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On the Papoulis sampling theorem: Some General Conditions.

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***Abstract*—Some general conditions for multichannel sampling are established for wide sense stationary sequences with spectral density. First, necessary and sufficient conditions are given for these processes so that they are linearly determined by the samples obtained from a multichannel sampling scheme. Some results are studied for stationary sequences, and then applied to the problem of sampling, not necessarily band limited, wide sense stationary processes. Conditions are also given for the existence of a frame sequence of the samples. In the case of frames, the condition that the spectral measure is absolutely continuous is proved to be necessary.**

***Index Terms*—Multi-channel sampling, stationary sequences, frames, Hilbert spaces.**

I. Introduction

Papoulis showed [21] that a band-limited function $f \in L^2(\mathbb{R})$ can be recovered from m output signals/functions obtained by filtering f with m appropriate time invariant linear filters, and then sampling these outputs at $\frac{1}{m}$ the Nyquist sampling rate associated to the input f. That paper explored sufficient conditions for a Shannon like recovery series expansion to hold. This original work extended an idea of Shannon and others of reconstructing a signal considering the samples of the signal and its derivatives. The novelty of this sampling scheme relied on that more general conditions on the m linear filters (or channels) were given, hence generalizing the derivation operations. However, Papoulis' conditions are more than sufficient. They are given in terms of interpolating formulas which involve the time and frequency domains at the same time, and stability conditions are not mentioned. Also, in in [22], straightforward extensions are given for wide-sense stationary (w.s.s.) random processes. Brown, in [5] gave a more detailed study of this result for $L^2(\mathbb{R})$ band limited signals. He realized that the conditions can be given in a clearer matrix form in the frequency domain. With this formulation, some sufficient stability conditions in terms of the spectrum of these matrices arise in a natural way. The formulation of Brown, given in terms of the eigenvalues, more specifically the determinant, of these functional matrices resembles and is related to the theory of shift invariant subspaces (SIS) of $L^2(\mathbb{R})$ with a finite number of generators and Riesz bases [7]. This relationship between Papoulis and Brown's results and the theory of Riesz bases is treated explicitly in [28], where a more general result for shift invariant subspaces (SIS) is obtained (see also [8]). There, the results are given in terms of some sufficient conditions, and those results are extended for some classes of non band limited $L^2(\mathbb{R})$ functions. In [8], some simple extensions for the case of w.s.s. random processes are mentioned. Finally, another interesting extension of Papoulis' work is Hoskins' *et al.*, [13], where some general conditions for the convergence of the sampling expansions of $L^p(\mathbb{R})$ $(p \neq 2)$ for generalized, band limited, functions are given. The problem is also closely related to U-invariant sampling [9], [10], [23].

In this line, here we study some necessary and sufficient conditions for multichannel sampling of wide sense stationary (w.s.s.) sequences, and then we relate these results to the problem of sampling multi-band w.s.s. processes. We also study conditions for stability in terms of Hilbert space frames.

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II. Preliminaries

In this section, some known results and notation are introduced in order to make this presentation self-contained.

A. Stationary Processes

This brief description follows closely [25], [26]. Let $\mathcal{X} = \{X_k\}_{k\in\mathbb{Z}} \subset L^2(\Omega, \mathcal{F}, \mathbf{P})$ be a zero mean, complex, w.s.s. random sequence over a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In this case, it is known by Bochner's Theorem that for some finite Borel measure ν (*spectral measure*) the correlation function can be written as:

$$\langle X_k, X_j\rangle_{L^2(\Omega,\mathcal{F},\mathbf{P})} = \mathbf{E}(X_k\overline{X}_j) = \int_{[-\pi,\pi]} e^{i\lambda(k-j)} d\nu(\lambda)\,.$$

In the case that ν is absolutely continuous with respect to the Lebesgue measure, then, there exists its *Radon-Nikodym (RN) derivative* ϕ, i.e. the *spectral density* of $\mathcal{X}$, such that for any measurable subset A we have $\nu(A) = \int_A \phi(\lambda)d\lambda$. For this and other measure theoretical aspects of this paper we refer the reader to [12].

If $H(\mathcal{X}) = \overline{span}\,\mathcal{X} \subset L^2(\Omega, \mathcal{F}, \mathbf{P})$, the mean square estimation theory for stationary sequences is mainly based on *Kolmogorov's isomorphism*:

$$I : L^2([-\pi,\pi], d\nu) \longrightarrow H(\mathcal{X}) \tag{1}$$

given by the formula:

$$I(f) = \int_{[-\pi,\pi]} f(\lambda)dM(\lambda)\,,$$

where M is the (orthogonal) random measure associated to $\mathcal{X}$. Finally, if A is a Borel subset, then ν, M, and X_k are related by the following formulas:

$$\mathbf{E}|I(\mathbf{1}_A)|^2 = \mathbf{E}|M(A)|^2 = \nu(A)$$

and

$$X_k = \int_{[-\pi,\pi]} e^{ik\lambda} dM(\lambda)\,. \tag{2}$$

Analogous results hold for $\{X_t\}_{t\in\mathbb{R}}$ a w.s.s. random process with index time $t \in \mathbb{R}$ but with the integrals being taken over the whole real line $\mathbb{R}$. Indeed, with appropriate restrictions, the whole theory can be constructed for stationary processes indexed over more general Locally Compact Abelian (LCA) Groups. Moreover, from the spectral theory of unitary operators in a Hilbert space [1], analogous representations to (2) are valid on a Hilbert space $\mathcal{H}$, and therefore none of the results presented here are essentially changed if one considers a stationary sequence $\mathcal{X} = \{X_k\}_{k\in\mathbb{Z}} \subset \mathcal{H}$ with $\mathcal{H}$ a complex Hilbert space. However, the reader should keep in mind the case $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbf{P})$ which is the main interest in applications.
Following [26], linear time invariant filtering operations on $\mathcal{X}$ are defined by:

$$Y_k = \int_{[-\pi,\pi]} f(\lambda) e^{ik\lambda} dM(\lambda)\,,\ f \in L^2([-\pi,\pi], d\nu),$$

so that the resulting stationary sequence $\mathcal{Y} = \{Y_k\}_{k\in\mathbb{Z}}$ can be thought as the output of a linear system with a frequency response given by f (i.e. *filter*) and input $\mathcal{X} = \{X_k\}_{k\in\mathbb{Z}}$.

B. Miscelaneous notation and remarks

If A is any subset, its associated indicator function will be noted $\mathbf{1}_A$. Whenever it is mentioned that a property holds "a.e." (or *almost everywhere*) without mention of the underlying measure, it will be implicitly understood that we are referring to the usual Lebesgue measure, otherwise if it is not the case it will be clear from the context. For a Lebesgue measurable subset $A \subseteq \mathbb{R}$ its (Lebesgue) measure will be denoted $|A|$. According to the motivation of this work all other involved measures μ, ν, etc. are supposed to be finite and if μ is absolutely continuous with respect to ν its Radon-Nikodym (RN) derivative will be denoted $\frac{d\mu}{d\nu}$. Some properties will be described in terms of some particular mappings between the intervals $[-\pi,\pi)$, $[-\pi, -\pi + \frac{2\pi}{m})$, $(m \in \mathbf{N})$ and $\mathbb{R}$. These mappings are respectively defined as:

$$\varphi_m(\lambda) : [-\pi,\pi) \longrightarrow [-\pi, -\pi + \frac{2\pi}{m}),$$

$$\varphi_m(\lambda) = \sum_{k=0}^{m-1} \left(\lambda - \frac{2\pi k}{m}\right) \mathbf{1}_{[-\pi+\frac{2\pi k}{m}, -\pi+\frac{2\pi(k+1)}{m})}(\lambda)\,,$$

and

$$\varphi(\lambda) : \mathbb{R} \longrightarrow [-\pi,\pi),$$

$$\varphi(\lambda) = \sum_{k\in\mathbb{Z}} (\lambda - 2\pi k)\, \mathbf{1}_{I_k}(\lambda)\,, I_k = [-\pi,\pi) + 2\pi k\,. \tag{3}$$

If $\mathbf{A} \in \mathbb{C}^{n\times n}$ then its set of eigenvalues is denoted $\sigma(\mathbf{A})$.

