

FREE ALGEBRAS IN VARIETIES OF BL-ALGEBRAS GENERATED BY A BL_n -CHAIN

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Abstract

Free algebras with an arbitrary number of free generators in varieties of BL-algebras generated by one BL-chain that is an ordinal sum of a finite MV-chain L_n and a generalized BL-chain B are described in terms of weak Boolean products of BL-algebras that are ordinal sums of subalgebras of L_n and free algebras in the variety of basic hoops generated by B . The Boolean products are taken over the Stone spaces of the Boolean subalgebras of idempotents of free algebras in the variety of MV-algebras generated by L_n .

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Introduction

Basic Fuzzy Logic (BL for short) was introduced by Hájek (see [19] and the references given there) to formalize fuzzy logics in which the conjunction is interpreted by a continuous t-norm on the real segment $[0, 1]$ and the implication by its corresponding adjoint. He also introduced BL-algebras as the algebraic counterpart of these logics. BL-algebras form a variety (or equational class) of residuated lattices [19]. More precisely, they can be characterized as *bounded basic hoops* [1, 7]. Subvarieties of the variety of BL-algebras are in correspondence with axiomatic extensions of BL. Important examples of subvarieties of BL-algebras are MV-algebras (that correspond to Łukasiewicz many-valued logics, see [14]), linear Heyting algebras (that correspond to the superintuitionistic logic characterized by the axiom $(P \Rightarrow Q) \vee (Q \Rightarrow P)$, see [25] for a historical account about this logic), PL-algebras (that correspond to the

logic determined by the t-norm given by the ordinary product on $[0, 1]$, see [15]), and also Boolean algebras (that correspond to classical logic).

Since the propositions under BL equivalence form a free BL-algebra, descriptions of free algebras in terms of functions give concrete representations of these propositions. Such descriptions are known for some subvarieties of BL-algebras. The best known example is the representation of classical propositions by Boolean functions. Free MV-algebras have been described in terms of continuous piecewise linear functions by McNaughton [22] (see also [14]). Finitely generated free linear Heyting algebras were described by Horn [20], and a description of finitely generated free PL-algebras was given in [15]. Linear Heyting algebras and PL-algebras are examples of varieties of BL-algebras satisfying the *Boolean retraction property*. Free algebras in these varieties were completely described in [17].

In [10] the first author described the finitely generated free algebras in the varieties of BL-algebras generated by a single BL-chain which is an ordinal sum of a finite MV-chain \mathbf{L}_n and a generalized BL-chain \mathbf{B} . We call these chains BL_n -chains. The aim of this paper is to extend the results of [10] considering the case of infinitely many free generators. The results of [10] were heavily based on the fact that the Boolean subalgebras of finitely generated algebras in the varieties generated by BL_n -chains are finite. Therefore the methods of [10] cannot be applied to the general case.

As a preliminary step we characterize the Boolean algebra of idempotent elements of a free algebra in \mathcal{MV}_n , the variety of MV-algebras generated by the finite MV-chain \mathbf{L}_n . It is the free Boolean algebra over a poset which is the cardinal sum of chains of length $n - 1$. In the proof of this result a central role is played by the Moisil algebra reducts of algebras in \mathcal{MV}_n .

Free algebras in varieties of BL-algebras generated by a single BL_n -chain $\mathbf{L}_n \uplus \mathbf{B}$ are then described in terms of weak Boolean products of BL-algebras that are ordinal sums of subalgebras of \mathbf{L}_n and free algebras in the variety of basic hoops generated by \mathbf{B} . The Boolean products are taken over the Stone spaces of the Boolean algebras of idempotent elements of free algebras in \mathcal{MV}_n . An important intermediate step is the characterization of the variety of generalized BL-algebras generated by \mathbf{B} (Corollary 3.5).

The paper is organized as follows. In the first section we recall, for further reference, some basic notions on BL-algebras and on the varieties \mathcal{MV}_n . We also recall some facts about the representation of free algebras in varieties of BL-algebras as weak Boolean products. The only new result is given in Theorem 1.5. In Section 2, after giving the necessary background on Moisil algebra reducts of algebras in \mathcal{MV}_n , we characterize the Boolean algebras of idempotent elements of free algebras in \mathcal{MV}_n . These results are used in Section 3 to give the mentioned description of free algebras in the varieties of BL-algebras generated by a BL_n -chain. Finally in Section 4 we give some examples and we compare our results with those of [10] and [17].

1. Preliminaries

1.1. BL-algebras: basic notions A *hoop* [7] is an algebra $\mathbf{A} = (A, *, \rightarrow, \top)$ of type $(2, 2, 0)$, such that $(A, *, \top)$ is a commutative monoid and for all $x, y, z \in A$:

- (1) $x \rightarrow x = \top$,
- (2) $x * (x \rightarrow y) = y * (y \rightarrow x)$,
- (3) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$.

A *basic hoop* [1] or a *generalized BL-algebra* [18], is a hoop that satisfies the equation

$$(1.1) \quad (((x \rightarrow y) \rightarrow z) * ((y \rightarrow x) \rightarrow z)) \rightarrow z = \top.$$

It is shown in [1] that generalized BL-algebras can be characterized as algebras $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, \top)$ of type $(2, 2, 2, 2, 0)$ such that

- (1) $(A, *, \top)$, is an commutative monoid,
- (2) $\mathbf{L}(\mathbf{A}) := (A, \wedge, \vee, \top)$, is a lattice with greatest element \top ,
- (3) $x \rightarrow x = \top$,
- (4) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z$,
- (5) $x \wedge y = x * (x \rightarrow y)$,
- (6) $(x \rightarrow y) \vee (y \rightarrow x) = \top$.

A *BL-algebra* or *bounded basic hoop* is a bounded generalized BL-algebra, that is, it is an algebra $\mathbf{A} = (A, \wedge, \vee, *, \rightarrow, \perp, \top)$ of type $(2, 2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, *, \rightarrow, \top)$ is a generalized BL-algebra, and \perp is the lower bound of $\mathbf{L}(\mathbf{A})$. In this case, we define the unary operation \neg by the equation $\neg x = x \rightarrow \perp$. The BL-algebra with only one element, that is, $\perp = \top$, is called the *trivial BL-algebra*. The varieties of BL-algebras and of generalized BL-algebras will be denoted by \mathcal{BL} and \mathcal{GBL} , respectively.

In every generalized BL-algebra \mathbf{A} we denote by \leq the (partial) order defined on A by the lattice $\mathbf{L}(\mathbf{A})$, that is, for $a, b \in A$, $a \leq b$ if and only if $a = a \wedge b$ if and only if $b = a \vee b$. This order is called the *natural order* of \mathbf{A} . When this natural order is total (that is, for each $a, b \in A$, $a \leq b$ or $b \leq a$), we say that \mathbf{A} is a *generalized BL-chain* (*BL-chain* in case \mathbf{A} is a BL-algebra). The following theorem makes obvious the importance of BL-chains and can be easily derived from [19, Lemma 2.3.16].

THEOREM 1.1. *Each BL-algebra is a subdirect product of BL-chains.*

In every BL-algebra \mathbf{A} we define a binary operation $x \oplus y = \neg(\neg x * \neg y)$. For each positive integer k , the operations x^k and kx are inductively defined as follows:

- (a) $x^1 = x$ and $x^{k+1} = x^k * x$,
- (b) $1x = x$ and $(k+1)x = (kx) \oplus x$.

MV-algebras, the algebras of Łukasiewicz infinite-valued logic, form a subvariety of \mathcal{BL} , which is characterized by the equation $\neg\neg x = x$ (see [19]). The variety of MV-algebras is denoted by \mathcal{MV} . Totally ordered MV-algebras are called *MV-chains*. For each BL-algebra \mathbf{A} , the set

$$MV(\mathbf{A}) := \{x \in A : \neg\neg x = x\}$$

is the universe of a subalgebra $\mathbf{MV}(\mathbf{A})$ of \mathbf{A} which is an MV-algebra (see [18]).

A *PL-algebra* is a BL-algebra that satisfies the two axioms:

- (1) $(\neg\neg z * ((x * z) \rightarrow (y * z))) \rightarrow (x \rightarrow y) = \top$,
- (2) $x \wedge \neg x = \perp$.

PL-algebras correspond to *product fuzzy logic*, see [15] and [19].

It follows from Theorem 1.1 that for each BL-algebra \mathbf{A} the lattice $\mathbf{L}(\mathbf{A})$ is distributive. The complemented elements of $L(\mathbf{A})$ form a subalgebra $\mathbf{B}(\mathbf{A})$ of \mathbf{A} which is a Boolean algebra. Elements of $B(\mathbf{A})$ are called *Boolean elements* of \mathbf{A} .

1.2. Implicative filters

DEFINITION 1.2. An *implicative filter* of a BL-algebra \mathbf{A} is a subset $F \subseteq A$ satisfying the conditions

- (1) $\top \in F$.
- (2) If $x \in F$ and $x \rightarrow y \in F$, then $y \in F$.

