CYCLIC HOMOLOGY OF CLEFT EXTENSIONS OF ALGEBRAS

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ABSTRACT. Let k be a commutative algebra with $\mathbb{Q} \subseteq k$ and let (E, p, i) be a cleft extension of A. We obtain a new mixed complex, simpler than the canonical one, giving the Hochschild and cyclic homologies of E relative to $\ker(p)$. This complex resembles the canonical reduced mixed complex of an augmented algebra. We begin the study of our complex showing that it has a harmonic decomposition like to the one considered by Cuntz and Quillen for the normalized mixed complex of an algebra.

Introduction

Let k be a commutative ring such that $\mathbb{Q} \subseteq k$, and let A be an associative unital k-algebra. A *cleft extension* of A is a triple (E, p, i), consisting of a k-algebra E and k-algebra morphisms $i : A \to E$ and $p : E \to A$, such that $p \circ i = \mathrm{id}_A$. When applications i and p are evident we simply say that E is a cleft extension of A instead of saying that (E, p, i) it is. A *morphism* $f : (E, p, i) \longrightarrow (F, q, j)$, of a cleft extension of an algebra A to a cleft extension of an algebra B, is a morphisms of algebras $f : E \to F$ such that there exists a (necessarily unique) morphism $\varphi : A \to B$ satisfying $f \circ i = j \circ \varphi$ and $q \circ f = \varphi \circ p$.

For example, given a graded algebra $E = \bigoplus_{i \geq 0} A_i$, each subalgebra of E including A_0 is a cleft extension of A_0, and the same is true for each quotient E/I, of E by a two sided ideal $I \subseteq A_+ := \bigoplus_{i > 0} A_i$. This includes a lot of well known examples of constructive nature, such as tensor algebras, truncated tensor algebras, truncated quiver algebras, symmetric algebras, exterior algebras, general quadratic algebras, monomial algebras, Rees algebras, etcetera. Let R be a commutative ring such that $k \subseteq R$. As was point out in [3], also there are examples appearing for structural reasons. For instance:

(1) An R-algebra E is a cleft extension of $A := E/\mathrm{J}(E)$, where $\mathrm{J}(E)$ denotes the Jacobson radical of E, when:

    (i) E is a basic semiperfect algebra or A is R-projective, $\mathrm{J}(E)$ is nilpotent and the Hochschild dimension of A is lesser or equal than 1 (see [19] and [21]),

    (ii) R is a field and $\dim_R(E) < \infty$ (see [8] and [18]).

(2) If R is a field, then, by the Levi-Malcev Theorem (see [1]), the universal enveloping algebra $U(L)$, of a finite-dimensional Lie algebra L, is a cleft extension of $U(L/J)$, where J is the radical of L.

(3) If A is quasi-free, then any nilpotent extension of A is cleft (see [6]).

A special interesting type of cleft extensions are the split-by-nilpotent extension algebras (see [2], [16] and [17]).

There is a canonical way to construct a cleft extension of an algebra A. Given a an A-bimodule M endowed with an associative A-bimodule map

$$M \otimes_A M \longrightarrow M,$$
$$m \otimes m' \longmapsto m \triangledown m'$$

the triple $(A \ltimes_\triangledown M, i, p)$, where $A \ltimes_\triangledown M$ is the algebra with underlying vector space $A \times M$ and multiplication map

$$(a, m)(a', m') \coloneqq (aa', am' + ma' + m \triangledown m'),$$

$i \colon A \longrightarrow A \ltimes_\triangledown M$ is the map defined by $i(a) \coloneqq (a, 0)$ and $p \colon A \ltimes_\triangledown M \longrightarrow A$ is the map defined by $p(a, m) \coloneqq a$, is a cleft extension of A, that is called *the $\triangledown$-extension* of A or the *cleft extension of A associated with* $\triangledown$. Moreover it is not difficult to see that each clef extension is isomorphic to one of this type.

Let k be a commutative ring and let C be a k-algebra. If K is a subalgebra of C, we will say that C is a K-algebra. In this paper we begin the study of the Hochschild, cyclic, negative and periodic homologies of cleft extensions (E, p, i) of a K-algebra A. It is easy to see that these homologies are the direct sum of the corresponding homologies of the K-algebra A and the corresponding homologies of the K-algebra E, relative to $\ker(p)$. So we restrict our attention to the last ones. Moreover, by the previous discussion, we can assume that E is a cleft extension $A \ltimes_\triangledown M$ associated with an associative A-bimodule map $\triangledown$. Our main result is Theorem 2.2, in which we obtain a double mixed complex $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$, given these homologies, whose associated mixed complex $(\check{X}, \check{b}, \check{B})$ is simpler than the canonical mixed complex of $A \ltimes_\triangledown M$ relative to M. We also introduce an begin the study of an harmonic decomposition of this complex. In a forthcoming paper we are going to obtain an still simpler complex under the hypothesis that M is isomorphic to $A \otimes_k V$ as a left A-module. We hope that this allows us to get explicit computations.

The paper is organized in the following way:

In Section 1 we recall some well known definitions and results. Among them, the perturbation lemma, which we will use again and again in the rest of the paper, and the definition of double mixed complex, which we got from [4].

Section 2 is devoted to establishing the main results in this paper. In fact, $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ can be thought as a double mixed complex associated to the 3-tuple $(A, M, \triangledown)$, and this association is functorial in an evident sense. For $0 \le 2w \le v$, let X_{vw} be the direct sum of all the tensor products $X_0 \otimes \cdots \otimes X_n$ such that $X_0 = M$, $X_i = M$ for w indices $i > 0$ and $X_i = \overline{A}$ for the other ones, where $n = v + w$ and $\overline{A} = A/k$. Let $b \colon X_{vw} \longrightarrow X_{v-1,w}$ be the map given by the same formula as the Hochschild boundary map of an algebra, where the meaning of the concatenation $x_i x_{i+1}$ of two consecutive factors in a simple tensor is the one given in item (3) of Notation 0.2. Let $t \colon X_{vw} \longrightarrow X_{vw}$ be the map defined by

$$t(x_0 \otimes \cdots \otimes x_n) = (-1)^{in} x_i \otimes \cdots \otimes x_n \otimes x_0 \otimes \cdots \otimes x_{i-1},$$

where i denotes the last index such that $x_i \in M$ and let $N = \mathrm{id} + t + t^2 + \cdots + t^w$. The double mixed complex $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ has objects $\hat{X}_{vw} = X_{vw} \oplus X_{v-1,w}$. The boundary maps are given by

$$\hat{b}(\mathbf{x}, \mathbf{y}) = \big(b(\mathbf{x}) + (\mathrm{id} - t)(\mathbf{y}), -b(\mathbf{y})\big) \qquad \text{and} \qquad \hat{d}(\mathbf{x}, \mathbf{y}) = \big(d(\mathbf{x}), -d'(\mathbf{y})\big),$$

where $d, d' \colon X_{vw} \longrightarrow X_{v,w-1}$ are maps depending on the map $\triangledown$, and the Connes operator is given by $\hat{B}(\mathbf{x}, \mathbf{y}) = (0, N(\mathbf{x}))$. So, it resembles the reduced mixed complex of an augmented algebra. Since the maps t and N satisfy

$$\mathrm{Im}(1 - t) = \ker(N) \qquad \text{and} \qquad \mathrm{Im}(N) = \ker(1 - t), \tag{0.1}$$

the cyclic homology of E relative to M is the homology of the quotient complex of (X, b, d) by the image of $\mathrm{id} - t$. Indeed, this also follows from the fact that $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ satisfies the Connes property ([7]), which is another consequence of the equalities (0.1).

Our purpose in Section 3 is to show that $(\check{X}, \check{b}, \check{B})$ has a harmonic decomposition like the one studied in [7]. In order to carry out this task we need to define a de Rham coboundary map and a Karoubi operator on $(\check{X}, \check{b})$. Actually it will be convenient for us to work with a new double mixed complex, namely $(\ddot{X}, \ddot{d}, \ddot{b}, \ddot{B})$, whose associated mixed complex is also $(\check{X}, \check{b}, \check{B})$. As in [7] the Karoubi operator $\ddot{\kappa}$ of $(\ddot{X}, \ddot{d}, \ddot{b})$ commutes with $\ddot{b}$ and $\ddot{d}$ and satisfies a polynomial equation $P_w(\kappa)$ on each $\ddot{X}_{vw}$. Thus we have the harmonic decomposition $\ddot{X} = P(\ddot{X}) \oplus P^{\perp}(\ddot{X})$, where P is the spectral projection onto the generalized nullspace for $\mathrm{id} - \ddot{\kappa}$ and $P^{\perp} = 1 - P$. The first component of this decomposition is $\ddot{B}$-acyclic and the second one is $\ddot{d}$-acyclic and killed by $\ddot{B}$. Hence $(\ddot{X}, \ddot{d}, \ddot{b})$ has the Connes property. We finish the section by giving an explicit description of $P(\ddot{X})$.

Next we introduce some notations that we will use throughout this paper.

Notations 0.1. Let V be a K-bimodule and let M be a C-bimodule.

(1) We set $\overline{C} := C/K$. Moreover, given $c \in C$, we also denote by c the class of c in $\overline{C}$.

(2) We use the unadorned tensor symbol $\otimes$ to denote the tensor product $\otimes_K$.

(3) We let $V^{\otimes^n}$ denote the n-fold power tensor of V.

(4) Given $c_0, \dots, c_r \in C$ and $i < j$, we write $\mathbf{c}_i^j := c_i \otimes \cdots \otimes c_j$.

(5) Given a K-bimodule M, we let $M\otimes$ denote the quotient $M/[M, K]$, where $[M, K]$ is the k-vector subspace of M generated by all the commutators $m\lambda - \lambda m$, with $m \in M$ and $\lambda \in K$. Moreover, for $m \in M$, we let $[m]$ denote the class of m in $M\otimes$.

Notations 0.2. Let $E := A \ltimes_{\triangledown} M$ be a cleft extension.

(1) We let
$$\pi_A : E \longrightarrow A \qquad \text{and} \qquad \pi_M : E \longrightarrow M$$
denote the maps defined by $\pi_A(a, m) := a$ and $\pi_M(a, m) := m$, respectively.

(2) Given $x, y \in A \bigcup M$ we set
$$x \triangledown y := \begin{cases} x \triangledown y & \text{if } x, y \in M, \\ 0 & \text{otherwise.} \end{cases}$$
We extend this definition for $x, y \in E$ by linearity.

(3) Given $x, y \in A \bigcup M$ we set
$$xy := \begin{cases} \text{the product of } x \text{ and } y \text{ in } A & \text{if } x, y \in A, \\ \text{the left action of } x \text{ on } y & \text{if } x \in A \text{ and } y \in M, \\ \text{the right action of } y \text{ on } x & \text{if } x \in M \text{ and } y \in A, \\ 0 & \text{if } x, y \in M. \end{cases}$$
We extend this definition for $x, y \in E$ by linearity.

(4) For $0 \le w \le n$, let $B_w^n \subseteq \overline{E}^{\otimes^n}$ be the k-submodule spanned by the simple tensors $\mathbf{x}_1^n$ such that exactly w of the x_i's belong to M, while the other ones belong to $\overline{A}$. To unify expressions we make the convention that $B_0^0 := k$ and $B_w^n := 0$, for $w < 0$ or $n < w$.

(5) We will say that a simple tensor $\mathbf{x}_0^n \in E \otimes \overline{E}^{\otimes^n} \otimes$ is very simple if $x_i \in A \cup M$ for all i.

(6) For $\mathbf{x}_0^n \in \left(M \otimes B_w^n \otimes\right) \cup \left(A \otimes B_{w+1}^n \otimes\right)$ and $0 \le l \le n$, we define $\mu_j(\mathbf{x}_0^n)$ by

$$\mu_j(\mathbf{x}_0^n) := \begin{cases} (-1)^j \mathbf{x}_0^{j-1} \otimes x_j x_{j+1} \otimes \mathbf{x}_{j+2}^n & \text{if } 0 \le j < n, \\ (-1)^n x_n x_0 \otimes \mathbf{x}_1^{n-1} & \text{If } j = n. \end{cases}$$

Moreover we set

$$\mu_0^A(\mathbf{x}_0^n) := \pi_A(x_0 x_1) \otimes \mathbf{x}_2^n, \qquad \mu_n^A(\mathbf{x}_0^n) = (-1)^n \pi_A(x_n x_0) \otimes \mathbf{x}_1^{n-1},$$
$$\mu_0^M(\mathbf{x}_0^n) := \pi_M(x_0 x_1) \otimes \mathbf{x}_2^n, \qquad \mu_n^M(\mathbf{x}_0^n) = (-1)^n \pi_M(x_n x_0) \otimes \mathbf{x}_1^{n-1}.$$

(7) For $\mathbf{x}_0^n \in E \otimes \overline{E}^{\otimes^n} \otimes$ and $0 \le j \le n$, we define $\varrho_j(\mathbf{x}_0^n)$ by

$$\varrho_j(\mathbf{x}_0^n) = \begin{cases} (-1)^j \mathbf{x}_0^{j-1} \otimes x_j \triangledown x_{j+1} \otimes \mathbf{x}_{j+2}^n & \text{if } 0 \le j < n, \\ (-1)^n x_n \triangledown x_0 \otimes \mathbf{x}_1^{n-1} & \text{If } j = n. \end{cases}$$

(8) Let $\mathbf{x}_0^n \in E \otimes \overline{E}^{\otimes^n} \otimes$ be a very simple tensor. We let $i(\mathbf{x}_0^n)$ and denote the last index i such that $x_i \in M$.

(9) For a very simple tensor $\mathbf{x}_0^n \in E \otimes \overline{E}^{\otimes^n} \otimes$, we define

$$t(\mathbf{x}_0^n) := (-1)^{i(\mathbf{x}_0^n)n} \mathbf{x}_{i(\mathbf{x}_0^n)}^n \otimes \mathbf{x}_0^{i(\mathbf{x}_0^n)-1}.$$

We extend this definition to $E \otimes \overline{E}^{\otimes^n} \otimes$ by linearity.

1 Preliminaries

In this section we recall some well known definitions and results that we will use in the rest of the paper. Let C be a k-algebra.