C. Frames, Riesz Bases and Minimality

Let us review some of the basic results about frames and Hilbert spaces which are going to be used here. For a comprehensive basic reference about the general theory of this topic see e.g. [7], [11].

Definition 1. *A sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \|f\|^2_{\mathcal{H}} \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle_{\mathcal{H}}|^2 \leq C_2 \|f\|^2_{\mathcal{H}}\ ,$$

for every $f \in \mathcal{H}$.

Frames, in our context, provide very useful tools for sampling and reconstructing signals. Frames are some times referred as over complete (unconditional) bases. On the other hand, frames provide several iterative reconstruction methods for sampled signals among other interesting properties [11].

Definition 2. A Riesz basis *for $\mathcal{H}$ is a family $\{f_n\}_{n\in\mathbb{N}}$ of the form $f_n = Ue_n$, where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded linear bijective operator.*

Theorem II.1. *A sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ is a* Riesz basis *for $\mathcal{H}$ if and only if it is complete and there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \sum_{n=1}^{\infty} |c_n|^2 \leq \left\| \sum_{n=1}^{\infty} c_n f_n \right\|^2_{\mathcal{H}} \leq C_2 \sum_{n=1}^{\infty} |c_n|^2\ ,$$

for every $(c_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$.

These two notions are closely related:

Theorem II.2. *If $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$ then it is a frame for $\mathcal{H}$.*

We recall the following definition:

Definition 3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in a Hilbert space $\mathcal{H}$. We say that $\{f_n\}_{n\in\mathbb{N}}$ is minimal if $f_j \notin \overline{\text{span}}\{f_k\}_{k\neq j}$.*

There is an interesting relationship between minimal sequences and frames:

Theorem II.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a frame for a Hilbert space $\mathcal{H}$, then the following are equivalent:*
i) $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis for $\mathcal{H}$.
ii) If $\sum_n c_n f_n = 0$ for $(c_n)_n \in l^2$ then $c_n = 0$ for all n.
iii) $\{f_n\}_{n\in\mathbb{N}}$ is minimal.

The following result by the authors [19] gives a necessary and sufficient condition for a vector stationary sequence to be

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a frame or a Riesz basis of the closed linear span of its scalar values. In [20] a similar result is proved for scalar sequences.

Theorem II.4. *Let* $\mathcal{X} = \{X_k^j\}_{k\in\mathbb{Z}}^{j=1\ldots n}$ *be (w.s.s.) stationary in the variable k. Then: a)* $\mathcal{X}$ *is a frame for its span* $H(\mathcal{X})$ *(in* $L^2(\Omega,\mathcal{F},\mathbf{P})$*) with constants* C_1, C_2 *if and only if the (cross) spectral measures* $\mu_{i\,j}$ *verify the following conditions: (i)* $\mu_{i\,j}$ *is absolutely continuous with respect to the Lebesgue measure for every* i, j*, (ii) the spectral densities matrix* $\mathbf{D}$ *verifies* $\sigma(\mathbf{D})(\lambda) \subseteq \{0\} \cup [C_1, C_2]$ *for almost all* $\lambda \in [-\pi,\pi)$*. b)* $\mathcal{X}$ *is a Riesz basis of* $H(\mathcal{X})$ *with constants* C_1, C_2 *if and only if: (i)* $\mu_{i\,j}$ *is absolutely continuous with respect to the Lebesgue measure for every* i, j*, (ii) the spectral densities matrix* $\mathbf{D}$ *verifies* $\sigma(\mathbf{D})(\lambda) \subseteq [C_1, C_2]$ *for almost all* $\lambda \in [-\pi,\pi)$*.*

Frequencies are referred to the interval $[-\pi,\pi)$. However as the $\mu_{i\,j}$'s are absolutely continuous, the boundaries of the interval have null measure. This result resembles some known characterizations of frames in terms of the Fourier transform of the generators of a SIS of $L^2(\mathbb{R})$ [3], [4], [6], [24]. On the other hand, referring frequencies to $[-\pi,\pi)$ gives a rather simple condition for minimality for a stationary sequence or to be a Riesz basis. The study of these conditions for w.s.s. sequences goes back to Kolmogorov e.g. [27], [29], [18].

III. Outline of the Paper and review of some existing related results

With the considerations previously introduced, we may state more precisely the problem described in the introduction: Given $m \in \mathbb{N}$, find under which conditions for the filters $f_1,\ldots,f_m$ the (under) sampled set of filtered sequences $\mathcal{Y} = \{Y_{mk}^r\}_{k\in\mathbb{Z}}^{r=1,\ldots,m}$, with

$$Y_k^r = \int_{[-\pi,\pi]} f_r(\lambda) e^{ik\lambda} dM(\lambda)$$

completely determines $\mathcal{X}$ in a linear form.

The first condition we may establish is that $\mathcal{Y} = \{Y_{mk}^r\}_{k\in\mathbb{Z}}^{r=1,\ldots,m}$ should be complete in $H(\mathcal{X})$. In this case the necessary and sufficient conditions are given for w.s.s. sequences with spectral density. The singular case will be discussed in further work. However a complete system may not constitute an adequate framework for applications since it may be impossible to have stable representations. An appropriate tool to ensure stability is the use of frames. However, frames could provide redundant representations and if additionally uniqueness is required, one should study conditions to ensure the existence of an unconditional Riesz basis. Necessary and sufficient conditions will be established for ν and f_r so that $\mathcal{Y} = \{Y_{mk}^r\}_{k\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame or Riesz basis. In this case, the condition of the existence of an spectral density is *proved to be necessary*. A similar result has been given for stationary sequences in [20], [19], for scalar and vector sequences respectively (see Theorem II.4 above), see also [9], [23]. In Section VII these results are related to the problem of reconstructing a continuous time, *not necessarily band limited* multi-band w.s.s. process from m under-sampled measurements.

Recalling the isomorphism (1) of section II, if ν is a finite Borel measure on $[-\pi,\pi)$, these problems can be restated as finding conditions under which $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete or a frame of $L^2([-\pi,\pi], d\nu)$ respectively, which may be an interesting function spaces problem on its own. Following this fact, no further mention on the underlying process X_k or $H(\mathcal{X})$ will be done, but the original motivation should remain clear from the context.

IV. Completeness Conditions

Assuming the existence of the spectral density ϕ associated to the spectral measure ν of $\mathcal{X}$ we shall give some conditions on $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ to be complete. This problem is related to the sampling and reconstruction of a stationary sequence from periodic observations. First, we need:

Lemma IV.1. *Let* $f \in L^1([-\pi,\pi])$ *and* $m \in \mathbb{Z}$*, then:*

$$\int_{[-\pi,\pi]} f(\lambda) e^{i\lambda mn} d\lambda = 0 \ \textit{for all } n \in \mathbb{Z},$$

if and only if

$$\sum_{k=0}^{m-1} f\left(\lambda + \frac{2\pi}{m}k\right) = 0$$

a.e. in $[-\pi, -\pi + \frac{2\pi}{m}]$*.*

Proof: Define $I_k = \left[-\pi + \frac{2\pi}{m}k, -\pi + \frac{2\pi}{m}(k+1)\right)$, $k = 0,\ldots,m-1$.

$$\int_{[-\pi,\pi]} f(\lambda) e^{i\lambda mn} d\lambda = \sum_{k=0}^{m-1} \int_{I_k} f(\lambda) e^{i\lambda mn} d\lambda$$

$$= \int_{[-\pi,-\pi+\frac{2\pi}{m}]} \sum_{k=0}^{m-1} f\left(\lambda + \frac{2\pi}{m}k\right) e^{i(\lambda+\frac{2\pi}{m}k)mn} d\lambda\,,$$

$$= \int_{[-\pi,-\pi+\frac{2\pi}{m}]} \sum_{k=0}^{m-1} f\left(\lambda + \frac{2\pi}{m}k\right) e^{i\lambda mn} d\lambda\,,$$