An implicative filter is called *proper* provided that $F \neq A$. If W is a subset of a BL-algebra \mathbf{A} , the implicative filter generated by W will be denoted by $\langle W \rangle$. If U is a filter of the Boolean subalgebra $\mathbf{B}(\mathbf{A})$, then the implicative filter $\langle U \rangle$ is called *Stone filter* of \mathbf{A} . An implicative filter F of a BL-algebra \mathbf{A} is called *maximal* if and only if it is proper and no proper implicative filter of \mathbf{A} strictly contains F .

Implicative filters characterize congruences in BL-algebras. Indeed, if F is an implicative filter of a BL-algebra \mathbf{A} it is well known (see [19, Lemma 2.3.14]), that the binary relation \equiv_F on A defined by

$$x \equiv_F y \quad \text{if and only if} \quad x \rightarrow y \in F \quad \text{and} \quad y \rightarrow x \in F$$

is a congruence of \mathbf{A} . Moreover, $F = \{x \in A : x \equiv_F \top\}$. Conversely, if \equiv is a congruence relation on A , then $\{x \in A : x \equiv \top\}$ is an implicative filter, and $x \equiv y$ if and only if $x \rightarrow y \equiv \top$ and $y \rightarrow x \equiv \top$. Therefore, the correspondence $F \mapsto \equiv_F$ is a bijection from the set of implicative filters of \mathbf{A} onto the set of congruences of \mathbf{A} .

LEMMA 1.3 (see [17]). *Let \mathbf{A} be a BL-algebra, and let F be a filter of $\mathbf{B}(\mathbf{A})$. Then $(\equiv_F) = \{(a, b) \in A \times A : a \wedge c = b \wedge c \text{ for some } c \in F\}$ is a congruence relation on \mathbf{A} that coincides with the congruence relation given by the implicative filter $\langle F \rangle$ generated by F .*

1.3. MV_n -algebras For $n \geq 2$, we define:

$$L_n = \left\{ \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-1}{n-1} \right\}.$$

The set L_n equipped with the operations $x * y = \max(0, x + y - 1)$, $x \rightarrow y = \min(1, 1 - x + y)$, and with $\perp = 0$ defines a finite MV -algebra, which shall be denoted by \mathbf{L}_n . Clearly $B(\mathbf{L}_n) = \{0, 1\}$.

A BL-algebra \mathbf{A} is said to be *simple* provided it is nontrivial and the only proper implicative filter of \mathbf{A} is the singleton $\{\top\}$. In [14], it is proved that \mathbf{L}_n is a simple MV -algebra for each integer n .

We shall denote by \mathcal{MV}_n the subvariety of \mathcal{MV} generated by \mathbf{L}_n . The elements of \mathcal{MV}_n are called *MV_n -algebras*. A finite MV -chain \mathbf{L}_m belongs to \mathcal{MV}_n if and only if $m - 1$ is a divisor of $n - 1$. Therefore it is not hard to corroborate that every MV_n -algebra is a subdirect product of a family of algebras $(\mathbf{L}_{m_i}, i \in I)$ where $m_i - 1$ divides $n - 1$ for each $i \in I$.

It can be deduced from [14, Corollary 8.2.4 and Theorem 8.5.1] that \mathcal{MV}_n is the proper subvariety of \mathcal{MV} characterized by the equations

$$(\alpha_n) \quad x^{(n-1)} = x^n,$$

and if $n \geq 4$, for every integer $p = 2, \dots, n - 2$ that does not divide $n - 1$

$$(\beta_n) \quad (p x^{p-1})^n = n x^p.$$

If \mathbf{A} is an MV_n -algebra, it is not hard to verify that for each $x \in A \setminus \{\top\}$, $x^n = \perp$ and for each $y \in A \setminus \{\perp\}$, $n y = \top$.

1.4. Ordinal sum and decomposition of BL-chains Let $\mathbf{R} = (R, *_R, \rightarrow_R, \top)$ and $\mathbf{S} = (S, *_S, \rightarrow_S, \top)$ be two hoops such that $R \cap S = \{\top\}$. Following [7] we can define the *ordinal sum* $\mathbf{R} \uplus \mathbf{S}$ of these two hoops as the hoop given by $(R \cup S, *, \rightarrow, \top)$ where the operations $(*, \rightarrow)$ are defined as follows:

$$x * y = \begin{cases} x *_R y & \text{if } x, y \in R, \\ x *_S y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \rightarrow y = \begin{cases} \top & \text{if } x \in R \setminus \{\top\}, y \in S, \\ x \rightarrow_R y & \text{if } x, y \in R, \\ x \rightarrow_S y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

If $R \cap S \neq \{\top\}$, \mathbf{R} and \mathbf{S} can be replaced by isomorphic copies whose intersection is $\{\top\}$, thus their ordinal sum can be defined. When \mathbf{R} is a generalized BL-chain and \mathbf{S} is a generalized BL-algebra, the hoop resulting from their ordinal sum satisfies equation (1.1). Thus $\mathbf{R} \uplus \mathbf{S}$ is a generalized BL-algebra. Moreover, if \mathbf{R} is a BL-chain, then $\mathbf{R} \uplus \mathbf{S}$ is a BL-algebra, where $\perp = \perp_{\mathbf{R}}$. If S is totally ordered it is obvious that the chain $\mathbf{R} \uplus \mathbf{S}$ is subdirectly irreducible if and only if \mathbf{S} is subdirectly irreducible. Notice also that for any generalized BL-algebra \mathbf{S} , $\mathbf{L}_2 \uplus \mathbf{S}$ is the BL-algebra that arises from adjoining a bottom element to \mathbf{S} .

Given a BL-algebra \mathbf{A} , we can consider the set $D(\mathbf{A}) := \{x \in \mathbf{A} : \neg x = \perp\}$. It is shown in [18], that $\mathbf{D}(\mathbf{A}) = (D(\mathbf{A}), \wedge, \vee, *, \rightarrow, \top)$ is a generalized BL-algebra.

THEOREM 1.4 (see [10]). *For each BL-chain \mathbf{A} , we have that $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$.*

THEOREM 1.5. *Let \mathbf{A} be a BL-algebra such that $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_n$ for some integer n . Then $\mathbf{A} \cong \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A}) \cong \mathbf{L}_n \uplus \mathbf{D}(\mathbf{A})$.*

PROOF. From Theorem 1.1, we can think of each non trivial BL-algebra \mathbf{A} as a subdirect product of a family $(\mathbf{A}_i, i \in I)$ of non trivial BL-chains, that is, there exists an embedding $e : \mathbf{A} \rightarrow \prod_{i \in I} \mathbf{A}_i$, such that $\pi_i(e(\mathbf{A})) = \mathbf{A}_i$ for each $i \in I$, where π_i denotes each projection. We shall identify \mathbf{A} with $e(\mathbf{A})$. Then each element of A is a tuple \mathbf{x} and coordinate i is $x_i \in \mathbf{A}_i$. With this notation we have that for each $\mathbf{x} \in A$, $\pi_i(\mathbf{x}) = x_i$. We will prove the following items:

(1) *For each $i \in I$, $\mathbf{MV}(\mathbf{A}_i)$ is isomorphic to \mathbf{L}_n .*

Since for each $i \in I$, π_i is a homomorphism and $\pi_i(MV(\mathbf{A})) \subseteq \mathbf{A}_i$, we have that $\pi_i(MV(\mathbf{A})) \subseteq MV(\mathbf{A}_i)$. Then $\pi_i(\mathbf{MV}(\mathbf{A}))$ is a subalgebra of $\mathbf{MV}(\mathbf{A}_i)$. On the other hand, given $i \in I$, let $x_i \in MV(\mathbf{A}_i)$. Then $\neg \neg x_i = x_i$ and there exists an element $\mathbf{x} \in A$ such that $\pi_i(\mathbf{x}) = x_i$. Taking $\mathbf{y} = \neg \neg \mathbf{x} \in MV(\mathbf{A})$ we have that $\pi_i(\mathbf{y}) = x_i$ and $x_i \in \pi_i(MV(\mathbf{A}))$. Hence $MV(\mathbf{A}_i) \subseteq \pi_i(MV(\mathbf{A}))$.

In conclusion $\mathbf{MV}(\mathbf{A}_i) = \pi_i(\mathbf{MV}(\mathbf{A})) = \pi_i(\mathbf{L}_n) = \mathbf{L}_n$, because \mathbf{L}_n is simple.

