STUDIA MATHEMATICA

Online First version

Harmonic analysis on some generalized Gelfand pairs attached to Heisenberg groups

by

FERNANDO LEVSTEIN and LINDA SAAL (Córdoba)

Abstract. Let H_n be the $2n+1$-dimensional Heisenberg group. We consider the generalized Gelfand pairs $(\mathbb{R}^* \ltimes H_1, \mathbb{R}^*)$ and $((\mathbb{R}_{>0} \times \mathrm{SO}(n)) \ltimes H_n, \mathbb{R}_{>0} \times \mathrm{SO}(n))$ for $n \geq 2$. We describe the spherical distributions corresponding to these pairs and we obtain inversion formulæ in terms of them for the spaces of Schwartz functions on $\mathbb{R}^{2n}$ and H_n. We use the Tengstrand transform to compute the spherical distributions for $n = 1$ explicitly.

1. Introduction. Let G be a unimodular Lie group. Given a unitary representation $(\pi, \mathcal{H})$ of G on a Hilbert space $\mathcal{H}$, a vector $v \in \mathcal{H}$ is called a $\mathcal{C}^\infty$-*vector* if $\pi_v : g \mapsto \pi(g)v$ is a $\mathcal{C}^\infty$ map from G into $\mathcal{H}$. We denote by $\mathcal{H}^\infty$ the space of $\mathcal{C}^\infty$-vectors endowed with a natural Sobolev topology that makes it into a Fréchet space. For X in the Lie algebra of G, and $v \in \mathcal{H}^\infty$, we set

$$\pi(X)v = \frac{d}{dt}\bigg|_{t=0} \pi(\exp tX)v.$$

The seminorms are defined by

$$p_m(v) = \sum_{|\alpha| \leq m} \|\pi(X_1)^{\alpha_1} \dots \pi(X_k)^{\alpha_k}(v)\|_{\mathcal{H}}$$

where $X_1, \dots, X_k$ is a basis of the Lie algebra of G, and $|\alpha| = \alpha_1 + \dots + \alpha_k$.

$\mathcal{H}^{-\infty}$ will denote the antidual space consisting of continuous conjugate linear functionals on $\mathcal{H}^\infty$. Thus $\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}$. The elements of $\mathcal{H}^{-\infty}$ are called *distribution vectors*. The action of G on $\mathcal{H}^\infty$ gives a corresponding action on $\mathcal{H}^{-\infty}$,

$$\langle \pi_{-\infty}(g)\phi, v \rangle = \langle \phi, \pi_\infty(g)v \rangle, \quad g \in G,\ \phi \in \mathcal{H}^{-\infty},\ v \in \mathcal{H}^\infty.$$

2010 *Mathematics Subject Classification*: Primary 43A80; Secondary 35A08.
Key words and phrases: Heisenberg group, spherical distribution, generalized Gelfand pair.
Received 15 April 2016; revised 25 May 2017.
Published online *.

DOI: 10.4064/sm8592-5-2017

Let $K \subset G$ be a closed unimodular subgroup and let

$$\mathcal{H}_1^{-\infty} = \{\phi \in \mathcal{H}^{-\infty} : \pi_{-\infty}(k)\phi = \phi \text{ for all } k \in K\},$$

the space of distribution vectors fixed by K. Then a pair (G, K) is called a *generalized Gelfand pair* if for each irreducible unitary representation $(\pi, \mathcal{H})$ of G the space $\mathcal{H}_1^{-\infty}$ is at most one-dimensional (see for instance [vD]).

We recall that when K is compact and (G, K) is a Gelfand pair, a spherical function ζ of positive type is written as

$$\zeta(g) = \langle \pi(g)v, v\rangle,$$

where π is an irreducible unitary representation of G and v is a vector fixed by K.

When K is no longer compact and π admits a distribution vector $\phi \in \mathcal{H}_1^{-\infty}$ fixed by K, then, for f smooth on G, we have $\pi_{-\infty}(f)\phi \in \mathcal{H}^{\infty}$, and so we can associate to ϕ the distribution

$$\Phi_\pi(f) = \langle \phi, \pi_{-\infty}(f)\phi\rangle. \tag{1}$$

This is a positive type K-biinvariant distribution on G, and since π is irreducible, it is an extremal point of the cone of positive type K-biinvariant distributions on G (see [F]). Following Molchanov [Mo], we call Φ_π a *spherical distribution*.

In this work we will consider pairs $(K \ltimes H_n, K)$ (also denoted by (K, H_n)), where H_n denotes the $2n+1$-dimensional Heisenberg group. For $n \geq 2$, $K = \mathbb{R}_{>0} \times \mathrm{SO}(n)$ and the action considered is

$$(r, A).(x, y, t) = (rAx, r^{-1}Ay, t) \quad \text{ for } r \in \mathbb{R}_{>0},\ A \in \mathrm{SO}(n).$$

For $n = 1, K = \mathbb{R}_{>0} \times \mathrm{O}(1) \simeq \mathbb{R}^*$ and the action is

$$r.(x, y, t) = (rx, r^{-1}y, t) \quad \text{ for } r \in \mathbb{R}^*.$$

With these actions the corresponding $K \ltimes H_n$ are unimodular.

In [LS2] it was shown that for $n \geq 2$, $(K \ltimes H_n, K)$ is a generalized Gelfand pair. There it was mistakenly stated that $(\mathbb{R}_{>0} \ltimes H_1, \mathbb{R}_{>0})$ is a generalized Gelfand pair.

In Section 2 we will see that if we consider instead $K = \mathbb{R}_{>0} \times \mathrm{O}(1)$, then $(K \ltimes H_1, K)$ is a generalized Gelfand pair.

In Section 3 we consider the pairs $(K \ltimes H_n, K)$ for $n \geq 2$, and describe the spherical distributions attached to vector distributions fixed by K. These spherical distributions depend on a fundamental parameter λ, representing a character of H_n in the central variable t. We obtain inversion formulæ in terms of them for the spaces of Schwartz functions on H_n and $\mathbb{R}^{2n}$.

Finally in Section 4 we develop the spherical analysis related to $(\mathbb{R}^*, H_1)$ by using the Tengstrand transform.

2. Preliminaries. Let us consider the Heisenberg group $H_n = \{(x,y,t) : x,y \in \mathbb{R}^n,\ t \in \mathbb{R}\}$ with multiplication given by

$$(x_1,y_1,t)(x_2,y_2,s) = \big(x_1+x_2, y_1+y_2, t+s+\tfrac{1}{2}(\langle x_1,y_2\rangle - \langle y_1,x_2\rangle)\big)$$

where $\langle x,y\rangle$ denotes the standard inner product on $\mathbb{R}^n$.

The irreducible unitary representations of H_n which are non-trivial on the center are determined up to equivalence by their central characters, and thus parametrized by $\lambda \in \mathbb{R}^*$. A realization of them is given by the Schrödinger model defined on $\mathcal{H}_\lambda = L^2(\mathbb{R}^n)$ and denoted by $(\pi_\lambda, \mathcal{H}_\lambda)$. For those acting trivially on the center, there is a correspondence with the characters $\chi_{\xi,\eta}$ of $\mathbb{R}^{2n}$, $\xi,\eta \in \mathbb{R}^n$.

The natural action of $\mathrm{Sp}(n,\mathbb{R})$ on $\mathbb{R}^{2n}$ extends to an action on H_n by automorphisms fixing every element of the center. For $k \in \mathrm{Sp}(n,\mathbb{R})$, let $\pi_\lambda^k(x,y,t) = \pi_\lambda(k(x,y),t)$ for $(x,y,t) \in H_n$. Then π_λ^k is equivalent to π_λ and there exists a unitary operator $\omega(k)$ that intertwines π_λ^k and π_λ. This defines a projective representation of $\mathrm{Sp}(n,\mathbb{R})$ on $L^2(\mathbb{R}^n)$, called the *metaplectic representation.*

For $n=1$ the group K is $\mathbb{R}^*$ and acts by the automorphisms $r.(x,y,t) = (rx, r^{-1}y, t)$ for $r \in \mathbb{R}^*$, and $\omega(r)f(x) = |r|^{-1/2} f(r^{-1}x)$.

For $n \ge 2$, the group $K = \mathbb{R}_{>0} \times \mathrm{SO}(n)$ acts on H_n by $(r,A).(x,y,t) = (rAx, r^{-1}Ay, t)$ for $r \in \mathbb{R}_{>0}$, $A \in \mathrm{SO}(n)$, and

$$\omega(r,A)f(x) = r^{-n/2} f(r^{-1}g^{-1}x)$$

gives a well defined unitary representation of K.

Let $\widehat{K}$ be the set of irreducible unitary representations of K up to equivalence. According to Mackey's theory (see [Ma]), the elements of $\widehat{K \ltimes H_n}$ are of two types:

- $\pi_{\lambda,\tau}(k,h) = \tau(k) \otimes \omega(k)\pi_\lambda(h)$ where $k \in K$, $h \in H_n$ and $\tau \in \widehat{K}$,
- $\rho_{\xi,\eta} = \mathrm{Ind}_{K_{\xi,\eta} \ltimes H_n}^{K \ltimes H_n}(\sigma \otimes \chi_{\xi,\eta})$ where $K_{\xi,\eta}$ is the stabilizer of (ξ,η) in K and $\sigma \in \widehat{K}_{\xi,\eta}$.

This second type corresponds precisely to $\widehat{K \ltimes \mathbb{R}^{2n}}$.

Since K is reductive, any unitary representation π decomposes in a unique way into a direct integral of irreducible unitary representations,

$$\pi = \int_{\widehat{K}} m_\pi(\tau)\tau\, d\mu(\tau),$$

where μ is a Borel measure on $\widehat{K}$ and $m_\pi : \widehat{K} \to \mathbb{N} \cup \{\infty\}$ is the multiplicity.

Recall that a unitary representation of a group K on a separable Hilbert space $\mathcal{H}$ is *multiplicity free* if the ring of continuous endomorphisms commuting with K, $\mathrm{End}_K(\mathcal{H})$, is commutative [K, pp. 503–504]. Also, the following are equivalent:

(i) $\mathrm{End}_K(\mathcal{H})$ is commutative.
(ii) $m_\pi(\tau) \le 1$ for μ-almost all $\tau \in \widehat{K}$.

Notice that $\pi_{\lambda,\sigma}$ has a distribution vector fixed by K if and only if σ appears in the decomposition of $\omega|_K$ [MT, Th. 2.1]. Moreover, by using Frobenius reciprocity, it is not difficult to see that $(K, \mathbb{R}^{2n})$ is always a generalized Gelfand pair. Thus (K, H_n) is a generalized Gelfand pair if and only if $\omega|_K$ is multiplicity free [MT, Prop. 3.1 and Th. 3.2].