But, by the uniqueness of the Fourier series coefficients, this last integral equals 0 for all $n \in \mathbb{Z}$ if and only if

$$\sum_{k=0}^{m-1} f\left(\lambda + \frac{2\pi}{m}k\right) = 0$$

a.e. in $[-\pi, -\pi + \frac{2\pi}{m}]$. ∎

From this result we can prove:

Theorem IV.1. *Let* ν *be absolutely continuous with respect to the Lebesgue measure with spectral density* ϕ*, let* A *be the support of* ν *and* $f_1,\ldots, f_m \in L^2([-\pi,\pi], d\nu)$*.*
Then:
$\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ *is complete in* $L^2([-\pi,\pi], d\nu)$ *if and*

only if $det(\mathbf{F}(\lambda)) \neq 0$ *a.e. in* $\mathcal{A} = \varphi_m(A) = \bigcap_{k=0}^{m-1} (A \cap I_k) - \frac{2\pi}{m}k$*, where* $(\mathbf{F}(\lambda))_{r\,j} = f_r\left(\lambda + \frac{2\pi}{m}(j-1)\right)$*, with* $r, j = 1, \ldots, m$*,* $\lambda \in \mathcal{A} \subseteq [-\pi, -\pi + \frac{2\pi}{m}]$ *and the* I_k*'s are defined as in Lemma IV.1.*

Proof: Let us prove the "if" part. Let $g \in L^2([-\pi,\pi], d\nu)$ be such that:

$$\int\limits_{[-\pi,\pi]} f_r(\lambda)\overline{g(\lambda)}e^{i\lambda mn}\phi(\lambda)d\lambda = 0 \;\; \forall\, n \in \mathbb{Z}, r = 1 \ldots m\,. \tag{4}$$

By Lemma IV.1:

$$\sum_{k=0}^{m-1} f_r\left(\lambda + \frac{2\pi}{m}k\right)\phi\left(\lambda + \frac{2\pi}{m}k\right)\overline{g}\left(\lambda + \frac{2\pi}{m}k\right) = 0 \;\; \forall\, r \tag{5}$$

and almost all $\lambda \in [-\pi, -\pi + \frac{2\pi}{m}]$. If we define

$$\mathbf{H}(\lambda)_{r\,k} = f_r\left(\lambda + \frac{2\pi}{m}(k-1)\right)\phi\left(\lambda + \frac{2\pi}{m}(k-1)\right)\,, \tag{6}$$

then:

$$det(\mathbf{H}(\lambda)) = \prod_{j=1}^{m} \phi\left(\lambda + \frac{2\pi}{m}(j-1)\right) det(\mathbf{F}(\lambda)))\,. \tag{7}$$

Hence if $det(\mathbf{F}) \neq 0$ a.e. in $\mathcal{A}$ then $det(\mathbf{H}) \neq 0$ a.e. in $\mathcal{A}$, which together with (5), implies that, for all k: $\overline{g}\left(\lambda + \frac{2\pi}{m}k\right) = 0$ for almost all $\lambda \in \mathcal{A} = \varphi_m(A)$ and then $g = 0$ a.e. in A. To prove the "only if" part suppose that $|B| > 0$, where $B = \{\lambda : det(\mathbf{F}(\lambda)) = 0\} \bigcap \mathcal{A} = \{\lambda : det(\mathbf{H}(\lambda)) = 0\} \bigcap \mathcal{A}$, see (7). Then, there exist non null $x_1(\lambda), \ldots x_m(\lambda)$ measurable functions such that

$$\mathbf{H}x(\lambda) = 0,\; \forall\, \lambda \in B$$

where $x = (x_1, \ldots, x_m)^t$. Hence, if we take $y_k = x_k \mathbf{1}_B$, then:

$$\mathbf{H}(\lambda)y(\lambda) = 0,\;\; \forall\, \lambda \in \mathcal{A}\,. \tag{8}$$

Now, consider the (non identically zero) function:

$$g(\lambda) = \sum_{k=0}^{m-1} y_k(\lambda - \frac{2\pi}{m}k)\mathbf{1}_{[-\pi+\frac{2\pi}{m}k, -\pi+\frac{2\pi}{m}(k+1))}(\lambda)\,.$$

Then:

$$\int\limits_{[-\pi,\pi]} f_r(\lambda)g(\lambda)e^{imn\lambda}\phi(\lambda)d\lambda$$

$$= \int\limits_{[-\pi,-\pi+\frac{2\pi}{m})} (\mathbf{H}(\lambda)y(\lambda))_r e^{i\lambda mn}d\lambda = 0$$

Then $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is not complete. ∎

V. Frames and Stationary Sequences

In the previous section we assumed the absolute continuity of ν. However, completeness alone may be of no practical value and then some additional stability conditions could be required. One way to have stable representations is by means of frames. We shall see that the absolute continuity of ν is a necessary condition for $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ to constitute a frame sequence.

Theorem V.1. *If* $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ *is a frame for* $L^2([-\pi,\pi], d\nu)$ *for some* $f_r \in L^2([-\pi,\pi], d\nu)$*, for* $r = 1, \ldots, m$*, then* ν *is absolutely continuous with respect the Lebesgue measure, i.e. there exists a spectral density.*

Proof: Note that by Definition 1, $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame if and only if there exists some $0 < C_1 \leq C_2$ such that $\forall\, g \in L^2([-\pi,\pi], d\nu)$:

$$C_1 \int\limits_{[-\pi,\pi]} |g(\lambda)|^2 d\nu(\lambda)$$

$$\leq \sum_{r=1}^{m}\sum_{n\in\mathbb{Z}} \left| \int\limits_{[-\pi,\pi]} f_r(\lambda)\overline{g(\lambda)}e^{imn\lambda}d\nu(\lambda) \right|^2 \tag{9}$$

$$\leq C_2 \int\limits_{[-\pi,\pi]} |g(\lambda)|^2 d\nu(\lambda)\,.$$

First, let's check that $\nu(\{\pi\}) = 0$. Suppose this is not the case. Then, from the completeness condition, we have that for some f_r: $f_r(\pi) \neq 0$ it is:

$$\int\limits_{[-\pi,\pi]} \mathbf{1}_{\{\pi\}}(\lambda)f_r(\lambda)e^{imn\lambda}d\nu(\lambda) = (-1)^{mn} f_r(\pi)\nu(\pi)$$

Then, if we take $g = \mathbf{1}_{\{\pi\}}$ in (9) we obtain:

$$C_2\nu(\{\pi\})$$

$$\geq \sum_{r=1}^{m}\sum_{n\in\mathbb{Z}} \left| \int\limits_{[-\pi,\pi]} f_r(\lambda)\mathbf{1}_{\{\pi\}}(\lambda)e^{imn\lambda}d\nu(\lambda) \right|^2 = \infty\,,$$

a contradiction. Hence, the boundaries of the interval have zero ν-measure and the same holds for every atomic set, and then, ν is a non-atomic measure.