1.1 Double and triple complexes

A double complex $\mathcal{X} := (X, d^v, d^h)$ of C-modules, is a family $(X_{pq})_{p,q \in \mathbb{Z}}$ of C-modules, together with C-linear maps

$$d^h: X_{pq} \longrightarrow X_{p-1,q} \qquad \text{and} \qquad d^v: X_{pq} \longrightarrow X_{p,q-1},$$

such that $d^h \circ d^h = 0$, $d^v \circ d^v = 0$ and $d^v \circ d^h + d^h \circ d^v = 0$. The total complex of (X, d^v, d^h) is the complex $\text{Tot}(\mathcal{X}) = (X, d)$, in which

$$X_n := \prod_p X_{p,n-p} \qquad \text{and} \qquad d := d^v + d^h.$$

A morphism of double complexes $f: (X, d^v, d^h) \longrightarrow (Y, \delta^v, \delta^h)$ is a family of maps $f: X_{pq} \longrightarrow Y_{pq}$, such that $\delta^v \circ f = f \circ d^v$ and $\delta^h \circ f = f \circ d^h$. The morphism from $\text{Tot}(X, d^v, d^h)$ to $\text{Tot}(Y, \delta^v, \delta^h)$ induced by f will be denoted $\text{Tot}(f)$.

Similarly, one can give the notions of triple complex $\mathcal{X} := (X, d^v, d^h, d^d)$ and of morphism of triple complexes. For a triple complex $\mathcal{X}$, there are three ways for constructing a double complex by taking total complexes of double complexes. We call each one of these double complexes a partial total complex of $\mathcal{X}$. The total complex $\text{Tot}(\mathcal{X})$ of $\mathcal{X}$, is the total complex of any of its partial total complexes. Of course, $\text{Tot}(\mathcal{X})$ is independently of the chosen way to construct it.

1.2 Mixed complexes

In this subsection we recall briefly the notion of mixed complex. For more details about this concept we refer to [3] and [15].

A *mixed complex* (X, b, B) is a graded k-module $(X_n)_{n \geq 0}$, endowed with morphisms

$$b \colon X_n \longrightarrow X_{n-1} \qquad \text{and} \qquad B \colon X_n \longrightarrow X_{n+1},$$

such that

$$b \circ b = 0, \quad B \circ B = 0 \quad \text{and} \quad B \circ b + b \circ B = 0.$$

A *morphism of mixed complexes* $f \colon (X, b, B) \longrightarrow (Y, d, D)$ is a family of maps $f \colon X_n \longrightarrow Y_n$, such that $d \circ f = f \circ b$ and $D \circ f = f \circ B$. Let u be a degree 2 variable. A mixed complex $\mathcal{X} \coloneqq (X, b, B)$ determines a double complex

$$\mathrm{BP}(\mathcal{X}) =$$

where $b(\mathbf{x}u^i) \coloneqq b(\mathbf{x})u^i$ and $B(\mathbf{x}u^i) \coloneqq B(\mathbf{x})u^{i-1}$. By deleting the positively numbered columns we obtain a subcomplex $\mathrm{BN}(\mathcal{X})$ of $\mathrm{BP}(\mathcal{X})$. Let $\mathrm{BN}'(\mathcal{X})$ be the kernel of the canonical surjection from $\mathrm{BN}(\mathcal{X})$ to (X, b). The quotient double complex $\mathrm{BP}(\mathcal{X})/\mathrm{BN}'(\mathcal{X})$ is denoted by $\mathrm{BC}(\mathcal{X})$. The homology groups $\mathrm{HC}_*(\mathcal{X})$, $\mathrm{HN}_*(\mathcal{X})$ and $\mathrm{HP}_*(\mathcal{X})$, of the total complexes of $\mathrm{BC}(\mathcal{X})$, $\mathrm{BN}(\mathcal{X})$ and $\mathrm{BP}(\mathcal{X})$ respectively, are called the *cyclic*, *negative* and *periodic homology groups* of $\mathcal{X}$. The homology $\mathrm{HH}_*(\mathcal{X})$, of (X, b), is called the *Hochschild homology* of $\mathcal{X}$. Finally, it is clear that a morphism $f \colon \mathcal{X} \to \mathcal{Y}$ of mixed complexes induces a morphism from the double complex $\mathrm{BP}(\mathcal{X})$ to the double complex $\mathrm{BP}(\mathcal{Y})$.

Following [4] by a double mixed complex we will understand a bigraded module X equipped with three k-linear maps of degree ± 1: b, that lowers the first index and fixes the second one, β, that fixes the first index and lowers the second one, and B, that fixes the first index and increases the second one. These maps satisfy

$$0 = b^2 = \beta^2 = B^2 = \beta \circ b + b \circ \beta = \beta \circ B + B \circ \beta = b \circ B + B \circ b.$$

The *total mixed complex* of a double mixed complex (X, b, β, B) is the mixed complex $(X, b + \beta, B)$, where $(X, b + \beta) \coloneqq \mathrm{Tot}(X, b, \beta)$ and $B_n \coloneqq \bigoplus_{i+j=n} B_{ij}$. By definition, the Hochschild, cyclic, periodic and negative homologies of (X, b, β, B) are the Hochschild, cyclic, periodic and negative homologies of $(X, b + \beta, B)$, respectively.

A morphism of double mixed complexes $f \colon (X, b, \beta, B) \longrightarrow (Y, d, \delta, D)$ is a family $f \colon X_{ij} \longrightarrow Y_{ij}$, such that $d \circ f = f \circ b$, $\delta \circ f = f \circ \beta$ and $D \circ f = f \circ B$. It is obvious that the correspondence $(X, b, \beta, B) \mapsto (X, b + \beta, B)$ is functorial.

1.3   The relative Hochschild and cyclic homologies

Let C be a K-algebra. By definition, the *normalized mixed complex of the K-algebra C* is the mixed complex $(C \otimes \overline{C}^{\otimes^*} \otimes, b, B)$, where b is the canonical Hochschild boundary map and the Connes operator B is given by

$$B([\mathbf{c}_{0r}]) := \sum_{i=0}^{r} [(-1)^{ir} \otimes \mathbf{c}_{ir} \otimes \mathbf{c}_{0,i-1}].$$

The *cyclic, negative, periodic* and *Hochschild homology groups* $\mathrm{HC}_*^K(C)$, $\mathrm{HN}_*^K(C)$, $\mathrm{HP}_*^K(C)$ and $\mathrm{HH}_*^K(C)$ of C are the respective homology groups of $(C \otimes \overline{C}^{\otimes^*} \otimes, b, B)$.

Let I be a two sided ideal of C and let $D := C/I$. The *cyclic, negative, periodic and Hochschild homologies* $\mathrm{HC}_*^K(C, I)$, $\mathrm{HN}_*^K(C, I)$, $\mathrm{HP}_*^K(C, I)$ and $\mathrm{HH}_*^K(C, I)$, *of the K-algebra C relative to I,* are by definition the respective homologies of the mixed complex

$$\ker\Big((C \otimes \overline{C}^{\otimes^*} \otimes, b, B) \xrightarrow{\;\;\pi\;\;} (D \otimes \overline{D}^{\otimes^*} \otimes, b, B) \Big),$$

where π is the map induced by the canonical projection from C onto D.

1.4   The perturbation lemma

Next, we recall the perturbation lemma. We present the version given in [5].

A *homotopy equivalence data*

$$(\mathrm{b}) \qquad (Y, \partial) \underset{i}{\overset{p}{\longleftarrow\!\!\!\longrightarrow}} (X, d) \qquad X_* \xrightarrow{\;\;h\;\;} X_{*+1}$$

consists of the following:

(1) Chain complexes (Y, ∂), (X, d) and quasi-isomorphisms i, p between them,

(2) A homotopy h from $i \circ p$ to id.

A *perturbation* of (b) is a map $\delta \colon X_* \longrightarrow X_{*-1}$ such that $(d + \delta)^2 = 0$. We call it *small* if $\mathrm{id} - \delta \circ h$ is invertible. In this case we write $A = (\mathrm{id} - \delta \circ h)^{-1} \circ \delta$ and we consider de diagram

$$(\mathrm{c}) \qquad (Y, \partial^1) \underset{i^1}{\overset{p^1}{\longleftarrow\!\!\!\longrightarrow}} (X, d) \qquad X_* \xrightarrow{\;\;h^1\;\;} X_{*+1}$$

where

$$\partial^1 := \partial + p \circ A \circ i, \quad i^1 := i + h \circ A \circ i, \quad p^1 := p + p \circ A \circ h, \quad h^1 := h + h \circ A \circ h.$$

A *deformation retract* is a homotopy equivalence data such that $p \circ i = \mathrm{id}$. A deformation retract is called *special* if $h \circ i = 0$, $p \circ h = 0$ and $h \circ h = 0$.

In all the cases considered in this paper the morphism $\delta \circ h$ is locally nilpotent. Consequently, $(\mathrm{id} - \delta \circ h)^{-1} = \sum_{n=0}^{\infty} (\delta \circ h)^n$.

Theorem 1.1 ([5])**.** *If δ is a small perturbation of* (b)*, then the diagram* (c) *is an homotopy equivalence data. Furthermore, if* (b) *is a special deformation retract, then so it is* (c)*.*

1.5 The suspension

The *suspension of a chain complex* (X, b) is the complex $(X, b)[1] := (X[1], b[1])$, defined by

$$X[1]_* := X_{*-1} \qquad \text{and} \qquad b[1]_* := -b_{*-1}.$$

Similarly, the *suspension of a chain double complex* (X, b, β) is the complex

$$(X, b, \beta)[0, 1] := (X[0, 1], b[0, 1], \beta[0, 1]),$$

defined by $X[0, 1]_{**} := X_{*, *-1}$, $b[0, 1]_{**} := -b_{*, *-1}$ and $\beta[0, 1]_{**} := -\beta_{*, *-1}$.

2 The relative cyclic homology of a cleft extension

Let C be a k-algebra and let $p \colon C \to A$ be a morphism of k-algebras. Assume there exists a k-algebra morphism $i \colon A \to C$ such that $p \circ i = \mathrm{id}$. Then $M := \ker(p)$ is naturally an A-bimodule endowed with an associative multiplication $\triangledown \colon M \otimes_A M \longrightarrow M$ and C is isomorphic to the cleft extension $E := A \ltimes_\triangledown M$. Let K be a k-subalgebra of A and let

$$(\check{\mathfrak{X}}, \check{\mathfrak{b}}, \check{\mathfrak{B}}) := \ker\big((E \otimes \overline{E}^{\otimes^*} \otimes, b, B) \xrightarrow{\ \pi\ } (A \otimes \overline{A}^{\otimes^*} \otimes, b, B)\big),$$

where π is the morphism induced by the canonical surjection of E on A. Using that E is a cleft extension of A, it is easy to see that the short exact sequence

$$0 \longrightarrow (\check{\mathfrak{X}}, \check{\mathfrak{b}}, \check{\mathfrak{B}}) \longrightarrow (E \otimes \overline{E}^{\otimes^*} \otimes, b, B) \xrightarrow{\ \pi\ } (A \otimes \overline{A}^{\otimes^*} \otimes, b, B) \longrightarrow 0$$

splits. Hence,

$$\mathrm{HH}^K(E) = \mathrm{HH}^K(A) \oplus \mathrm{HH}^K_*(E, M),$$
$$\mathrm{HC}^K(E) = \mathrm{HC}^K(A) \oplus \mathrm{HC}^K_*(E, M),$$
$$\mathrm{HP}^K(E) = \mathrm{HP}^K(A) \oplus \mathrm{HP}^K_*(E, M),$$
$$\mathrm{HN}^K(E) = \mathrm{HN}^K(A) \oplus \mathrm{HN}^K_*(E, M).$$

So in order to compute the type cyclic homologies of the K-algebra E, it suffices to calculate those of the K-algebra A and those of the K-algebra E relative to M. With this in mind, in this section we obtain a double mixed complex, simpler than the canonical one, giving the Hochschild, cyclic, periodic and negative homologies of the K-algebra E relative to M. Then we show that the cyclic homology of the K-algebra E relative to M is also given by a still simpler complex. From now on we often will use indices v, w and n, which always will satisfy the relation $n = v + w$.

Let $\hat{\mathfrak{X}}_{vw} := (M \otimes B^n_w \otimes) \oplus (A \otimes B^n_{w+1} \otimes)$. It is evident that $(\check{\mathfrak{X}}, \check{\mathfrak{b}})$ is the total complex of the second quadrant double complex

$$
\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & \\
\cdots \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{22} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{21} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{20} & \\
& \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & \\
\cdots \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{12} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{11} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{10} & \\
& \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & & \downarrow{\scriptstyle\hat{\mathfrak{b}}} & \\
\cdots \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{02} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{01} & \xrightarrow{\hat{\partial}} & \hat{\mathfrak{X}}_{00}, &
\end{array}
$$

where the boundary maps

$$\hat{\mathfrak{b}} \colon \hat{\mathfrak{X}}_{vw} \longrightarrow \hat{\mathfrak{X}}_{v-1,w} \qquad \text{and} \qquad \hat{\mathfrak{d}} \colon \hat{\mathfrak{X}}_{vw} \longrightarrow \hat{\mathfrak{X}}_{v,w-1}$$

are defined by the formulas

$$\hat{\mathfrak{b}}(\mathbf{x}, \mathbf{y}) \coloneqq (\mathfrak{b}^0(\mathbf{x}) + \alpha(\mathbf{y}), \mathfrak{b}^1(\mathbf{y})) \qquad \text{and} \qquad \hat{\mathfrak{d}}(\mathbf{x}, \mathbf{y}) \coloneqq \sum_{j=0}^{n} (\varrho_j(\mathbf{x}), \varrho_j(\mathbf{y})),$$

in which

$$\mathfrak{b}^0 \coloneqq \sum_{j=0}^{n} \mu_l, \qquad \mathfrak{b}^1 \coloneqq \mu_0^A + \sum_{j=1}^{n-1} \mu_l + \mu_n^A \qquad \text{and} \qquad \alpha \coloneqq \mu_0^M + \mu_n^M.$$

Furthermore $(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}})$, where $\hat{\mathfrak{B}}$ is defined by

$$\hat{\mathfrak{B}}(\mathbf{x}, \mathbf{y}) \coloneqq (0, B(\mathbf{x}) + B(\mathbf{y})),$$

is a double mixed complex and its total mixed complex is $(\check{\mathfrak{X}}, \check{\mathfrak{b}}, \check{\mathfrak{B}})$.

2.1   Complexes for the relative homologies

For $v, w \geq 0$, let $X_{vw} \coloneqq M \otimes B_w^n \otimes$. By convenience we put $X_{vw} \coloneqq 0$, otherwise. Consider the triple diagram

where

$$b(\mathbf{x}_0^n) \coloneqq \sum_{j=0}^{n} \mu_j(\mathbf{x}_0^n), \qquad\qquad d'(\mathbf{x}_0^n) \coloneqq \sum_{j=0}^{n-1} \varrho_j(\mathbf{x}_0^n),$$

$$d(\mathbf{x}_0^n) \coloneqq d'(\mathbf{x}_0^n) + \varrho_n(\mathbf{x}_0^n), \qquad N(\mathbf{x}_0^n) \coloneqq \sum_{l=0}^{w} t^l(\mathbf{x}_0^n),$$

the middle face (X, b, d) is the 0-th face and the bottom row is the 0-th row. Note that the map $N \colon X_{v0} \longrightarrow X_{v0}$ is the identity map and the map $\mathrm{id} - t \colon X_{v0} \longrightarrow X_{v0}$ is the zero map.