(2) *If $\mathbf{x} \in A$, then $\mathbf{x} \in MV(\mathbf{A}) \cup D(\mathbf{A})$.*

Let $\mathbf{x} \in A$ and let $\mathbf{y} = n(\neg \mathbf{x})$. If $x_i \in L_n \setminus \{\top\}$, then $\neg x_i \in L_n \setminus \{\perp\}$. From equation (α_n) we obtain that $y_i = n(\neg x_i) = \top$. On the other hand, if $\neg x_i = \perp$, then $y_i = n(\neg x_i) = \perp$. Now let $\mathbf{z} = (\neg \neg \mathbf{x})^n$. If $x_i \in L_n \setminus \{\top\}$, then $z_i = \perp$, but if $\neg \neg x_i = \top$, then $z_i = \top$.

Suppose there exists $\mathbf{x} \in A$ such that $\mathbf{x} \notin MV(\mathbf{A})$ and $\mathbf{x} \notin D(\mathbf{A})$. It follows from Theorem 1.4 that for each $i \in I$, $\mathbf{A}_i = \mathbf{MV}(\mathbf{A}_i) \uplus \mathbf{D}(\mathbf{A}_i)$. Then there exist $i, j \in I$, such that $x_i \in MV(\mathbf{A}_i) \setminus \{\top\} = L_n \setminus \{\top\}$ and $x_j \in D(\mathbf{A}_j) \setminus \{\top\}$.

Let $\mathbf{y} = n(\neg \mathbf{x})$. Then $y_i = \top$, $y_j = \perp$, and $y_k \in \{\perp, \top\}$ for each $k \in I \setminus \{i, j\}$. Now let $\mathbf{z} = (\neg \neg \mathbf{x})^n$. We have that $z_j = \top$, $z_i = \perp$, and $z_k \in \{\perp, \top\}$ for each

$k \in I \setminus \{i, j\}$. It follows that \mathbf{y} and \mathbf{z} are elements in the chain $MV(\mathbf{A}) = L_n$, which are not comparable, and this is a contradiction.

(3) *If $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ and $\mathbf{y} \in D(\mathbf{A})$, then $\mathbf{x} < \mathbf{y}$.*

The statement is clear if $x_i \in MV(\mathbf{A}_i) \setminus \{\top\}$ for every $i \in I$ or if $y_i = \top$ for each $i \in I$. Otherwise, suppose $x_i = \top$ for some $i \in I$. Since $\mathbf{x} \neq \top$, there must exist $j \in I$ such that $x_j \neq \top$. If $y_i = \top$ for each $i \in I$ such that $x_i = \top$, then $\mathbf{x} < \mathbf{y}$. If not, let $\mathbf{z} = \mathbf{x} \wedge \mathbf{y}$. Since operations are coordinatewise, $z_j \in MV(\mathbf{A}_j) \setminus \{\top\}$ and $z_i \in D(\mathbf{A}_i) \setminus \{\top\}$, for some $i \in I$. Hence $\mathbf{z} \notin MV(\mathbf{A})$ and $\mathbf{z} \notin D(\mathbf{A})$, contradicting the previous item.

(4) *If $\mathbf{x} \in MV(\mathbf{A}) \setminus \{\top\}$ and $\mathbf{y} \in D(\mathbf{A})$, then $\mathbf{y} \rightarrow \mathbf{x} = \mathbf{x}$ and $\mathbf{y} * \mathbf{x} = \mathbf{x}$.*

Since $\neg \mathbf{y} = \perp$ we have that

$$\begin{aligned} \mathbf{y} \rightarrow \mathbf{x} &= \mathbf{y} \rightarrow \neg \neg \mathbf{x} = \mathbf{y} \rightarrow (\neg \mathbf{x} \rightarrow \perp) = \neg \mathbf{x} \rightarrow (\mathbf{y} \rightarrow \perp) \\ &= \neg \mathbf{x} \rightarrow \perp = \neg \neg \mathbf{x} = \mathbf{x}, \end{aligned}$$

and

$$\mathbf{x} = \mathbf{y} \wedge \mathbf{x} = \mathbf{y} * (\mathbf{y} \rightarrow \mathbf{x}) = \mathbf{y} * \mathbf{x}.$$

From the previous items it follows that $\mathbf{A} \cong MV(\mathbf{A}) \uplus D(\mathbf{A}) = L_n \uplus D(\mathbf{A})$. □

1.5. Free algebras in varieties of BL-algebras generated by a BL_n -chain Recall that an algebra \mathbf{A} in a variety \mathcal{K} is said to be *free over a set Y* if and only if for every algebra \mathbf{C} in \mathcal{K} and every function $f : Y \rightarrow \mathbf{C}$, f can be uniquely extended to a homomorphism of \mathbf{A} into \mathbf{C} . Given a variety \mathcal{K} of algebras, we denote by $\mathbf{Free}_{\mathcal{K}}(X)$ the free algebra in \mathcal{K} over X . As mentioned in the introduction, we define a BL_n -chain as a BL-chain that is an ordinal sum of the MV-chain L_n and a generalized BL-chain. Once we fixed the generalized BL-chain \mathbf{B} , we study the free algebra $\mathbf{Free}_{\mathcal{V}}(X)$, where \mathcal{V} is the variety of BL-algebras generated by the BL_n -chain

$$\mathbf{T}_n := L_n \uplus \mathbf{B}.$$

Notice that $MV(\mathbf{T}_n) \cong L_n$ and if $x \notin MV(\mathbf{T}_n) \setminus \{\top\}$, then $x \in D(\mathbf{T}_n) = \mathbf{B}$.

Recall that a *weak Boolean product* of a family $(A_y, y \in Y)$ of algebras over a Boolean space Y is a subdirect product \mathbf{A} of the given family such that the following conditions hold:

- (1) If $a, b \in A$, then $[a = b] = \{y \in Y : a_y = b_y\}$ is open.
- (2) If $a, b \in A$ and Z is a clopen in X , then $a|_Z \cup b|_{X \setminus Z} \in A$.

Since the variety \mathcal{BL} is congruence distributive, it has the Boolean Factor Congruence property. Therefore each nontrivial BL-algebra can be represented as a weak Boolean product of directly indecomposable BL-algebras (see [5] and [23]). The

explicit representation of each BL-algebra as a weak Boolean product of directly indecomposable algebras is given in [17] by the following lemma.

LEMMA 1.6. *Let \mathbf{A} be a BL-algebra and let $\text{Sp } \mathbf{B}(\mathbf{A})$ be the Boolean space of ultrafilters of the Boolean algebra $\mathbf{B}(\mathbf{A})$. The correspondence*

$$a \mapsto (a/\langle U \rangle)_{U \in \text{Sp } \mathbf{B}(\mathbf{A})}$$

gives an isomorphism of \mathbf{A} onto the weak Boolean product of the family

$$(\mathbf{A}/\langle U \rangle) : U \in \text{Sp } \mathbf{B}(\mathbf{A})$$

over the Boolean space $\text{Sp } \mathbf{B}(\mathbf{A})$. This representation is called the Pierce representation. Any other representation of \mathbf{A} as a weak Boolean product of a family of directly indecomposable algebras is equivalent to the Pierce representation.

Therefore, to describe $\mathbf{Free}_V(X)$ we need to describe $\mathbf{B}(\mathbf{Free}_V(X))$ and the quotients $\mathbf{Free}_V(X)/\langle U \rangle$ for each $U \in \text{Sp } \mathbf{B}(\mathbf{Free}_V(X))$.

In Section 2 we obtain a characterization of the Boolean algebra $\mathbf{B}(\mathbf{Free}_V(X))$. Once this aim is achieved, we consider the quotients $\mathbf{Free}_V(X)/\langle U \rangle$.

2. $\mathbf{B}(\mathbf{Free}_V(X))$

The next two results can be found in [18].

THEOREM 2.1. *For each BL-algebra \mathbf{A} , $\mathbf{B}(\mathbf{A}) \cong \mathbf{B}(\mathbf{MV}(\mathbf{A}))$.*

THEOREM 2.2. *For each variety \mathcal{K} of BL-algebras and each set X*

$$\mathbf{MV}(\mathbf{Free}_{\mathcal{K}}(X)) \cong \mathbf{Free}_{\mathcal{MV} \cap \mathcal{K}}(\neg \neg X).$$

THEOREM 2.3. *$\mathcal{V} \cap \mathcal{MV}$ is the variety \mathcal{MV}_n .*

PROOF. Since $\mathbf{L}_n \cong \mathbf{MV}(\mathbf{T}_n)$ is in $\mathcal{V} \cap \mathcal{MV}$, we have that $\mathcal{MV}_n \subseteq \mathcal{V} \cap \mathcal{MV}$. On the other hand, let \mathbf{A} be an MV-algebra in $\mathcal{V} \cap \mathcal{MV}$. Suppose \mathbf{A} is not in \mathcal{MV}_n . Then there exists an equation $e(x_1, \dots, x_p) = \top$ that is satisfied by \mathbf{L}_n and is not satisfied by \mathbf{A} , that is, there exist elements a_1, \dots, a_p in \mathbf{A} such that $e(a_1, \dots, a_p) \neq \top$. Since $(\neg \neg b_1, \dots, \neg \neg b_p)$ is in $(L_n)^p$, for each tuple (b_1, \dots, b_p) in $(T_n)^p$, the equation $e'(x_1, \dots, x_p) = e(\neg \neg x_1, \dots, \neg \neg x_p) = \top$ is satisfied in \mathcal{V} . Since $\mathbf{A} \in \mathcal{V} \cap \mathcal{MV}$, it follows that $\top = e'(a_1, \dots, a_p) = e(\neg \neg a_1, \dots, \neg \neg a_p) = e(a_1, \dots, a_p) \neq \top$, a contradiction. Hence $\mathcal{MV}_n = \mathcal{V} \cap \mathcal{MV}$. \square

From these results we obtain the following theorem.