Next, for $l = 1, \ldots, m$, $j = 0, \ldots, m-1$, taking $g = sign(f_l)\mathbf{1}_{I_j}$ with $I_j = \left[-\pi + \frac{2\pi}{m}j, -\pi + \frac{2\pi}{m}(j+1)\right)$, replacing this g in (9), we obtain that for each j, r:

$$\sum_{n\in\mathbb{Z}} \left| \int\limits_{I_j} |f_r(\lambda)|e^{imn\lambda}d\nu(\lambda) \right|^2 \leq C_2\nu(I_j) < \infty\,.$$

This implies that the Fourier coefficients of the measure $|\rho|_{r\,j}$, defined for each Borel subset $A \subseteq I_j$ by:

$$|\rho_{r\,j}|(A) = \int_A |f_r| d\nu$$

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are in $l^2(\mathbb{Z})$. Then, these are the Fourier coefficients of a function in $L^2(I_j)$, and then also in $L^1(I_j)$. By the uniqueness of the Fourier transform this implies that there exists $\phi_{r\,j} \in L^1(I_j)$ such that $|\rho_{r\,j}|(A) = \int_A \phi_{r\,j}(\lambda)d\lambda$. and then, we may define for each measurable subset $A \subset [-\pi, \pi)$:

$$|\rho_r|(A) = \sum_{j=0}^{m-1} |\rho_{r\,j}|(A \cap I_j) = \int_A \sum_j \phi_{r\,j} d\lambda$$

But on the other hand

$$|\rho_r|(A) = \int_A |f_r| d\nu$$

and then the signed measure $\rho_r(A) = \int_A f_r d\nu$ is absolutely continuous with respect to the Lebesgue measure. Hence, there exists $\widetilde{f_r} \in L^1$ such that for each Borel subset $A \subseteq [-\pi, \pi)$:

$$\rho_r(A) = \int_A f_r d\nu = \int_A \widetilde{f}_r \, d\lambda \,, \tag{10}$$

and then:

$$\int_A f_r(\lambda)e^{imn\lambda} d\nu = \int_A \widetilde{f}_r(\lambda)e^{imn\lambda} d\lambda \,. \tag{11}$$

Now, considering a measurable subset A such that $|A| = 0$ and setting $g = \mathbf{1}_A$ in (9), by (11) we obtain

$$C_1 \nu(A) \leq \sum_{r=1}^{m} \sum_{n\in\mathbb{Z}} \left| \int_A \widetilde{f}_r(\lambda)e^{imn\lambda} d\lambda \right|^2 = 0 \,,$$

which gives the absolute continuity of ν. ■

Remark: (10) extends from characteristic functions $\mathbf{1}_A$ to simple functions and by an approximation argument (11) holds.

Theorem V.2. *Let $f_1, \ldots, f_m \in L^2([-\pi, \pi], d\nu)$ and let A be the support of ν. Then $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame for $L^2([-\pi, \pi], d\nu)$ if and only if ν is absolutely continuous with respect to the Lebesgue measure and there exists some constants $0 < C_1 \leq C_2$ such that*

$$\sigma(\mathbf{H}(\phi)(\lambda)\mathbf{H}^H(\phi)(\lambda)) \subseteq [C_1, C_2] \tag{12}$$

for almost all $\lambda \in \mathcal{A} = \varphi_m(A)$, where ϕ is the RN derivative of ν and

$$(\mathbf{H}(\phi)(\lambda))_{r\,j} = f_r\left(\lambda + \frac{2\pi}{m}(j-1)\right)\sqrt{\phi\left(\lambda + \frac{2\pi}{m}(j-1)\right)},$$

$k = 1, \ldots, m$.

Some Remarks: As in Brown's original formulation, one can easily check that the condition given in (12) holds if and only if for some positive constants C_1, C_2: $C_1 \leq det(\mathbf{H}(\phi)(\lambda)) \leq C_2$ for almost all $\lambda \in \mathcal{A} = \varphi_m(A)$. Additionally, note that $\varphi_m(A) \subseteq [-\pi, -\pi + \frac{2\pi}{m}]$ so, in some way, this result indicates that the bandwidth can been "reduced" by a factor of $1/m$. As far as the authors know this result cannot be derived, at least directly, from Theorem II.4 (Theorem 3.1 of [19]), or from similar results in, for example [9], [23]. Hence, we include a complete proof of this formulation, which is also more suitable for the subsequent analysis on sampling presented here below. Note that, applying the result of [19], combined with Theorem V.1 and the next Lemma V.1, we obtain the similar but less informative condition: $\sigma(\mathbf{H}(\phi)(\lambda)\mathbf{H}^H(\phi)(\lambda)) \subseteq [C_1, C_2] \cup \{0\}$ a.e.

Proof:

Proof (of Theorem V.2)

First, we observe that by Definition 1, $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame if and only if there exists some $0 < C_1 \leq C_2$ such that eq. 9 of Theorem V.1 holds. On the other hand, if we assume that $d\nu = \phi d\lambda$, and repeating an argument similar to that in the proof of Lemma IV.1 we get that:

$$\int\limits_{[-\pi,\pi]} f_r(\lambda)\overline{g(\lambda)}e^{imn\lambda} d\nu(\lambda)$$

$$= \int\limits_{[-\pi,-\pi+\frac{2\pi}{m}]} \sum_{k=0}^{m-1} f_r\left(\lambda + \frac{2\pi}{m}k\right) \ldots$$

$$\phi\left(\lambda + \frac{2\pi}{m}k\right) \overline{g\left(\lambda + \frac{2\pi}{m}k\right)} e^{i\lambda mn} d\lambda \,,$$

From Parseval's Theorem and if A is a support, this implies that the inner integral of the double inequality in(9) equals:

$$\sum_{r=1}^{m} \int\limits_{\varphi_m(A)} \left| \sum_{k=0}^{m-1} f_r\left(\lambda + \frac{2\pi}{m}k\right) \ldots \right.$$

$$\left. \phi\left(\lambda + \frac{2\pi}{m}k\right) \overline{g\left(\lambda + \frac{2\pi}{m}k\right)} \right|^2 d\lambda \,,$$

and in a similar way:

$$\int\limits_{[-\pi,\pi]} |g(\lambda)|^2 d\nu(\lambda)$$

$$= \int\limits_{\varphi_m(A)} \sum_{k=0}^{m-1} \left| g\left(\lambda + \frac{2\pi}{m}k\right) \right|^2 \phi\left(\lambda + \frac{2\pi}{m}k\right) d\lambda \,.$$

Now, we are in condition to prove the "only if" part of the Theorem. The absolute continuity of ν was proved in the previous Theorem V.1. Now, given $x \in \mathbb{C}^m$, $\lambda_0 \in \left(-\pi, -\pi + \frac{2\pi}{m}\right)$ and $\epsilon > 0$, define:

$$g_\epsilon(\lambda) = \frac{\mathbf{1}_A(\lambda)}{\sqrt{\phi((\lambda)}} \sum_{k=0}^{m-1} \overline{x}_{k+1} \mathbf{1}_{B(\lambda_0,\epsilon)} \left(\lambda - \frac{2\pi}{m}k\right) \,,$$

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thus if $\lambda \in \mathcal{A}$,

$$\sum_{k=0}^{m-1} f_r\left(\lambda + \frac{2\pi}{m}k\right)\phi\left(\lambda + \frac{2\pi}{m}k\right)\overline{g_\epsilon\left(\lambda + \frac{2\pi}{m}k\right)}$$

$$= \left(\sum_{k=0}^{m-1} f_r\left(\lambda + \frac{2\pi}{m}k\right)\sqrt{\phi\left(\lambda + \frac{2\pi}{m}k\right)}x_{j+1}\right)\dots \quad (13)$$

$$\mathbf{1}_{B(\lambda_0,\epsilon)\cap\mathcal{A}}(\lambda) = (\mathbf{H}(\phi)(\lambda)x)_r$$

since $\mathbf{1}_{B(\lambda_0,\epsilon)}\left(\lambda - \frac{2\pi}{m}k\right) = 0$ if $\lambda \in (-\pi, -\pi + \frac{2\pi}{m})$, $j \neq k$ and $\epsilon > 0$ is such that $B(\lambda_0,\epsilon) \subseteq (-\pi, -\pi + \frac{2\pi}{m})$. By a similar argument if $\lambda \in \mathcal{A}$:

$$\sum_{k=0}^{m-1}\left|g_\epsilon\left(\lambda + \frac{2\pi}{m}k\right)\right|^2\phi\left(\lambda + \frac{2\pi}{m}k\right)$$

$$= \sum_{k=0}^{m-1}\frac{|x_{k+1}|^2}{\phi\left(\lambda + \frac{2\pi}{m}k\right)}\phi\left(\lambda + \frac{2\pi}{m}k\right)\mathbf{1}_{B(\lambda_0,\epsilon)\cap\mathcal{A}}(\lambda) \quad (14)$$

$$= \sum_{k=1}^{m}|x_k|^2\mathbf{1}_{B(\lambda_0,\epsilon)\cap\mathcal{A}}(\lambda)\,.$$