For $l \in \mathbb{Z}$, let $\tau^l(\mathcal{X})$ be the subdiagram of $\mathcal{X}$ obtaining by deleting the faces (X, b, d) and $(X, -b, d')$ placed in columns with index greater than l, and let $\tau_0(\mathcal{X})$ and $\tau_0^1(\mathcal{X})$ be the quotient triple diagrams of $\mathcal{X}$ by $\tau^{-1}(\mathcal{X})$ and $\tau^1(\mathcal{X})$ by $\tau^{-1}(\mathcal{X})$, respectively.

Theorem 2.1. $\mathcal{X}$ *is a triple complex. Moreover*

$$\mathrm{HH}_*^K(E, M) = \mathrm{H}_*(\mathrm{Tot}(\tau_0^1(\mathcal{X}))),$$
$$\mathrm{HC}_*^K(E, M) = \mathrm{H}_*(\mathrm{Tot}(\tau_0(\mathcal{X}))),$$
$$\mathrm{HP}_*^K(E, M) = \mathrm{H}_*(\mathrm{Tot}(\mathcal{X})),$$
$$\mathrm{HN}_*^K(E, M) = \mathrm{H}_*(\mathrm{Tot}(\tau^1(\mathcal{X}))).$$

Consequently, the following equalities hold:

$$d \circ b = -b \circ d, \qquad\qquad d' \circ b = -b \circ d', \qquad d' \circ N = N \circ d,$$
$$d \circ (\mathrm{id} - t) = (\mathrm{id} - t) \circ d', \qquad b \circ N = N \circ b, \qquad t \circ b = b \circ t.$$

We will use these equalities freely throughout the paper.

Theorem 2.1 is a consequence of Theorem 2.2, which we enounce below and whose proof will be relegated to Appendix A. For $v, w \geq 0$, let $\hat{X}_{vw} := X_{vw} \oplus X_{v-1,w}$. Consider the diagram $(\hat{X}, \hat{b}, \hat{d})$ where the maps

$$\hat{b} \colon \hat{X}_{vw} \longrightarrow \hat{X}_{v-1,w} \qquad \text{and} \qquad \hat{d} \colon \hat{X}_{vw} \longrightarrow \hat{X}_{v,w-1}$$

are defined by

$$\hat{b}(\mathbf{x}, \mathbf{y}) := \big(b(\mathbf{x}) + (\mathrm{id} - t)(\mathbf{y}), -b(\mathbf{y})\big) \qquad \text{and} \qquad \hat{d}(\mathbf{x}, \mathbf{y}) := \big(d(\mathbf{x}), -d'(\mathbf{y})\big).$$

Note that $(\hat{X}, \hat{b}, \hat{d})$ is one of the partial total complexes of the triple complex $\tau_0^1(\mathcal{X})$, and so $\mathrm{Tot}(\tau_0^1(\mathcal{X})) = \mathrm{Tot}(\hat{X}, \hat{b}, \hat{d})$ (but we have not proven that $\tau_0^1(\mathcal{X})$ is a triple complex, yet). Let

$$\hat{B} \colon \hat{X}_{vw} \longrightarrow \hat{X}_{v+1,w}, \qquad \hat{\theta} \colon \hat{\mathfrak{X}}_{vw} \longrightarrow \hat{X}_{vw} \qquad \text{and} \qquad \hat{\vartheta} \colon \hat{X}_{vw} \longrightarrow \hat{\mathfrak{X}}_{vw}$$

be the maps defined by

$$\hat{B}(\mathbf{x}, \mathbf{y}) := (0, N(\mathbf{x})),$$
$$\hat{\theta}(\mathbf{x}, \mathbf{y}) := (\mathbf{x} + t(\mathbf{y}), \mu_0^M(\mathbf{y})),$$
$$\hat{\vartheta}(\mathbf{x}, \mathbf{y}) := \sum_{l=0}^{n-i(\mathbf{y})} \big(\mathbf{x}, 1 \otimes \mathsf{t}^l(\mathbf{y})\big),$$

where $\mathsf{t}(y_0 \otimes \cdots \otimes y_{n-1}) := (-1)^{n-1} y_{n-1} \otimes y_0 \otimes \cdots \otimes y_{n-2}$.

Theorem 2.2. *The following assertions hold:*

(1) $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ *is a double mixed complex.*

(2) *Let* $(\check{X}, \check{b})$ *be the total complex of* $(\hat{X}, \hat{b}, \hat{d})$. *The maps*

$$\hat{\vartheta} \colon (\hat{X}, \hat{b}, \hat{d}, \hat{B}) \longrightarrow (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}) \qquad and \qquad \hat{\theta} \colon (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}) \longrightarrow (\hat{X}, \hat{b}, \hat{d}, \hat{B}),$$

are morphisms of double mixed complexes such that $\hat{\theta} \circ \hat{\vartheta} = \mathrm{id}$. *Moreover* $\check{\vartheta} \circ \check{\theta}$ *is homotopic to the identity map, where*

$$\check{\vartheta} \colon (\check{X}, \check{b}) \longrightarrow (\check{\mathfrak{X}}, \check{\mathfrak{b}}) \qquad and \qquad \check{\theta} \colon (\check{\mathfrak{X}}, \check{\mathfrak{b}}) \longrightarrow (\check{X}, \check{b}),$$

are the morphisms induced by $\hat{\vartheta}$ *and* $\hat{\theta}$, *respectively.*

(3) *The Hochschild, cyclic, periodic and negative homologies of* $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ *are the Hochschild, cyclic, periodic and negative homologies of the K-algebra E relative to M, respectively.*

Proof. See Appendix A. $\qquad\square$

Proof of Theorem 2.1. Let $(\breve{X}, \breve{b}, \breve{B})$ be the mixed complex associated with $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$. Theorem 2.1 follows immediately from Theorem 2.2 and the fact that

$$\mathrm{Tot}(\tau_0^1(\mathcal{X})) = (\breve{X}, \breve{b}),$$
$$\mathrm{Tot}(\tau_0(\mathcal{X})) = \mathrm{Tot}(\mathrm{BC}(\breve{X}, \breve{b}, \breve{B})),$$
$$\mathrm{Tot}(\mathcal{X}) = \mathrm{Tot}(\mathrm{BP}(\breve{X}, \breve{b}, \breve{B})),$$
$$\mathrm{Tot}(\tau^1(\mathcal{X})) = \mathrm{Tot}(\mathrm{BN}(\breve{X}, \breve{b}, \breve{B})),$$

which can be easily checked. $\qquad\square$

The following result was proven in [12].

Lemma 2.3. *The rows of the triple complex $\mathcal{X}$ are contractible.*

Proof. For $v, w \ge 0$, let $\sigma, \sigma' \colon X_{vw} \longrightarrow X_{vw}$ be the maps defined by

$$\sigma := \frac{1}{w+1}\,\mathrm{id} \qquad \text{and} \qquad \sigma' := \sum_{j=0}^{w-1} \frac{w-j}{w+1} t^j.$$

A direct computation shows that:

$$\sigma \circ N = N \circ \sigma = \frac{1}{w+1} N, \tag{2.2}$$

$$(\mathrm{id} - t) \circ \sigma' = \sigma' \circ (\mathrm{id} - t) = \sum_{j=0}^{w-1} \frac{w-j}{w+1} t^j - \sum_{j=1}^{w} \frac{w-j+1}{w+1} t^j = \mathrm{id} - \frac{1}{w+1} N. \tag{2.3}$$

The result follows immediately from these equalities. $\qquad\square$

Theorem 2.4. *Let $(\overline{X}, \overline{b}, \overline{d})$ be the cokernel of $\mathrm{id} - t \colon (X, -b, -d') \longrightarrow (X, b, d)$. The relative cyclic homology $\mathrm{HC}_*^K(E, M)$ is the homology of $(\overline{X}, \overline{b}, \overline{d})$.*

Proof. This follows immediately from Theorem 2.1 and Lemma 2.3. $\qquad\square$

2.2   Graded algebras

Let $E = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded algebra. Let $M := A_1 \oplus A_2 \oplus \cdots$ be the augmentation ideal of E. Clearly E is a cleft extension of $A := A_0$ by M. Let K be a k-subalgebra of A. As it is well known, the double mixed complex $(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}})$ decompose as the direct sum

$$(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}) = \bigoplus_{i \in \mathbb{N}} (\hat{\mathfrak{X}}^{(i)}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}), \tag{2.4}$$

where $\hat{\mathfrak{X}}_{vw}^{(i)}$ is the k-submodule of $\hat{\mathfrak{X}}_{vw}$ generated by the simple tensors $a_0 \otimes \cdots \otimes a_n$ such that each a_j is homogeneous of degree $|a_j|$ and $\sum_j |a_j| = i$. Similarly, the double mixed complex $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ decompose as the direct sum

$$(\hat{X}, \hat{b}, \hat{d}, \hat{B}) = \bigoplus_{i \in \mathbb{N}} (\hat{X}^{(i)}, \hat{b}, \hat{d}, \hat{B}),$$

where $\hat{X}^{(i)}$ is defined in the same manner as $\hat{\mathfrak{X}}_{vw}^{(i)}$, and the morphisms $\hat{\vartheta}$ and $\hat{\theta}$ are compatible with these decompositions. Moreover these maps induce homotopy equivalence between the complexes $(\hat{X}^{(i)}, \hat{b}, \hat{d}, \hat{B})$ and $(\hat{\mathfrak{X}}^{(i)}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}})$. In order to check this fact it suffices to note that the homotopy $\breve{\epsilon}$, introduced in item 4) of Lemma A.3, is compatible with the decomposition (2.4).

3 The harmonic decomposition

As in the proof of Theorem 2.1, we let $(\breve{X}, \breve{b}, \breve{B})$ denote the mixed complex associated with the double mixed complex $(\hat{X}, \hat{b}, \hat{d}, \hat{B})$, introduced in Theorem 2.2. Our purpose in this section is to show that $(\breve{X}, \breve{b}, \breve{B})$ has a harmonic decomposition like the one studied in [7]. In order to carry out this task we need to define a de Rham coboundary map and a Karoubi operator on $(\breve{X}, \breve{b})$. As we said in the introduction we are going to work with a new double mixed complex $(\ddot{X}, \ddot{b}, \ddot{d}, \ddot{B})$, whose associated mixed complex is also $(\breve{X}, \breve{b}, \breve{B})$. In the first three subsections we follow closely the exposition of [7].

3.1 The Rham coboundary map and the Karoubi operator

It is easy to see that $\tau_0^1(\mathcal{X})$ is the total complex of the double complex

$$
(\ddot{X}, \ddot{b}, \ddot{d}) :=
$$

$$
\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & \\
\cdots \xrightarrow{\ddot{d}} & \ddot{X}_{22} & \xrightarrow{\ddot{d}} & \ddot{X}_{21} & \xrightarrow{\ddot{d}} & \ddot{X}_{20} & \\
& \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & \\
\cdots \xrightarrow{\ddot{d}} & \ddot{X}_{12} & \xrightarrow{\ddot{d}} & \ddot{X}_{11} & \xrightarrow{\ddot{d}} & \ddot{X}_{10} & \\
& \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & & \downarrow{\scriptstyle\ddot{b}} & \\
\cdots \xrightarrow{\ddot{d}} & \ddot{X}_{02} & \xrightarrow{\ddot{d}} & \ddot{X}_{01} & \xrightarrow{\ddot{d}} & \ddot{X}_{00}, &
\end{array}
$$

where $\ddot{X}_{vw} := X_{vw} \oplus X_{v,w-1}$ and the boundary maps are defined by

$$\ddot{b}(\mathbf{x}, \mathbf{y}) := (b(\mathbf{x}), -b(\mathbf{y})) \qquad \text{and} \qquad \ddot{d}(\mathbf{x}, \mathbf{y}) := (d(\mathbf{x}) + (\mathrm{id} - t)(\mathbf{y}), -d'(\mathbf{y})).$$

The de Rham coboundary map $\ddot{d}_R \colon \ddot{X}_{vw} \longrightarrow \ddot{X}_{v,w+1}$ is defined by $\ddot{d}_R(\mathbf{x}, \mathbf{y}) := (0, \mathbf{x})$. It is obvious that $(\ddot{X}, \ddot{d}_R)$ is acyclic. We now define the Karoubi operator of $\ddot{X}$. Let

$$\ddot{\kappa}(0) \colon \ddot{X}_{vw} \longrightarrow \ddot{X}_{vw} \qquad \text{and} \qquad \ddot{\kappa}(1) \colon \ddot{X}_{vw} \longrightarrow \ddot{X}_{vw}$$

be the maps defined by

$$\ddot{\kappa}(0)(\mathbf{x}, \mathbf{y}) := (t(\mathbf{x}), t(\mathbf{y})) \qquad \text{and} \qquad \ddot{\kappa}(1)(\mathbf{x}, \mathbf{y}) := \big(0, d'(\mathbf{x}) - d(\mathbf{x})\big).$$

The Karoubi operator $\ddot{\kappa}$ of $\ddot{X}$ is the degree zero operator defined by

$$\ddot{\kappa} := \ddot{\kappa}(0) + \ddot{\kappa}(1).$$

Let $\breve{d}_R \colon \breve{X}_n \longrightarrow \breve{X}_{n+1}$ and $\breve{\kappa} \colon \breve{X}_n \to \breve{X}_n$ be the maps defined by

$$\breve{d}_{R_n} := \bigoplus_{w=0}^{n} \ddot{d}_{R_{n-w,w}} \qquad \text{and} \qquad \breve{\kappa}_n := \bigoplus_{w=0}^{n} \ddot{\kappa}_{n-w,w},$$

respectively. A direct computation shows that

$$\mathrm{id} - \ddot{\kappa} = \ddot{d} \circ \ddot{d}_R + \ddot{d}_R \circ \ddot{d} \qquad \text{and} \qquad 0 = \ddot{b} \circ \ddot{d}_R + \ddot{d}_R \circ \ddot{b}. \tag{3.5}$$