For $n \ge 2$, (K, H_n) is a generalized Gelfand pair since, for ω the metaplectic representation, $\omega{\downarrow}^{\mathrm{Sp}(n,\mathbb{R})}_{\mathrm{SO}(n)\times\mathbb{R}_{>0}} = \bigoplus_k \int_{-\infty}^{\infty} \tau_k \otimes s^{i\alpha - n/2}\, d\alpha$, where (τ_k, Y_k) denotes the irreducible representation of $\mathrm{SO}(n)$ on the space of spherical harmonics of degree k.

We can show now that $(\mathbb{R}^*, H_1)$ is a generalized Gelfand pair.

Proposition 1. *The metaplectic action on $L^2(\mathbb{R})$ is multiplicity free with the decomposition*

$$L^2(\mathbb{R}) = \int_{-\infty}^{\infty} |x|^{i\alpha - 1/2}\, d\alpha \oplus \int_{-\infty}^{\infty} \mathrm{sg}(x)|x|^{i\alpha-1/2}\, d\alpha. \tag{2}$$

Proof. According to Mackey, the representations of $G = \mathbb{R}^* \ltimes H_1$ are either induced by characters of H_1 or given by

$$\pi_{\lambda,\alpha}(s, (x,y,t)) = |s|^{i\alpha} \pi_\lambda(x,y,t)\omega(s)$$

where π_λ is the Schrödinger representation of H_1 and $(\omega(s)f)(x) = |s|^{-1/2} f(s^{-1}x)$. Hence, for $f_\alpha(x) = |x|^{i\alpha-1/2}$ we have $\omega(s)f_\alpha = |s|^{-i\alpha} f_\alpha$.

The Mellin transform is the Fourier transform adapted to $\mathbb{R}_{>0}$, and it is defined by $Mf(\lambda) = \int_0^\infty f(s)s^{i\lambda}\,\frac{ds}{s}$. The action of $\mathbb{R}_{>0}$ on $L^2\big(\mathbb{R}_{>0}, \frac{ds}{s}\big)$ given by $\delta_t f(s) = f(t^{-1}s)$ decomposes, via the Mellin transform, as

$$L^2\left(\mathbb{R}_{>0}, \frac{ds}{s}\right) = \int_{-\infty}^{\infty} F_\lambda\, d\lambda$$

where F_λ is the $\mathbb{C}$-vector space generated by $s^{i\lambda}$ [Ta, p. 168].

Let Ψ be an even function. For $u \ge 0$ let $g(u) = u^{1/2}\Psi(u)$. Then

$$g(u) = \int_{-\infty}^{\infty} \left(\int_0^\infty g(v) v^{i\alpha} \frac{dv}{v} \right) u^{-i\alpha}\, d\alpha.$$

Thus

$$\Psi(u) = \int_{-\infty}^{\infty} \left(\int_0^\infty \Psi(v) v^{i\alpha - 1/2} \frac{dv}{v} \right) u^{-i\alpha - 1/2}\, d\alpha \qquad \forall u \ge 0.$$

Analogously, for $u < 0$,

$$\Psi(u) = \Psi(-u) = \int_{-\infty}^{\infty}\Big(\int_{0}^{\infty}\Psi(v)v^{i\alpha-1/2}\,\frac{dv}{v}\Big)(-u)^{-i\alpha-1/2}\,d\alpha$$
$$= \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{0}\Psi(-w)(-w)^{i\alpha-1/2}\,\frac{dw}{-w}\Big)(-u)^{-i\alpha-1/2}\,d\alpha.$$

Since Ψ is even and $\Psi = \Psi\chi_{(0,\infty)} + \Psi\chi_{(-\infty,0)}$, we obtain

$$\Psi(u) = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\Psi(v)|v|^{i\alpha-1/2}\,\frac{dv}{|v|}\Big)|u|^{-i\alpha-1/2}\,d\alpha.$$

For Ψ odd, we apply the formula obtained above to the function $\Phi(u) = \mathrm{sg}(u)\Psi(u)$ and obtain

$$\Psi(u) = \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\infty}\Psi(v)\mathrm{sg}(v)|v|^{i\alpha-1/2}\,\frac{dv}{|v|}\Big)\,\mathrm{sg}(u)|u|^{-i\alpha-1/2}\,d\alpha.$$

Since any function can be written as the sum of an even function and an odd function, the proposition follows. ∎

Remark 2. Let $f_\alpha(x)$ be as above and let $g_\alpha(x) = \mathrm{sg}(x)|x|^{i\alpha-1/2}$. Then f_α and g_α are distribution vectors fixed by $\sigma_{\lambda,\alpha}|_{\mathbb{R}_{>0}}$ but only f_α is fixed by $\sigma_{\lambda,\alpha}|_{\mathbb{R}^*}$.

Remark 3. $\omega(s)f_\alpha = |s|^{-i\alpha}f_\alpha$ and $\omega(s)g_\alpha = \mathrm{sg}(s)|s|^{-i\alpha}g_\alpha$.

Remark 4. The set of characters of $\mathbb{R}^*$ is $\{|s|^{-i\alpha},\ \mathrm{sg}(s)|s|^{-i\alpha}\}$.

Remark 5. If K is compact and (G,K) is a Gelfand pair, then (G,K_0) is also a Gelfand pair, where K_0 denotes the connected component of K. In the non-compact case, the pair $(\mathbb{R}^*, H_1)$ gives an example of a generalized Gelfand pair such that the connected component of $\mathbb{R}^*$ does not give a generalized Gelfand pair. Indeed, according to the decomposition (2) and Remark 2, the metaplectic representation is not multiplicity free.

3. Spherical analysis on $(\mathbb{R}_{>0} \times \mathrm{SO}(n), H_n)$ for $n \geq 2$

3.1. K-invariant distribution vectors attached to $\pi_{\lambda,\tau}$. Let $K = \mathbb{R}_{>0} \times \mathrm{SO}(n)$ and $G = K \ltimes H_n$. Recall that G is the set of pairs $(k,h) \in K \times H_n$ with product given by $(k_1,h_1)(k_2,h_2) = (k_1k_2, h_1(k_1 \cdot h_2))$, where the dot denotes the action of K on H_n.

We observe that a K-invariant distribution ϕ on H_n gives rise to a K-biinvariant distribution Φ on G by the rule

$$\langle \Phi, f\rangle_G = \langle \phi, f_0\rangle_{H_n}, \quad \text{where} \quad f_0(h) = \int_K f(k.(e_K,h))\,dk. \tag{3}$$

Conversely, let Φ be a K-biinvariant distribution on G. Since the map $(k,h) \mapsto (e_K, h)(k, e_{H_n})$ is a diffeomorphism, the composition gives a distribution Ψ on $K \times H_n$, which is right K-invariant. Thus $\Psi = 1 \otimes \phi$ with ϕ a K-invariant distribution on H_n. Moreover Φ is of positive type if and only if ϕ is.

Thus the spherical distributions are the extremal points of the cone of K-invariant distributions of positive type on H_n (see [F]).

For $\lambda \neq 0$, we denote by (π_λ, H_λ) the Schrödinger representation of H_n. We recall that H_λ^∞ is the Schwartz space $S(\mathbb{R}^n)$, and thus $H_\lambda^{-\infty}$ is $S'(\mathbb{R}^n)$.

Let K be a compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$ such that (K, H_n) is a Gelfand pair. When $\lambda \neq 0$, the set of spherical functions can be given by $\{\varphi_{\lambda,\tau}(h) = \operatorname{tr} \pi_\lambda(h)|_{V_\tau}\}$, where (τ, V_τ) is an irreducible representation of K that appears in the multiplicity free action of K on H_λ.

In our case, K is not compact but we will obtain a similar formula. In [LS1] it was proved that the algebra of polynomials invariant under the action of K on $\mathbb{R}^{2n}$ is generated by

$$s(x,y) = \langle x, y\rangle \text{ and } q(x,y) = x_1y_2 - x_2y_1 \quad \text{for } x = (x_1,x_2),\ y = (y_1,y_2)$$

when $n = 2$, and by

$$s(x,y) = \langle x, y\rangle \text{ and } q(x,y) = |x|^2|y|^2 \quad \text{for } x = (x_1,\ldots,x_n),\ y = (y_1,\ldots,y_n)$$

when $n \geq 3$.

Let $\{X_1,\ldots,X_n,Y_1,\ldots,Y_n,T\}$ be the standard basis of the Lie algebra of H_n, that is, $[X_j, Y_j] = T$ and all other brackets are zero.

Lemma 6. *Let $E = \sum_{j=1}^n Y_jX_j$, $\Delta_X = \sum_{j=1}^n X_j^2$ and $\Delta_Y = \sum_{j=1}^n Y_j^2$. Then the algebra of left K-invariant differential operators on H_n is generated by T, E and $X_1Y_2 - X_2Y_1$ when $n = 2$, and by T, E and $\Delta_X\Delta_Y$ when $n \geq 3$.*

Proof. Let σ be the symmetrization map. By a well known result [V, p. 180], the algebra of left K-invariant differential operators on H_n is generated by T, $\sigma(E)$ and $\sigma(\Delta_X\Delta_Y)$ when $n \geq 3$, and by T, E and $\sigma(X_1Y_2 - X_2Y_1) = X_1Y_2 - X_2Y_1$ when $n = 2$. Thus we can assume $n \geq 3$.

Since $X_jY_j - Y_jX_j = T$, we have

$$\sigma\Big(\sum_{j=1}^n X_jY_j\Big) = \sum_{j=1}^n Y_jX_j + \frac{n}{2}T.$$

So

$$\sigma(E) = \frac{n}{2}T + E. \tag{4}$$

Now, since X_j commutes with Y_i when $i \neq j$,

$$\sigma(\Delta_X\Delta_Y) = \sum_{i\neq j} X_j^2Y_i^2 + \sigma\Big(\sum_{i=1}^n X_i^2Y_i^2\Big). \tag{5}$$

We will use the following identities:

(i) $X_iY_iX_iY_i = X_i^2Y_i^2 - X_iY_iT$.
(ii) $Y_iX_iX_iY_i = X_iY_iX_iY_i - X_iY_iT = X_i^2Y_i^2 - 2X_iY_iT$, where in the second equality we use (i).
(iii) $Y_iX_iY_iX_i = Y_iX_iX_iY_i - Y_iX_iT = X_i^2Y_i^2 - 2X_iY_iT - (X_iY_iT - T^2) = X_i^2Y_i^2 - 3X_iY_iT + T^2$, where in the second equality we use (ii).
(iv) $Y_iY_iX_iX_i = Y_iX_iY_iX_i - Y_iX_iT = X_i^2Y_i^2 - 3X_iY_iT + T^2 - (X_iY_iT - T^2) = X_i^2Y_i^2 - 4X_iY_iT + 2T^2$, where in the second equality we used (iii).
(v) $X_iY_iY_iX_i = X_iY_iX_iY_i - X_iY_iT = X_i^2Y_i^2 - 2X_iY_iT$, where in the second equality we use (i).