Thus, recalling (9) and replacing g by g_ϵ, then, by (13) and (14), using the Lebesgue differentiation Theorem, we obtain that there exists a Lebesgue set $F_x \subseteq \mathcal{A}$ such that $|F_x^c \cap \mathcal{A}| = 0$ and that for each $\lambda_0 \in F_x$:

$$C_1\sum_{k=1}^{m}|x_k|^2 \leq \|\mathbf{H}(\phi)(\lambda_0)x\|_2^2$$

$$= \lim_{\epsilon \longrightarrow 0}\frac{1}{2\epsilon}\int_{B(\lambda_0,\epsilon)}\sum_{r=1}^{m}|(\mathbf{H}(\phi)(\lambda)x)_r|^2\,d\lambda \leq C_2\sum_{k=1}^{m}|x_k|^2\,.$$

If we consider a countable dense subset $\mathcal{D} \subset \mathbb{C}^m$, then

$$C_1\sum_{k=1}^{m}|x_k|^2 \leq \|\mathbf{H}(\phi)(\lambda)x\|_2^2 \leq C_2\sum_{k=1}^{m}|x_k|^2$$

for each $\lambda \in F = \bigcap_{x\in\mathcal{D}} F_x \subseteq \mathcal{A}$, and then this inequality holds for almost all $\lambda \in \mathcal{A}$ and all $x \in \mathcal{D}$. But for each λ and $k > 0$ the map $x \mapsto \|\mathbf{H}(\phi)x\|_2^2 - k\,\|x\|_2^2$ is continuous, and then the inequality holds for every x.
The "if" part is obtained replacing in the last inequality $x_k = g(\lambda + \frac{2\pi}{m}k)$, and reversing the previous arguments and (9). ∎

A. Minimality and Riesz Bases

Frames can provide stable representations, but they may not be minimal. We recall that a minimal frame is a Riesz basis (Theorem II.3) We need an auxiliary Lemma:

Lemma V.1. *Let $m \in \mathbb{N}$ and let M be the random measure associated to $\mathcal{X}$ (see equation (2), Section II). If its spectral measure ν is absolutely continuous with respect to the Lebesgue measure with spectral density ϕ and $f_r \in L^2([-\pi,\pi],\nu)$, then the vector process $\mathcal{Y} = \{Y_{mk}^r\}_{k\in\mathbb{Z}}^{r=1,\dots,m}$ defined by*

$$Y_{mk}^r = \int_{[-\pi,\pi]} f_r(\lambda)e^{imk\lambda}dM(\lambda)\,, k \in \mathbb{Z}, r = 1,\dots,m,$$

is stationarity correlated in the variable $k \in \mathbb{Z}$ and has its spectral density matrix, for $r, l = 1,\dots,m$, determined by the coefficients:

$$(\mathbf{D}(\lambda))_{r\,l} = \sum_{n=0}^{m-1} f_r(\lambda + \frac{2\pi n}{m})\overline{f_l}(\lambda + \frac{2\pi n}{m})\phi(\lambda + \frac{2\pi n}{m})\,, \quad (15)$$

with $r, l = 1,\dots,m,\ \lambda \in [-\pi, -\pi + \frac{2\pi}{m})$.

Remark: Usually, following the original definition of spectral measure given in Section II, the resulting spectral density of $\mathcal{Y}$ could be expressed as a function defined over $[-\pi,\pi]$. However, the alternative scaled formulation of (15) seems to be more compact and easier to handle in the forthcoming proofs, as it is also reflected in related works such as [14], and moreover, it clearly reflects again the effect of contracting the original "bandwidth" of $\mathcal{X}$ by a factor of $\frac{1}{m}$.

Proof: The derivation of this matrix is very close to the proof of Lemma 4.1 of [15] so we omit the details. By a change of variable and as the spectral density is uniquely determined by the cross-correlations:

$$\mathbf{E}(Y_{mk}^r\overline{Y}_{mj}^l) = \int_{[-\pi,\pi)} e^{im(k-j)\lambda}f_r(\lambda)\overline{f_l}(\lambda)\phi(\lambda)d\lambda$$

$$= \int_{[-\pi,-\pi+\frac{2\pi}{m})}\left(\sum_{n=0}^{m-1} f_r(\lambda + \frac{2\pi n}{m})\overline{f_l}(\lambda + \frac{2\pi n}{m})\dots\right.$$

$$\left.\phi(\lambda + \frac{2\pi n}{m})\right)e^{im(k-j)\lambda}d\lambda\,,$$

the result follows from these observations. ∎

Now, the following is obtained almost directly from Theorem II.4:

Theorem V.3. *Let $f_1,\dots,f_m \in L^2([-\pi,\pi],d\nu)$. Then $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\dots,m}$ is a Riesz basis for $L^2([-\pi,\pi],d\nu)$ if and only if ν is absolutely continuous with respect to the Lebesgue measure and there exists some constants $0 < C_1 \leq C_2$ such that*

$$\sigma(\mathbf{H}(\phi)(\lambda)\mathbf{H}^H(\phi)(\lambda)) \subseteq [C_1, C_2] \quad (16)$$

for almost all $\lambda \in [-\pi, -\pi + \frac{2\pi}{m})$, where ϕ is the RN derivative of ν and

$$(\mathbf{H}(\phi)(\lambda))_{r\,j} = f_r\left(\lambda + \frac{2\pi}{m}(j-1)\right)\sqrt{\phi\left(\lambda + \frac{2\pi}{m}(j-1)\right)},$$

$k = 1,\dots,m$.

Proof:

First, supposing that $d\nu = \phi d\lambda$, by Lemma V.1 we have an spectral density matrix given by (15). So if $\mathbf{D}$ exists is easy to check that

$$\mathbf{D} = \mathbf{H}(\phi)\mathbf{H}(\phi)^H\,.$$

Now, suppose that $\{f_r(\lambda)e^{imn\lambda}\}_{n\in\mathbb{Z}}^{r=1,\dots,m}$ is a Riesz basis then, in particular, it is also a frame, and then by Theorem V.1 there

exists a spectral density ϕ, and thus we can infer that the vector process $\mathcal{Y} = \{Y^r_{mk}\}^{r=1\ldots m}_{k\in\mathbb{Z}}$ has an spectral density matrix given by (15). Then by Theorem II.4 $\sigma(\mathbf{D}(\lambda)) \subseteq [C'_1, C'_2]$ a.e. in $[-\pi, -\pi + \frac{2\pi}{m})$.
Conversely, if a spectral density ϕ exists then by Lemma V.1 we have a spectral density matrix given by equation (15) as $\mathbf{D} = \mathbf{H}(\phi)\mathbf{H}(\phi)^H$. Then by Theorem II.4 $\{f_r(\lambda)e^{imn\lambda}\}^{r=1,\ldots,m}_{n\in\mathbb{Z}}$ is a Riesz basis. ∎

B. Some Examples

1) A complete system which is not a frame.: Take $\phi(\lambda) = e^{-\frac{1}{|\lambda|}}$ and $f_1 = \mathbf{1}_{[-\pi,0)}$ and $f_2 = \mathbf{1}_{[0,\pi]}$. In this case we get that:

$$\mathbf{F}(\lambda) = \begin{pmatrix} f_1(\lambda) & f_1(\lambda+\pi) \\ f_2(\lambda) & f_2(\lambda+\pi) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so by Theorem IV.1 $\{f_1(\lambda)e^{i2n\lambda}, f_2(\lambda)e^{i2n\lambda}\}_{n\in\mathbb{Z}}$ is complete, but is not a frame since:

$$\mathbf{H}(\phi)(\lambda) = \begin{pmatrix} f_1(\lambda)\sqrt{\phi(\lambda)} & f_1(\lambda+\pi)\sqrt{\phi(\lambda+\pi)} \\ f_2(\lambda)\sqrt{\phi(\lambda)} & f_2(\lambda+\pi)\sqrt{\phi(\lambda+\pi)} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-\frac{1}{|\lambda|}} & 0 \\ 0 & e^{-\frac{1}{|\lambda+\pi|}} \end{pmatrix},$$

then $\mathbf{H}(\phi)(\lambda)\mathbf{H}^H(\phi)(\lambda)$ has eigenvalues $e^{-\frac{2}{|\lambda|}}$ and $e^{-\frac{2}{|\lambda+\pi|}}$ and the result follows from Theorem V.2.