In particular, $\ddot{\kappa}$ is homotopic to the identity with respect to either of the differentials $\ddot{d}$, $\ddot{d}_R$, and so it commutes with them. From (3.5) it follows that

$$\mathrm{id} - \breve{\kappa} = \breve{b} \circ \breve{d}_R + \breve{d}_R \circ \breve{b}.$$

Consequently $\breve{\kappa}$ commutes with $\breve{b}$ and $\breve{d}_R$. Hence $\ddot{\kappa}$ also commutes with $\ddot{b}$ (which can be also proven by a direct computation). Let $\ddot{B}\colon \ddot{X}_{vw} \longrightarrow \ddot{X}_{v,w+1}$ be the map defined by $\ddot{B}(\mathbf{x},\mathbf{y}) \coloneqq (0, N(\mathbf{x}))$. An easy computation shows that $(\ddot{X}, \ddot{b}, \ddot{d}, \ddot{B})$ is a double mixed complex and that its associated mixed complex is $(\breve{X}, \breve{b}, \breve{B})$. Furthermore,

$$\ddot{B}(\mathbf{x}) = \sum_{i=0}^{w} \ddot{\kappa}^i \circ \ddot{d}_R(\mathbf{x}) \qquad \text{for all } \mathbf{x} \in \ddot{X}_{vw}.$$

Using this we obtain

$$\ddot{B} \circ \ddot{\kappa} = \ddot{\kappa} \circ \ddot{B} = \ddot{B} \qquad \text{and} \qquad \ddot{d}_R \circ \ddot{B} = \ddot{B} \circ \ddot{d}_R = 0.$$

3.2   The harmonic decomposition

From the definition of $\ddot{\kappa}$ it follows immediately that

$$(\ddot{\kappa}^w - \mathrm{id}) \circ (\ddot{\kappa}^{w+1} - \mathrm{id})(\ddot{X}_{vw}) \subseteq (\kappa^w - \mathrm{id})(0 \oplus X_{v,w-1}) = 0.$$

This implies that $\ddot{\kappa}$ satisfies the polynomial equation $P_w(\ddot{\kappa}) = 0$ on $\ddot{X}_{vw}$, where

$$P_w = (X^{w+1} - 1)(X^w - 1).$$

The roots of P_w are the r-th roots of unity, with $r = w + 1$ and $r = w$. Moreover, 1 is a double root and the all other roots are simple. Consequently $\ddot{X}_{vw}$ decomposes into the direct sum of the generalized eigenspace $\ker(\ddot{\kappa} - \mathrm{id})^2$ and its complement $\mathrm{Im}(\ddot{\kappa} - \mathrm{id})^2$. Combining this for all v, w we obtain the following decomposition

$$\ddot{X} = \ker(\ddot{\kappa} - \mathrm{id})^2 \oplus \mathrm{Im}(\ddot{\kappa} - \mathrm{id})^2,$$

Each of these generalized subspaces is stable under any operator commuting with $\ddot{\kappa}$, for instance, $\ddot{b}$, $\ddot{d}$, $\ddot{d}_R$ and $\ddot{B}$.

3.3   The harmonic projection and the Green operator

Let P be the harmonic projection operator, which is the identity map on $\ker(\ddot{\kappa} - \mathrm{id})^2$ and the zero map on $\mathrm{Im}(\ddot{\kappa} - \mathrm{id})^2$. Thus we have

$$\ddot{X} = P(\ddot{X}) \oplus P^{\perp}(\ddot{X}),$$

where $P^{\perp} \coloneqq \mathrm{id} - P$. It is clear that $(P(\ddot{X}), \ddot{b}, \ddot{d}, \ddot{B})$ and $(P^{\perp}(\ddot{X}), \ddot{b}, \ddot{d}, \ddot{B})$ are double mixed sub-complexes of $(\ddot{X}, \ddot{b}, \ddot{d}, \ddot{B})$. On $P^{\perp}(\ddot{X})$ the operator

$$\mathrm{id} - \ddot{\kappa} = \ddot{d} \circ \ddot{d}_R + \ddot{d}_R \circ \ddot{d}$$

is both invertible and homotopic to zero with respect to either differential $\ddot{d}$ and $\ddot{d}_R$. Hence the complexes $(P^{\perp}(\ddot{X}), \ddot{d})$ and $(P^{\perp}(\ddot{X}), \ddot{d}_R)$ are acyclic. Let

$$P(\breve{X}_n) = \bigoplus_{w=0}^{n} P(\ddot{X}_{n-w,w}) \qquad \text{and} \qquad P^{\perp}(\breve{X}_n) = \bigoplus_{w=0}^{n} P^{\perp}(\ddot{X}_{n-w,w}).$$

The same argument shows that $(P^{\perp}(\breve{X}), \breve{b})$ and $(P^{\perp}(\breve{X}), \breve{d}_R)$ are also acyclic. The Green operator $G\colon \ddot{X} \to \ddot{X}$ is defined to be zero on $P(\ddot{X})$ and the inverse of $\mathrm{id} - \ddot{\kappa}$ on $P^{\perp}(\ddot{X})$. It is clear that

$$G \circ P = P \circ G = 0 \qquad \text{and} \qquad P^{\perp} = G \circ (\mathrm{id} - \ddot{\kappa}) = G \circ (\ddot{d} \circ \ddot{d}_R + \ddot{d}_R \circ \ddot{d}). \tag{3.6}$$

Moreover P and G commute with each operator that commutes with $\ddot{\kappa}$.

Proposition 3.1. *The following equality holds:*

$$P^\perp(\ddot X) = \dot d \circ P^\perp(\ddot X) \oplus \ddot d_R \circ P^\perp(\ddot X).$$

Furthermore $\ddot d_R$ maps $\dot d \circ P^\perp(\ddot X)$ isomorphically onto $\ddot d_R \circ P^\perp(\ddot X)$ with inverse $G \circ \dot d$ and $\dot d$ maps $\ddot d_R \circ P^\perp(\ddot X)$ isomorphically onto $\dot d \circ P^\perp(\ddot X)$ with inverse $G \circ \ddot d_R$.

Proof. The proof of [7, Proposition 2.1] works in our setting. □

The above proposition gives a new proof that $(P^\perp(\ddot X), \dot d)$ and $(P^\perp(\ddot X), \ddot d_R)$ are acyclic.

Proposition 3.2. *An element $\mathbf{x} \in \ddot X_{vw}$ belongs to $P(\ddot X_{vw})$ if and only if $\ddot d_R(\mathbf{x})$ and $\ddot d_R \circ \dot d(\mathbf{x})$ are $\ddot\kappa$-invariant.*

Proof. The proof of [7, Proposition 2.2] works in our setting. □

For $v, w \ge 0$ let $\grave X_{vw}$ and $\acute X_{vw}$ be the image of the canonical inclusions of X_{vw} into $\ddot X_{vw}$ and $\ddot X_{v,w+1}$ respectively, and let $\acute\kappa \colon \acute X_{vw} \longrightarrow \acute X_{vw}$ be the map induced by $\ddot\kappa$. For $\mathbf{x} = (\mathbf{x}_0, 0) \in \grave X_{vw}$ we write

$$\grave b(\mathbf{x}) := (b(\mathbf{x}_0), 0) \qquad \text{and} \qquad \grave d(\mathbf{x}) := (d(\mathbf{x}_0), 0),$$

and, for $\mathbf{y} = (0, \mathbf{y}_0) \in \acute X_{vw}$, we write

$$\acute b(\mathbf{y}) := (0, b(\mathbf{y}_0)), \qquad \acute{d'}(\mathbf{y}) := (0, d'(\mathbf{y}_0)) \qquad \text{and} \qquad \acute t(\mathbf{y}) := (0, t(\mathbf{y}_0)).$$

It is obvious that $\acute\kappa$ coincides with $\acute t$. Note that $\ddot\kappa$ has finite order on $\ddot d_R(\ddot X) = \acute X$ in each degree. In fact $\ddot\kappa^{w+1} = \mathrm{id}$ on $\acute X_{vw}$. By the discussion in the page 86 of [7],

$$\acute X_{vw} = \ker(\mathrm{id} - \acute\kappa) \oplus \mathrm{Im}(\mathrm{id} - \acute\kappa), \tag{3.7}$$

$$P(\ddot d_R(\grave X_{vw})) = P(\acute X_{vw}) = \ker(\mathrm{id} - \acute\kappa), \tag{3.8}$$

$$P^\perp(\ddot d_R(\grave X_{vw})) = P^\perp(\acute X_{vw}) = \mathrm{Im}(\mathrm{id} - \acute\kappa), \tag{3.9}$$

and the maps $P_{\acute X_{vw}}$ and $G_{\acute X_{vw}}$, defined as the projection onto $\ker(\mathrm{id} - \acute\kappa)$ associated with the decomposition (3.7) and the Green operator for $\mathrm{id} - \acute\kappa \colon \acute X_{vw} \longrightarrow \acute X_{vw}$, respectively, satisfy:

$$P_{\acute X_{vw}} = \frac{1}{w+1} \sum_{i=0}^{w} \acute\kappa^i \qquad \text{and} \qquad G_{\acute X_{vw}} = \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i\right) \acute\kappa^i. \tag{3.10}$$

Consequently, for all $\mathbf{x} \in \ddot X_{vw}$,

$$P \circ \ddot d_R(\mathbf{x}) = \frac{1}{w+1} \sum_{i=0}^{w} \ddot\kappa^i \circ \ddot d_R(\mathbf{x}) = \frac{1}{w+1} \ddot B(\mathbf{x}) \tag{3.11}$$

and

$$G \circ \ddot d_R(\mathbf{x}) = \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i\right) \ddot\kappa^i \circ \ddot d_R(\mathbf{x}). \tag{3.12}$$

Formula (3.11) implies that

$$\ddot B(P^\perp(\ddot X)) = 0 \tag{3.13}$$

and

$$\ddot B(\mathbf{x}) = (w+1) \ddot d_R(P(\mathbf{x})) \qquad \text{for all } \mathbf{x} \in \ddot X_{vw}.$$

So, since $(P^\perp(\ddot X), \ddot d_R)$ is acyclic,

$$H_*(P(\ddot X), \ddot B) = H_*(P(\ddot X), \ddot d_R) = H_*(\ddot X, \ddot d_R) = 0. \tag{3.14}$$

In the terminology of [7] this says that $(P(\ddot X), \dot d, \ddot b, \ddot B)$ is $\ddot B$-acyclic. Lastly, (3.10) combined with (3.12) and the second formula of (3.6), allows us to obtain an explicit formula for P. In

fact, for $\mathbf{x} \in \acute{X}_{vw}$, this is given by the first equality in (3.10). Hence, assume $\mathbf{x} \in \grave{X}_{vw}$. Since by (3.12) we know that $G \circ \ddot{d}_R(\mathbf{x}) \in \acute{X}_{vw}$, we have:

$$G \circ \ddot{d} \circ \ddot{d}_R(\mathbf{x}) = \ddot{d} \circ G \circ \ddot{d}_R(\mathbf{x}) = -\acute{d}' \circ G \circ \ddot{d}_R(\mathbf{x}) + \mathrm{sw} \circ (\mathrm{id} - \acute{t}) \circ G \circ \ddot{d}_R(\mathbf{x}),$$

where $\mathrm{sw} \colon \acute{X}_{vw} \longrightarrow \grave{X}_{vw}$ is the map defined by $\mathrm{sw}(0, \mathbf{x}) := (\mathbf{x}, 0)$. Using this, the second equalities in (3.6) and (3.10), and the fact that $\acute{t} \circ \ddot{\kappa}^i \circ \ddot{d}_R(\mathbf{x}) = \ddot{\kappa}^{i+1} \circ \ddot{d}_R(\mathbf{x})$, we obtain:

(1) If $w = 0$, then $P(\mathbf{x}) = \mathbf{x}$.

(2) If $w > 0$, then

$$\begin{aligned}
P(\mathbf{x}) &= \mathbf{x} - G \circ \ddot{d}_R \circ \grave{d}(\mathbf{x}) - G \circ \ddot{d} \circ \ddot{d}_R(\mathbf{x}) \\
&= \mathbf{x} - \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \ddot{\kappa}^i \circ \ddot{d}_R \circ \grave{d}(\mathbf{x}) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \acute{d}' \circ \ddot{\kappa}^i \circ \ddot{d}_R(\mathbf{x}) \\
&\quad - \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \mathrm{sw} \circ (\mathrm{id} - \acute{t}) \circ \ddot{\kappa}^i \circ \ddot{d}_R(\mathbf{x}) \\
&= \frac{1}{w+1} \mathrm{sw} \circ \ddot{B}_n(\mathbf{x}) - \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \ddot{\kappa}^i \circ \ddot{d}_R \circ \grave{d}(\mathbf{x}) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \acute{d}' \circ \ddot{\kappa}^i \circ \ddot{d}_R(\mathbf{x}).
\end{aligned}$$

Summarizing,

$$P(0, \mathbf{y}) = \frac{1}{w}(0, N(\mathbf{y})) \qquad\qquad \text{for } (0, \mathbf{y}) \in \ddot{X}_{vw} \text{ with } w > 0, \qquad (3.15)$$

$$P(\mathbf{x}, 0) = (\mathbf{x}, 0) \qquad\qquad \text{for } (\mathbf{x}, 0) \in \ddot{X}_{v0}, \qquad (3.16)$$

$$\begin{aligned}
P(\mathbf{x}, 0) &= \frac{1}{w+1}\big(N(\mathbf{x}), 0\big) \\
&\quad - \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \big(0, t^i(d(\mathbf{x}))\big) \qquad \text{for } (\mathbf{x}, 0) \in \ddot{X}_{vw} \text{ with } w > 0. \qquad (3.17) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \big(0, d'(t^i(\mathbf{x}))\big).
\end{aligned}$$

Remark 3.3. Take $\mathbf{x} := (\mathbf{x}_0, \mathbf{x}_1) \in \ddot{X}_{vw}$, with $\mathbf{x}_0 \in X_{vw}$ and $\mathbf{x}_1 \in X_{v,w-1}$. By Proposition 3.2 we know that $\mathbf{x} \in P(\ddot{X})$ if and only if $\mathbf{x}_0$ and $d(\mathbf{x}_0) + (\mathrm{id} - t)(\mathbf{x}_1)$ are t-invariant. From this it follows immediately that if $\mathbf{x} \in P(\ddot{X})$, then $(\mathbf{x}_0, \mathbf{x}_1') \in P(\ddot{X})$ for all $\mathbf{x}_1' \in X_{v,w-1}$ such that $\mathbf{x}_1' - \mathbf{x}_1$ is a t-invariant element. Conversely, if $\mathbf{x}$ and $(\mathbf{x}_0, \mathbf{x}_1')$ belong to $P(\ddot{X})$, then $\mathbf{z} := (\mathrm{id} - t)(\mathbf{x}_1' - \mathbf{x}_1)$ is t-invariant, but this implies that

$$w\mathbf{z} = N(\mathbf{z}) = N \circ (\mathrm{id} - t)(\mathbf{x}_1' - \mathbf{x}_1) = 0.$$

In other words, that $\mathbf{x}_1' - \mathbf{x}_1$ is t-invariant (note that if $w = 0$, then $\mathbf{x}_1' = \mathbf{x}_1 = 0$). Let ${}^t P(\ddot{X}_{vw})$ be the set of all elements of the shape $(0, \mathbf{x}_1) \in \ddot{X}_{vw}$ with $\mathbf{x}_1$ a t-invariant element. By the previous computations

$${}^t P(\ddot{X}_{vw}) = P(\ddot{X}_{vw}) \cap \acute{X}_{v,w-1}.$$

It is evident that $({}^t P(\ddot{X}), -\acute{b}, -\acute{d}')$ is a subcomplex of $(P(\ddot{X}), \ddot{b}, \ddot{d})$.