Thus

$$\begin{aligned}\sigma(X_i^2Y_i^2) &= \tfrac{1}{6}(X_i^2Y_i^2 + X_iY_iX_iY_i + Y_iX_iX_iY_i \\ &\quad + Y_iX_iY_iX_i + Y_iY_iX_iX_i + X_iY_iY_iX_i) \\ &= X_i^2Y_i^2 - 2X_iY_iT + \tfrac{1}{2}T^2.\end{aligned}$$

Consequently, $\sigma(\sum_{i=1}^n X_i^2Y_i^2) = \sum_{i=1}^n X_i^2Y_i^2 - 2T\sum_{i=1}^n X_iY_i + \frac{n}{2}T^2$. Finally,

$$\sigma(\Delta_X\Delta_Y) = \Delta_X\Delta_Y - 2ET - \frac{3n}{2}T^2, \tag{6}$$

and the proof is complete. ∎

Lemma 7. *Let $\phi \in S'(\mathbb{R}^n)$ be a joint eigendistribution of the operators $d\pi_\lambda(E)$ and $d\pi_\lambda(\Delta_X\Delta_Y)$. Then there exist $\gamma \in \mathbb{C}$ and a harmonic polynomial p_k of degree $k \in \mathbb{N}_0$ such that*

$$\langle\phi, f\rangle = \int_{\mathbb{R}^n} f(u)p_k(u/|u|)|u|^\gamma\,du.$$

Proof. We have $d\pi_\lambda(E) = i\lambda\sum_j u_j\partial_{u_j}$ and $d\pi_\lambda(\Delta_y\Delta_x) = \lambda^2|u|^2\Delta_u$. It is well known that a function is an eigenfunction of the Euler operator with eigenvalue γ if and only if it is homogeneous of degree γ. Analogously, using polar coordinates one can show that a distribution is an eigendistribution of $d\pi_\lambda(E)$ with eigenvalue γ if and only if it is homogeneous of degree γ, and it is given by

$$\langle\phi, f\rangle_{\mathbb{R}^n} = \int_0^\infty \langle\psi, f(r\cdot)\rangle_{S^{n-1}} r^{\gamma+n-1}\,dr, \qquad \text{where } \psi \in \mathcal{D}'(S^{n-1}).$$

Moreover, if $\operatorname{Re}\gamma + n \le 0$, then ψ must satisfy $\langle\psi, Y_m\rangle = 0$ for $m \le -(\operatorname{Re}\gamma + n)$, where Y_m is the space of spherical harmonics of degree m.

In polar coordinates, $\sum_j u_j\partial_{u_j} = r\partial_r$ and

$$|u|^2\Delta_u = r^2\partial_r^2 + (n-1)r\partial_r + \Delta_\varpi \tag{7}$$

$$= (r\partial_r)^2 + (n-2)r\partial_r + \Delta_\varpi, \tag{8}$$

where Δ_ϖ denotes the Laplacian on S^{n-1}. Thus ψ is an eigendistribution of Δ_ϖ, that is, a spherical harmonic of degree k for some $k \in \mathbb{N}_0$. ∎

We now assume $n \geq 3$. Let $\{p_i : 1 \leq i \leq \dim Y_k\}$ be an orthonormal basis of the spherical harmonics of degree k, and define $q_i(u) = |u|^\gamma p_i(u/|u|)$. Then q_i is a distribution vector of H_λ, so for F smooth on H_n, $\pi_\lambda^{-\infty}(F)q_i \in H_\lambda^\infty$ and $\langle \Phi, F\rangle := \sum_{i=1}^{\dim Y_k} \langle q_i, \pi_\lambda^{-\infty}(F)q_i\rangle$ defines a tempered distribution on H_n. Since $\pi_\lambda(XF) = -\pi_\lambda(F)d\pi_\lambda(X)$ for every X in the Lie algebra of H_n, we see that Φ is a joint eigendistribution of E and $\Delta_X\Delta_Y$. Moreover, $\pi_\lambda(F * F^*) = \pi_\lambda(F)\pi_\lambda(F^*)$ and $\pi_\lambda(F^*) = \pi_\lambda(F)^*$, thus Φ is of positive type.

Let us see that Φ is K-invariant if and only if $\gamma = -n/2 + i\alpha$.

Indeed, for $(s,h) \in K$, we have $\langle \Phi^{(s,h)}, F\rangle = \langle \Phi, F^{(s,h)}\rangle$, where $F^{(s,h)}(x,y,t) = F(shx, s^{-1}hy, t)$.

Since $\pi_\lambda^{-\infty}(F^{(s,h)}) = \omega((s,h))\pi_\lambda^{-\infty}(F)\omega((s,h)^{-1})$ and the sum defining Φ is invariant under any orthonormal basis, we require that the action of $\omega((s,h))$ be unitary on q_i. The action of $\mathrm{SO}(n)$ always is. But $\omega(s,1)f(x) = s^{-n/2}f(s^{-1}x)$, which forces $\gamma = -n/2 + i\alpha$ for some $\alpha \in \mathbb{R}$.

Finally, for $k \in N_0$, $\alpha \in \mathbb{R}$, $F \in S(\mathbb{H}_n)$, let

$$\langle \Phi_{\lambda,\alpha,k}, F\rangle := \sum_{i=1}^{\dim Y_k} \langle q_{i,\alpha}^k, \pi_\lambda^{-\infty}(F) q_{i,\alpha}^k\rangle$$

where $q_{i,\alpha}^k(u) = |u|^{i\alpha - n/2} p_i(u/|u|)$ and $\{p_i : 1 \leq i \leq \dim Y_k\}$ is an orthonormal basis of the spherical harmonics of degree k. Then $\Phi_{\lambda,\alpha,k}$ are spherical distributions. Since an eigenfunction of E and $X_1Y_2 - X_2Y_1$ is also an eigenfunction of E and $\Delta_X\Delta_Y$, the same argument as above holds for $n = 2$. Observe that $\dim Y_k = 1$ for every $k \in \mathbb{N}_0$, and

$$\langle \Phi_{\lambda,\alpha,k}, F\rangle := \langle q_\alpha^k, \pi_\lambda^{-\infty}(F) q_\alpha^k\rangle$$

where $q_{\alpha,k}(u) = |u|^{i\alpha - n/2 - k}(u_1 + iu_2)^k$.

Finally, Mackey theory ensures that this set exhausts the set of spherical distributions attached to $\pi_{\lambda,\tau}$.

3.2. K-invariant distribution vectors attached to $\rho_{\xi,\eta}$. We now consider the irreducible unitary representations of G attached to characters of $\mathbb{R}^{2n}$. Notice that they are in correspondence with the irreducible unitary representations of $K \ltimes \mathbb{R}^{2n}$, with the obvious action of K on $\mathbb{R}^{2n}$, since the center of $\mathbb{H}_n$ plays no role. In fact, we are dealing with the spherical analysis on the generalized Gelfand pair $(\mathbb{R}_{>0} \times \mathrm{SO}(n), \mathbb{R}^{2n})$. Thus, given a unitary character $\chi_{\xi,\eta}(x,y) = e^{i\langle(\xi,\eta),(x,y)\rangle}$ of $\mathbb{R}^{2n}$, let $K_{\xi,\eta} \subset \mathrm{SO}(n)$ be the stabilizer of (ξ,η) in K. We extend $\chi_{\xi,\eta}$ trivially to $K_{\xi,\eta}$, and for $(\tau, V_\tau) \in \widehat{K}_{\xi,\eta}$ we find that the representations $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^G(\tau \otimes \chi_{\xi,\eta})$, together with the representations described above, exhaust the irreducible unitary representations of G. We will need the following result.

LEMMA 8.

(i) *If the representation* $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^{G}(\tau\otimes\chi_{\xi,\eta})$ *has a distribution vector fixed by* K *then* τ *is trivial.*
(ii) $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^{G}(1\otimes\chi_{\xi,\eta})$ *has a distribution vector fixed by* K *if and only if either* $\xi=0$ *and* $\eta=0$, *or* $\xi\neq 0$ *and* $\eta\neq 0$.

Proof. We recall that $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^{G}(\tau\otimes\chi_{\xi,\eta})$ is represented on the completion of the set of functions $f\in\mathcal{C}(K\ltimes\mathbb{R}^{2n},V_\tau)$ satisfying

$$f((1,h,u,v)g)=\tau(h)\chi_{\xi,\eta}(u,v)f(g),\quad \forall h\in K_{\xi,\eta},\ u,v\in\mathbb{R}^{2n},\ g\in K\ltimes\mathbb{R}^{2n},$$

where $K\ltimes\mathbb{R}^{2n}$ acts by right multiplication.

Since $(s,k,x,y)=(1,1,x,y)(s,h,0,0)$, any f in this space can be written as $f(s,k,x,y)=\chi_{\xi,\eta}(x,y)\tilde f(s,h)$ where $\tilde f((1,h)(s,k))=\tau(h)\tilde f(s,k)$. Thus as a K-module, $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^{G}(\tau\otimes\chi_{\xi,\eta})$ is isomorphic to $\mathrm{Ind}_{K_{\xi,\eta}}^{K}(\tau)$. Since $K=\mathbb{R}_{>0}\times\mathrm{SO}(n)$, we have $\mathrm{Ind}_{K_{\xi,\eta}}^{K}(\tau)=L^2(\mathbb{R}_{>0})\otimes\mathrm{Ind}_{K_{\xi,\eta}}^{\mathrm{SO}(n)}(\tau)$. Thus if ϕ is a distribution vector fixed by K, then $\phi=\phi_0\otimes 1$ with ϕ_0 a distribution vector fixed by the right representation of $\mathbb{R}_{>0}$ on $L^2(\mathbb{R}_{>0},ds/s)$. But by Frobenius reciprocity, the trivial representation appears in $\mathrm{Ind}_{K_{\xi,\eta}}^{\mathrm{SO}(n)}(\tau)$ only if τ is trivial. This proves the first assertion.