THEOREM 2.4. $\mathbf{B}(\mathbf{Free}_V(X)) \cong \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg \neg X))$.

2.1. n-valued Moisil algebras Boolean elements of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ depend on some operators that can be defined on each \mathcal{MV}_n -algebra. Such operators provide each \mathcal{MV}_n -algebra with an n -valued Moisil algebra structure, in the sense of the following definition.

DEFINITION 2.5. For each integer $n \geq 2$, an n -valued Moisil algebra ([8, 11]) or n -valued Łukasiewicz algebra ([4, 12, 13]) is an algebra

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$$

of type $(2, 2, 1, \dots, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a distributive lattice with unit 1 and zero 0, and $\neg, \sigma_1^n, \dots, \sigma_{n-1}^n$ are unary operators defined on A that satisfy the following conditions:

- (1) $\neg\neg x = x$,
- (2) $\neg(x \vee y) = \neg x \wedge \neg y$,
- (3) $\sigma_i^n(x \vee y) = \sigma_i^n x \vee \sigma_i^n y$,
- (4) $\sigma_i^n x \vee \neg\sigma_i^n x = 1$,
- (5) $\sigma_i^n \sigma_j^n x = \sigma_j^n x$, for $i, j = 1, 2, \dots, n-1$,
- (6) $\sigma_i^n(\neg x) = \neg(\sigma_{n-i}^n x)$,
- (7) $\sigma_i^n x \vee \sigma_{i+1}^n x = \sigma_{i+1}^n x$, for $i = 1, 2, \dots, n-2$,
- (8) $x \vee \sigma_{n-1}^n x = \sigma_{n-1}^n x$,
- (9) $(x \wedge \neg\sigma_i^n x \wedge \sigma_{i+1}^n y) \vee y = y$, for $i = 1, 2, \dots, n-2$.

Properties and examples of n -valued Moisil algebras can be found in [4] and [8]. The variety of n -valued Moisil algebras will be denoted \mathcal{M}_n .

THEOREM 2.6 (see [11]). Let \mathbf{A} be in \mathcal{M}_n . Then $x \in B(\mathbf{A})$ if and only if

$$\sigma_{n-1}^n(x) = x.$$

Furthermore,

$$\sigma_{n-1}^n(x) = \min\{b \in B(\mathbf{A}) : x \leq b\} \quad \text{and} \quad \sigma_1^n(x) = \max\{a \in B(\mathbf{A}) : a \leq x\}.$$

DEFINITION 2.7. For each integer $n \geq 2$, a Post algebra of order n is a system

$$\mathbf{A} = (A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, e_1, \dots, e_{n-1}, 0, 1)$$

such that $(A, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$ is an n -valued Moisil algebra and e_1, \dots, e_{n-1} are constants that satisfy the equations:

$$\sigma_i^n(e_j) = \begin{cases} 0 & \text{if } i + j < n, \\ 1 & \text{if } i + j \geq n. \end{cases}$$

For every $n \geq 2$, we can define one-variable terms $\sigma_1^n(x), \dots, \sigma_{n-1}^n(x)$ in the language $(\neg, \rightarrow, \top)$ such that evaluated on the algebras \mathbf{L}_n give

$$\sigma_i^n\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n, \end{cases}$$

for $i = 1, \dots, n-1$ (see [13] or [24]). It is easy to check that

$$\mathbf{M}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$$

is a n -valued Moisil algebra. Since these algebras are defined by equations and \mathbf{L}_n generates the variety \mathcal{MV}_n , we have that each $\mathbf{A} \in \mathcal{MV}_n$ admits a structure of an n -valued Moisil algebra, denoted by $\mathbf{M}(\mathbf{A})$. The chain $\mathbf{M}(\mathbf{L}_n)$ plays a very important role in the structure of n -valued Moisil algebras, since each n -valued Moisil algebra is a subdirect product of subalgebras of $\mathbf{M}(\mathbf{L}_n)$ (see [4] or [12]). If we add to the structure $\mathbf{M}(\mathbf{L}_n)$ the constants $e_i = i/(n-1)$, for $i = 1, \dots, n-1$, then $\mathbf{PT}(\mathbf{L}_n) = (L_n, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, e_1, \dots, e_{n-1}, 0, 1)$ is a Post algebra.

Not every n -valued Moisil algebra has a structure of \mathbf{MV}_n -algebra (see [21]). For example, a subalgebra of $\mathbf{M}(\mathbf{L}_n)$ may not be a subalgebra of \mathbf{L}_n as \mathbf{MV}_n -algebra. For instance, the set

$$C = \left\{ \frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{4}{4} \right\}$$

is the universe of a subalgebra of $\mathbf{M}(\mathbf{L}_5)$, but not the universe of a subalgebra of \mathbf{L}_5 . On the other hand, every Post algebra has a structure of \mathbf{MV}_n -algebra (see [24, Theorem 10]).

The next example will play an important role in what follows.

EXAMPLE 2.8. Let $\mathbf{C} = (C, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. We define

$$C^{[n]} := \{\mathbf{z} = (z_1, \dots, z_{n-1}) \in C^{n-1} : z_1 \leq z_2 \leq \dots \leq z_{n-1}\}.$$

For each $\mathbf{z} = (z_1, \dots, z_{n-1}) \in C^{[n]}$, we define

$$\neg_n \mathbf{z} = (\neg z_{n-1}, \dots, \neg z_1),$$

$$\mathbf{0} = (0, \dots, 0),$$

$$\mathbf{1} = (1, \dots, 1),$$

$$\sigma_i^n(\mathbf{z}) = (z_i, z_i, \dots, z_i) \quad \text{for } i = 1, \dots, n-1.$$

With \wedge and \vee defined coordinatewise, $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \dots, \sigma_{n-1}^n, \mathbf{0}, \mathbf{1})$ is an n -valued Moisil algebra (see [8, Chapter 3, Example 1.10]). If we define $\mathbf{e}_j = (e_{j,1}, \dots, e_{j,n-1})$ by

$$e_{j,i} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i \geq j, \end{cases}$$

then $\mathbf{C}^{[n]} = (C^{[n]}, \wedge, \vee, \neg_n, \sigma_1^n, \dots, \sigma_{n-1}^n, \mathbf{e}_1, \dots, \mathbf{e}_{n-1}, \mathbf{0}, \mathbf{1})$ is a Post algebra. Consequently, $\mathbf{C}^{[n]}$ has a structure of \mathbf{MV}_n -algebra.

It is easy to see that for each \mathbf{MV}_n -algebra \mathbf{A} , $\mathbf{B}(\mathbf{A}) = \mathbf{B}(\mathbf{M}(\mathbf{A}))$.

We need to show that the Boolean elements of the \mathbf{MV}_n -algebra generated by a set G coincide with the Boolean elements of the n -valued Moisil algebra generated by the same set. In order to prove this result it is convenient to consider the following operators on each n -valued Moisil algebra \mathbf{A} . For each $i = 0, \dots, n-1$,

$$J_i(x) = \sigma_{n-i}^n(x) \wedge \neg \sigma_{n-i-1}^n(x),$$

where $\sigma_0^n(x) = 0$ and $\sigma_n^n(x) = 1$. In $\mathbf{M}(\mathbf{L}_n)$ we have

$$J_i\left(\frac{j}{(n-1)}\right) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

LEMMA 2.9. *Let \mathbf{A} be an \mathbf{MV}_n -algebra, and let $G \subset A$. If $\langle G \rangle_{\mathbf{MV}_n}$ is the subalgebra of \mathbf{A} generated by the set G and $\langle G \rangle_{\mathbf{M}_n}$ is the subalgebra of $\mathbf{M}(\mathbf{A})$ generated by G , then $\mathbf{B}(\langle G \rangle_{\mathbf{MV}_n}) = \mathbf{B}(\langle G \rangle_{\mathbf{M}_n})$.*

PROOF. Since $\langle G \rangle_{\mathbf{M}_n}$ is always a subalgebra of $\mathbf{M}(\langle G \rangle_{\mathbf{MV}_n})$, we have that $\mathbf{B}(\langle G \rangle_{\mathbf{M}_n})$ is a subalgebra of $\mathbf{B}(\langle G \rangle_{\mathbf{MV}_n})$.