Proposition 3.4. *Assume that $w > 0$. Then*

$$P(t(\mathbf{x}), 0) = P(\mathbf{x}, 0) + \frac{1}{w}(0, N \circ \varrho_n(\mathbf{x})) - \frac{1}{w(w+1)}(0, N \circ d(\mathbf{x})) \quad and \quad P(0, t(\mathbf{y})) = P(0, \mathbf{y})$$

for all $(\mathbf{x}, 0), (0, \mathbf{y}) \in \ddot{X}_{vw}$.

Proof. By equality (3.16), for $(0, \mathbf{y}) \in \ddot{X}_{vw}$ we have

$$P(0, \mathbf{y}) = \frac{1}{w}(0, N(\mathbf{y})) = \frac{1}{w}(0, N(t(\mathbf{y}))) = P(0, t(\mathbf{y})).$$

Next we obtain the formula for $P(t(\mathbf{x}), 0)$. By equality (3.17),

$$\begin{aligned}
P(t(\mathbf{x}), 0) - P(\mathbf{x}, 0) &= \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \left(0, (t^i - t^{i+1}) \circ d(\mathbf{x}) \right) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \left(0, d' \circ (t^{i+1} - t^i)(\mathbf{x}) \right) \\
&= \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \left(0, (t^i - t^{i+1}) \circ d(\mathbf{x}) \right) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \left(0, d \circ (t^{i+1} - t^i)(\mathbf{x}) \right) \\
&\quad - \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \left(0, \varrho_n \circ (t^{i+1} - t^i)(\mathbf{x}) \right) \\
&= \frac{1}{w} \sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i \right) \left(0, (\mathrm{id} - t) \circ t^i \circ d(\mathbf{x}) \right) \\
&\quad + \frac{1}{w+1} \sum_{i=0}^{w} \left(\frac{w}{2} - i \right) \left(0, (t - \mathrm{id}) \circ d' \circ t^i(\mathbf{x}) \right) \\
&\quad + \left(0, \varrho_n(\mathbf{x}) \right) - \frac{1}{w+1} \left(0, \varrho_n \circ N(\mathbf{x}) \right),
\end{aligned}$$

because

$$\sum_{i=0}^{w} \left(\frac{w}{2} - i \right) (t^{i+1} - t^i) = -(w+1)t^0 + N.$$

Hence

$$P(t(\mathbf{x}), 0) - P(\mathbf{x}, 0) = \left(0, (t - \mathrm{id})(\mathbf{x}') \right) + \left(0, \varrho_n(\mathbf{x}) \right) - \frac{1}{w+1} \left(0, \varrho_n \circ N(\mathbf{x}) \right),$$

for some $\mathbf{x}' \in X_{v,w-1}$. But by Remark 3.3 we know that

$$(t - \mathrm{id})(\mathbf{x}') + \varrho_n(\mathbf{x}) - \frac{1}{w+1} \varrho_n \circ N(\mathbf{x})$$

is t-invariant, and so

$$\begin{aligned}
(t - \mathrm{id})(\mathbf{x}') + \varrho_n(\mathbf{x}) - \frac{1}{w+1} \varrho_n \circ N(\mathbf{x}) &= \frac{1}{w} N \circ \varrho_n(\mathbf{x}) - \frac{1}{w(w+1)} N \circ \varrho_n \circ N(\mathbf{x}) \\
&= \frac{1}{w} N \circ \varrho_n(\mathbf{x}) - \frac{1}{w(w+1)} N \circ d(\mathbf{x}).
\end{aligned}$$

The desired formula for $P(t(\mathbf{x}), 0)$ follows immediately from this fact. $\square$

Proposition 3.5. *For $1 \le i \le w$ and $(\mathbf{x}, 0) \in \ddot{X}_{vw}$, we have*

$$P(t^i(\mathbf{x}), 0) = P(\mathbf{x}, 0) + \frac{1}{w} \sum_{j=0}^{i-1} (0, N \circ \varrho_n \circ t^j(\mathbf{x})) - \frac{i}{w(w+1)} (0, N \circ d(\mathbf{x})). \tag{3.18}$$

Proof. By induction on i. Case $i = 1$ is Proposition 3.4. Assume $0 < i < w$ and that formula (3.18) is valid for i. Then, again by Proposition 3.4,

$$P(t^{i+1}(\mathbf{x}), 0) = P(t^i(\mathbf{x}), 0) + \frac{1}{w}(0, N \circ \varrho_n \circ t^i(\mathbf{x})) - \frac{1}{w(w+1)}(0, N \circ d \circ t^i(\mathbf{x}))$$

$$= P(t^i(\mathbf{x}), 0) + \frac{1}{w}(0, N \circ \varrho_n \circ t^i(\mathbf{x})) - \frac{1}{w(w+1)}(0, N \circ d(\mathbf{x})),$$

where the last equality follows from the fact that $N \circ d \circ t^i = d' \circ N \circ t^i = d' \circ N = N \circ d$. Consequently, by the inductive hypothesis,

$$P(t^{i+1}(\mathbf{x}), 0) = P(\mathbf{x}, 0) + \frac{1}{w} \sum_{j=0}^{i} (0, N \circ \varrho_n \circ t^j(\mathbf{x})) - \frac{i+1}{w(w+1)}(0, N \circ d(\mathbf{x})),$$

as desired. $\qquad\square$

Corollary 3.6. *For $w \geq 1$ and $(\mathbf{x}, 0) \in \ddot{X}_{vw}$, we have*

$$P(\mathbf{x}, 0) = \frac{1}{w+1}(N(\mathbf{x}), 0) + \sum_{j=0}^{w-1} \sum_{i=0}^{w} \frac{2j + 2i - 2w + 1}{2w(w+1)}(0, t^j \circ \varrho_n \circ t^i(\mathbf{x})).$$

Proof. By Proposition 3.5,

$$P(\mathbf{x}, 0) = \frac{1}{w+1}P(N(\mathbf{x}), 0) - \frac{1}{(w+1)w} \sum_{i=0}^{w} (w-i)(0, N \circ \varrho_n \circ t^i(\mathbf{x})) + \frac{1}{2(w+1)}(0, N \circ d(\mathbf{x}))$$

$$= \frac{1}{w+1}P(N(\mathbf{x}), 0) + \sum_{i=0}^{w} \frac{2i - w}{2w(w+1)}(0, N \circ \varrho_n \circ t^i(\mathbf{x})).$$

On the other hand, by formula (3.17)

$$\frac{1}{w+1}P(N(\mathbf{x}), 0) = \frac{1}{w+1}(N(\mathbf{x}), 0) - \frac{1}{(w+1)w} \sum_{j=0}^{w-1} \left(\frac{w-1}{2} - j\right)(0, t^j \circ d \circ N(\mathbf{x}))$$

$$+ \frac{1}{(w+1)^2} \sum_{j=0}^{w} \left(\frac{w}{2} - j\right)(0, d' \circ N(\mathbf{x}))$$

$$= \frac{1}{w+1}(N(\mathbf{x}), 0) - \frac{1}{(w+1)w} \sum_{j=0}^{w-1} \left(\frac{w-1}{2} - j\right)(0, t^j \circ d' \circ N(\mathbf{x}))$$

$$- \frac{1}{(w+1)w} \sum_{j=0}^{w-1} \left(\frac{w-1}{2} - j\right)(0, t^j \circ \varrho_n \circ N(\mathbf{x}))$$

$$= \frac{1}{w+1}(N(\mathbf{x}), 0) - \frac{1}{(w+1)w} \sum_{j=0}^{w-1} \left(\frac{w-1}{2} - j\right)(0, t^j \circ \varrho_n \circ N(\mathbf{x})),$$

where the second equality follows from the facts that $d = d' + \varrho_n$ and $\sum_{i=0}^{w} \left(\frac{w}{2} - i\right) = 0$; and the last equality, from the facts that $t^i \circ d' \circ N = t^i \circ N \circ d = N \circ d$ and $\sum_{i=0}^{w-1} \left(\frac{w-1}{2} - i\right) = 0$. So,

$$P(\mathbf{x}, 0) = \frac{1}{w+1}(N(\mathbf{x}), 0) + \sum_{j=0}^{w-1} \sum_{i=0}^{w} \frac{2j + 2i - 2w + 1}{2w(w+1)}(0, t^j \circ \varrho_n \circ t^i(\mathbf{x})),$$

as desired $\qquad\square$

We now consider the chain complex $(\ddot{X}, \ddot{b}, \ddot{d})$ and we denote by $\ker(\ddot{B})$ and $\mathrm{Im}(\ddot{B})$ the kernel and image of $\ddot{B}$ respectively. These are subcomplexes of $(\ddot{X}, \ddot{b}, \ddot{d})$. By (3.13) and (3.14), we have

$$\ker(\ddot{B}) / \mathrm{Im}(\ddot{B}) = P^{\perp}(\ddot{X}).$$

Consequently,

$$H_*(\ker(\ddot{B})/\operatorname{Im}(\ddot{B}),\ddot{b},\ddot{d}) = 0.$$

That is, the double mixed complex $(\ddot{X},\ddot{d},\ddot{b},\ddot{B})$ has the Connes property ([7]).

Let us define the reduced cyclic complex $\overline{C}_X^\lambda$ to be the quotient double complex

$$\overline{C}_X^\lambda := \ddot{X}/\ker(\ddot{B}).$$

It is easy to check that

$$\overline{C}_X^\lambda = \frac{P(\ddot{X}) \oplus P^\perp(\ddot{X})}{\operatorname{Im}(\ddot{B}) \oplus P^\perp(\ddot{X})} = \frac{P(\ddot{X})}{\operatorname{Im}(\ddot{B})}$$

and that $\breve{B}$ induces the isomorphism of complexes $\operatorname{Tot}(\overline{C}_X^\lambda)[1] \simeq \operatorname{Im}(\breve{B})$. So, we have the short exact sequence of double complexes

$$0 \longrightarrow \overline{C}_X^\lambda[0,1] \overset{i}{\longrightarrow} P(\ddot{X}) \overset{j}{\longrightarrow} \overline{C}_X^\lambda \longrightarrow 0$$

where j is the canonical surjection and i is induced by $\breve{B}$. Taking total complexes we obtain the short exact sequence

$$0 \longrightarrow \operatorname{Tot}(\overline{C}_X^\lambda)[1] \overset{i}{\longrightarrow} P(\breve{X}) \overset{j}{\longrightarrow} \operatorname{Tot}(\overline{C}_X^\lambda) \longrightarrow 0$$

3.4 A description of $P(\ddot{X})$

In this subsection we obtain a precise description of the double mixed complex $(P(\ddot{X}),\breve{b},\breve{d},\breve{B})$. The main result is Theorem 3.9. We relegate its proof to Appendix B.

Recall from Theorem 2.4 that $\overline{X}_{vw}$ is the cokernel of $\operatorname{id} - t\colon X_{vw} \longrightarrow X_{vw}$. Consider the double complex

$$(\widetilde{\mathfrak{X}},\widetilde{\mathfrak{b}},\widetilde{\mathfrak{d}}) :=$$

where $\widetilde{\mathfrak{X}}_{vw} := \overline{X}_{vw} \oplus \overline{X}_{v,w-1}$ and the boundary maps are defined by

$$\widetilde{\mathfrak{b}}(\mathbf{x},\mathbf{y}) := (\overline{b}(\mathbf{x}),-\overline{b}(\mathbf{y})) \qquad \text{and} \qquad \widetilde{\mathfrak{d}}(\mathbf{x},\mathbf{y}) := (\overline{d}(\mathbf{x}),-\overline{d}(\mathbf{y})),$$

respectively. Let $\mathfrak{p}\colon X_{vw} \longrightarrow \overline{X}_{vw}$ be the map defined by

$$\mathfrak{p}(\mathbf{x}) := \frac{1}{w+1}[\mathbf{x}] \qquad \text{for each very simple tensor } \mathbf{x} \in X_{vw},$$

where $[\mathbf{x}]$ denotes the class of $\mathbf{x}$ in $\overline{X}_{vw}$. Let $\overline{N}\colon \overline{X}_{vw} \longrightarrow X_{vw}$ be the map induced by N. It is easy to check that

$$\overline{N} \circ \overline{b} = b \circ \overline{N}, \qquad \overline{N} \circ \overline{d} = d' \circ \overline{N} \qquad \text{and} \qquad \mathfrak{p} \circ \overline{N} = \operatorname{id}. \tag{3.19}$$

Let

$$\widetilde{\xi}\colon \widetilde{X}_{vw} \longrightarrow \widetilde{X}_{v,w-1}$$

be the maps defined by $\widetilde{\overline{\xi}}(\mathbf{x}, \mathbf{y}) := (0, \overline{\xi}(\mathbf{x}))$, where $\overline{\xi} \colon \overline{X}_{vw} \longrightarrow \overline{X}_{v,w-2}$ is given by

$$
\overline{\xi}(\mathbf{x}) := \begin{cases} \frac{1}{w+1} \mathfrak{p} \circ d' \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) & \text{if } w > 1, \\ 0 & \text{if } w \le 1, \end{cases}
$$

where $\sigma' \colon X_{v,w-1} \longrightarrow X_{v,w-1}$ is the morphism introduced in the proof of Lemma 2.3.