2) A simple frame.: Let's modify the previous example. Take any density $\phi \in L^1[-\pi,\pi]$. If A is a support then define $f_1 = \frac{\mathbf{1}_{[-\pi,0)\cap A}}{\sqrt{\phi}}$ and $f_2 = \frac{\mathbf{1}_{[0,\pi]\cap A}}{\sqrt{\phi}}$ thus

$$\mathbf{H}(\phi)(\lambda) = \begin{pmatrix} f_1(\lambda)\sqrt{\phi(\lambda)} & f_1(\lambda+\pi)\sqrt{\phi(\lambda+\pi)} \\ f_2(\lambda)\sqrt{\phi(\lambda)} & f_2(\lambda+\pi)\sqrt{\phi(\lambda+\pi)} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{1}_A(\lambda),$$

and then by Theorem V.2 $\{f_1(\lambda)e^{i2n\lambda}, f_2(\lambda)e^{i2n\lambda}\}_{n\in\mathbb{Z}}$ is a frame.

VI. SOME PROJECTIONS AND ISOMORPHISMS.

In this section some linear operators between $L^2(\mathbb{R}, d\mu)$ and $L^2([-\pi,\pi), d\nu)$ are studied, where ν is a *periodization* of the spectral measure μ, $\nu := \mu_\varphi$. Equivalently, ν is the measure induced by the map φ previously defined in (3). We recall that if μ is absolutely continuous then ν has a spectral density defined for $\lambda \in [-\pi,\pi]$ as $\widetilde{\phi}(\lambda) = \sum_k \phi(\lambda + 2\pi k)$. The following gives a complete characterization [19]:

Proposition VI.1. *Let μ be a complex Borel measure over $\mathbb{R}$. Then, the induced measure over $[-\pi,\pi)$ by φ, $\nu := \mu_\varphi$ is absolutely continuous with respect to Lebesgue measure if and only if the same holds for μ. If ϕ is the RN derivative with respect to the Lebesgue measure of the absolutely continuous part of μ, then $\widetilde{\phi} = \sum_k \phi(\,.\,+2\pi k)$ is the RN derivative of the absolutely continuous part of ν.*

This is the basic result for the multichannel sampling analysis in the following section. As noted in [17] there is a relationship between the sampling problem and the characterization of certain subspaces of periodic functions. We begin with a definition: we shall denote with $\mathcal{P}$ the following closed (in $L^2(\mathbb{R}, d\mu)$) subspace of "2π-periodic functions":

$$\mathcal{P} = \{f \in L^2(\mathbb{R}, d\mu) : f = g\,a.e.[\mu] \text{ for some } g \in L^2(\mathbb{R}, d\mu),$$
$$\ldots\; g(\lambda) = g(\lambda + 2\pi k), \forall\, k \in \mathbb{Z},\; \lambda \in \mathbb{R}\,\}$$

Theorem VI.1. *Let $f \in L^2(\mathbb{R}, d\mu)$, then the formula given for each $\lambda \in [-\pi,\pi)$ by:*

$$(Tf)(\lambda) = \sum_{k\in\mathbb{Z}} f(\lambda + 2\pi k)\frac{d\nu_k}{d\nu}(\lambda), \tag{17}$$

defines an onto linear mapping $T : L^2(\mathbb{R}, d\mu) \longrightarrow L^2([-\pi,\pi), d\nu)$, such that $\|T\|_{op} = 1$, where the measures ν_k and ν are defined for every Borel subset of $A \subseteq [-\pi,\pi)$ by the equations:

$$\nu_k(A) = \mu(A + 2\pi k)\,,\;\; \nu = \mu_\varphi = \sum_{k\in\mathbb{Z}} \nu_k\,.$$

Proof: First note that if we apply Hölder's inequality with measure $m(C) = \sum_{k\in C} \frac{d\nu_k}{d\nu}$ then:

$$|Tf(\lambda)|^2 \le \sum_{k\in\mathbb{Z}} |f(\lambda + 2\pi k)|^2 \frac{d\nu_k}{d\nu}(\lambda)\,,$$

since $\sum_{k\in\mathbb{Z}} \frac{d\nu_k}{d\nu} = 1$ a.e. $[\nu]$ and then, by the monotone convergence Theorem:

$$\int_{[-\pi,\pi)} |Tf(\lambda)|^2 d\nu \le \int_{[-\pi,\pi)} \sum_{k\in\mathbb{Z}} |f(\lambda+2\pi k)|^2 \frac{d\nu_k}{d\nu}(\lambda) d\nu(\lambda)$$

$$\int_{[-\pi,\pi)} \sum_{k\in\mathbb{Z}} |f(\lambda+2\pi k)|^2 d\nu_k(\lambda) = \int_{\mathbb{R}} |f(\lambda)|^2 d\mu(\lambda)\,.$$

To check that T is onto, let's observe that if g is such that $g(\lambda) = g(\lambda + 2\pi)$ for every $\lambda \in \mathbb{R}$, then: $(Tg)(\lambda) = g(\lambda)$ for almost every $\lambda \in [-\pi,\pi)$ $[\mu]$. Thus, given $f \in L^2([-\pi,\pi), d\nu)$, let's define for $\lambda \in \mathbb{R}$:

$$g(\lambda) = \sum_{k\in\mathbb{Z}} f(\lambda - 2\pi k)\mathbf{1}_{I_k}(\lambda)\,,$$

with $I_k = [-\pi,\pi) + 2\pi k$. This g verifies that $(Tg)(\lambda) = g(\lambda) = f(\lambda)$ for almost all $[\mu]$ $\lambda \in [-\pi,\pi)$, and in particular $\|Tg\|_{L^2([-\pi,\pi],d\nu)} = \|g\|_{L^2(\mathbb{R},d\mu)}$, so that $\|T\|_{op} = 1$. ∎

Since the following chain rule for RN derivatives holds, see e.g. Theorem A p.133 [12]: $\frac{d\nu_k}{d\nu}\frac{d\nu}{d\lambda} = \frac{d\nu_k}{d\lambda}$, then, if μ has a support A and μ is absolutely continuous with respect to the Lebesgue measure with RN derivative given by $\frac{d\mu}{d\lambda} = \phi$, then, in this case, we have the following alternative expression for Tf:

$$(Tf)(\lambda) = \frac{\sum\limits_{k\in\mathbb{Z}} f(\lambda+2\pi k)\phi(\lambda+2\pi k)}{\sum\limits_{k\in\mathbb{Z}} \phi(\lambda+2\pi k)} \mathbf{1}_{\bigcup_k A+2\pi k}(\lambda) \qquad a.e. \tag{18}$$

Theorem VI.2. *Let $f \in L^2([-\pi,\pi), d\nu)$ then the formula given for $\lambda \in \mathbb{R}$ by:*

$$(Sf)(\lambda) = \sum_{k\in\mathbb{Z}} f(\lambda - 2\pi k)\mathbf{1}_{I_k}(\lambda)\,, \tag{19}$$

with $I_k = [-\pi,\pi) + 2\pi k$ defines a one to one linear mapping $S : L^2([-\pi,\pi), d\nu) \longrightarrow \mathcal{P}$, such that $\|S\|_{op} = 1$.

