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Lazer–Leach conditions for coupled Gompertz-like delayed systems

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A coupled Gompertz-like system of delay differential equations is considered. We prove the existence of T-periodic solutions under resonance assuming a Lazer–Leach type condition.

1. Introduction

One of the most popular nonlinear models for self-limiting cell population growth is the equation introduced by Gompertz in [1], namely

$$N'(t) = -\alpha N(t) \ln(K/N(t)),$$

where $N(t)$ is the density of the population, α is a positive constant called the intrinsic growth rate and the positive constant K is usually referred to as the environment carrying capacity or saturation level. The model was derived by Gompertz in 1825 and used in the context of actuarial statistics. In 1932, Winsor [2] found out that it provides a good empirical description of decelerating tumor growth. In 1964 [3], the same

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model was used by Laird in the description of tumor growth, and later in 1965 [4], for the first time Laird fitted the experimental data to the Gompertz curve. Since that time the Gompertz equation has been often used in the formulation of equations modeling the population dynamics and to describe the inner growth of a tumor and other processes of nature, e.g. [5–7].

To better reflect the reality some of the past states of the systems may be included, that is, ideally, a more realistic system can be modeled by differential equations with a time delay:

$$N'(t) = -\alpha N(t) \ln(K/N(t-\tau)). \tag{1}$$

It is clear that (1) has a unique positive equilibrium $N \equiv K$. It is observed, furthermore, that if K is replaced by a positive continuous function of (minimal) period T, then for small values of τ the problem admits a positive T-solution N of minimal period T. This is readily verified by several means: for instance, it suffices to apply the implicit function theorem to the mapping

$$F(N,\tau)(t) := N'(t) + \alpha N(t) \ln(K(t)/N(t-\tau))$$

defined over appropriate spaces, at the point $(K, 0)$. However, since we are looking for solutions $N > 0$, a more straightforward argument follows from the fact that the substitution $u(t) := \ln(N(t))$ transforms the equation into a linear one, namely

$$u'(t) = \alpha u(t-\tau) + p(t),$$

where $p(t) := -\alpha \ln(K(t))$. Thus, a simple computation shows that, in fact, the problem has a unique T-periodic solution for almost all values of τ; specifically, this happens for all values except for a finite number, if we assume w.l.o.g. that $\tau < T$. In order to make the statement more precise, consider the space of continuous T-periodic functions

$$C_T := \{u \in C(\mathbb{R},\mathbb{R}) : u(t+T) = u(t) \text{ for all } t\}.$$

Observe that any solution $u \in C_T$ has minimal period T because if $u(t+S) = u(t)$ for all t, then $p(t+S) = p(t)$. For convenience, denote $\omega := \frac{2\pi}{T}$, then it is seen that the homogeneous equation

$$u'(t) = \alpha u(t-\tau)$$

admits nontrivial solutions only when

$$\cos(k\omega\tau) = 0 = k\omega + \alpha \sin(k\omega\tau)$$

for some $k \in \mathbb{N}$, that is:

$$k\omega = \alpha, \qquad \tau = \frac{m - \frac{1}{4}}{k} T, \quad m = 1, \dots, k. \tag{2}$$

Except for these specific choices of the parameters, a (unique) T-periodic positive solution exists for each p. When (2) holds, the problem is called resonant and has (infinitely many) solutions if and only if p is orthogonal in the L^2 sense to the kernel of the operator $Lu(t) := u'(t) - \alpha u(t-\tau)$ or, in terms of the original equation:

$$\int_0^T \ln(K(t)) e^{ik\omega t}\, dt = 0.$$

We are interested in extending the previous ideas to a system in which Gompertz equation is coupled with a second equation and serves as a model in many biological situations (see e.g. [8–11], and for DDEs see e.g. [12–14] and the references therein):

$$\begin{cases} N'(t) = -\alpha N(t) \ln(K(t)/N(t-\tau_1)) + N(t) f(N(t-\tau_1), v(t-\tau_2)) \\ v'(t) = \beta v(t) + g(t, N(t-\tau_1), v(t-\tau_2)). \end{cases} \tag{3}$$

We shall assume that $f : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded, g is T-periodic in t, $\beta \neq 0$ and, as before, α is a positive constant and K a positive continuous T-periodic function. We look for T-periodic solutions (N, v) of system (3) with $N > 0$.