Proposition 3.7. *For $\mathbf{x} \in \overline{X}_{vw}$ with $w > 1$, we have*

$$
\overline{\xi}(\mathbf{x}) = -\frac{1}{w+1} \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ \varrho_n \circ \overline{N}(\mathbf{x}).
$$

Proof. In fact

$$
(w+1)\overline{\xi}(\mathbf{x}) = \mathfrak{p} \circ d \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) - \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) = -\mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}),
$$

since

$$
\begin{aligned}
\mathfrak{p} \circ d \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) &= \frac{w}{w-1} \overline{d} \circ \mathfrak{p} \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) \\
&= \sum_{j=0}^{w-2} \frac{w-j-1}{w-1} \overline{d} \circ \mathfrak{p} \circ t^j \circ d \circ \overline{N}(\mathbf{x}) \\
&= \sum_{j=0}^{w-2} \frac{w-j-1}{w(w-1)} \overline{d} \circ \overline{d}\big([\overline{N}(\mathbf{x})]\big) \\
&= 0,
\end{aligned}
$$

where $[\overline{N}(\mathbf{x})]$ denotes the class of $\overline{N}(\mathbf{x}) \in X_{vw}$ in $\overline{X}_{vw}$. But

$$
\mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ d \circ \overline{N}(\mathbf{x}) = \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ d' \circ \overline{N}(\mathbf{x}) + \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ \varrho_n \circ \overline{N}(\mathbf{x}) = \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ \varrho_n \circ \overline{N}(\mathbf{x}),
$$

since

$$
\begin{aligned}
\mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ d' \circ \overline{N}(\mathbf{x}) &= \mathfrak{p} \circ \varrho_{n-1} \circ \sigma' \circ \overline{N} \circ \overline{d}(\mathbf{x}) \\
&= \frac{w-1}{2} \mathfrak{p} \circ \varrho_{n-1} \circ \overline{N} \circ \overline{d}(\mathbf{x}) \\
&= \frac{1}{2} \overline{d} \circ \overline{d}(\mathbf{x}) \\
&= 0,
\end{aligned}
$$

which finishes the proof. $\qquad\square$

Proposition 3.8. *Let $\mathbf{x}_0^n \in \overline{X}_{vw}$ with $w > 1$ and let $0 = i_0 < \cdots < i_w \le n$ be the indices such that $x_{i_j} \in M$. Then*

$$
\overline{\xi}(\mathbf{x}_0^n) = \sum_{0 \le j < l \le w} (-1)^{(i_l+1)(n+1)+i_j} \frac{(w+2j-2l+1)}{(w-1)w(w+1)} \big[\varrho_{i_j} \circ \varrho_{i_l}(\mathbf{x}_o^n)\big],
$$

where $\big[\varrho_{i_j} \circ \varrho_{i_l}(\mathbf{x}_o^n)\big]$ denotes the class of $\varrho_{i_j} \circ \varrho_{i_l}(\mathbf{x}_o^n) \in X_{v,w-2}$ in $\overline{X}_{v,w-2}$.

Proof. This follows from Proposition 3.7. We leave the details to the reader. $\qquad\square$

Given a t-invariant element $\mathbf{x} \in X_{vw}$, let

$$
\Upsilon(\mathbf{x}) := \begin{cases} (-\operatorname{id} + \overline{N} \circ \mathfrak{p}) \circ \sigma' \circ d(\mathbf{x}) \in X_{v,w-1} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases}
$$

Theorem 3.9. *Let* $\widetilde{\mathfrak{B}}\colon \widetilde{X}_{vw} \longrightarrow \widetilde{X}_{v,w+1}$ *be the map defined by* $\widetilde{\mathfrak{B}}(\mathbf{x},\mathbf{y}) := (0,\mathbf{x})$. *The diagram* $(\widetilde{X},\widetilde{\mathfrak{b}},\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}})$ *is a mixed double complex and the map*

$$\Psi\colon (\widetilde{X},\widetilde{\mathfrak{b}},\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}}) \longrightarrow (P(\ddot{X}),\ddot{b},\ddot{d},\ddot{B}),$$

defined by

$$\Psi_{vw}(\mathbf{x},\mathbf{y}) := \frac{1}{w+1}\big(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\big) + \big(0, \overline{N}(\mathbf{y})\big),$$

is an isomorphism of double mixed complexes.

Proof. See Appendix B. $\qquad\square$

Proposition 3.10. *The following assertions hold:*

(1) *The map*

$$\Gamma\colon \operatorname{Tot}(\overline{X},\overline{b},\overline{d}) \longrightarrow \operatorname{Tot}\mathrm{BC}(\widetilde{X},\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}}),$$

defined by

$$\Gamma_{vw}(\mathbf{x}) := \big((\mathbf{x},0),(-\overline{\xi}(\mathbf{x}),0),(\overline{\xi}^2(\mathbf{x}),0),(-\overline{\xi}^3(\mathbf{x}),0),(\overline{\xi}^4(\mathbf{x}),0),\dots\big),$$

is a morphism of complexes.

(2) *The map*

$$\Pi\colon \operatorname{Tot}\mathrm{BC}(\widetilde{X},\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}}) \longrightarrow \operatorname{Tot}(\overline{X},\overline{b},\overline{d}),$$

defined by

$$\Pi_{vw}\big((\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),(\mathbf{x}_3,\mathbf{y}_3),(\mathbf{x}_4,\mathbf{y}_4),\dots\big) := \mathbf{x}_0,$$

is a morphism of complexes.

(3) $\Pi \circ \Gamma = \mathrm{id}$ *and* $\Gamma \circ \Pi$ *is homotopic to the identity map. A homotopy is the family of maps*

$$\Xi\colon \operatorname{Tot}\mathrm{BC}(\widetilde{X},\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}})_w \longrightarrow \operatorname{Tot}\mathrm{BC}(\widetilde{X},\widetilde{\mathfrak{b}}+\widetilde{\mathfrak{d}}+\widetilde{\xi},\widetilde{\mathfrak{B}})_{w+1},$$

defined by

$$\Xi_{vw}(\mathbf{z}) := \big((0,0),(-\overline{\mathbf{y}}_0,0),(-\overline{\mathbf{y}}_1,0),(-\overline{\mathbf{y}}_2,0),(-\overline{\mathbf{y}}_3,0),\dots\big),$$

where

$$\mathbf{z} := \big((\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),(\mathbf{x}_3,\mathbf{y}_3),\dots\big) \qquad and \qquad \overline{\mathbf{y}}_i := \sum_{j=0}^{i}(-1)^j \overline{\xi}^j(\mathbf{y}_{i-j}).$$

Proof. Consider the following special deformation retract

$$\operatorname{Tot}(\overline{X},\overline{b},\overline{d}) \xleftarrow[\ \Gamma'\ ]{\ \Pi'\ } \operatorname{Tot}\mathrm{BC}(\widetilde{X},\widetilde{\mathfrak{d}}+\widetilde{\mathfrak{d}},\widetilde{\mathfrak{B}}), \qquad\qquad \Xi',$$

where Γ', Π' and Ξ' are given by

$$\Gamma'_{vw}(\mathbf{x}) := \big((\mathbf{x},0),(0,0),(0,0),(0,0),(0,0),\dots\big),$$
$$\Pi'_{vw}\big((\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),(\mathbf{x}_3,\mathbf{y}_3),(\mathbf{x}_4,\mathbf{y}_4),\dots\big) := \mathbf{x}_0$$

and

$$\Xi'_{vw}\big((\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),\dots\big) := \big((0,0),(-\mathbf{y}_0,0),(-\mathbf{y}_1,0),(-\mathbf{y}_2,0),\dots\big),$$

endowed with the perturbation

$$\Omega'\colon \bigoplus_{i\geq 0}\widetilde{X}_{*,*-i} \longrightarrow \bigoplus_{i\geq 0}\widetilde{X}_{*,*-1-i},$$

given by

$$\Omega'_{vw}\big((\mathbf{x}_0,\mathbf{y}_0),(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2),\dots\big) := \big((0,\overline{\xi}(\mathbf{x}_0)),(0,\overline{\xi}(\mathbf{x}_1)),(0,\overline{\xi}(\mathbf{x}_2)),\dots\big).$$

The proposition follows applying the perturbation lemma to this datum. $\qquad\square$

Let $i\colon \overline{X}_{v,w-1} \longrightarrow \widetilde{X}_{vw}$ and $\pi\colon \widetilde{X}_{vw} \longrightarrow \overline{X}_{vw}$ be the canonical maps. The short exact sequence of double complexes

$$0 \longrightarrow (\overline{X}_{*,*-1}, \overline{b}, \overline{d}) \overset{i}{\longrightarrow} (\widetilde{X}_{**}, \widetilde{\mathfrak{b}}, \widetilde{\mathfrak{d}} + \widetilde{\xi}) \overset{\pi}{\longrightarrow} (\overline{X}_{**}, \overline{b}, \overline{d}) \longrightarrow 0 \qquad (3.20)$$

splits in each level via the maps $s\colon \overline{X}_{vw} \longrightarrow \widetilde{X}_{vw}$ and $r\colon \widetilde{X}_{vw} \longrightarrow \overline{X}_{v,w-1}$, given by $s(\mathbf{x}) \coloneqq (\mathbf{x}, 0)$ and $r(\mathbf{x}, \mathbf{y}) \coloneqq \mathbf{y}$. From this it follows immediately that the connection map of the homology long exact sequence associated with (3.20) is induced by the morphism of double complexes

$$\overline{\xi}\colon (\overline{X}_{**}, \overline{b}, \overline{d}) \longrightarrow (\overline{X}_{*,*-2}, \overline{b}, \overline{d}).$$

Proposition 3.11. *The maps*

$$S_n\colon \mathrm{HC}_n(E, M) \longrightarrow \mathrm{HC}_{n-2}(E, M),$$
$$B_n\colon \mathrm{HC}_n(E, M) \longrightarrow \mathrm{HH}_{n+1}(E, M)$$

and

$$i_n\colon \mathrm{HH}_n(E, M) \longrightarrow \mathrm{HC}_n(E, M),$$

are induced by $-\overline{\xi}$, i *and* π, *respectively.*

Proof. This follows by a direct computation using Proposition 3.10. We leave the details to the reader. $\qquad\square$

Proposition 3.12. *Let* $\widetilde{P}\colon \ddot{X} \to \widetilde{X}$ *be the map* $\Psi^{-1} \circ P$. *We have*

$$\widetilde{P}(0, \mathbf{y}) = \frac{1}{w}(0, [\mathbf{y}]) \qquad \text{for } (0, \mathbf{y}) \in \ddot{X}_{vw} \text{ with } w > 0. \qquad (3.21)$$

and

$$\widetilde{P}(\mathbf{x}, 0) = \begin{cases} ([\mathbf{x}], 0) & \text{if } (\mathbf{x}, 0) \in \ddot{X}_{n0}, \\ ([\mathbf{x}], 0) + \sum_{i=0}^{w} \dfrac{2i - w}{2w(w+1)}\big(0, [\varrho_n \circ t^i(\mathbf{x})]\big) & \text{if } (\mathbf{x}, 0) \in \ddot{X}_{vw} \text{ with } w > 0. \end{cases} \qquad (3.22)$$

Proof. See Appendix B. $\qquad\square$

Remark 3.13. Let M be an associative nonunital k-algebra. Let E be the augmented algebra obtained adjoin the unit of k to M. In [7] the authors introduced and studied the the harmonic decomposition of the mixed complex of E. Applying Theorem 3.9 to E we obtain a complex isomorphic to the harmonic part of of this decomposition.

Appendix A

This appendix is devoted to prove Theorem 2.2.

Let $E \coloneqq A \ltimes_\triangledown M$ be a cleft extension and let K be a k-subalgebra of A. The w-th column $(\hat{\mathfrak{X}}_{*w}, \hat{\mathfrak{b}})$ of the double complex $(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$ introduced at the beginning of Section 2 is the total

complex of the double complex

$$(\mathfrak{X}_w, \mathfrak{b}, \alpha) := \quad \begin{array}{ccc} \vdots & & \vdots \\ \downarrow{\mathfrak{b}^0} & & \downarrow{\mathfrak{b}^1} \\ \mathfrak{X}_{2w}^0 & \xleftarrow{\ \alpha\ } & \mathfrak{X}_{2w}^1 \\ \downarrow{\mathfrak{b}^0} & & \downarrow{\mathfrak{b}^1} \\ \mathfrak{X}_{1w}^0 & \xleftarrow{\ \alpha\ } & \mathfrak{X}_{1w}^1 \\ \downarrow{\mathfrak{b}^0} & & \downarrow{\mathfrak{b}^1} \\ \mathfrak{X}_{0w}^0 & \xleftarrow{\ \alpha\ } & \mathfrak{X}_{0w}^1, \end{array}$$

where $\mathfrak{X}_{vw}^0 := M \otimes B_w^n \otimes$ and $\mathfrak{X}_{v-1,w}^1 := A \otimes B_{w+1}^n \otimes$.

Lemmas (A.1) and (A.2) below was proven in [12, Appendix A].

Lemma A.1. *Let* $\theta^1 \colon (\mathfrak{X}_{*w}^1, \mathfrak{b}^1) \longrightarrow (X_{*w}, -b)$ *and* $\vartheta^1 \colon (X_{*w}, -b) \longrightarrow (\mathfrak{X}_{*w}^1, \mathfrak{b}^1)$ *be the morphisms of complexes given by*

$$\theta^1(\mathbf{y}_0^{n+1}) := \mu_0^M(\mathbf{y}_0^{n+1}) \qquad and \qquad \vartheta^1(\mathbf{y}_0^n) := \sum_{l=0}^{n-i(\mathbf{y}_0^n)} 1 \otimes \mathfrak{t}^l(\mathbf{y}_0^n),$$

where $\mathfrak{t}(y_0 \otimes \cdots \otimes y_n) := (-1)^n y_n \otimes y_0 \otimes \cdots \otimes y_{n-1}$. *Then,* $\theta^1 \circ \vartheta^1 = \mathrm{id}$ *and* $\vartheta^1 \circ \theta^1$ *is homotopic to* id. *A homotopy is the family of maps* $\epsilon_{*w} \colon \mathfrak{X}_{*-1,w}^1 \longrightarrow \mathfrak{X}_{*w}^1$, *defined by*

$$\epsilon(\mathbf{y}_0^n) := -\sum_{l=0}^{n-i(\mathbf{y}_0^n)} 1 \otimes \mathfrak{t}^l(\mathbf{y}_0^n).$$

Lemma A.2. *For* $w \geq 0$, *let* $\tau_0^1(\mathcal{X}_{*w})$ *be the double diagram with two columns*

$$(X_{*w}, b) \xleftarrow{\ \mathrm{id}\,-t\ } (X_{*w}, -b).$$

The following assertions hold:

(1) $\tau_0^1(\mathcal{X}_{*w})$ *is a double complex.*

(2) *The map* $\vartheta \colon \tau_0^1(\mathcal{X}_{*w}) \longrightarrow (\mathfrak{X}_{*w}, \mathfrak{b}, \alpha)$, *defined by* $\vartheta := (\vartheta^0, \vartheta^1)$, *where* ϑ^0 *is the identity map and* ϑ^1 *is as in Lemma A.1, is a morphism of double complexes.*

(3) *The map* $\hat{\theta} \colon (\hat{\mathfrak{X}}_{*w}, \hat{\mathfrak{b}}) \longrightarrow \mathrm{Tot}(\tau_0^1(\mathcal{X}_{*w}))$, *defined by*

$$\hat{\theta}(\mathbf{x}_0^n, \mathbf{y}_0^n) := (\mathbf{x}_0^n + t(\mathbf{y}_0^n), \mu_0^M(\mathbf{y}_0^n)),$$

is a morphism of complexes.