We will see that $\mathbf{B}(\langle G \rangle_{\mathbf{MV}_n}) \subseteq \mathbf{B}(\langle G \rangle_{\mathbf{M}_n})$. The case $G = \emptyset$ is clear. Suppose that G is a finite set of cardinality $p \geq 1$. Since \mathbf{MV}_n -algebras are locally finite (see [9, Chapter II, Theorem 10.16]), we obtain that $\langle G \rangle_{\mathbf{MV}_n}$ is a finite \mathbf{MV}_n -algebra. Since finite \mathbf{MV}_n -algebras are direct products of simple algebras, there exists a finite $k \geq 1$ such that $\langle G \rangle_{\mathbf{MV}_n} = \prod_{i=1}^k \mathbf{L}_{m_i}$, where each $m_i - 1$ divides $n - 1$, for each $i = 1, \dots, k$. If $k = 1$, then $\langle G \rangle_{\mathbf{M}_n}$ and $\langle G \rangle_{\mathbf{MV}_n}$ are finite chains whose only Boolean elements are their extremes. Otherwise, we can think of the elements of $\langle G \rangle_{\mathbf{MV}_n}$ as k -tuples, that is, if $\mathbf{x} \in \langle G \rangle_{\mathbf{MV}_n}$, then $\mathbf{x} = (x_1, \dots, x_k)$. We shall denote by $\mathbf{1}^j$ the k -tuple given by

$$(\mathbf{1}^j)_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

It is clear that for each $j = 1, \dots, k$, $\mathbf{1}^j$ is in $\langle G \rangle_{\mathbf{MV}_n}$. From this it follows that for every pair $i \neq j$, $i, j \in \{1, \dots, k\}$, there exists an element $\mathbf{x} \in G$ such that $x_j \neq x_i$. Indeed, suppose on the contrary that there exist $i, j \leq k$ such that $x_i = x_j$, for every $\mathbf{x} \in G$. Then for every $\mathbf{z} \in \langle G \rangle_{\mathbf{MV}_n}$ we would have $z_j = z_i$ contradicting the fact that $\mathbf{1}^i$ is in $\langle G \rangle_{\mathbf{MV}_n}$.

To see that every Boolean element in $\langle G \rangle_{\mathbf{MV}_n}$ is also in $\langle G \rangle_{\mathbf{M}_n}$, it is enough to prove that $\mathbf{1}^j$ is in $\langle G \rangle_{\mathbf{M}_n}$ for every $j = 1, \dots, k$. For a fixed j , for each $i \neq j$,

$i = 1, \dots, k$, we choose $\mathbf{x}^i \in G$ such that $x_j^i \neq x_i^i$. Let j_i be the numerator of $x_j^i \in L_n$. It is not hard to verify that

$$\mathbf{1}^j = \bigwedge_{i=1, i \neq j}^k J_{j_i}(\mathbf{x}^i).$$

Therefore $\mathbf{1}^j \in \langle G \rangle_{\mathcal{M}_n}$ and $B(\langle G \rangle_{\mathcal{M}\mathcal{V}_n}) \subseteq B(\langle G \rangle_{\mathcal{M}_n})$.

If G is not finite, let \mathbf{y} be a Boolean element in $\langle G \rangle_{\mathcal{M}\mathcal{V}_n}$. Hence, there exists a finite subset $G_{\mathbf{y}}$ of G such that \mathbf{y} belongs to the subalgebra of $\langle G \rangle_{\mathcal{M}\mathcal{V}_n}$ generated by $G_{\mathbf{y}}$. Therefore, since \mathbf{y} is Boolean, \mathbf{y} belongs to the subalgebra of $\langle G \rangle_{\mathcal{M}_n}$ generated by $G_{\mathbf{y}}$, and we conclude that $B(\langle G \rangle_{\mathcal{M}\mathcal{V}_n}) \subseteq B(\langle G \rangle_{\mathcal{M}_n})$ for all sets G . \square

Given an algebra \mathbf{A} in a variety \mathcal{K} , a subalgebra \mathbf{S} of \mathbf{A} , and an element $x \in A$, we shall denote by $\langle \mathbf{S}, x \rangle_{\mathcal{K}}$ the subalgebra of \mathbf{A} generated by the set $S \cup \{x\}$ in \mathcal{K} .

LEMMA 2.10. *Let \mathbf{C} be in \mathcal{M}_n and $x \in C$. Let \mathbf{S} be a subalgebra of \mathbf{C} such that $\sigma_i^n(x)$ belongs to $B(\mathbf{S})$ for each $i = 1, \dots, n-1$. Then $\mathbf{B}(\langle \mathbf{S}, x \rangle_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S})$.*

PROOF. Clearly $\mathbf{B}(\mathbf{S})$ is a subalgebra of $\mathbf{B}(\langle \mathbf{S}, x \rangle_{\mathcal{M}_n})$. It is left to check that $B(\langle \mathbf{S}, x \rangle_{\mathcal{M}_n}) \subseteq B(\mathbf{S})$. To achieve this aim, we shall study the form of the elements in $\langle \mathbf{S}, x \rangle_{\mathcal{M}_n}$. We define for each $s \in S$,

$$\begin{aligned} \alpha(s) &= s \wedge x, \\ \beta(s) &= s \wedge \neg x, \\ \gamma_i(s) &= s \wedge \sigma_i^n(x), \quad \text{for } i = 1, \dots, n-1, \\ \delta_i(s) &= s \wedge \neg \sigma_i^n(x), \quad \text{for } i = 1, \dots, n-1. \end{aligned}$$

For all $s \in S$ we have that $\gamma_i(s)$ and $\delta_i(s)$ are in S for $i = 1, \dots, n-1$. Let

$$M := \left\{ y = \bigvee_{j=1}^{k_y} \bigwedge_{i=1}^{p_j} f_i(s_i) : f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \dots, \gamma_{n-1}, \delta_{n-1}\} \text{ and } s_i \in S \right\}.$$

We shall see that $\langle \mathbf{S}, x \rangle_{\mathcal{M}_n} = \mathbf{M} = (M, \wedge, \vee, \neg, \sigma_1^n, \dots, \sigma_{n-1}^n, 0, 1)$. Indeed, for all $s \in S$, $s = \gamma_1(s) \vee \delta_1(s)$, and then $S \subseteq M$. Besides, $x \in M$ because $x = \alpha(1)$. Lastly, it is easy to see that M is closed under the operations of n -valued Moisil algebra. Thus $\langle \mathbf{S}, x \rangle_{\mathcal{M}_n}$ is a subalgebra of \mathbf{M} . From the definition of M , it is obvious that $M \subseteq \langle \mathbf{S}, x \rangle_{\mathcal{M}_n}$, and the equality follows.

Now let $z \in B(\langle \mathbf{S}, x \rangle_{\mathcal{M}_n})$. By Theorem 2.6, $\sigma_{n-1}^n(z) = z$ and $z = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i)$ with $f_i \in \{\alpha, \beta, \gamma_1, \delta_1, \dots, \gamma_{n-1}, \delta_{n-1}\}$ and $s_i \in S$. Then we have

$$z = \sigma_{n-1}^n(z) = \sigma_{n-1}^n \left(\bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} f_i(s_i) \right) = \bigvee_{j=1}^{k_z} \bigwedge_{i=1}^{p_j} \sigma_{n-1}^n(f_i(s_i)),$$

is in $B(\mathbf{S})$ because $\sigma_{n-1}^n(f_i(s_i)) = \gamma_k(\sigma_{n-1}^n(s_i))$ or $\sigma_{n-1}^n(f_i(s_i)) = \delta_k(\sigma_{n-1}^n(s_i))$, for some $k = 1, \dots, n-1$. \square

THEOREM 2.11. *Let \mathbf{C} be an MV_n -algebra and $x \in \mathbf{C}$. Let \mathbf{S} be a subalgebra of \mathbf{C} such that $\sigma_i^n(x)$ belongs to $B(\mathbf{S})$ for each $i = 1, \dots, n-1$. Then*

$$\mathbf{B}(\langle \mathbf{S}, x \rangle_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{S}).$$

PROOF. By Lemma 2.9 and Lemma 2.10 we obtain $\mathbf{B}(\langle \mathbf{S}, x \rangle_{\mathcal{MV}_n}) = \mathbf{B}(\langle \mathbf{S}, x \rangle_{\mathcal{M}_n}) = \mathbf{B}(\mathbf{S})$. \square

2.2. Boolean elements in $\mathbf{Free}_{\mathcal{MV}_n}(\mathbf{Z})$ Recall that a Boolean algebra \mathbf{B} is said to be *free over a poset Y* if for each Boolean algebra \mathbf{C} and for each non-decreasing function $f : Y \rightarrow \mathbf{C}$, f can be uniquely extended to a homomorphism from \mathbf{B} into \mathbf{C} .