When the problem is non-resonant, the existence of such solutions follows from a direct application of Schauder's Theorem; thus, we shall focus on the case in which the resonance conditions (2) are satisfied for the first equation and $\tau = \tau_1$. We shall prove the existence of a constant $R > 0$ such that if the limits

$$f_0 := \lim_{u\to 0^+} f(u, v), \qquad f_\infty := \lim_{u\to +\infty} f(u, v)$$

exist uniformly for $|v| \leq R$ and $f_0 \neq f_\infty$, then the problem admits at least one T-periodic solution when the projection (in the L^2 sense) of the function $\ln(K(t))$ to the kernel of the operator L is sufficiently small. In more precise terms, the existence of solutions is guaranteed under the following Lazer–Leach type condition [15]:

$$\left| \int_0^T \ln(K(t)) e^{ik\omega t}\, dt \right| < \frac{2}{\alpha} |f_\infty - f_0|. \tag{4}$$

Theorem 1.1. *Let $R := \frac{\|g\|_\infty}{|\beta|}$ and assume that the previous limits f_0, f_∞ exist uniformly for $|v| \leq R$. Then (3) has at least one T-periodic solution, provided that (4) holds.*

The proof will follow from a slightly more general result (see Theorem 2.2).

2. A general abstract system

Let $C_T^1 := C_T \cap C^1(\mathbb{R}, \mathbb{R})$ and consider the linear operator $L : C_T^1 \to C_T$ given by $Lu(t) := u'(t) - Au(t) - Bu(t - \tau)$. Writing $u = \sum_{n\in\mathbb{Z}} a_n e^{in\omega t}$, a simple computation shows that $\mathrm{Ker}(L)$ is nontrivial if and only if

$$A + B\cos(k\omega\tau) = k\omega + B\sin(k\omega\tau) = 0$$

for some $k \in \mathbb{N}_0$. When $k = 0$, the condition simply says that $A + B = 0$; for $k > 0$, resonance situation may occur only if $|A| < |B|$ and $T = \frac{2\pi k}{\sqrt{B^2 - A^2}}$, which is satisfied for finitely many values of $\tau \in [0, T)$. Observe, furthermore, that in this case

$$\mathrm{Ker}(L) = \mathrm{span}\{\cos(k\omega t), \sin(k\omega t)\}$$

and $\mathrm{Im}(L) = \mathrm{Ker}(L)^\perp$, where orthogonality is understood in the L^2 sense, namely:

$$\mathrm{Ker}(L)^\perp = \left\{ u \in C_T : \int_0^T u(t) e^{ik\omega t}\, dt = 0 \right\}.$$

In particular, $\mathrm{Ker}(L) = \mathrm{Ker}(L^*)$, where the adjoint operator L^* is given by

$$L^* u(t) = -u'(t) - Au(t) - Bu(t + \tau).$$

Moreover, it is clear that $L : C_T^1 \cap \mathrm{Ker}(L)^\perp \to \mathrm{Im}(L)$ is bijective; thus, by the open mapping theorem we deduce the existence of a constant c such that $\|u\|_{C^1} \leq c\|Lu\|_\infty$ for all $u \in C_T^1 \cap \mathrm{Ker}(L)^\perp$, namely, $c = \|L^{-1}\|$.

Now consider the more general system

$$\begin{cases} L_1 u(t) = g_1(u(t - \tau_1), v(t - \tau_2)) + p(t) \\ L_2 v(t) = g_2(t, u_t, v_t) \end{cases} \tag{5}$$

with $g_1 : \mathbb{R}^2 \to \mathbb{R}$, $g_2 : \mathbb{R} \times C([-\tau_1, 0]) \times C([-\tau_2, 0]) \to \mathbb{R}$ bounded continuous functions and $L_j u(t) := u'(t) - a_j u(t) - b_j u(t - \tau_j)$. Assume that $|a_1| < |b_1|$ and set $T = \frac{2\pi k}{\sqrt{b_1^2 - a_1^2}}$ for some $k \in \mathbb{N}$. Define as before $\omega = \frac{2\pi}{T}$ and fix $\tau_1 < T$ such that

$$\cos(k\omega\tau_1) = -a_1/b_1, \qquad \sin(k\omega\tau_1) = -k\omega/b_1.$$

Moreover, we shall assume that $a_2 + b_2 \neq 0$ and that the previous relations do not hold for the delay $\tau_2 < T$; thus, $\mathrm{Ker}(L_2) = \{0\}$ and it is deduced as before that

$$\|u\|_{C^1} \leq c_2 \|L_2 u\|_\infty \tag{6}$$

for all $u \in C_T^1$.