If $(\xi,\eta)=(0,0)$ and $K_{\xi,\eta}=K$, the induced representation $\mathrm{Ind}_{K\times\mathbb{R}^{2n}}^{G}(\tau\otimes 1)$ is just $\tau\otimes 1$, and it has a vector fixed by K only if τ is trivial. Thus the corresponding spherical distribution is $\Phi_0\equiv 1$.

Assume that $\xi\neq 0$ or $\eta\neq 0$. We observe that $\mathrm{Ind}_{K_{\xi,\eta}\times\mathbb{R}^{2n}}^{G}(1\otimes\chi_{\xi,\eta})$ can be realized on

$$\mathcal{H}=L^2(\mathbb{R}_{>0}\times K_{\xi,\eta}\backslash\mathrm{SO}(n))$$

with the action given by

$$\rho(s,h,(x,y))F(r,\bar k)=e^{i\langle (r,k)\cdot(x,y),(\xi,\eta)\rangle}F(rs,\bar k h),$$

where $F\in\mathcal{H}$, $(r,k)\cdot(x,y)=(rkx,r^{-1}ky)$ and $\bar k=K_{\xi,\eta}k$. Since $K_{\xi,\eta}$ fixes (ξ,η), the representation ρ is well defined.

We now describe the space of $\mathcal{C}^\infty$-vectors: since $\frac{\partial}{\partial x_j}\rho(1,1,x,0)F(r,1)=ir\xi_jF(r,1)$ and $\frac{\partial}{\partial y_j}\rho(1,1,0,y)F(r,1)=ir^{-1}\eta_jF(r,1)$, we see that for $F\in\mathcal{H}^\infty$ we have $r^m|F(r)|\le c_m$ for all $m\in\mathbb{Z}$. Thus $\langle\phi_0,F\rangle=\int_0^\infty F(r)\,\frac{dr}{r}$ defines a distribution vector fixed by $\mathbb{R}_{>0}$, and so $\phi=\phi_0\otimes 1$ is a distribution vector fixed by K.

When $\xi=0$ (resp. $\eta=0$), the condition $r^m|F(r)|\le c_m$ for all $m\in\mathbb{N}$ (resp. $r^{-m}|F(r)|\le c_{-m}$ for all $m\in\mathbb{N}$) is no longer valid, so ϕ_0 is not well defined. Assume there exists a distribution vector ψ fixed by K. By the proof of the first part of the lemma we can write $\psi = \psi_0 \otimes 1$. Thus the existence of an invariant distribution vector fixed by K would imply the existence of a translation invariant distribution $\chi \in \mathcal{D}'(\mathbb{R})$. Thus $\chi' = 0$ and so $\chi = c1$, which is absurd. ∎

Thus, we will consider only the case $\xi \neq 0$ and $\eta \neq 0$. In this case $K_{\xi,\eta}$ is isomorphic to either $\mathrm{SO}(n-2)$ or $\mathrm{SO}(n-1)$. We observe that in the first case the Stiefel manifold $\mathrm{SO}(n-2)\backslash\mathrm{SO}(n)$ can be described as $\{(\zeta_1,\zeta_2) : \zeta_1,\zeta_2$ are ortonormal vectors in $\mathbb{R}^n\}$.

Now we introduce on $\mathbb{R}^{2n}$ a new system of coordinates

$$(x,y) = \left(|x|\frac{x}{|x|}, \frac{\langle x,y\rangle}{|x|}\frac{x}{|x|} + y'\right) = (t\zeta_1, u\zeta_1 + v\zeta_2)$$

where

$$t = |x|, \qquad v = |y'|, \qquad u = \frac{\langle x,y\rangle}{|x|}, \qquad \zeta_1 = \frac{x}{|x|}, \qquad \zeta_2 = \frac{y'}{|y'|}, \qquad \langle \zeta_1,\zeta_2\rangle = 0.$$

We can also write

$$(x,y) = (tke_1, uke_1 + vke_2), \qquad k \in \mathrm{SO}(n)/\mathrm{SO}(n-2),\ t,v \in \mathbb{R}^{>0},\ u \in \mathbb{R}.$$

In fact, we can take

$$k = e^{\sum_{j=2}^{n}\theta_j A_{1j}} e^{\sum_{l=3}^{n}\phi_l A_{2l}},$$

where $A_{ij} = E_{ij} - E_{ji}$ for $i < j$ and $E_{ij} = (\delta_{i\alpha}\delta_{j\beta})$. Now we will compute the Jacobian of the map

$$F(t,\theta_2,\ldots,\theta_n,u,v,\phi_3,\ldots,\phi_n) = (tke_1, uke_1 + vke_2).$$

We obtain

$$\begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial \theta_2} & \cdots & \frac{\partial x}{\partial \theta_n} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \phi_3} & \cdots & \frac{\partial x}{\partial \phi_n} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial \theta_2} & \cdots & \frac{\partial y}{\partial \theta_n} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial \phi_3} & \cdots & \frac{\partial y}{\partial \phi_n} \end{pmatrix}$$
$$= \begin{pmatrix} ke_1 & t\frac{\partial ke_1}{\partial \theta_2} & \cdots & t\frac{\partial ke_1}{\partial \theta_n} & 0 & 0 & 0 & \cdots & 0 \\ 0 & * & \cdots & * & ke_1 & ke_2 & v\frac{\partial y}{\partial \phi_3} & \cdots & v\frac{\partial y}{\partial \phi_n} \end{pmatrix}.$$

So the Jacobian is $t^{n-1}v^{n-2}\mu(\theta_2,\ldots,\theta_n,\phi_3,\ldots,\phi_n)$, where

$$\mu(\theta_2,\ldots,\theta_n,\phi_3,\ldots,\phi_n) = \left|ke_1\frac{\partial ke_1}{\partial \theta_2}\cdots\frac{\partial ke_1}{\partial \theta_n}\right| \left|ke_1 ke_2 \frac{\partial y}{\partial \phi_3}\cdots\frac{\partial y}{\partial \phi_n}\right|,$$

and the integral becomes

$$\int f(x,y)\,dx\,dy = \int f\circ F(t,\theta,u,v,\phi)t^{n-1}v^{n-2}\mu(\theta,\phi)\,dt\,d\theta\,du\,dv\,d\phi.$$

By a change of variables, it is easy to see that the measure $\mu(\theta,\phi)\,d\theta\,d\phi$ on $\mathrm{SO}(n-2)\backslash\mathrm{SO}(n)$ is $\mathrm{SO}(n)$-invariant.

We now look for the distribution vector corresponding to the representation induced by the character $\chi_{\xi,\eta}$ with $\xi,\eta \in \mathbb{R}^n \setminus \{0\}$.

We set $M = \mathrm{SO}(n-2)\backslash\mathrm{SO}(n)$ or $M = \mathrm{SO}(n-2)\backslash\mathrm{SO}(n-1)$, according to whether ξ, η are linearly independent or not. For $h \in \mathrm{SO}(n)$, $\overline{h}$ will denote the equivalence class and $d\bar{h}$ the $\mathrm{SO}(n)$-invariant measure of M.

By Lemma 8, the distribution vector fixed by K is given by

$$\langle \phi, F\rangle = \int_{\mathbb{R}_{>0}\times M} F(s, \mathrm{SO}(n-2)h)\, \frac{ds}{s}\, d\bar{h},$$

and, for $f \in \mathcal{C}_c^\infty(K \ltimes \mathbb{R}^{2n})$, the spherical distribution of the pair $(K, \mathbb{R}^{2n})$ is

$$\Psi_{\xi,\eta}(f) = \langle \phi, \rho(f)\phi\rangle.$$

Thus,

$$\langle \rho(f)\phi, F\rangle = \langle \phi, \rho(f)F\rangle = \int_{\mathbb{R}_{>0}\times M} \int_{K\ltimes\mathbb{R}^{2n}} f(g)\rho(g)F(s,\bar{h})\, dg\, \frac{ds}{s}\, d\bar{h}.$$

Taking coordinates $g = (t, h', x', y')$ in $K \ltimes \mathbb{R}^{2n}$ we have

$$\begin{aligned}
&\int_{K\ltimes\mathbb{R}^{2n}} f(g)\rho(g)F(s,\bar{h})\, dg \\
&\qquad = \int_{K\ltimes\mathbb{R}^{2n}} f(t,h',x',y') e^{i\langle (s,\bar{h})\cdot(x',y'),(\xi,\eta)\rangle} F(st, \bar{h}h')\, \frac{dt}{t}\, dh'\, dx'\, dy' \\
&\qquad = \int_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \mathcal{F}f(t,h',(s,\bar{h}^{-1})\cdot(\xi,\eta)) F(st,\bar{h}h')\, \frac{dt}{t}\, dh'
\end{aligned}$$

where we have performed the Fourier transform in the last $2n$ variables (x', y') at the point $((s, h^{-1}) \cdot (\xi, \eta))$. Therefore,

$$\begin{aligned}
&\langle \rho(f)\phi, F\rangle \\
&\quad = \int_{\mathbb{R}_{>0}\times M} \int_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \mathcal{F}f\big(t,h',(s,\bar{h}^{-1})\cdot(\xi,\eta)\big) F(st,\bar{h}h')\, \frac{dt}{t}\, dh'\, \frac{ds}{s}\, d\bar{h} \\
&\quad = \int_{\mathbb{R}_{>0}\times M} F(r',\bar{k}) \int_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \mathcal{F}f\big(t,h',(rt^{-1},h'\bar{k}^{-1})\cdot(\xi,\eta)\big)\, \frac{dt}{t}\, dh'\, \frac{dr}{r}\, d\bar{k};
\end{aligned}$$

in the last equality we made the change of variables $r = st$, $\bar{k} = \bar{h}h'$. So,

$$\rho(f)\phi(r,k) = \int_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \mathcal{F}f\big(t,h',(rt^{-1},(\bar{k}h'^{-1})^{-1})\cdot(\xi,\eta)\big)\, \frac{dt}{t}\, dh'.$$

Finally,

$$
\begin{aligned}
&\langle \phi, \rho(f)\phi\rangle \\
&\quad = \int\limits_{\mathbb{R}_{>0}\times M} \int\limits_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \mathcal{F}f\big(t, h', (rt^{-1}, (\bar{k}h'^{-1})^{-1})\cdot(\xi,\eta)\big)\,\frac{dt}{t}\,dh'\,\frac{dr}{r}\,d\bar{k} \\
&\quad = \int\limits_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} \int\limits_{\mathbb{R}_{>0}\times M} \mathcal{F}f\big(t, h', (r, \bar{k}^{-1})\cdot(\xi,\eta)\big)\,\frac{dt}{t}\,dh'\,\frac{dr}{r}\,d\bar{k} \\
&\quad = \int\limits_{\mathbb{R}_{>0}\times M} \mathcal{F}\bar{f}\big((r, \bar{k}^{-1})\cdot(\xi,\eta)\big)\,\frac{dr}{r}\,d\bar{k}
\end{aligned}
$$

where $\bar{f} = \int_{\mathbb{R}_{>0}\times \mathrm{SO}(n)} f(s,h,x,y)\,\frac{ds}{s}\,dh$.

