minimal.

In order to be more precise, we fix some terminology. Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$ and let $\mathcal{H}$ be the Hilbert space $L^2(\mu)$. Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space; if $z : \Omega \to \mathbb{R}$ is P-measurable then the *expectation* of z is $E(z) = \int_\Omega z(\omega) dP(\omega)$.

Let $\delta x : \mathbb{R} \times \Omega \to \mathbb{R}$ be a $\mu \times P$-measurable function such that:

(1) for almost every $t \in \mathbb{R}$, $E(\delta x(t, .)) = 0$,
(2) for almost every $\omega \in \Omega$, $\delta x(., \omega) \in \mathcal{H}$,
(3) $E(\|\delta x\|^2) = \int_\Omega \int_\mathbb{R} |\delta x(\omega, t)|^2 d\mu(t) dP(\omega) < \infty$.

The *variance operator* $A \in L(\mathcal{H})^+$ of δx is defined by

$$Ax = E(\langle x, \delta x \rangle \delta x) = \int_\Omega \delta x(\omega, .) \int_\mathbb{R} \delta x(\omega, t) x(t) d\mu(t) dP(\omega),$$

for every $x \in \mathcal{H}$. As it is shown in [16, Lemma 2], the variance operator A is positive trace class operator.

Problem 6.1. *Let $\mathcal{H} = L^2(\mathbb{R})$, $F, H \in L(\mathcal{K}, \mathcal{H})$ be closed range operators such that $\mathcal{H} = R(F) + N(H^*)$. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, let $\mathcal{J} = \{X \in \mathcal{CS}(F, H) : E(FXH^*(f + \delta f)) = f$ for every $f \in \mathcal{M}\}$. Find $X_0 \in \mathcal{J}$ such that $E(\|FX_0H^*\delta f\|^2) \leq E(\|FX_0H^*\delta f\|^2)$ for every $X \in \mathcal{J}$.*

Remark 6.2. *Observe that, since $E(FXH^*(f+\delta f)) = FXH^*f+E(FXH^*\delta f) = FXH^*f$, Problem 6.1 has a solution if every signal in $\mathcal{M}$ can be perfectly recovered. Then, by Corollary 5.2, a necessary condition is that $\mathcal{M} \subseteq R(F)$ and $c_0(\mathcal{M}, R(F) \cap N(H^*)) < 1$.*

Problem 6.1 is related with the *V-approximation processes* studied by A. Sard in [16] and generalized in [1]:

Definition 4. *Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$ and δf with the above assumptions and variance operator $V \in L(\mathcal{H})^+$, let $\mathcal{U} = \{T \in L(\mathcal{H}) : E(T(f + \delta f) = f$, for every $f \in \mathcal{M}\}$. Then $T \in \mathcal{U}$ is a V-*approximation process *over $\mathcal{M}$ if $E\|T\delta f\|^2 \leq E\|U\delta f\|^2$ for every $U \in \mathcal{U}$.*

In [1, Theorem 3.4], it is shown that $T \in L(\mathcal{H})$ is a V-approximation process over $\mathcal{M}$ if and only if T can be decomposed as $T = I - P^* + W$, for some $P \in \mathcal{P}(V, \mathcal{M}^\perp)$ and $W \in L(N(V) \cap \mathcal{M}^\perp, \mathcal{M}^\perp)$. In the following we will assume that $N(V) = \{0\}$, then T is a V-approximation process over $\mathcal{M}$ if and only if $T = I - P^*$, for some $P \in \mathcal{P}(V, \mathcal{M}^\perp)$.

Theorem 6.3. *Let $F, H \in L(\mathcal{K}, \mathcal{H})$ be closed range operators such that $\mathcal{H} = R(F)+N(H^*)$ and let $\mathcal{M} \subseteq R(F)$ be a closed subspace such that $c_0(\mathcal{M}, N(H^*)) < 1$. Suppose that δf is a stochastic process (with the above assumptions) and variance $V \in L(\mathcal{H})^+$, with $N(V) = \{0\}$. Then Problem 6.1 has a solution if $\mathcal{M} \dot{+} N(H^*) = \mathcal{H}$ and $R(H) \subseteq V^{-1}(\mathcal{M})$.*

Proof. Since $\mathcal{M} \subseteq R(F)$ and $c_0(\mathcal{M}, N(H^*)) < 1$, by Corollary 5.2 it follows that there exists $X \in \mathcal{CS}(F, H)$ such that $\mathcal{M} \subseteq R(FXH^*)$, i.e., for every $f \in \mathcal{M}$, $FXH^*f = f$. Then, by Remark 6.2, it follows that $E(FXH^*(f + \delta f)) = f$, for every $f \in \mathcal{M}$. Furthermore, by Proposition 4.1, $I - FXH^* \in \mathcal{P}(A, N(H^*))$ with $A \in \mathcal{A}_H(R(F))$. Then, it follows that $FXH^* = P_{\mathcal{M}//N(H^*)}$, because $\mathcal{M} \dot{+} N(H^*) = \mathcal{H}$.