(4) *Let* $\hat{\vartheta} \colon \mathrm{Tot}(\tau_0^1(\mathcal{X}_{*w})) \longrightarrow (\hat{\mathfrak{X}}_{*w}, \hat{\mathfrak{b}})$ *be the map induced by* ϑ. *It is true that* $\hat{\theta} \circ \hat{\vartheta} = \mathrm{id}$ *and that* $\hat{\vartheta} \circ \hat{\theta}$ *is homotopic to the identity map. A homotopy is the family of maps*

$$\hat{\epsilon}_{v+1,w} \colon \mathfrak{X}_{vw}^0 \oplus \mathfrak{X}_{v-1,w}^1 \longrightarrow \mathfrak{X}_{v+1,w}^0 \oplus \mathfrak{X}_{vw}^1 \qquad (v \geq 0),$$

defined by $\hat{\epsilon}(\mathbf{x}_0^n, \mathbf{y}_0^n) := (0, \epsilon(\mathbf{y}_0^n))$, *where* ϵ *is the homotopy introduced in Lemma A.1.*

Recall from the proof of Lemma 2.1 that $(\breve{X}, \breve{b}) := \mathrm{Tot}(\hat{X}, \hat{b}, \hat{d})$ and from the beginning of this section that $(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}) := \mathrm{Tot}(\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$. The first item of the following lemma is part of item (1) of Theorem 2.2.

Lemma A.3. *The following assertions hold:*

(1) *The diagram $(\hat{X}, \hat{b}, \hat{d})$, introduced above Theorem 2.2, is a double complex.*

(2) *The map $\hat{\vartheta} \colon (\hat{X}, \hat{b}, \hat{d}) \longrightarrow (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}})$ is a morphism of double complexes.*

(3) *The map $\hat{\theta} \colon (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}) \longrightarrow (\hat{X}, \hat{b}, \hat{d})$ is a morphism of double complexes.*

(4) *Let $\breve{\vartheta} \colon (\breve{X}, \breve{b}) \longrightarrow (\breve{\mathfrak{X}}, \breve{\mathfrak{b}})$ and $\breve{\theta} \colon (\breve{\mathfrak{X}}, \breve{\mathfrak{b}}) \longrightarrow (\breve{X}, \breve{b})$ be the maps induced by $\hat{\vartheta}$ and $\hat{\theta}$, respectively. It is true that $\breve{\theta} \circ \breve{\vartheta} = \mathrm{id}$ and that $\breve{\vartheta} \circ \breve{\theta}$ is homotopic to the identity map. A homotopy is the family of maps $\breve{\epsilon}_{n+1} \colon \breve{\mathfrak{X}}_n \longrightarrow \breve{\mathfrak{X}}_{n+1}$, defined by $\breve{\epsilon}_{n+1} := \bigoplus_{w=0}^{n} \hat{\epsilon}_{n+1-w,w}$, where $\hat{\epsilon}_{n+1-w,w}$ is as in Lemma A.2.*

Proof. By Lemma A.2 we have the following special deformation retract

$$\bigoplus_{w \geq 0} (\hat{X}_{*w}, \hat{b}) \xleftarrow[\hat{\vartheta}]{\hat{\theta}} \bigoplus_{w \geq 0} (\hat{\mathfrak{X}}_{*w}, \hat{\mathfrak{b}}), \qquad \hat{\epsilon},$$

where $\hat{\vartheta} := \bigoplus_{w \geq 0} \hat{\vartheta}_{*w}$, $\hat{\theta} := \bigoplus_{w \geq 0} \hat{\theta}_{*w}$ and $\hat{\epsilon} := \bigoplus_{w \geq 0} \hat{\epsilon}_{*w}$. Consider the perturbation $\hat{\mathfrak{d}} := \bigoplus_{w \geq 0} \hat{\mathfrak{d}}_{*w}$. Applying the perturbation lemma to this datum, we obtain a special deformation retract

$$(\breve{X}, \overline{b}) \xleftarrow[\overline{\vartheta}]{\overline{\theta}} (\breve{\mathfrak{X}}, \breve{\mathfrak{b}}), \qquad \overline{\epsilon}_{*+1} \colon \breve{\mathfrak{X}}_* \longrightarrow \breve{\mathfrak{X}}_{*+1}.$$

To finish the proof it remains to check that $\overline{b} = \breve{b}$, $\overline{\vartheta} = \breve{\vartheta}$, $\overline{\theta} = \breve{\theta}$ and $\overline{\epsilon} = \breve{\epsilon}$, for which it suffices to check that

$$\hat{\theta} \circ \hat{\mathfrak{d}} \circ \hat{\vartheta} = \hat{d}, \qquad \hat{\epsilon} \circ \hat{\mathfrak{d}} \circ \hat{\vartheta} = 0, \qquad \hat{\theta} \circ \hat{\mathfrak{d}} \circ \hat{\epsilon} = 0 \qquad \text{and} \qquad \hat{\epsilon} \circ \hat{\mathfrak{d}} \circ \hat{\epsilon} = 0,$$

which follows by a direct computation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma A.4. *Let $\hat{B}$ be as in Theorem 2.2. The following assertions hold:*

(1) *$(\hat{X}, \hat{b}, \hat{d}, \hat{B})$ is a double mixed complex.*

(2) *The maps*

$$\hat{\vartheta} \colon (\hat{X}, \hat{b}, \hat{d}, \hat{B}) \longrightarrow (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}) \qquad and \qquad \hat{\theta} \colon (\hat{\mathfrak{X}}, \hat{\mathfrak{b}}, \hat{\mathfrak{d}}, \hat{\mathfrak{B}}) \longrightarrow (\hat{X}, \hat{b}, \hat{d}, \hat{B})$$

are morphisms of double mixed complexes.

(3) *For W equals C, N and P, let*

$$\breve{\vartheta} \colon \mathrm{Tot}\, \mathrm{BW}(\breve{X}, \breve{b}, \breve{B}) \longrightarrow \mathrm{Tot}\, \mathrm{BW}(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}, \breve{\mathfrak{B}})$$

and

$$\breve{\theta} \colon \mathrm{Tot}\, \mathrm{BW}(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}, \breve{\mathfrak{B}}) \longrightarrow \mathrm{Tot}\, \mathrm{BW}(\breve{X}, \breve{b}, \breve{B})$$

be the maps induced by the morphisms $\breve{\vartheta}$ and $\breve{\theta}$ introduced in item 4) of Lemma A.3. It is true that $\breve{\theta} \circ \breve{\vartheta} = \mathrm{id}$ and $\breve{\vartheta} \circ \breve{\theta}$ is homotopic to the identity map. A homotopy is the family of maps

$$\breve{\epsilon} \colon \mathrm{Tot}\, \mathrm{BW}(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}, \breve{\mathfrak{B}})_* \longrightarrow \mathrm{Tot}\, \mathrm{BW}(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}, \breve{\mathfrak{B}})_{*+1},$$

defined by applying $\breve{\epsilon}$ on each component.

Proof. 1) From the fact that $\hat{b} \circ \hat{b} = 0$ it follows easily that $t \circ b = b \circ t$. Thus we obtain that $b \circ N = N \circ b$, which implies that $\hat{b} \circ \hat{B} + \hat{B} \circ \hat{b} = 0$. To prove that $\hat{d} \circ \hat{B} + \hat{B} \circ \hat{d} = 0$ we must check that $d' \circ N = N \circ d$. Let $\mathbf{x}_0^n \in X_{vw}$ be a very simple tensor. Let $0 = i_0 < i_1 < \cdots < i_w \leq n$ be the indices such that $x_{i_j} \in M$ and let $i_{w+1} = n + 1$. A direct computation shows that for $0 \leq l \leq n$ and $0 \leq j < w$,

$$t^j \circ \varrho_l(\mathbf{x}_0^n) = \begin{cases} \varrho_{l+n+1-i_{w+1-j}} \circ t^j(\mathbf{x}_0^n) & \text{if } l < i_{w+1-j} - 1, \\ \varrho_{l-i_{w-j}} \circ t^{j+1}(\mathbf{x}_0^n) & \text{if } l \geq i_{w+1-j} - 1. \end{cases}$$

Hence

$$N \circ d(\mathbf{x}_0^n) = \sum_{j=0}^{w-1} \sum_{l=0}^{n} t^j \circ \varrho_l(\mathbf{x}_0^n) = \sum_{l=0}^{n-1} \sum_{j=0}^{w} \varrho_l \circ t^j(\mathbf{x}_0^n) = d' \circ N(\mathbf{x}_0^n),$$

as we want.

2) and 3) From Lemma A.3 we get a special deformation retract between the total complexes of the double complexes $\mathrm{BC}(\breve{X}, \breve{b}, 0)$ and $\mathrm{BC}(\breve{\mathfrak{X}}, \breve{\mathfrak{b}}, 0)$. Consider the perturbation induced by $\breve{\mathfrak{B}}$. The result it follows by applying the perturbation lemma to this setting, and using that $\breve{B} = \breve{\theta} \circ \breve{\mathfrak{B}} \circ \breve{\vartheta}$, $\breve{\mathfrak{B}} \circ \breve{\epsilon} = 0$ and $\breve{\epsilon} \circ \breve{\mathfrak{B}} = 0$. $\qquad\square$

Proof of Theorem 2.2. This follows immediately from Lemma A.4. $\qquad\square$

Appendix B

Recall from the discussion above Proposition 3.7, that for each t-invariant element $\mathbf{x} \in X_{vw}$,

$$\Upsilon(\mathbf{x}) := \begin{cases} (-\operatorname{id} + \overline{N} \circ \mathfrak{p}) \circ \sigma' \circ d(\mathbf{x}) \in X_{v,w-1} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases}$$

We claim that $\Upsilon(\mathbf{x})$ is univocally determined by the following properties:

$$\Upsilon(\mathbf{x}) \in \ker(\mathfrak{p}) \qquad \text{and} \qquad (\mathbf{x}, \Upsilon(\mathbf{x})) \in P(\ddot{X}).$$

First note that $\mathfrak{p} \circ \Upsilon = 0$, since $\mathfrak{p}$ is a retraction of $\overline{N}$. By Remark 3.3, in order to prove that $(\mathbf{x}, \Upsilon(\mathbf{x})) \in P(\ddot{X}_{vw})$ it suffices to see that $d(\mathbf{x}) + (\operatorname{id} - t) \circ \Upsilon(\mathbf{x})$ is t-invariant. This is evident for $w = 0$ and it is true for $w > 0$, since

$$d(\mathbf{x}) + (\operatorname{id} - t) \circ \Upsilon(\mathbf{x}) = d(\mathbf{x}) + (t - \operatorname{id}) \circ \sigma' \circ d(\mathbf{x}) = \frac{1}{w} N \circ d(\mathbf{x}), \tag{B.23}$$

where the first equality follows from the fact that $t \circ \overline{N} = \overline{N}$ and the second one, from equality (2.3). Conversely, assume that $(\mathbf{x}, \mathbf{y}) \in P(\ddot{X}_{vw})$ and let $[\mathbf{y} - \Upsilon(\mathbf{x})]$ denote the class of $\mathbf{y} - \Upsilon(\mathbf{x})$ in $\overline{X}_{v,w+1}$. Again by Remark 3.3, we know that $\mathbf{y} - \Upsilon(\mathbf{x})$ is t-invariant, and so, if $\mathbf{y} \in \ker(\mathfrak{p})$, then

$$[\mathbf{y} - \Upsilon(\mathbf{x})] = \mathfrak{p} \circ \overline{N}([\mathbf{y} - \Upsilon(\mathbf{x})]) = \mathfrak{p} \circ N(\mathbf{y} - \Upsilon(\mathbf{x})) = w\mathfrak{p}(\mathbf{y} - \Upsilon(\mathbf{x})) = 0,$$

which implies that $\mathbf{y} = \Upsilon(\mathbf{x})$ (note that if $w = 0$, then $\mathbf{y} = \Upsilon(\mathbf{x}) = 0$). Recall also from Remark 3.3, that ${}^t P(\ddot{X}_{vw})$ is the set of all elements of the shape $(0, \mathbf{x})$ that belongs to $P(\ddot{X}_{vw})$, and that

$${}^t P(\ddot{X}_{vw}) = \{(0, \mathbf{x}) : \mathbf{x} \in X_{v,w-1} \text{ and } t(\mathbf{x}) = \mathbf{x}\} = P(\ddot{X}_{vw}) \cap \acute{X}_{v,w-1}.$$

Let

$${}^e P(\ddot{X}_{vw}) := \{(\mathbf{x}, \Upsilon(\mathbf{x})) \in \ddot{X}_{vw} : \mathbf{x} \text{ is } t\text{-invariant}\}.$$

Clearly, $P(\ddot{X}) = {}^e P(\ddot{X}) \oplus {}^t P(\ddot{X})$. We assert that $\ddot{b}({}^e P(\ddot{X}_{vw})) \subseteq {}^e P(\ddot{X}_{v-1,w})$, for each $v > 0$. In order to prove this we will need the following result.