THEOREM 2.12. *$\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free Boolean algebra over the poset $Z' := \{\sigma_i^n(z) : z \in Z, i = 1, \dots, n-1\}$.*

PROOF. Let \mathbf{S} be the subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ generated by Z' . Let \mathbf{C} be a Boolean algebra and let $f : Z' \rightarrow \mathbf{C}$ be a non-decreasing function. The monotonicity of f implies that the prescription

$$f'(z) = (f(\sigma_1^n(z)), \dots, f(\sigma_{n-1}^n(z)))$$

defines a function $f' : Z \rightarrow \mathbf{C}^{[n]}$. Since $\mathbf{C}^{[n]} \in \mathcal{MV}_n$, there is a unique homomorphism $h' : \mathbf{Free}_{\mathcal{MV}_n}(Z) \rightarrow \mathbf{C}^{[n]}$ such that $h'(z) = f'(z)$ for every $z \in Z$. Let $\pi : \mathbf{C}^{[n]} \rightarrow \mathbf{C}$ be the projection over the first coordinate. The composition $\pi \circ h'$ restricted to \mathbf{S} is a homomorphism $h : \mathbf{S} \rightarrow \mathbf{C}$, and for $y = \sigma_j^n(z) \in Z'$ we have

$$\begin{aligned} h(y) &= \pi(h'(\sigma_j^n(z))) = \pi(\sigma_j^n(h'(z))) = \pi(\sigma_j^n(f'(z))) \\ &= \pi(\sigma_j^n(f(\sigma_1^n(z)), \dots, f(\sigma_{n-1}^n(z)))) \\ &= \pi(f(\sigma_j^n(z)), \dots, f(\sigma_j^n(z))) = f(\sigma_j^n(z)) = f(y). \end{aligned}$$

Hence \mathbf{S} is the free Boolean algebra over the poset Z' . However, since $\sigma_j^n(z)$ is in \mathbf{S} for all $z \in Z$ and $j = 1, \dots, n-1$, Theorem 2.11 asserts that

$$\mathbf{S} = \mathbf{B}(\mathbf{S}) = \mathbf{B}(\langle \mathbf{S}, z \rangle_{\mathcal{MV}_n})$$

for every $z \in Z$. From the fact that \mathbf{S} is a subalgebra of $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ we obtain

$$\mathbf{S} = \mathbf{B}(\langle \mathbf{S}, Z \rangle_{\mathcal{MV}_n}) = \mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$$

that is, $\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(Z))$ is the free Boolean algebra generated by the poset Z' . \square

From Theorem 2.4 we obtain the following result.

COROLLARY 2.13. $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra generated by the poset $Y := \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$.

REMARK 2.14. If $n = 2$, that is, the variety considered \mathcal{V} is generated by a \mathbf{BL}_2 -chain, then $\sigma_1^2(x) = x$ for each $x \in X$. Therefore, in this case, $Y = \{\neg\neg x : x \in X\}$, and the cardinality of Y equals the cardinality of X . It follows that $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra over the set Y .

3. $\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle$

Following the program established at the end of Section 2, our next aim is to describe $\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle$ for each ultrafilter U in the free Boolean algebra generated by $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$, where $\langle U \rangle$ is the BL-filter generated by the Boolean filter U .

The plan is to prove that $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle)$ is a subalgebra of \mathbf{L}_n and then, using Theorem 1.5, decompose each quotient $\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle$ into an ordinal sum. To accomplish this we need the following results.

THEOREM 3.1. Let \mathbf{A} be a BL-algebra and $U \in \text{Sp } \mathbf{B}(\mathbf{A})$. Then

$$\mathbf{MV}(\mathbf{A}/\langle U \rangle) \cong \mathbf{MV}(\mathbf{A})/(\langle U \rangle \cap \mathbf{MV}(\mathbf{A})).$$

PROOF. Let $V =: \langle U \rangle \cap \mathbf{MV}(\mathbf{A})$ and let $f : \mathbf{MV}(\mathbf{A})/V \rightarrow \mathbf{MV}(\mathbf{A}/\langle U \rangle)$ be given by $f(a/V) = a/\langle U \rangle$, for each $a \in \mathbf{MV}(\mathbf{A})$. It is easy to see that f is a homomorphism into $\mathbf{MV}(\mathbf{A}/\langle U \rangle)$. We have that

(1) f is injective.

Let $a/\langle U \rangle = b/\langle U \rangle$, with $a, b \in \mathbf{MV}(\mathbf{A})$. From Lemma 1.3, we know that there exists $u \in U$ such that $a \wedge u = b \wedge u$. Since $U \subseteq \mathbf{MV}(\mathbf{A})$, we have that $u \in V$. From the fact that u is Boolean (see [17, Lemma 2.2]), we have that $a * u = a \wedge u = b \wedge u \leq b$, thus $u \leq a \rightarrow b$ and similarly $u \leq b \rightarrow a$. Then $a \rightarrow b$ and $b \rightarrow a$ are in V and this means that $a/V = b/V$.

(2) f is surjective.

Let $a/\langle U \rangle \in \mathbf{MV}(\mathbf{A}/\langle U \rangle)$. Then $a/\langle U \rangle = \neg\neg(a/\langle U \rangle) = \neg\neg a/\langle U \rangle$, and since $\neg\neg a \in \mathbf{MV}(\mathbf{A})$ we obtain that $f(\neg\neg a/V) = a/\langle U \rangle$. \square

By Theorem 2.4, if $U \in \text{Sp } \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, then U is an ultrafilter in

$$\mathbf{B}(\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)).$$

Moreover, $\langle U \rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)) = \langle U \rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ is the Stone ultrafilter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$ generated by U . From [14, Chapter 6.3], we have that

$$\langle U \rangle \cap \mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$$

is a maximal filter of $\mathbf{Free}_{\mathcal{MV}_n}(\neg\neg X)$. It follows from [14, Corollary 3.5.4] that the MV-algebra $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X))/(\langle U \rangle \cap \mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X)))$ is an MV-chain in \mathcal{MV}_n , thus from Theorem 3.1 we have the following result.

THEOREM 3.2. $\mathbf{MV}(\mathbf{Free}_{\mathcal{V}}(X))/\langle U \rangle \cong \mathbf{L}_s$ with $s - 1$ dividing $n - 1$.

From Theorems 1.5 and 3.2 we obtain the next result.

THEOREM 3.3. For each $U \in \mathbf{SpB}(\mathbf{Free}_{\mathcal{V}}(X))$, we have that

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle \cong \mathbf{L}_s \uplus \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle)$$

for some $s - 1$ dividing $n - 1$.

In order to complete the description of $\mathbf{Free}_{\mathcal{V}}(X)$ we have to find a description of $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U \rangle)$ for each $U \in \mathbf{SpB}(\mathbf{Free}_{\mathcal{V}}(X))$. This last description depends on the characterization of the variety \mathcal{W} of generalized BL-algebras generated by the generalized BL-chain \mathbf{B} . Therefore, we shall firstly consider such variety.

3.1. The subvariety of \mathcal{GBL} generated by \mathbf{B} We recall that \mathcal{V} is the variety of BL-algebras generated by the BL-chain $\mathbf{T}_n = \mathbf{L}_n \uplus \mathbf{B}$. Let \mathcal{W} be the variety of generalized BL-algebras generated by the chain \mathbf{B} .

Let $\{e_i, i \in I\}$ be the set of equations that define \mathcal{MV}_n as a subvariety of \mathcal{BL} , and $\{d_j, j \in J\}$ be the set of equations that define \mathcal{W} as a subvariety of \mathcal{GBL} . For each $i \in I$, let e'_i be the equation that results from substituting $\neg\neg x$ for each variable x in e_i , and for each $j \in J$, let d'_j be the equation that results from substituting $\neg\neg x \rightarrow x$ for each variable x in the equation d_j . Let \mathcal{V}' denote the variety of BL-algebras characterized by the equations of BL-algebras plus the equations $\{e'_i, i \in I\} \cup \{d'_j, j \in J\}$.

THEOREM 3.4. $\mathcal{V}' \subseteq \mathcal{V}$.