On the other hand, $Q = I - (FXH^*)^* = I - P_{R(H)//\mathcal{M}^\perp} = P_{\mathcal{M}^\perp//R(H)}$. Since $R(H) \subseteq V^{-1}(\mathcal{M})$, by Lemma 2.2 it follows that $Q \in \mathcal{P}(V, \mathcal{M}^\perp)$; then $FXH^* = I - Q^*$ with $Q \in \mathcal{P}(V, \mathcal{M}^\perp)$ and by [1, Theorem 3.4] the projection FXH^* is a V-approximation process over $\mathcal{M}^\perp$. Therefore FXH^* is a solution of Problem 6.1.

□

References

[1] Corach G., Giribet J. I., Maestripieri A., *Sard's approximation processes and oblique projections*, Studia Math. 194 (2009), 65–80.

[2] Corach G., Maestripieri A., *Weighted generalized inverses, oblique projections and least squares problems*, Numer. Funct. Anal. Optim. 26 No. 6 (2005), 659–673.

[3] Corach G., Maestripieri A., *Redundant decompositions, angles between subspaces and oblique projections*, Publicacions Matemàtiques, Volume 54, Number 2 (2010), 461-484.

[4] Corach G., Maestripieri A., Stojanoff D., *Oblique projections and Schur complements*, Acta Sci. Math. (Szeged) 67 (2001), 337–256.

[5] Corach G., Maestripieri A., Stojanoff D., *Oblique projections and abstract splines.* J. Approx. Theory 117 (2002), 189–206.

[6] Douglas R. G., *On majorization, factorization and range inclusion of operators in Hilbert spaces*, Proc. Amer. Math. Soc. 17 (1966), 413–416.

[7] Eldar Y., *Sampling and Reconstruction in Arbitrary Spaces and Oblique Dual Frame Vectors*, J. Fourier Anal. Appl. 1, (2003), 77–96.

[8] Eldar Y., Werther T., *General framework for consistent sampling in Hilbert spaces*, Int. J. Wavelets Multiresolut. Inf. Process. 3 (2005), 497–509.

[9] Hirabayashi A., Unser M., *Consistent Sampling and Signal Recovery*, IEEE Trans. on Signal Proc., Vol. 55, Issue 8 (2007),4104–4115.

[10] Hogan J. A., Lakey J., *Sampling and aliasing without translation-invariance*, Proc. Conf. SampTA (2001), 61–66.

[11] Jenssen A. J. E. M., *The Zak transform and sampling theorems for wavelets subspaces*, IEEE Trans. Signal Proc. 41, (1993), 3360–3364.

[12] Knyazev A. V., Argentati M. E., *Principal angles between subspaces in an A-based scalar product: algoritheorems and perturbation estimates*, SIAM J. Sci. Comput. 23 (2002), 2008–2040.

[13] Odzijewicz A., *On reproducing kernels and quantization of states*, Comm. Math. Phys. 114 (1988), 577–597.

[14] Odzijewicz A., *Coherent states and geometric quantization*,. Comm. Math. Phys. 150 (1992), 385–413.

[15] Pasternak-Winiarski Z., *On the dependence of the orthogonal projector on deformations of the scalar product*, Studia Math. 128 (1998), 1–17.

[16] Sard A., *Approximation and variance*. Trans. Amer. Math. Soc. 73 (1952), 428–446.

[17] Unser M. *Sampling 50 Years After Shannon*, Proceedings of the IEEE 88, (2000) 569–587.

[18] Unser, M., Aldroubi, A., *A general sampling theory for nonideal acquisition devices*, IEEE Trans. on Signal Proc., 42, (1994), 2915–2925.

Instituto Argentino de Matemática - CONICET and, Facultad de Ingeniería, Universidad de Buenos Aires, Argentina.

E-mail address: gcorach@fi.uba.ar

Instituto Argentino de Matemática - CONICET and, Universidad Nacional de Quilmes, Buenos Aires, Argentina.

E-mail address: jgiribet@fi.uba.ar