Remark 2.1. In particular, when $b_2 = 0 \neq a_2$, it is readily verified that $c_2 = \frac{1}{|a_2|}$. Indeed, if $|u(t)|$ achieves its absolute maximum value at $t = t_0$ then $L_2 u(t_0) = -a_2 u(t_0)$ which, in turn, implies: $|a_2| \|u\|_\infty \leq \|L_2 u\|_\infty$.

In order to state our Lazer–Leach type condition, we shall assume that the limits

$$g_1^{inf}(\pm\infty) := \liminf_{u \to \pm\infty} g_1(u, v) \tag{7}$$

$$g_1^{sup}(\pm\infty) := \limsup_{u \to \pm\infty} g_1(u, v) \tag{8}$$

exist uniformly for $|v| \leq c_2 \|g_2\|_\infty$ and that the following technical condition holds:

$$\int_0^T [g_1(\rho\cos(k\omega t) + \psi(t), \phi(t)) - g_1(\rho\cos(k\omega t) + \psi(t), 0)] \sin(k\omega t)\, dt \to 0 \tag{9}$$

as $\rho \to +\infty$ uniformly for $\|\phi\|_\infty \leq c_2 \|g_2\|_\infty$ and $\|\psi\|_\infty \leq c_1(\|g_1\|_\infty + \|p\|_\infty)$ where, as before, c_1 is the norm of the right inverse of L_1. For example, (9) is satisfied when $g_1(\pm\infty) := \lim_{u \to \pm\infty} g_1(u, v)$ exist uniformly for $|v| \leq c_2 \|g_2\|_\infty$.

In particular, setting $u := \ln N$ and $g_1(u, v) := f(e^u, v)$ and taking into account Remark 2.1, the existence of T-periodic solutions of (3) with $N > 0$ is an immediate consequence of the following main result of the paper:

Theorem 2.2. *Let $|a_1| < |b_1|$ and fix T, τ_1 and τ_2 as before. Assume that g_2 is T-periodic in its first coordinate, $p_1 \in C_T$, the limits* (7) and (8) *exist uniformly for $|v| \leq c_2 \|g_2\|_\infty$ and* (9) *is satisfied.*

If

$$\left| \int_0^T p(t) e^{ik\omega t}\, dt \right| < 2(g_1^{inf}(+\infty) - g_1^{sup}(-\infty)) \tag{10}$$

or

$$\left| \int_0^T p(t) e^{ik\omega t}\, dt \right| < 2(g_1^{inf}(-\infty) - g_1^{sup}(+\infty)) \tag{11}$$

then the problem admits at least one solution $u \in C_T$. If furthermore p has minimal period T, then u has minimal period T.

Proof. According to the standard continuation method adapted to this context (see e.g. [16, Thm 2.1]), let us firstly prove that the solutions $(u, v) \in C_T \times C_T$ of the system

$$\begin{cases} u'(t) = a_1 u(t) + b_1 u(t-\tau_1) + \lambda[g_1(u(t-\tau_1), v(t-\tau_2)) + p(t)] \\ v'(t) = a_2 v(t) + b_2 v(t-\tau_2) + \lambda g_2(t, u_t, v_t) \end{cases} \tag{12}$$

with $\lambda \in (0,1)$ are *a priori* bounded. Indeed, suppose that (u_n, v_n) is a sequence of solutions of (12) for some $\lambda_n \in (0,1)$, then we already know from (6) that

$$\|v_n\|_\infty \le \|v_n\|_{C^1} \le c_2\|L_2 v_n\|_\infty \le c_2\|g_2\|_\infty.$$