Further, for $f \in \mathcal{S}(\mathbb{R}^{2n})$, let

$$
(\Psi_{\xi,\eta}, f) = \int\limits_{\mathbb{R}_{>0}\times M} \mathcal{F}\bar{f}((r, \bar{k}^{-1})\cdot(\xi,\eta))\,\frac{dr}{r}\,d\bar{k}. \tag{9}
$$

Thus, the spherical distribution of the pair $(\mathbb{R}^{2n}, K)$ is the integral of the Fourier transform of $\bar{f}$ along the orbit of (ξ,η) under the action of K.

For $f \in \mathcal{S}(\mathbb{H}_n)$, let $f_t(x,y) = f(x,y,t)$, and

$$
(\Phi_{\xi,\eta}, f) = \int\limits_{\mathbb{R}} \langle \Psi_{\xi,\eta}, f_t\rangle\,dt = \langle \Psi_{\xi,\eta}\otimes 1, f\rangle.
$$

It is easy to see that for points (ξ,η) in the same K-orbit the corresponding induced representations are equivalent. Also from

$$
\rho|_{\mathbb{R}^{2n}} = \int\limits_{K/K_{\xi,\eta}} \chi_{k\xi,k\eta}\,d\bar{k},
$$

it follows that points in different K-orbits correspond to non-equivalent representations. Let $\tilde{u} = |\xi|u$ and $\tilde{v} = |\xi|v$. Then we can write, for ξ,η linearly independent,

$$
(\xi,\eta) = (|\xi|, k)\cdot(e_1, \tilde{u}e_1 + \tilde{v}e_2), \qquad k \in \mathrm{SO}(n-2)\backslash\mathrm{SO}(n),\ \tilde{u}\in\mathbb{R},\ \tilde{v}\in\mathbb{R}_{>0}.
$$

Thus, in this case, the orbits under $\mathbb{R}_{>0}\times\mathrm{SO}(n)$ are parametrized by $(u,v) \in \mathbb{R}\times\mathbb{R}_{>0}$.

Similarly, when there is a linear dependence between ξ and η, the orbits are parametrized by $(u,0)\in\mathbb{R}\times\{0\}$. This is the only possible case when $n=2$. Thus we have the following result.

THEOREM 9. *A complete set of spherical distributions for* $((\mathbb{R}_{>0}\times\mathrm{SO}(n)), H_n)$ *is given by* $\Phi_0 = 1$ *corresponding to the trivial representation, and*

$$\Phi_{\lambda,\alpha,k}f = \sum_{i=1}^{\dim Y_k} \langle q^k_{i,\alpha}, \pi_\lambda^{-\infty}(f) q^k_{i,\alpha})\rangle \quad \textit{for } \lambda, \alpha \in \mathbb{R},\ \lambda \neq 0,\ k \in \mathbb{N}_0,$$
$$\Phi_{u,v}(f) = \langle \Psi_{u,v} \otimes 1, f\rangle$$

for $u, v \in \mathbb{R}$, $(u,v) \neq (0,0)$, $v \geq 0$, *and* $\Psi_{u,v}$ *is as in* (9).

3.3. Eigenvalues. For $n = 2$ let $D_1 = \sigma(E)$ and $D_2 = X_1Y_2 - X_2Y_1$. In [LS1] it was shown that

$$D_1\Phi_{\lambda,\alpha,k} = -\lambda\alpha\Phi_{\lambda,\alpha,k} \quad \text{and} \quad D_2\Phi_{\lambda,\alpha,k} = \lambda k\Phi_{\lambda,\alpha,k}.$$

For $n \geq 3$ we consider the symmetrized operators $D_1 = \sigma(E) = (n/2)T + E$ and $D_2 = \sigma(\Delta_X\Delta_Y) = \Delta_X\Delta_Y - 2ET - (3n/2)T^2$.

Since $X_j = \frac{\partial}{\partial x_j} - \frac{x_j}{2}\frac{\partial}{\partial t}$ and $Y_j = \frac{\partial}{dy_j} + \frac{y_j}{2}\frac{\partial}{dt}$, we obtain

$$D_1\Phi_{u,v} = \sum_{j=1}^n \frac{\partial^2}{dy_j\partial x_j}\Phi_{u,v} = -(u\|\zeta_1\|^2 + v\langle\zeta_1,\zeta_2\rangle)\Phi_{u,v} = -u\Phi_{u,v}.$$

Also, $\Delta_X\Delta_Y(\Phi_{u,v}) = (u^2+v^2)\Phi_{u,v}$, and thus using $T\Phi_{u,v} = 0$, by (6),

$$D_2\Phi_{u,v} = (u^2+v^2)\Phi_{u,v}.$$

To compute the eingenvalues of $\Phi_{\lambda,\alpha,k}$, it is enough to know the eigenvalues of $d\pi_\lambda(E)q^k_{i,\alpha}$ and $d\pi_\lambda(\Delta_y\Delta_x)q^k_{i,\alpha}$.

A computation gives $d\pi_\lambda(E)q^k_{i,\alpha} = i\lambda(i\alpha - n/2)q^k_{i,\alpha}$, and by (7),

$$d\pi_\lambda(\Delta_y\Delta_x)q^k_{i,\alpha} = -\lambda^2[(i\alpha - n/2)^2 + (n-2)(i\alpha - n/2) - k(k+n-2)]q^k_{i,\alpha},$$

where we have used $\Delta_\varpi Y_k = -k(k+n-2)Y_k$.

Thus,

$$D_1\Phi_{\lambda,\alpha,k} = -\lambda\alpha\Phi_{\lambda,\alpha,k}, \quad D_2\Phi_{\lambda,\alpha,k} = \lambda^2(\alpha^2 + n^2/4 + k(k+n-2) - 3n/2).$$

3.4. Inversion formulæ. The aim of this section is to obtain an inversion formula for a Schwartz function on $\mathbb{R}^{2n}$ in terms of the $\{\Phi_{u,v}\}$. Let $h \in S(\mathbb{R}^{2n})$. Then

$$\begin{aligned}(h * \Phi_{u,v})(x,y) &= \langle \Phi_{u,v}, L_{(x,y)}\check{h}\rangle \\ &= \int\limits_{\mathbb{R}_{>0}\times M} (L_{(x,y)}\check{h})\hat{}\,((r,\bar{k}^{-1})\cdot(e_1, ue_1+ve_2))\,\frac{dr}{r}\,d\bar{k} \\ &= \int\limits_{\mathbb{R}_{>0}\times M} e^{i\langle (x,y), r\bar{k}^{-1}\cdot(e_1,ue_1+ve_2)\rangle}\hat{h}(-(r,\bar{k}^{-1})\cdot(e_1,ue_1+ve_2))\,\frac{dr}{r}\,d\bar{k}.\end{aligned}$$

Recall that we can take the following coordinates in $\mathbb{R}^{2n}$:

$$(x,y) = (s\zeta_1, s^{-1}\zeta_2(ue_1+ve_2)) = (s\bar{k}^{-1}e_1, s^{-1}\bar{k}^{-1}(ue_1+ve_2))$$

with $s \in \mathbb{R}_{>0}$, $(u,v) \in \mathbb{R}^2$, and $\bar{k} = (\zeta_1,\zeta_2) \in \mathrm{SO}(n-2)k$.

Also, up to a set of measure zero, $\mathrm{SO}(n-2)\backslash\mathrm{SO}(n)$ can be parametrized by global coordinates, with $d\bar{k}$ the $\mathrm{SO}(n)$-invariant measure.

Let $V(s,\bar{k},u,v) := -(s\bar{k}^{-1}e_1, s^{-1}\bar{k}^{-1}(ue_1+ve_2))$. By a similar computation, we find that the Jacobian of V is v^{n-2}/s.

Consider the measure on $\mathbb{R}^2$ given by $d\mu(u,v) = v^{n-2}\,du\,dv$. Then by a change of variables,

$$\begin{aligned} h(x,y) &= \int\limits_{\mathbb{R}^{2n}} \hat{h}(a,b)e^{i\langle (x,y),(a,b)\rangle}\,da\,db \\ &= \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}_{>0}\times M} \hat{h}\circ V(s,\bar{k},u,v)e^{i\langle (x,y),r\bar{k}^{-1}\cdot(e_1,ue_1+ve_2)\rangle}v^{n-2}\,\frac{ds}{s}\,d\bar{k}\,du\,dv \\ &= \int\limits_{\mathbb{R}^2} (h*\Phi_{u,v})(x,y)\,d\mu(u,v). \end{aligned}$$

Now we obtain an inversion formula for a Schwartz function f on the Heisenberg group in terms of the spherical distributions $\{\Phi_{\lambda,\alpha,k}\}$. We recall that $\pi_\lambda(f) = \int_{H_n} f(w)\pi_\lambda(w^{-1})\,dw$, $\pi_\lambda(L_w\check{f}) = \pi_\lambda(f)\pi_\lambda(w^{-1})$ and $\pi_\lambda^{-\infty}(f)q_{i,\alpha}^k \in H_\lambda^\infty$.

Lemma 10. *Let $\{p_i\}_{i=1}^{\dim Y_k}$ be a basis of Y_k, and let $q_{i,\alpha}^k(u) = |u|^{i\alpha-n/2}p_i(u/|u|)$. Then for $\varphi\in S(\mathbb{R}^n)$,*

$$\varphi(u) = \sum_{k,i}\int\limits_{-\infty}^{\infty} \langle\varphi, q_{i,\alpha}^k\rangle q_{i,\alpha}^k(u)\,d\alpha.$$

Proof. Taking polar coordinates, we see that for each ξ in S^{n-1} the function $r\mapsto r^{n/2}\varphi(r\xi)$ is in $L^2\big(\mathbb{R}_{>0},\frac{dr}{r}\big)$.