PROOF. Let \mathbf{A} be a subdirectly irreducible BL-algebra in \mathcal{V}' . From Theorem 1.1, \mathbf{A} is a BL-chain, and by Theorem 1.4, $\mathbf{A} = \mathbf{MV}(\mathbf{A}) \uplus \mathbf{D}(\mathbf{A})$. Since for each $x \in MV(\mathbf{A})$, we have $\neg\neg x = x$, $\mathbf{MV}(\mathbf{A})$ satisfies equations $\{e_i, i \in I\}$. Then $\mathbf{MV}(\mathbf{A})$ is a chain in \mathcal{MV}_n , that is, $\mathbf{MV}(\mathbf{A}) \cong \mathbf{L}_s$, with $s - 1$ dividing $n - 1$. Moreover, since for each $x \in D(\mathbf{A})$, we have $\neg\neg x \rightarrow x = x$, $\mathbf{D}(\mathbf{A})$ satisfies equations $\{d_j, j \in J\}$. Hence

$\mathbf{D}(\mathbf{A}) = \mathbf{C}$ is a generalized BL-chain in \mathcal{W} . Since \mathbf{A} is subdirectly irreducible, \mathbf{C} is also subdirectly irreducible, and since \mathcal{GBL} is a congruence distributive variety, we can apply Jónsson's Lemma (see [9]) to conclude that $\mathbf{C} \in \mathbf{HSP}_u(\mathbf{B})$. Hence there is a set $J \neq \emptyset$ and an ultrafilter U over J such that \mathbf{C} is a homomorphic image of a subalgebra of \mathbf{B}^J/U . From the proof of [2, Proposition 3.3] it follows that $(\mathbf{L}_n \uplus \mathbf{B})^J/U = \mathbf{L}_n^J/U \uplus \mathbf{B}^J/U$, and since \mathbf{L}_n is finite, $\mathbf{L}_n^J/U \cong \mathbf{L}_n$. Now it is easy to see that $\mathbf{A} = \mathbf{L}_s \uplus \mathbf{C} \in \mathbf{HSP}_u(\mathbf{L}_n \uplus \mathbf{B}) \subseteq \mathcal{V}$. \square

The next corollary states the main result of this section.

COROLLARY 3.5. *The variety \mathcal{W} of generalized BL-algebras generated by \mathbf{B} consists of the generalized BL-algebras \mathbf{C} such that $\mathbf{L}_n \uplus \mathbf{C}$ belongs to \mathcal{V} .*

PROOF. Given $\mathbf{C} \in \mathcal{W}$, $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}' \subseteq \mathcal{V}$. On the other hand, if \mathbf{C} is a generalized BL-algebra such that $\mathbf{L}_n \uplus \mathbf{C} \in \mathcal{V}$, then the elements of \mathbf{C} satisfy equations d'_j for each $j \in J$ and since $\neg\neg x \rightarrow x = x$ for each $x \in \mathbf{C}$, the elements of \mathbf{C} satisfy equations d_j for each $j \in J$. Hence \mathbf{C} is in \mathcal{W} . \square

3.2. $\mathbf{D}(\mathbf{Free}_\mathcal{V}(X)/\langle U \rangle)$ We know that the ultrafilters of a Boolean algebra are in bijective correspondence with the homomorphisms from the algebra into the two elements Boolean algebra, $\mathbf{2}$. Since every upwards closed subset of the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$ is in correspondence with an increasing function from Y onto $\mathbf{2}$, and every increasing function from Y can be extended to a homomorphism from $\mathbf{B}(\mathbf{Free}_\mathcal{V}(X))$ into $\mathbf{2}$, the ultrafilters of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(X))$ are in correspondence with the upwards closed subsets of Y . This is summarized in the following lemma.

LEMMA 3.6. *Consider the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n-1\}$. The correspondence that assigns to each upwards closed subset $S \subseteq Y$ the Boolean filter U_S generated by the set $S \cup \{\neg y : y \in Y \setminus S\}$, defines a bijection from the set of upwards closed subsets of Y onto the ultrafilters of $\mathbf{B}(\mathbf{Free}_\mathcal{V}(X))$.*

We shall refer to each member of $\mathbf{Sp} \mathbf{B}(\mathbf{Free}_\mathcal{V}(X))$ by U_S making explicit reference to the upwards closed subset S that corresponds to it.

LEMMA 3.7. *Let \mathbf{F}_S be the subalgebra of the generalized BL-algebra*

$$\mathbf{D}(\mathbf{Free}_\mathcal{V}(X)/\langle U_S \rangle)$$

generated by the set $X_S := \{x/\langle U_S \rangle : x \in X, \neg\neg x \in \langle U_S \rangle\}$. Then

$$\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_\mathcal{V}(X)/\langle U_S \rangle).$$

PROOF. $\mathbf{Free}_V(X)/\langle U_S \rangle$ is the BL-algebra generated by the set $Z_S = \{x/\langle U_S \rangle : x \in X\}$. From Theorem 3.3, there exists an integer m such that

$$\mathbf{Free}_V(X)/\langle U_S \rangle = \mathbf{L}_m \uplus \mathbf{D}(\mathbf{Free}_V(X)/\langle U_S \rangle).$$

Hence each element of Z_S is either in $L_m \setminus \{\top\}$ or it is in $D(\mathbf{Free}_V(X)/\langle U_S \rangle)$.

If $X_S = \emptyset$, then $F_S = D(\mathbf{Free}_V(X)/\langle U_S \rangle) = \{\top\}$. So let us suppose $X_S \neq \emptyset$. Let $y \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$. Recalling that \mathbf{F}_S is the generalized BL-algebra generated by X_S , we will check that y is in F_S . Since $y \in \mathbf{Free}_V(X)/\langle U_S \rangle$, y is given by a term on the elements $x/\langle U_S \rangle \in Z_S$. By induction on the complexity of y , we have:

- If y is a generator, that is, $y = x/\langle U_S \rangle$ for some $x/\langle U_S \rangle \in Z_S$, since $y \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$, we have that $\top = \neg\neg y = \neg\neg(x/\langle U_S \rangle) = (\neg\neg x)/\langle U_S \rangle$. This happens only if $\neg\neg x \in \langle U_S \rangle$.
- Suppose that for each element $z \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$ of complexity less than k , z can be written as a term in the variables $x/\langle U_S \rangle$ in X_S . Let $y \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$ be an element of complexity k . The possible cases are the following:

- (1) $y = a \rightarrow b$ for some elements a, b of complexity $< k$. In this case the possibilities are
 - (a) $a \leq b$. This means $a \rightarrow b = \top$ and y can be written as $x/\langle U_S \rangle \rightarrow x/\langle U_S \rangle$ for any $x/\langle U_S \rangle \in X_S$, and thus $y \in F_S$,
 - (b) $a \not\leq b$. Since $y = a \rightarrow b$ is in $D(\mathbf{Free}_V(X)/\langle U_S \rangle)$, the only possibility is that $a, b \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$ and by inductive hypothesis y is in F_S .
- (2) $y = a * b$ for some elements a, b of complexity $< k$. In this case necessarily $a, b \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$ and by inductive hypothesis y is in F_S .

Then for each $y \in D(\mathbf{Free}_V(X)/\langle U_S \rangle)$, y can be written as a term on the elements of X_S . Therefore $y \in F_S$ and we conclude that $\mathbf{F}_S = \mathbf{D}(\mathbf{Free}_V(X)/\langle U_S \rangle)$. \square

With the notation of the previous lemma, we have the following theorem.

THEOREM 3.8. *For each U_S in $\mathbf{Sp} \mathbf{B}(\mathbf{Free}_V(X))$,*

$$\mathbf{D}(\mathbf{Free}_V(X)/\langle U_S \rangle) \cong \mathbf{Free}_W(X_S).$$

PROOF. From Theorem 2.6 and Lemma 3.6 we can deduced that $\neg\neg x \in \langle U_S \rangle$ if and only if $\sigma_1^n(\neg\neg x) \in S$ if and only if $\sigma_i^n(\neg\neg x) \in S$ for $i = 1, \dots, n-1$. Hence if $\neg\neg x \notin \langle U_S \rangle$ there is a j such that $\sigma_j^n(\neg\neg x) \notin S$. We define, for each $x \in X$,

$$j_x = \begin{cases} \perp & \text{if } \neg\neg x \in \langle U_S \rangle, \\ \max\{i \in \{1, \dots, n-1\} : \sigma_i^n(\neg\neg x) \notin S\} & \text{otherwise.} \end{cases}$$

Let $\mathbf{C} \in \mathcal{W}$ and let $\mathbf{C}' = \mathbf{L}_n \uplus \mathbf{C}$. From Theorem 3.5, \mathbf{C}' is in \mathcal{V} . Given a function $f : X_S \rightarrow \mathbf{C}$, define $\hat{f} : X \rightarrow \mathbf{C}'$ by the prescriptions:

$$\hat{f}(x) = \begin{cases} f(x/\langle U_S \rangle) & \text{if } \neg\neg x \in \langle U_S \rangle, \\ (n - j_x - 1)/(n - 1) & \text{otherwise.} \end{cases}$$

There is a unique homomorphism $\hat{h} : \mathbf{Free}_{\mathcal{V}}(X) \rightarrow \mathbf{C}'$ such that $\hat{h}(x) = \hat{f}(x)$ for each $x \in X$. We have that $U_S \subseteq \hat{h}^{-1}(\{\top\})$. Indeed, if $\neg\neg x \in \langle U_S \rangle$, then $\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n(\neg\neg(\hat{h}(x))) = \sigma_i^n(\neg\neg f(x/\langle U_S \rangle)) = \sigma_i^n(\top) = \top$. If $\neg\neg x \notin \langle U_S \rangle$, then

$$\hat{h}(\sigma_i^n(\neg\neg x)) = \sigma_i^n\left(\neg\neg \frac{n - j_x - 1}{n - 1}\right) = \sigma_i^n\left(\frac{n - j_x - 1}{n - 1}\right) = \begin{cases} \perp & \text{if } i \leq j_x, \\ \top & \text{otherwise.} \end{cases}$$