Remark: Alternatively, if $\varphi(\lambda) = \sum\limits_{k\in\mathbb{Z}} (\lambda - 2\pi k)\mathbf{1}_{I_k}(\lambda)$ then Sf coincides a.e. $[\mu]$ with $f \circ \varphi$.

Proof: From the definition of S we have $Ran(S) = \mathcal{P}$. On the other hand,

$$\|Sf\|^2_{L^2(\mathbb{R},d\mu)} = \int\limits_{\mathbb{R}} |f\circ\varphi(\lambda)|^2\, d\mu(\lambda)$$

$$= \int\limits_{[-\pi,\pi)} |f(\lambda)|^2 d\mu_\varphi(\lambda) = \int\limits_{[-\pi,\pi)} |f(\lambda)|^2 d\nu(\lambda)\,.$$

■

Theorem VI.3. *Let T and S be as in Theorems VI.1 and VI.2 respectively. Then $P = S\circ T : L^2(\mathbb{R}, d\mu) \longrightarrow L^2(\mathbb{R}, d\mu)$ is the orthogonal projection over $\mathcal{P}$.*

Proof: From the definition $P^2 = P$. The fact that $Ran(P) = \mathcal{P}$ follows from T being a surjective map and $Ran(S) = \mathcal{P}$, so one only has to prove that $\langle Pf, g\rangle = \langle f, Pg\rangle$.

$$\langle Pf, g\rangle = \int\limits_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} f(\varphi(\lambda)+2\pi k)\frac{d\nu_k}{d\nu}(\varphi(\lambda))\right)\overline{g}(\lambda)d\mu(\lambda)$$

$$= \sum_{j\in\mathbb{Z}} \int\limits_{I_j} \left(\sum_{k\in\mathbb{Z}} f(\varphi(\lambda)+2\pi k)\frac{d\nu_k}{d\nu}(\varphi(\lambda))\right)\overline{g}(\lambda)d\mu(\lambda)$$

$$= \sum_{j\in\mathbb{Z}} \int\limits_{[-\pi,\pi)} \left(\sum_{k\in\mathbb{Z}} f(\varphi(\lambda+2\pi j)+2\pi k)\frac{d\nu_k}{d\nu}(\varphi(\lambda+2\pi j))\right)$$

$$\ldots \overline{g}(\lambda+2\pi j)d\nu_j(\lambda)$$

(since $\varphi(\lambda) = \varphi(\lambda+2\pi j)$ for all λ and $\varphi(\lambda) = \lambda$ if $\lambda \in [-\pi,\pi)$)

$$= \sum_{j\in\mathbb{Z}} \int\limits_{[-\pi,\pi)} \left(\sum_{k\in\mathbb{Z}} f(\lambda+2\pi k)\frac{d\nu_k}{d\nu}(\lambda)\right)$$

$$\ldots \overline{g}(\lambda+2\pi j)\frac{d\nu_j}{d\nu}(\lambda)d\nu(\lambda)$$

$$= \sum_{k\in\mathbb{Z}} \int\limits_{[-\pi,\pi)} f(\lambda+2\pi k)\frac{d\nu_k}{d\nu}(\lambda)\left(\sum_{j\in\mathbb{Z}} \overline{g}(\lambda+2\pi j)\frac{d\nu_j}{d\nu}\right) d\nu(\lambda)\,.$$

Then, reversing the previous argument we obtain that the last equation equals $\langle f, Pg\rangle$. ■

Corollary VI.1. *Let T and S be as in Theorems VI.1 and VI.2 respectively. Then both restrictions: $T|_{\mathcal{P}} : \mathcal{P} \longrightarrow L^2([-\pi,\pi), d\nu)$ and $S|^{\mathcal{P}} : L^2([-\pi,\pi), d\nu) \longrightarrow \mathcal{P}$ define isometric isomorphisms.*

Finally, we recall a restatement of the main result of [17] (Theorem 1 there), which we will be useful in the following section.

Theorem VI.4. *If μ is finite Borel measure. $\mathcal{P} = L^2(\mathbb{R}, d\mu)$ if and only if there exists a support A of μ such that the translates $A + 2\pi k$ are mutually disjoint, $k \in \mathbb{Z}$.*

This result relates the problem of sampling to a condition of aliasing on the spectral measure of $\mathcal{X}$. A similar condition is used in e.g. [2], [16].

VII. Multichannel sampling of Continuous time processes.

Here, we give some results on multichannel sampling for, not necessarily band limited, multi band signals. The assumptions on μ and ν are the same as in the previous section. We shall see again, that the absolute continuities of μ and ν are necessary in the case of frame sequences.

Theorem VII.1. *Given $f_1, \ldots, f_m \in L^2(\mathbb{R}, d\mu)$, then: $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $\mathcal{P}$ if and only if $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $L^2([-\pi,\pi), d\nu)$.*

Proof: The result follows from corollary VI.1 and the fact that: $P(f_r e^{imn\,\cdot})(\lambda) = (S\circ T)(f_r e^{imn\,\cdot}) = (S\circ T)(f_r)(\lambda)e^{imn\lambda}$. The fact follows from:

$$T(f_r e^{imn\,\cdot})(\lambda) = T(f_r(\lambda)e^{imn\lambda})$$

$$= \sum_{k\in\mathbb{Z}} f_r(\lambda+2\pi k)e^{i(\lambda+2\pi k)mn}\frac{d\nu_k}{d\nu}(\lambda)$$

$$= e^{imn\lambda}\sum_{k\in\mathbb{Z}} f_r(\lambda+2\pi k)\frac{d\nu_k}{d\nu}(\lambda) = T(f_r)(\lambda)e^{imn\lambda}$$

and since $e^{imn\lambda} = e^{i\varphi(\lambda)mn}$ then:

$$(S\circ T)(f_r e^{imn\,\cdot})(\lambda) = S(T(f_r)e^{imn\,\cdot})(\lambda)$$

$$= T(f_r)(\varphi(\lambda))e^{i\varphi(\lambda)mn} = (Tf_r)(\varphi(\lambda))e^{imn\lambda}$$

$$= (S\circ T)(f_r)(\lambda)e^{imn\lambda} = P(f_r)(\lambda)e^{imn\lambda}$$

■

If in addition we assume that μ is absolutely continuous, then, we have the following corollary:

Corollary VII.1. *Given $f_1,\ldots,f_m \in L^2(\mathbb{R},d\mu)$ and μ a finite absolutely continuous measure, if A is the support of μ such that $\varphi_m^{-1}(\mathcal{A}) = \varphi(A)$ where $\mathcal{A} = \bigcap_{k=0}^{m-1}(\varphi(A)\cap I_k) - \frac{2\pi}{m}k$ and $I_k = [-\pi+\frac{2\pi}{m}k, -\pi+\frac{2\pi}{m}(k+1))$ then: $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $\mathcal{P}$ if and only if $det(\mathbf{F}(\lambda)) \neq 0$ a.e. on $\bigcup_{k\in\mathbb{Z}}[-\pi,-\pi+\frac{2\pi}{m})+2\pi k$, where: $(\mathbf{F}(\lambda))_{r\,j} = Pf_r\left(\lambda+\frac{2\pi}{m}(j-1)\right)$, $r,j = 1,\ldots,m$ and $\lambda\in[-\pi,-\pi+\frac{2\pi}{m}]$.*

Proof: Because of Theorem VII.1 the sequence $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $\mathcal{P}$ if and only if $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $L^2([-\pi,\pi],d\nu)$ with ν absolutely continuous by proposition VI.1. By Theorem IV.1 this holds if and only if $det(\widetilde{\mathbf{F}}(\lambda)) \neq 0$ a.e. on $\bigcap_{k=0}^{m-1}(\varphi(A)\cap I_k) - \frac{2\pi}{m}k$ where:

$$(\widetilde{\mathbf{F}}(\lambda))_{r\,j} = Tf_r\left(\lambda+\frac{2\pi}{m}(k-1)\right).$$