Thus by the inversion formula

$$\begin{aligned} \varphi(r\xi) &= \int\limits_{-\infty}^{\infty}\Big(\int\limits_0^{\infty} s^{n/2}\varphi(s\xi)s^{i\alpha}\,\frac{ds}{s}\Big)r^{-i\alpha-n/2}\,d\alpha \\ &= \int\limits_{-\infty}^{\infty}\Big(\int\limits_0^{\infty}\varphi(s\xi)s^{i\alpha-n/2}s^{n-1}\,ds\Big)r^{-i\alpha-n/2}\,d\alpha \\ &= \int\limits_{-\infty}^{\infty}\int\limits_0^{\infty}\Big[\sum_{k,i}\langle\varphi(s\xi),p_i\rangle_{L^2(S^{n-1})}p_i(\xi)\Big]s^{i\alpha}s^{n/2-1}\,ds\,r^{-i\alpha-n/2}\,d\alpha \\ &= \sum_{k,i}\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n}\varphi(x)\bar{q}_{i,-\alpha}^k(x)\,dx\Big)r^{-i\alpha-n/2}p_i(\xi)\,d\alpha \\ &= \sum_{k,i}\int\limits_{-\infty}^{\infty}\langle\varphi,q_{i,-\alpha}^k\rangle q_{i,-\alpha}^k(u)\,d\alpha. \quad\blacksquare \end{aligned}$$

Let $\{\varphi_m\}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$ and $f \in \mathcal{S}(H_n)$. Then

$$\begin{aligned}
\operatorname{tr}(\pi_\lambda(f)) &= \sum_m \langle \pi_\lambda(f)\varphi_m, \varphi_m\rangle_{L^2(\mathbb{R}^n)} = \sum_m \int_{\mathbb{R}^n} \pi_\lambda(f)\varphi_m(u)\overline{\varphi_m(u)}\,du \\
&= \sum_m \int_{\mathbb{R}^n} \sum_{k,i} \int_{-\infty}^{\infty} \langle \pi_\lambda(f)\varphi_m, q^k_{i,\alpha}\rangle q^k_{i,\alpha}(u)\overline{\varphi_m(u)}\,du\,d\alpha \\
&= \sum_m \int_{\mathbb{R}^n} \sum_{k,i} \int_{-\infty}^{\infty} \langle \varphi_m, \pi_\lambda^{-\infty}(f) q^k_{i,\alpha}\rangle q^k_{i,\alpha}(u)\overline{\varphi_m(u)}\,du\,d\alpha \\
&= \int_{\mathbb{R}^n} \sum_{k,i} \int_{-\infty}^{\infty} \Big[\sum_m \overline{\langle \pi_\lambda^{-\infty}(f)q^k_{i,\alpha}, \varphi_m\rangle}\,\overline{\varphi_m(u)}\Big] q^k_{i,\alpha}(u)\,du\,d\alpha \\
&= \int_{\mathbb{R}^n} \sum_{k,i} \int_{-\infty}^{\infty} \overline{\pi_\lambda^{-\infty}(f)q^k_{i,\alpha}(u)}\, q^k_{i,\alpha}(u)\,du\,d\alpha \\
&= \sum_{k,i} \int_{-\infty}^{\infty} \langle q^k_{i,\alpha}, \pi_\lambda^{-\infty}(f)q^k_{i,\alpha}\rangle\,d\alpha = \sum_{k,i} \int_{-\infty}^{\infty} \langle \pi_\lambda(f)q^k_{i,\alpha}, q^k_{i,\alpha}\rangle\,d\alpha,
\end{aligned}$$

that is,

$$\operatorname{tr}(\pi_\lambda(f)) = \sum_{k,i} \int_{-\infty}^{\infty} \langle \Phi_{\lambda,\alpha,k}, f\rangle\,d\alpha.$$

Since the space of $\mathcal{C}^\infty$-vectors is invariant under π_λ, a computation analogous to the above shows that for $w \in H_n$,

$$\begin{aligned}
\operatorname{tr}(\pi_\lambda(w)\pi_\lambda(f)) &= \sum_{k,i} \int_{-\infty}^{\infty} \langle \pi_\lambda(w)\pi_\lambda(f)q^k_{i,\alpha}, q^k_{i,\alpha}\rangle\,d\alpha \\
&= \sum_{k,i} \int_{-\infty}^{\infty} \langle \pi_\lambda(L_w f)q^k_{i,\alpha}, q^k_{i,\alpha}\rangle\,d\alpha \\
&= \sum_{k,i} \int_{-\infty}^{\infty} \langle \Phi_{\lambda,\alpha,k}, L_w f\rangle\,d\alpha = \sum_{k,i} \int_{-\infty}^{\infty} (\check{f} * \Phi_{\lambda,\alpha,k})(w)\,d\alpha,
\end{aligned}$$

where we have used the fact that $\pi_\lambda(L_w f) = \pi_\lambda(w)\pi_\lambda(f)$. Thus, by the inversion formula for Schwartz functions on H_n,

$$\begin{aligned}
f(w^{-1}) &= \int_{-\infty}^{\infty} \operatorname{tr}(\pi_\lambda(f)\pi_\lambda(w))|\lambda|^n\,d\lambda \\
&= \sum_k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\check{f} * \Phi_{\lambda,\alpha,k})(w)|\lambda|^n\,d\lambda\,d\alpha \qquad \forall f \in \mathcal{S}(H_n).
\end{aligned}$$

4. Spherical analysis for $(\mathbb{R}^*, H_1)$**.** We begin by describing the spherical distributions associated to the characters $\chi_{\xi,\eta}$ of $\mathbb{R}^2$. Since points (ξ,η) in the same K-orbit correspond to the same spherical distribution, they are parametrized by the orbits $\mathcal{O}_\beta = \{(\xi,\eta) : \xi\eta = \beta\}$ for $\beta \in \mathbb{R}^*$ and $\{(0,0)\}$.

Let $X = \frac{\partial}{\partial x} - x\frac{\partial}{\partial t}$, $Y = \frac{\partial}{dy} + y\frac{\partial}{dt}$, and $T = \frac{\partial}{dt}$ be the standard basis of the Lie algebra of H_1. The algebra of left K-invariant differential operators on H_1 is generated by T and $L = XY + YX$, which is the symmetrization of the K-invariant polynomial $p(x,y) = 2xy$.

A similar computation to one in Section 3.2 shows that for $\beta = 0$, $\Phi_0 \equiv 1$. Also, for $\beta \neq 0$ and $f \in \mathcal{S}(H_1)$,

$$(\Phi_\beta, f) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}^*} \mathcal{F} f_t \left(r, \frac{\beta}{r} \right) \frac{dr}{|r|} \right) dt,$$

where $f_t(x,y) = f(x,y,t)$.

A spherical distribution for $(\mathbb{R}^*, H_1)$ corresponding to (ξ,η) is an $\mathbb{R}^*$-invariant eigendistribution Φ of L such that $T\Phi = 0$. Thus we look for $\mathbb{R}^*$-invariant eigendistributions of $\frac{\partial^2}{\partial x \partial y}$.

We will obtain an explicit expression for Φ_β by using the Tengstrand transform.

4.1. Tengstrand transform for $\mathbb{R}^*$**.** In the 50's, Methée and de Rham characterized the distributions on $\mathbb{R}^m$ invariant under transformations of $\mathrm{SO}(p,q)$, $p+q=m$. Their description was improved by Gårding–Ross for the Lorentz group, and in general by Tengstrand in 1960. For $m=2$, the action of $\mathbb{R}^*$ on $\mathbb{R}^2$ is equivalent to the action of $\mathrm{SO}(1,1)$. We will adapt the notation accordingly.

We know that if

$$\mathcal{N} f(\tau) = \int_{\mathcal{O}_\tau} f(u)\, d\sigma(u)$$

with $d\sigma(u)$ the $\mathbb{R}^*$-invariant measure on the orbit, then the image under $\mathcal{N}$ of the Schwartz space $S(\mathbb{R}^2)$ is given by

$$\mathcal{T} = \{\phi(\tau) : \forall k \in \mathbb{N},\ \exists \text{ a polynomial } p_k(\tau) \text{ such that } \phi(\tau) - p_k(\tau)\log(|\tau|) \in \mathcal{C}^k(\mathbb{R})\},$$

where $p_k(\tau)$ denotes a polynomial of degree $\le k$.

Moreover by using a well known Borel lemma, it follows that

$$\mathcal{T} = \{\phi(\tau) = \phi_1(\tau) + \phi_2(\tau)\log|\tau|) : \phi_1, \phi_2 \in S(\mathbb{R})\}.$$

Thus any function $\phi \in \mathcal{T}$ has a unique expansion of the form

$$\phi(\tau) = \sum_{j\ge 0}^{n} B_j(\phi)\tau^j + \log(|\tau|) \sum_{j\ge 0}^{n} A_j(\phi)\tau^j + o(\tau^n)$$

where A_j and B_j come from the Taylor expansions of ϕ_1 and ϕ_2. Moreover, the maps $\phi \mapsto A_j(\phi)$ and $\phi \mapsto B_j(\phi)$ are in $\mathcal{T}'$, and in [Te, p. 208] the following result is proved.

Corollary 11. *Any $\theta \in \mathcal{T}'$ with support at $\tau = 0$ has the form $\sum(\alpha_j A_j + \beta_j B_j)$ where the sum is finite.*

We give a sketch of the proof of the fact that the image of $\mathcal{N}$ is $\mathcal{T}$:

For $f \in S(\mathbb{R}^m)$ we write $f = g + h$ with g of compact support and $h \equiv 0$ on a neighborhood of 0. Since $\mathcal{N}h$ is clearly in $S(\mathbb{R})$, it is enough to prove the result only for f of compact support.