Lemma B.1. *For $v > 0$, we have $\overline{b} \circ \mathfrak{p} = \mathfrak{p} \circ b$.*

Proof. This is trivial. $\qquad\square$

Proposition B.2. *Assume that $v > 0$ and let $\mathbf{x} \in X_{vw}$ be a t-invariant element. We have:*

$$\ddot{b}(\mathbf{x}, \Upsilon(\mathbf{x})) = (b(\mathbf{x}), \Upsilon \circ b(\mathbf{x})).$$

Proof. By definition

$$\ddot{b}(\mathbf{x}, \Upsilon(\mathbf{x})) = \big(b(\mathbf{x}), -b(\Upsilon(\mathbf{x}))\big).$$

Since $b(\mathbf{x})$ is t-invariant, in order to finish the proof it suffices to check that

$$\big(b(\mathbf{x}), -b(\Upsilon(\mathbf{x}))\big) \in P(\ddot{X}) \qquad \text{and} \qquad \mathfrak{p}(b(\Upsilon(\mathbf{x}))) = 0.$$

The first fact follows immediately from the fact that $P(\ddot{X})$ is a subcomplex of $(\ddot{X}, \ddot{b}, \ddot{d})$ and the second one follows easily from Lemma B.1. $\qquad\square$

For each $w > 0$, let

$${}^e d \colon {}^e P(\ddot{X}_{vw}) \to {}^e P(\ddot{X}_{v,w-1}) \qquad \text{and} \qquad {}^e \xi \colon {}^e P(\ddot{X}_{vw}) \to {}^t P(\ddot{X}_{v,w-1})$$

be the maps defined by

$${}^e d(\mathbf{x}, \Upsilon(\mathbf{x})) + {}^e \xi(\mathbf{x}, \Upsilon(\mathbf{x})) := \ddot{d}(\mathbf{x}, \Upsilon(\mathbf{x})).$$

We now want to compute these maps. To carry out this task we will need Proposition B.3 below. First note that if $\mathbf{x} \in X_{vw}$ is a t-invariant element, then $d'(\mathbf{x})$ also is, since

$$N \circ d(\mathbf{x}) = d' \circ N(\mathbf{x}) = (w+1)d'(\mathbf{x}). \tag{B.24}$$

For $w \geq 1$, let $\xi \colon {}^t X_{vw} \to X_{v,w-2}$ be the map defined by

$$\xi(\mathbf{x}) := -\frac{w+1}{w}\Upsilon \circ d'(\mathbf{x}) - d' \circ \Upsilon(\mathbf{x}),$$

where ${}^t X_{vw}$ denotes the set of t-invariant elements of X_{vw}.

Proposition B.3. *Assume that $w > 0$ and let $\mathbf{x} \in X_{vw}$ be a t-invariant element. Then,*

$${}^e d(\mathbf{x}, \Upsilon(\mathbf{x})) = \frac{w+1}{w}\big(d'(\mathbf{x}), \Upsilon \circ d'(\mathbf{x})\big) \qquad and \qquad {}^e \xi(\mathbf{x}, \Upsilon(\mathbf{x})) = (0, \xi(\mathbf{x})).$$

In particular $\xi(\mathbf{x})$ is t-invariant.

Proof. Since $d'(\mathbf{x})$ is t-invariant,

$$\big(d'(\mathbf{x}), \Upsilon \circ d'(\mathbf{x})\big) \in {}^e P(\ddot{X}_{v,w-1}). \tag{B.25}$$

Moreover, by equalities (B.23) and (B.24),

$$\begin{aligned}
\ddot{d}(\mathbf{x}, \Upsilon(\mathbf{x})) &= \left(\frac{1}{w}N \circ d(\mathbf{x}), -d' \circ \Upsilon(\mathbf{x})\right) \\
&= \left(\frac{w+1}{w}d'(\mathbf{x}), -d' \circ \Upsilon(\mathbf{x})\right) \\
&= \frac{w+1}{w}\big(d'(\mathbf{x}), \Upsilon \circ d'(\mathbf{x})\big) - \left(0, \frac{w+1}{w}\Upsilon \circ d'(\mathbf{x}) + d' \circ \Upsilon(\mathbf{x})\right) \\
&= \frac{w+1}{w}\big(d'(\mathbf{x}), \Upsilon \circ d'(\mathbf{x})\big) + (0, \xi(\mathbf{x})).
\end{aligned}$$

Consequently in order to finish the proof it suffices to note that

$$\frac{w+1}{w}\big(d'(\mathbf{x}), \Upsilon \circ d'(\mathbf{x})\big) \in {}^e P(\ddot{X}_{v,w-1}) \qquad \text{and} \qquad (0, \xi(\mathbf{x})) \in P(\ddot{X}),$$

since $P(\ddot{X})$ is a subcomplex of $(\ddot{X}, \ddot{b}, \ddot{d})$. $\qquad\square$

In Proposition B.5 below we will obtain another expression for ξ. In order to do this we will need the following result.

Lemma B.4. *Assume that $w > 0$ and let $\mathbf{x} \in X_{vw}$ be a t-invariant element. We have*

$$\overline{N} \circ \mathfrak{p} \circ \sigma' \circ d(\mathbf{x}) = \frac{(w-1)(w+1)}{2w} d'(\mathbf{x}).$$

Proof. In fact

$$\begin{aligned}
\overline{N} \circ \mathfrak{p} \circ \sigma' \circ d(\mathbf{x}) &= \sum_{j=0}^{w-2} \frac{w-j-1}{w} \overline{N} \circ \mathfrak{p} \circ t^j \circ d(\mathbf{x}) \\
&= \sum_{j=0}^{w-2} \frac{w-j-1}{w^2} \overline{N}([d(\mathbf{x})]) \\
&= \frac{w-1}{2w} N \circ d(\mathbf{x}) \\
&= \frac{(w-1)(w+1)}{2w} d'(\mathbf{x}),
\end{aligned}$$

where $[d(\mathbf{x})]$ denotes the class of $d(\mathbf{x})$ in $\overline{X}_{v,w-1}$ and the last equality follows from (B.24). $\square$

Proposition B.5. *Assume that $w > 1$ and let $\mathbf{x} \in X_{vw}$ be a t-invariant element. We have*

$$\xi(\mathbf{x}) = \frac{1}{w-1} N \circ d' \circ \sigma' \circ d(\mathbf{x}).$$

Proof. In fact, by Lemma B.4

$$d' \circ \overline{N} \circ \mathfrak{p} \circ \sigma' \circ d(\mathbf{x}) = \overline{N} \circ \mathfrak{p} \circ \sigma' \circ d \circ d'(\mathbf{x}) = 0.$$

Hence

$$\xi(\mathbf{x}) = -\frac{w+1}{w} \Upsilon \circ d'(\mathbf{x}) - d' \circ \Upsilon(\mathbf{x}) = \frac{w+1}{w} \sigma' \circ d \circ d'(\mathbf{x}) + d' \circ \sigma' \circ d(\mathbf{x}).$$

Combining this with the fact that $\xi(\mathbf{x})$ is t-invariant, we obtain

$$\begin{aligned}
\xi(\mathbf{x}) &= \frac{1}{w-1} N \circ \xi(\mathbf{x}) \\
&= \overline{N} \circ \mathfrak{p} \circ \xi(\mathbf{x}) \\
&= \frac{w+1}{w} \overline{N} \circ \mathfrak{p} \circ \sigma' \circ d \circ d'(\mathbf{x}) + \overline{N} \circ \mathfrak{p} \circ d' \circ \sigma' \circ d(\mathbf{x}) \\
&= \overline{N} \circ \mathfrak{p} \circ d' \circ \sigma' \circ d(\mathbf{x}) \\
&= \frac{1}{w-1} N \circ d' \circ \sigma' \circ d(\mathbf{x}),
\end{aligned}$$

where the forth equality follows from Lemma B.4. $\square$

Proof of Theorem 3.9. We claim that $\xi \circ \overline{N} = (w+1)\overline{N} \circ \overline{\xi}$ on $\overline{X}_{vw}$. This is obvious if $w \leq 1$, because both terms in this equality are zero. Assume that $w > 1$. Then, by Proposition B.5,

$$\xi \circ \overline{N} = \frac{1}{w-1} N \circ d' \circ \sigma' \circ d \circ \overline{N} = \overline{N} \circ \mathfrak{p} \circ d' \circ \sigma' \circ d \circ \overline{N} = (w+1)\overline{N} \circ \overline{\xi}.$$

Combining this with Proposition B.3 and the second equality in (3.19), we obtain that

$$
\begin{aligned}
\ddot{d} \circ \Psi_{vw}(\mathbf{x}, \mathbf{y}) &= \ddot{d}\left(\frac{1}{w+1}\left(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\right) + (0, \overline{N}(\mathbf{y}))\right) \\
&= \frac{1}{w+1}{}^{e}d\left(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\right) + \frac{1}{w+1}{}^{e}\xi\left(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\right) + (0, -d' \circ \overline{N}(\mathbf{y})) \\
&= \frac{1}{w}\left(d' \circ \overline{N}(\mathbf{x}), \Upsilon \circ d' \circ \overline{N}(\mathbf{x})\right) + \frac{1}{w+1}\left(0, \xi \circ \overline{N}(\mathbf{x})\right) - (0, d' \circ \overline{N}(\mathbf{y})) \\
&= \frac{1}{w}\left(\overline{N} \circ \overline{d}(\mathbf{x}), \Upsilon \circ \overline{N} \circ \overline{d}(\mathbf{x})\right) + (0, \overline{N} \circ \overline{\xi}(\mathbf{x})) - (0, \overline{N} \circ \overline{d}(\mathbf{y})) \\
&= \Psi_{v,w-1}\left((\overline{d}(\mathbf{x}), -\overline{d}(\mathbf{y})) + (0, \overline{\xi}(\mathbf{x}))\right) \\
&= \Psi_{v,w-1} \circ (\widetilde{\mathfrak{d}} + \widetilde{\xi})(\mathbf{x}, \mathbf{y}),
\end{aligned}
$$

and using Proposition B.2 and the first equality in (3.19), that

$$
\begin{aligned}
\ddot{b} \circ \Psi_{vw}(\mathbf{x}, \mathbf{y}) &= \ddot{b}\left(\frac{1}{w+1}\left(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\right) + (0, \overline{N}(\mathbf{y}))\right) \\
&= \frac{1}{w+1}(b \circ \overline{N}(\mathbf{x}), \Upsilon \circ b \circ \overline{N}(\mathbf{x})) - (0, b \circ \overline{N}(\mathbf{y})) \\
&= \frac{1}{w+1}(\overline{N} \circ \overline{b}(\mathbf{x}), \Upsilon \circ \overline{N} \circ \overline{b}(\mathbf{x})) - (0, \overline{N} \circ \overline{b}(\mathbf{y})) \\
&= \Psi_{v-1,w}(\overline{b}(\mathbf{x}), -\overline{b}(\mathbf{y})) \\
&= \Psi_{v-1,w} \circ \widetilde{\mathfrak{b}}(\mathbf{x}, \mathbf{y}).
\end{aligned}
$$

These facts show that Ψ is a morphism of double complexes. Since

$$
\begin{aligned}
\ddot{B} \circ \Psi_{vw}(\mathbf{x}, \mathbf{y}) &= \ddot{B}\left(\frac{1}{w+1}\left(\overline{N}(\mathbf{x}), \Upsilon \circ \overline{N}(\mathbf{x})\right) + (0, \overline{N}(\mathbf{y}))\right) \\
&= (0, \overline{N}(\mathbf{x})) = \Psi_{v,w+1}(0, \mathbf{x}) = \Psi_{v,w+1} \circ \widetilde{B}(\mathbf{x}, \mathbf{y}),
\end{aligned}
$$

Ψ is a morphism of mixed complexes. To finish the proof it suffices to note that Ψ is bijective. $\quad\square$

Proof of Proposition 3.12. By equality (3.15) and the definition of Ψ (see Theorem 3.9), we have

$$
P(0, \mathbf{y}) = \frac{1}{w}(0, N(y)) = \frac{1}{w}\Psi(0, [\mathbf{y}]),
$$

which proves equality (3.21). The case $w = 0$ of equality (3.22) follows from the fact that, by equality (3.16),

$$
P(\mathbf{x}, 0) = (\mathbf{x}, 0) = \Psi([\mathbf{x}], 0).
$$

Assume now that $w > 0$. Then, by Corollary 3.6,

$$
P(\mathbf{x}, 0) = \frac{1}{w+1}(N(\mathbf{x}), 0) + \sum_{j=0}^{w-1}\sum_{i=0}^{w}\frac{2j + 2i - 2w + 1}{2w(w+1)}\left(0, t^{j} \circ \varrho_{n} \circ t^{i}(\mathbf{x})\right)
$$

and by the definitions of Υ, σ' and Lemma B.4,

$$
\begin{aligned}
\Upsilon \circ \overline{N}([\mathbf{x}]) &= \frac{(w-1)(w+1)}{2w} d' \circ \overline{N}([\mathbf{x}]) - \sigma' \circ d \circ \overline{N}([\mathbf{x}]) \\
&= \frac{(w-1)(w+1)}{2w} d' \circ \overline{N}([\mathbf{x}]) - \sigma' \circ d' \circ \overline{N}([\mathbf{x}]) - \sigma' \circ \varrho_n \circ \overline{N}([\mathbf{x}]) \\
&= \frac{(w-1)(w+1)}{2w} \overline{N} \circ \overline{d}([\mathbf{x}]) - \sigma' \circ \overline{N} \circ \overline{d}([\mathbf{x}])) - \sigma' \circ \varrho_n \circ \overline{N}([\mathbf{x}]) \\
&= \frac{w-1}{2w} \overline{N} \circ \overline{d}([\mathbf{x}]) - \sum_{j=0}^{w-1} \frac{w-1-j}{w} t^j \circ \varrho_n \circ \overline{N}([\mathbf{x}]) \\
&= \frac{w-1}{2w} N \circ d(\mathbf{x}) - \sum_{j=0}^{w-1} \sum_{i=0}^{w} \frac{w-1-j}{w} t^j \circ \varrho_n \circ t^i(\mathbf{x}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
P(\mathbf{x},0) - \Psi([\mathbf{x}],0) &= P(\mathbf{x},0) - \frac{1}{w+1}\left(\overline{N}([\mathbf{x}]), \Upsilon \circ \overline{N}([\mathbf{x}])\right) \\
&= \sum_{j=0}^{w-1} \sum_{i=0}^{w} \frac{2i-1}{2w(w+1)}\left(0, t^j \circ \varrho_n \circ t^i(\mathbf{x})\right) - \frac{w-1}{2w(w+1)}\left(0, N \circ d(\mathbf{x})\right) \\
&= \sum_{i=0}^{w} \frac{2i-1}{2w(w+1)}\left(0, N \circ \varrho_n \circ t^i(\mathbf{x})\right) - \sum_{i=0}^{w} \frac{w-1}{2w(w+1)}\left(0, N \circ \varrho_n \circ t^i(\mathbf{x})\right) \\
&= \sum_{i=0}^{w} \frac{2i-w}{2w(w+1)}\left(0, N \circ \varrho_n \circ t^i(\mathbf{x})\right) \\
&= \sum_{i=0}^{w} \frac{2i-w}{2w(w+1)} \Psi\left(0, [\varrho_n \circ t^i(\mathbf{x})]\right),
\end{aligned}
$$

as desired. $\qquad\square$

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