But if $\lambda\in[-\pi,-\pi+\frac{2\pi}{m})$:

$$\begin{aligned} Pf_r\left(\lambda+\frac{2\pi}{m}(k-1)\right) &= Tf_r\left(\varphi\left(\lambda+\frac{2\pi}{m}(k-1)\right)\right)\\ &= Tf_r\left(\varphi(\lambda)+\frac{2\pi}{m}(k-1)\right)\end{aligned}$$

Hence, if $det(\mathbf{F}(\lambda)) \neq 0$ a.e. on $\bigcup_{k\in\mathbb{Z}}[-\pi,-\pi+\frac{2\pi}{m})+2\pi k = \varphi^{-1}([-\pi,-\pi+\frac{2\pi}{m}))$, then $det(\widetilde{\mathbf{F}}(\lambda)) = det(\mathbf{F}(\varphi(\lambda))) \neq 0$ on $[-\pi,-\pi+\frac{2\pi}{m})$, and then $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is complete in $L^2([-\pi,\pi],d\nu)$. ■

Lemma VII.1. *Given $f_1,\ldots,f_m \in L^2(\mathbb{R},d\mu)$, then: $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame sequence for $\mathcal{P}$ if and only if $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame sequence for $L^2([-\pi,\pi),d\nu)$.*

Proof: Since $Id = P|_{\mathcal{P}} = S\circ T$ and $S|^{\mathcal{P}}$ is an isometric isomorphism one gets for each $r = 1,\ldots,m$:

$$\begin{aligned}\sum_{n\in\mathbb{Z}}|\langle Pf_r e^{imn\,\cdot},g\rangle|^2_{L^2(\mathbb{R},d\mu)} &= \sum_{n\in\mathbb{Z}}|\langle Pf_r e^{imn\,\cdot},Pg\rangle|^2_{L^2(\mathbb{R},d\mu)}\\ &= \sum_{n\in\mathbb{Z}}|\langle Tf_r e^{imn\,\cdot},Tg\rangle|^2_{L^2([-\pi,\pi],d\nu)}.\end{aligned}$$

On the other hand:

$$\|g\|_{L^2(\mathbb{R},d\mu)} = \|Tg\|_{L^2([-\pi,\pi],d\nu)},$$

and then the result follows directly from the Definition 1. ■

Form this we prove:

Corollary VII.2. *Given $f_1,\ldots,f_m \in L^2(\mathbb{R},d\mu)$ and μ a finite absolutely continuous measure, if A is the support of μ such that $A\bigcap A+2\pi k = \oslash$ for all $k\in\mathbb{Z}$, then: $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame for $L^2(\mathbb{R},d\mu)$ if there exist some constants $0 < C_1 \leq C_2$ such that $\sigma(\mathbf{H}(\phi)(\lambda)\mathbf{H}^H(\phi)(\lambda)) \subseteq [C_1,C_2]$ a.e. in $\left(\varphi^{-1}(\varphi_m(\varphi(A)))\right)\bigcap A$, where ϕ is the RN derivative of μ and*

$$(\mathbf{H}(\phi)(\lambda))_{r\,j} = f_r\left(\lambda+\frac{2\pi}{m}(j-1)\right)\sqrt{\phi\left(\lambda+\frac{2\pi}{m}(j-1)\right)}$$

for $k = 1,\ldots,m$.

Proof: By Proposition VI.1, ν is absolutely continuous with spectral density $\widetilde{\phi}$, and by Lemma VII.1 $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame sequence in $\mathcal{P}$ if and only if $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame sequence in $L^2([-\pi,\pi],d\nu)$. But, the condition on the support A (Theorem VI.4) implies that $\mathcal{P} = L^2(\mathbb{R},d\mu)$. Hence, we just need to check that $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\ldots,m}$ is a frame sequence in $L^2([-\pi,\pi],d\nu)$. Taking in account that $\widetilde{\phi}(\lambda) = \sum_{k\in\mathbb{Z}}\phi(\lambda+2\pi k)$ a.e., if $J_k = [-\pi,\pi)+2\pi k$ then $\widetilde{\phi}(\varphi(\lambda)) = \sum_{k\in\mathbb{Z}}\widetilde{\phi}(\lambda-2\pi k)\mathbf{1}_{J_k}(\lambda)$ and therefore, if $\lambda\in A$:

$$\begin{aligned}\widetilde{\phi}(\varphi(\lambda)) &= \sum_{k\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}}\phi(\lambda+2\pi(k-j))\right)\mathbf{1}_{J_k}(\lambda)\\ &= \phi(\lambda)\sum_{k\in\mathbb{Z}}\mathbf{1}_{J_k}(\lambda)+\sum_{k\neq j}\left(\sum\phi(\lambda+2\pi(k-j))\right)\mathbf{1}_{J_k}(\lambda)\\ &= \phi(\lambda),\end{aligned} \tag{20}$$

since $\sum_{k\neq j}\left(\sum\phi(\lambda+2\pi(k-j))\right)\mathbf{1}_{J_k}(\lambda) = 0$ if $\lambda\in A$. By Theorem V.2 it suffices to check that $\sigma(\widetilde{\mathbf{H}}(\widetilde{\phi})(\lambda)) \subseteq [C_1,C_2]$ a.e. on $\varphi_m(\phi(A))$, with

$$\begin{aligned}&(\widetilde{\mathbf{H}}(\widetilde{\phi})(\lambda))_{r\,j}\\ &= Tf_r\left(\lambda+\frac{2\pi}{m}(j-1)\right)\sqrt{\widetilde{\phi}\left(\lambda+\frac{2\pi}{m}(j-1)\right)}\end{aligned}$$

But if $\lambda\in\left(\varphi^{-1}(\varphi_m(\varphi(A)))\right)\subseteq\bigcup_{k\in\mathbb{Z}}[-\pi,-\pi+\frac{2\pi}{m})+2\pi k$ then, by (20):

$$\begin{aligned}&(\widetilde{\mathbf{H}}(\widetilde{\phi})(\varphi(\lambda)))_{r\,j}\\ &= Tf_r\left(\varphi(\lambda)+\frac{2\pi}{m}(j-1)\right)\sqrt{\widetilde{\phi}\left(\varphi(\lambda)+\frac{2\pi}{m}(j-1)\right)}\\ &= Tf_r\left(\varphi\left(\lambda+\frac{2\pi}{m}(j-1)\right)\right)\sqrt{\widetilde{\phi}\left(\varphi\left(\lambda+\frac{2\pi}{m}(j-1)\right)\right)}\\ &= Pf_r\left(\lambda+\frac{2\pi}{m}(j-1)\right)\sqrt{\phi\left(\lambda+\frac{2\pi}{m}(j-1)\right)}\\ &= (\mathbf{H}(\phi)(\lambda))_{r\,j},\end{aligned}$$

and then the result follows form this. ■

Finally, we prove that the existence of a spectral density, in this case over $\mathbb{R}$, is a necessary condition for $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\dots,m}$ to be a frame.

Theorem VII.2. *Given* $f_1,\dots,f_m \in L^2(\mathbb{R},d\mu)$, *if* $\left\{(Pf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\dots,m}$ *is a frame for* $L^2(\mathbb{R},d\mu)$ *then* μ *is absolutely continuous with respect to the Lebesgue measure, i.e. there exists a spectral density associated to the spectral measure* μ*.*

Proof: The hypothesis implies, by Lemma VII.1, that $\left\{(Tf_r)(\lambda)e^{imn\lambda}\right\}_{n\in\mathbb{Z}}^{r=1,\dots,m}$ is a frame for $L^2([-\pi,\pi),d\nu)$ and then by Theorem V.2 (or V.1) ν is absolutely continuous, and then Proposition VI.1 gives the desired result. ■

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