Assume that $\operatorname{supp}(f) \subset \{(x, y) : x^2 + y^2 \le R^2\} = B_R(0)$. Then $B_R(0) \cap \mathcal{O}_\tau$ equals $\{(x, \tau/x) : x^2 + \tau^2/x^2 \le R^2\}$, or equivalently it is given by $x^4 - x^2R^2 + \tau^2 \le 0$. This forces $\sqrt{y_-} \le |x| \le \sqrt{y_+}$, where $\sqrt{y_\pm}$ are the roots of the equation $z^2 - zR^2 + \tau^2 = 0$. We also have $y_- \ge \tau^2/R^2$ and $y_+ \le R^2$. Thus,

$$\mathcal{N}f(\tau) = \int_{|\tau|/R}^{R} f\left(x, \frac{\tau}{x}\right) \frac{dx}{|x|} + \int_{-R}^{-|\tau|/R} f\left(x, \frac{\tau}{x}\right) \frac{dx}{|x|}.$$

Letting $\sigma = Rx$ we have

$$\mathcal{N}f(\tau) = \int_{|\tau|}^{R^2} f\left(\frac{\sigma}{R}, \frac{R\tau}{\sigma}\right) \frac{d\sigma}{|\sigma|} + \int_{-R^2}^{-|\tau|} f\left(\frac{\sigma}{R}, \frac{R\tau}{\sigma}\right) \frac{d\sigma}{|\sigma|}.$$

The key point to see that $\mathcal{N}f \in \mathcal{T}$ is to consider the Taylor series development of order $2n$,

$$f(x, y) = \sum_{\alpha+\beta\le 2n} a_{\alpha,\beta} x^\alpha y^\beta + R_{2n}(x, y),$$

where

$$R_{2n}(x, y) = \sum_{j=0}^{2n+2} \phi_j(\theta x, \theta y) x^j y^{2n+2-j}, \quad 0 \le \theta \le 1. \tag{10}$$

Integrating we have

$$\mathcal{N}f(\tau) = \psi_n(\tau) + \sum_{\alpha=0}^{n} a_{\alpha,\alpha} \tau^\alpha \log(|\tau|) + \mathcal{N}R_{2n}(\tau)$$

where $\psi_n \in \mathcal{C}^\infty(\mathbb{R})$.

To see that $\mathcal{N}R_{2n}(\tau) \in \mathcal{C}^n(\mathbb{R})$, we will study $\int_\tau^1 R_{2n}\left(x, \frac{\tau}{x}\right) \frac{dx}{x}$. For $j \ne 2n+2-j$ the integral of R_{2n} gives a $\mathcal{C}^\infty$ function in τ, therefore it is enough to consider $j = n + 1$ in (10). We will show that if $\phi \in \mathcal{C}^\infty(\mathbb{R}^2)$, then

$$h(\tau) = \tau^{n+1} \int_\tau^1 \phi\left(x, \frac{\tau}{x}\right) \frac{dx}{x}$$

belongs to $\mathcal{C}^n(\mathbb{R})$, defining $h(0) = 0$.

LEMMA 12.

$$F(\tau) = \tau^{n+k} \int_{\tau}^{1} \phi\left(x, \frac{\tau}{x}\right) \frac{dx}{x^k}$$

is in $\mathcal{C}^n(\mathbb{R})$ *for any* $\phi \in \mathcal{C}^\infty(\mathbb{R}^2)$.

Proof. The expression is clearly infinitely differentiable at every point except perhaps 0. When $n = 0$, the integrand $\phi(x, \frac{\tau}{x})\frac{\tau^k}{x^k}$ is dominated by $\|\phi\|_\infty$ and tends to 0 pointwise when $\tau \to 0$. By the dominated convergence theorem the integral tends to 0, and the expression belongs to $\mathcal{C}^0(\mathbb{R})$.

Taking derivative with respect to τ we obtain

$$\frac{dF(\tau)}{d\tau} = (n+k)\tau^{n+k-1} \int_{\tau}^{1} \phi\left(x, \frac{\tau}{x}\right) \frac{dx}{x^k} + \tau^{n+k} \int_{\tau}^{1} d\phi\left(x, \frac{\tau}{x}\right) \frac{dx}{x^{k+1}} - \tau^n \phi(\tau, 1).$$

By induction, the first two terms are in $\mathcal{C}^{n-1}(\mathbb{R})$ and the last one is in $\mathcal{C}^\infty(\mathbb{R})$. ∎

In [Te, Section 3] a topology is described that makes $\mathcal{T}$ a Fréchet space. In [Te, Th. 5.1] it is proved that the dual map of $\mathcal{N}$ is a linear homeomorphism from $\mathcal{T}'$ onto the space of $\mathbb{R}^*$-invariant tempered distributions on $\mathbb{R}^2$.

Let $L_0 = \frac{1}{2} \frac{d^2}{dx\, dy}$, $D = \tau \frac{d^2}{d\tau^2} + \frac{d}{d\tau}$ and $\Phi \in \mathcal{T}'$. Then

$$L_0 \mathcal{N}' \Phi = \mathcal{N}' D \Phi.$$

Notice that D is a symmetric operator:

$$\begin{aligned} \langle D\psi, \phi \rangle &= \left\langle \left(\tau \frac{d^2}{d\tau^2} + \frac{d}{d\tau}\right)\psi, \phi \right\rangle = \left\langle \frac{d^2}{d\tau^2}\psi, \tau\phi \right\rangle + (-1)\left\langle \psi, \frac{d}{d\tau}\phi \right\rangle \\ &= \left\langle \psi, \frac{d^2}{d\tau^2}(\tau\phi) \right\rangle - \left\langle \psi, \frac{d}{d\tau}\phi \right\rangle = \left\langle \psi, \tau \frac{d^2}{d\tau^2}\phi + \frac{d}{d\tau}\phi \right\rangle \\ &= \langle \psi, D\phi \rangle. \end{aligned}$$

4.2. Spherical distributions. We now look for the eigendistributions of D in $\mathcal{T}$. Since D is symmetric, the corresponding eigenvalues are real. For $\Phi \in \mathcal{T}'$, let $\langle \check{\Phi}, \psi \rangle = \langle \Phi, \check{\psi} \rangle$ where $\check{\psi}(\tau) = \psi(-\tau)$. Since $D\check{\Phi} = -(D\Phi)^\vee$, if $D\Phi = -\beta\Phi$ then $D\check{\Phi} = \beta\check{\Phi}$. Thus, it is enough to consider $\beta \geq 0$.

Case $\beta > 0$. Let J_0 and Y_0 be the solutions of Bessel's equation $\tau u'' + u' + \tau u = 0$ for $\tau > 0$, respectively known as Bessel functions of the first and second kind of order zero. Let $D_\beta \phi = D\phi + \beta\phi$, $U_\beta(\tau) = J_0((\beta\tau)^{1/2})$, and $V_\beta(\tau) = Y_0((\beta\tau)^{1/2})$. Thus, U_β and V_β are two linearly independent solutions for $\tau > 0$ of $D_\beta\phi = 0$.

We list the following facts [L, pp. 100, 101, 107, 134, 135]:

- $J_0'(\tau) = -J_1(\tau)$, so $U_\beta'(\tau) = -\frac{1}{2}\beta^{1/2} J_1((\beta\tau)^{1/2})\tau^{-1/2}$.
- $Y_0'(\tau) = -Y_1(\tau)$ and $Y_1(\tau) \sim -\frac{2}{\pi}\frac{1}{\tau}$ for $\tau \sim 0$.

- $Y_0(\tau) \sim \frac{2}{\pi}\log\left(\frac{\tau}{2}\right)$ for $\tau \sim 0$.
- $Y_0(x) \to 0$ when $x \to \infty$.

Let H be the Heaviside function and $\phi(\tau) = \phi_1(\tau) + \phi_2(\tau)\ln|\tau|$. Then

$$\begin{aligned}\langle D_\beta U_\beta H, \phi_1\rangle &= \int_0^\infty U_\beta(\tau) D_\beta \phi_1(\tau)\,d\tau \\ &= -(\tau U_\beta')\phi_1(\tau)|_0^\infty + \int_0^\infty D_\beta(U_\beta)(\tau)\phi_1(\tau)\,d\tau = 0.\end{aligned}$$

Now $D_\beta(\log(\tau)\phi_2(\tau)) = \log(\tau)D_\beta\phi_2(\tau) + 2\phi_2'$ is in $\mathcal{T}$ and

$$\begin{aligned}\langle D_\beta U_\beta H, \log(\tau)\phi_2\rangle &= \int_0^\infty U_\beta(\tau)(2\phi_2' + \log(\tau)D_\beta\phi_2(\tau)\,d\tau \\ &= (U_\beta\phi_2(\tau) - U_\beta'(\tau)\tau\log(\tau)\phi_2)|_0^\infty + \int_0^\infty \log(\tau)D_\beta(U_\beta)(\tau)\phi_2(\tau)\,d\tau \\ &= -\phi_2(0).\end{aligned}$$

Moreover

$$\begin{aligned}\langle D_\beta V_\beta H, \phi_1\rangle &= \int_0^\infty V_\beta(\tau) D_\beta \phi_1(\tau)\,d\tau \\ &= -(\tau V_\beta')\phi_1(\tau)|_0^\infty + \int_0^\infty D_\beta(V_\beta)(\tau)\phi_1(\tau)\,d\tau \\ &= \lim_{\epsilon\to 0} \epsilon V_\beta'(\epsilon)\phi_1(\epsilon) = \lim_{\epsilon\to 0}\epsilon\frac{1}{\pi\epsilon}\phi_1(\epsilon) = \frac{1}{\pi}\phi_1(0)\end{aligned}$$

and

$$\begin{aligned}\langle D_\beta V_\beta H, \log(\tau)\phi_2\rangle &= \int_0^\infty V_\beta(\tau)(2\phi_2' + \log(\tau)D_\beta\phi_2(\tau)\,d\tau \\ &= (V_\beta\phi_2(\tau) - V_\beta'(\tau)\tau\log(\tau)\phi_2)|_0^\infty + \int_0^\infty \log(\tau)D_\beta(V_\beta)(\tau)\phi_2(\tau)\,d\tau \\ &= \lim_{\epsilon\to 0}\big(V_\beta'(\epsilon)\epsilon\log(\epsilon)\phi_2(\epsilon) - V_\beta(\epsilon)\phi_2(\epsilon)\big) \\ &= \lim_{\epsilon\to 0}\frac{1}{\pi}\left(\log(\epsilon)\phi_2(\epsilon) - \frac{2}{\pi}\log\left(\frac{(\beta\epsilon)^{1/2}}{2}\right)\phi_2(\epsilon)\right) \\ &= -\frac{1}{\pi}\log\left(\frac{\beta}{4}\right)\phi_2(0).\end{aligned}$$

A solution of $D_\beta\phi = 0$ for $\tau < 0$ is the function $\tau \mapsto U_\beta(-\tau)$. We look now for a solution linearly independent from it on $\mathbb{R}_{<0}$.