Next, write

$$u_n = \tilde{u}_n + \rho_n \cos(k\omega t - \theta_n)$$

with $\tilde{u}_n \in \mathrm{Ker}(L)^\perp$, $\rho_n \ge 0$ and $\theta_n \in [0, 2\pi)$, then $\|\tilde{u}_n\|_{C^1} \le c_1(\|g_1\|_\infty + \|p\|_\infty)$. We conclude that $\rho_n \to +\infty$ and, moreover, since $\mathrm{Ker}(L^*) = \mathrm{Ker}(L)$ we deduce that $\int_0^T Lu_n(t)e^{ik\omega t}\,dt = 0$ for all n, that is

$$\int_0^T g_1(u_n(t-\tau_1), v_n(t-\tau_2))e^{ik\omega t}\,dt = -\int_0^T p(t)e^{ik\omega t}\,dt$$

for all n. Next, observe that the substitution $k\omega s := k\omega(t-\tau_1) - \theta_n$ and periodicity imply that the left-hand side term can be written as

$$e^{i(k\omega\tau_1+\theta_n)}\int_0^T g_1(\rho_n\cos(k\omega s) + \psi_n(s), \phi_n(s))e^{ik\omega s}\,ds,$$

where $|\psi_n| \le c_1\|g_1\|_\infty$ and $|\phi_n| \le c_2\|g_2\|_\infty$. For convenience, let us fix $A \in [0, 2\pi)$ such that $-e^{iA}\int_0^T p(t)e^{ik\omega t}\,dt = \left|\int_0^T p(t)e^{ik\omega t}\,dt\right|$, then the previous equality simply reads

$$\int_0^T g_1(\rho_n\cos(k\omega s) + \psi_n(s), \phi_n(s))\cos(k\omega s + R_n)\,ds = \left|\int_0^T p(t)e^{ik\omega t}\,dt\right|$$

where $R_n := k\omega\tau_1 + \theta_n + A$.

From the identity $\cos(k\omega s + R_n) = \cos(k\omega s)\cos(R_n) - \sin(k\omega s)\sin(R_n)$ and (9) we deduce, following the ideas of Lemma 3 in [17], that

$$\begin{aligned}\int_0^T & g_1(\rho_n\cos(k\omega s) + \psi_n(s), \phi_n(s))\sin(k\omega s)\,ds = \\ &\int_0^T [g_1(\rho_n\cos(k\omega s) + \psi_n(s), \phi_n(s)) - g_1(\rho_n\cos(k\omega s) + \psi_n(s), 0)]\sin(k\omega s)\,ds \\ &\quad + \int_0^T g_1(\rho_n\cos(k\omega s) + \psi_n(s), 0)\sin(k\omega s)\,ds \to 0\end{aligned}$$

as $n \to \infty$. Thus, assuming for example that (10) holds, by Fatou's lemma we conclude:

$$\left|\int_0^T p(t)e^{ik\omega t}\,dt\right| \ge g_1^{inf}(+\infty)\int_{I^+}\cos(k\omega s)\,ds - g_1^{sup}(-\infty)\int_{I^-}\cos(k\omega s)\,ds$$

where $I^\pm \subset [0, 2\pi]$ are respectively the positivity/negativity sets of $\cos(k\omega s)$. Hence

$$\left|\int_0^T p(t)e^{ik\omega t}\,dt\right| \ge 2[g_1^{inf}(+\infty) - g_1^{sup}(-\infty)],$$

a contradiction.

Next, identify Ker(L) with $\mathbb{C}$ by the mapping $\rho\cos(k\omega t-\theta)\mapsto z:=\rho e^{i\theta}$. We need to compute the degree of the mapping

$$\Phi(z):=\int_0^T g(\rho\cos(k\omega t-\theta))e^{ik\omega t}\,dt-\int_0^T p(t)e^{ik\omega t}\,dt$$

over large balls centered at 0. Now observe that

$$\int_0^T g(\rho\cos(k\omega t-\theta))e^{ik\omega t}\,dt=e^{i\theta}\int_0^T g(\rho\cos(k\omega t))e^{ik\omega t}\,dt$$

and, moreover, since $g(\rho\cos(k\omega t))$ is even, we obtain as before

$$\int_0^T g(\rho\cos(k\omega t))e^{ik\omega t}\,dt\geq[g_1^{inf}(+\infty)-g_1^{sup}(-\infty)]>\left|\int_0^T p(t)e^{ik\omega t}\,dt\right|$$

for $\rho\gg 0$. This implies

$$deg(\Phi,B_\rho(0),0)=\pm 1$$

and so completes the proof. □

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