Hence there is a unique homomorphism $h_1 : \mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle \rightarrow \mathbf{C}'$ such that $h_1(a/\langle U_S \rangle) = \hat{h}(a)$ for all $a \in \mathbf{Free}_{\mathcal{V}}(X)$. By Lemma 3.7, $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is the algebra generated by X_S . Then the restriction h of h_1 to $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle)$ is a homomorphism $h : \mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \rightarrow \mathbf{C}$, and for each x such that $\neg\neg x \in \langle U_S \rangle$,

$$h(x/\langle U_S \rangle) = h_1(x/\langle U_S \rangle) = \hat{h}(x) = \hat{f}(x) = f(x/\langle U_S \rangle).$$

Therefore we conclude that $\mathbf{D}(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) \cong \mathbf{Free}_{\mathcal{W}}(X_S)$. \square

THEOREM 3.9. *The free BL-algebra $\mathbf{Free}_{\mathcal{V}}(X)$ can be represented as a weak Boolean product of the family $(\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle) : U_S \in \mathbf{Sp} \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, where $\mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$ is the free Boolean algebra over the poset $Y = \{\sigma_i^n(\neg\neg x) : x \in X, i = 1, \dots, n - 1\}$. Moreover, for each $U_S \in \mathbf{Sp} \mathbf{B}(\mathbf{Free}_{\mathcal{V}}(X))$, there exists $m \geq 2$ such that $m - 1$ divides $n - 1$ and*

$$\mathbf{Free}_{\mathcal{V}}(X)/\langle U_S \rangle = \mathbf{L}_m \uplus \mathbf{Free}_{\mathcal{W}}(X_S),$$

where $X_S := \{x/\langle U_S \rangle : \neg\neg x \in \langle U_S \rangle\}$ and \mathcal{W} is the variety of generalized BL-algebras generated by \mathbf{B} .

4. Examples

4.1. PL-algebras Let \mathbf{G} be a lattice-ordered abelian group (ℓ -group), and $G^- = \{x \in G : x \leq 0\}$ its negative cone. For each pair of elements $x, y \in G^-$, we define the following operators:

$$x * y = x + y \quad \text{and} \quad x \rightarrow y = 0 \wedge (y - x).$$

Then $\mathbf{G}^- = (G^-, \wedge, \vee, *, \rightarrow, 0)$ is a generalized BL-algebra. The following result can be deduced from [3] (see also [6] and [15]).

THEOREM 4.1. *The following conditions are equivalent for a generalized BL-algebra \mathbf{A} :*

- (1) \mathbf{A} is a cancellative hoop.
- (2) There is an ℓ -group \mathbf{G} such that $\mathbf{A} \cong \mathbf{G}^-$.
- (3) \mathbf{A} is in the variety of generalized BL-algebras generated by \mathbf{Z}^- , where \mathbf{Z} denotes the additive group of integers with the usual order.

Let us consider \mathcal{W} , the variety of generalized BL-algebras generated by \mathbf{Z}^- , that is, the variety of cancellative hoops. In [16] a description of $\mathbf{Free}_{\mathcal{W}}(X)$ is given for any set X of free generators. Therefore we can have a complete description of free algebras in varieties of BL-algebras generated by the ordinal sum

$$\mathbf{PL}_n = \mathbf{L}_n \uplus \mathbf{Z}^-.$$

Indeed, if we denote by \mathcal{PL}_n the variety of BL-algebras generated by \mathbf{PL}_n , from Theorem 3.9 we obtain that $\mathbf{Free}_{\mathcal{PL}_n}(X)$ is a weak Boolean product of algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_{\mathcal{W}}(X')$ with $s - 1$ dividing $n - 1$ and some set X' of cardinality less or equal than X . Therefore, in the present case, the BL-algebra $\mathbf{Free}_{\mathcal{PL}_n}(X)$ can be completely described as a weak Boolean product of ordinal sums of two known algebras.

From [15, Theorem 2.8], \mathcal{PL}_2 is the variety of PL-algebras \mathcal{PL} . From Remark 2.14, $\mathbf{SpB}(\mathbf{Free}_{\mathcal{PL}}(X))$ is the Cantor space $2^{|X|}$. From Theorem 3.9, the free PL-algebra over a set X can be describe as a weak Boolean product over the Cantor space $2^{|X|}$ of algebras of the form $\mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{W}}(X')$ for some set X' of cardinality less or equal than X .

Given a BL-algebra \mathbf{A} , the radical $R(\mathbf{A})$ of \mathbf{A} is the intersection of all maximal implicative filters of \mathbf{A} . We have that $\mathbf{r}(\mathbf{A}) = (R(\mathbf{A}), *, \rightarrow, \wedge, \vee, \top)$ is a generalized BL-algebra. Let

$$\mathcal{PL}^r = \{\mathbf{R} : \mathbf{R} = \mathbf{r}(\mathbf{A}) \text{ for some } \mathbf{A} \in \mathcal{PL}\}.$$

\mathcal{PL}^r is a variety of generalized BL-algebras. In [17] a description of $\mathbf{Free}_{\mathcal{PL}}(X)$ is given. From Example 4.7 and Theorem 5.7 in the mentioned paper we obtain that $\mathbf{Free}_{\mathcal{PL}}(X)$ is the weak Boolean product of the family $(\mathbf{L}_2 \uplus \mathbf{Free}_{\mathcal{PL}^r}(S) : S \subseteq 2^{|X|})$ over the Cantor space $2^{|X|}$. In order to check that our description and the one given in [17] coincide it is only left to check that $\mathcal{PL}^r = \mathcal{W}$. From Corollary 3.5 we have that \mathcal{W} consist on the generalized BL-algebras \mathbf{C} such that $\mathbf{L}_2 \uplus \mathbf{C} \in \mathcal{PL}$.

THEOREM 4.2. $\mathcal{PL}^r = \mathcal{W}$.

PROOF. Let $\mathbf{C} \in \mathcal{PL}^r$. Then there exists a BL-algebra $\mathbf{A} \in \mathcal{PL}$ such that $\mathbf{r}(\mathbf{A}) = \mathbf{C}$. It is not hard to check that $\mathbf{L}_2 \uplus \mathbf{C}$ is a subalgebra of \mathbf{A} , thus $\mathbf{L}_2 \uplus \mathbf{C}$ is in \mathcal{PL} . It

follows that $\mathbf{C} \in \mathcal{W}$. On the other hand, let $\mathbf{C} \in \mathcal{W}$. Then $\mathbf{L}_2 \uplus \mathbf{C}$ is in \mathcal{PL} , and $\mathbf{C} \in \mathcal{PL}^r$. \square

4.2. Finitely generated free algebras As we mentioned in the introduction, when the set of generators X is finite, let us say of cardinality k , the algebra $\mathbf{Free}_V(X)$ is described in [10] as a direct product of algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_W(X')$, with $s - 1$ that divides $n - 1$ and some set X' of cardinality less than or equal to the cardinality of X , where W is again the subvariety of \mathcal{GBL} generated by \mathbf{B} . The method used to describe the algebras strongly relies on the fact that the Boolean elements of $\mathbf{Free}_V(X)$ form a finite Boolean algebra. Indeed, $\mathbf{Free}_V(X)$ is a direct product of n^k algebras obtained by taking the quotients by the implicative filters generated by the atoms of $\mathbf{B}(\mathbf{Free}_V(X))$. In this case, once you know the form of the atom that generates the ultrafilter U you also know the number s such that $\mathbf{MV}((\mathbf{Free}_V(X))/\langle U \rangle) = \mathbf{L}_s$.

When the set X of generators is finite, of cardinality k , then $Y = \{\sigma_i^n(\neg \neg x) : x \in X, i = 1, \dots, n - 1\}$ is the cardinal sum of k chains of length $n - 1$. Therefore the number of upwards closed subsets of Y is n^k . Since weak Boolean products over discrete finite spaces coincide with direct products, Theorem 3.9 asserts that $\mathbf{Free}_V(X)$ is a direct product of n^k BL-algebras of the form $\mathbf{L}_s \uplus \mathbf{Free}_W(Y)$, with $s - 1$ that divides $n - 1$ and some set Y of cardinality less than or equal to k .

Therefore the description given in the present paper coincides with the one in [10]. However, the description given in [10], based on a detailed analysis of the structure of the atoms of $\mathbf{B}(\mathbf{Free}_V(X))$ for a finite X , is more precise because it gives the number of factors of each kind appearing in the direct product representation.

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