Lemma 13. *The function*

$$Z_\beta(\tau) = U_\beta(\tau) \int_{-\infty}^{\tau} \frac{ds}{(U_\beta(s))^2 s}$$

is a solution for $\tau < 0$ *of*

$$\tau\phi'' + \phi' + \beta\phi = 0,$$

satisfying the following asymptotics:

$$\lim_{\tau\to 0} \frac{Z_\beta(\tau)}{\log(|\tau|)} = 1, \quad \lim_{\tau\to-\infty} Z_\beta(\tau) = 0, \quad \lim_{\tau\to 0} \tau Z'_\beta(\tau) = 1, \quad \lim_{\tau\to-\infty} Z'_\beta(\tau) = 0.$$

Proof. The integral is well defined since $U_\beta(\tau) \geq Ae^{B|\tau|^{1/2}}$ for $\tau < 0$. By L'Hôpital's rule we have

$$\begin{aligned}\lim_{\tau\to 0} \frac{Z_\beta(\tau)}{\log(|\tau|)} &= \lim_{\tau\to 0} \tau Z'_\beta(\tau) = \lim_{\tau\to 0} \left(\tau U'_\beta(\tau) \int_{-\infty}^{\tau} \frac{ds}{(U_\beta(s))^2 s} + \frac{1}{U_\beta(\tau)}\right)\\ &= 1 \quad \text{since } U_\beta(0) = 1.\end{aligned}$$

From this computation it also follows that $\lim_{\tau\to 0} \tau Z'_\beta(\tau) = 1$.

Since U_β is a decreasing function and $\frac{1}{sU_\beta(s)}$ is integrable on $(-\infty, 0)$,

$$|Z_\beta(\tau)| = \left| U_\beta(\tau) \int_{-\infty}^{\tau} \frac{ds}{(U_\beta(s))^2 s} \right| \leq \left| \frac{U_\beta(\tau)}{U_\beta(\tau)} \int_{-\infty}^{\tau} \frac{ds}{U_\beta(s) s} \right|.$$

Therefore, $\lim_{\tau\to-\infty} Z_\beta(\tau) = 0$.

Finally, for $\tau \to -\infty$, $U'_\beta(\tau)/U_\beta(\tau) \to 0$, so $Z'_\beta(\tau) \to 0$. ■

Let us compute

$$\langle D_\beta Z_\beta(1 - H), \phi_1\rangle = -\tau Z'_\beta(\tau)\phi_1(\tau)|_{\tau=-\infty}^{\tau=0} = -\phi_1(0)$$

and

$$\begin{aligned}&\langle D_\beta Z_\beta(1 - H), \log(|\tau|)\phi_2\rangle\\ &\qquad = -Z'_\beta(\tau)\tau \log(|\tau|)\phi_2(\tau)|_{\tau=-\infty}^{\tau=0} + Z_\beta(\tau)\phi_2(\tau)|_{\tau=-\infty}^{\tau=0}\\ &\qquad = \lim_{\epsilon\to 0} \big(-Z'_\beta(\epsilon)\epsilon \log(|\epsilon|)\phi_2(\epsilon) + Z_\beta(\epsilon)\phi_2(\epsilon)\big)\\ &\qquad = \lim_{\epsilon\to 0} \phi_2(\epsilon) \log(|\epsilon|)\left(-Z'_\beta(\epsilon)\epsilon + \frac{Z_\beta}{\log(|\epsilon|)}\right) = 0,\end{aligned}$$

since $Z'_\beta(\epsilon)\epsilon$ and $\frac{Z_\beta}{\log(|\epsilon|)}$ are differentiable, and take the value 1 at $\epsilon = 0$.

Proposition 14. *Since*

$$\begin{aligned}\langle D_\beta Z_\beta(1-H),\phi\rangle &= -\phi_1(0),\\ \langle D_\beta U_\beta H,\phi\rangle &= \phi_2(0),\\ \langle D_\beta V_\beta H,\phi\rangle &= \frac{1}{\pi}(\phi_1(0)-\log(\beta/4)\phi_2(0)),\end{aligned}$$

the function

$$\Phi = -\frac{1}{\pi}Z_\beta(1-H)+\frac{1}{\pi}\log(\beta/4)U_\beta H - V_\beta H$$

satisfies $D_\beta\Phi = 0$.

Case $\beta = 0$

Lemma 15. *We have*

$$\langle DH,\phi\rangle = -\phi_2(0),\qquad \langle D\log(|\tau|)H,\phi\rangle = \phi_1(0).$$

Therefore,

$$D(1)=0,\qquad D\big(H\log(|\tau|)+(1-H)\log(|\tau|)\big)=0.$$

Proof. Let us compute

$$\langle DH,\phi_1\rangle = -\tau\phi_1'(\tau)|_{\tau=0}^{\tau=-\infty} = 0,\qquad \langle DH,\log(|\tau|)\phi_2\rangle = -\phi_2(0).$$

Therefore $\langle DH,\phi\rangle = -\phi_2(0)$. We also have

$$\begin{aligned}\langle D\log(|\tau|)H,\phi_1\rangle &= \langle\log(|\tau|)H,(\tau\phi_1''+\phi_1'\rangle = -\int_0^\infty \ln(|\tau|)(\tau\phi_1''+\phi_1')\,d\tau\\ &= -\int_0^\infty(\ln(|\tau|)+1)\phi_1'\,d\tau + \int_0^\infty \ln(|\tau|)\phi_1'\,d\tau\\ &= -\int_0^\infty \phi_1'\,d\tau = \phi_1(0).\end{aligned}$$

Similarly,

$$\begin{aligned}\langle D\log(|\tau|)(1-H),\phi_1\rangle &= -\int_{-\infty}^0 \phi_1'\,d\tau = -\phi_1(0),\\ \langle D(\log(|\tau|)H),\log(|\tau|)\phi_2\rangle &= \tau(\log\tau)^2\phi_2'|_{\tau=0}^{\tau=\infty} = 0.\end{aligned}$$

Therefore, $D(1)=0$ and $D(H\log(|\tau|)+(1-H)\log(|\tau|))=0$. ∎

Lemma 16. *Let* Φ *be a distribution supported on* $\{0\}$. *Then*

(1) $D\Phi = aA_0 + bB_0$ *implies* $a=b=0$.
(2) $D\Phi = 0$ *implies* $\Phi = 0$.

Proof. We know from Corollary 11 that Φ is a finite sum of the form $\sum(\alpha_j A_j + \beta_j B_j)$, that is,

$$\langle \Phi, \phi \rangle = \sum(\alpha_j A_j + \beta_j B_j),$$

where $\phi(\tau) = \sum_{j\geq 0} B_j \tau^j + \log(|\tau|)\sum_{j\geq 0} A_j \tau^j$. Since

$$\tau\phi'' + \phi' = \sum_{j\geq 0}(j^2 B_j + 2jA_j)\tau^{j-1} + \log(|\tau|)\sum_{j\geq 0}(j^2 A_j)\tau^{j-1},$$

the terms with A_0 and B_0 disappear and therefore $D\Phi$ cannot include them. This proves (1).

Now, assuming that $D\Phi = 0$, we have

$$\begin{aligned}&\langle \Phi, \tau\phi'' + \phi' \rangle \\ &\quad = \Big\langle \sum(\alpha_k A_k + \beta_k B_k), \sum_{j\geq 0}(j^2 B_j + 2jA_j)\tau^{j-1} + \log(|\tau|)\sum_{j\geq 0} j^2 A_j \tau^{j-1}\Big\rangle \\ &\quad = \sum\big(\beta_k((k+1)^2 B_{k+1} - 2(k+1)A_{k+1}) - \alpha_k(k+1)^2 A_{k+1}\big) = 0.\end{aligned}$$

This implies

$$\beta_k(k+1)^2 = 0\ \forall k \geq 0 \quad \text{so} \quad \beta_k = 0\ \forall k \geq 0,$$

and hence in turn

$$\alpha_k(k+1)^2 = 0\ \forall k \geq 0 \quad \text{so} \quad \alpha_k = 0\ \forall k \geq 0.$$

Therefore the only solution is the trivial one. ∎

Proposition 17. *Every solution of $D\Phi = 0$ is of the form $\Phi = a + b\log|\tau|$.*

Proof. We have $\Phi|_{(0,\infty)} = a + b\log|\tau|$ and $\Phi|_{(-\infty,0)} = c + d\log|\tau|$. Then

$$S = \Phi - aH - bH\log|\tau| - c(1-H) - d(1-H)\log|\tau|$$

is supported in $\{0\}$ and satisfies

$$DS = (a-c)A_0 + (b-d)B_0.$$

According to Lemma 16, $a = c$ and $d = b$, therefore $S = 0$. ∎

Remark 18. Analogously, replacing D by D_β in the above arguments, one can show that, for $\beta \neq 0$, the unique solution of $D_\beta\Phi = 0$ is, up to a constant, the one found in Proposition 14.

We can now summarize the results of this section.

Theorem 19. *Up to a real constant multiple, the solutions in $\mathcal{T}'$ of $D\Psi = -\beta\Psi$ are*

- *for $\beta > 0$, $\Psi_\beta = -\frac{1}{\pi}Z_\beta(1-H) + \frac{1}{\pi}\log(\beta/4)U_\beta H - V_\beta H$ (see Prop. 14);*
- *for $\beta = 0$, $\Psi_0(\tau) = a + b\log|\tau|$;*
- *for $\beta < 0$, $\Psi_\beta = \check{\Psi}_{-\beta}$.*

Recall that to each $\beta \in \mathbb{R}$ there corresponds a unique, up to a positive constant, spherical distribution of the pair $(\mathbb{R}^*, \mathbb{R}^2)$. Thus for $\beta \neq 0$, $\mathcal{N}'\Psi_\beta$ or $-\mathcal{N}'\Psi_\beta$ is of positive type, and $\mathcal{N}' \log|\tau|$ is not of positive type. By abuse of terminology we call $\mathcal{N}'\Psi_\beta$ a distribution of positive type.

Remark 20 (Inversion formula). For $f \in S(\mathbb{R}^2)$,

$$(f * \mathcal{N}'\Psi_\tau)(x,y) = \int_{-\infty}^{\infty} \hat{f}(-s, -\tau/s) e^{-i(xs+y\tau/s)} \frac{ds}{|s|}.$$

So

$$\int_{-\infty}^{\infty} (f * \mathcal{N}'\Psi_\tau)(x,y)\, d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(s,t) e^{i(xs+yt)}\, ds\, dt = f(x,y).$$

Here we have first made the change of variable $t = \tau/s$ and then applied the Fourier inversion formula.

Taking into account that the spherical distribution corresponding to $(\pi_\lambda, \mathcal{H}_\lambda)$ was computed in [LS1, Theorem 4.2], we have

Theorem 21. *A complete set of spherical distributions attached to the pair* $(\mathbb{R}^*, H_1)$ *is given by:*

(1) *for* $\lambda = 0$,

- (i) $\Phi_\beta = \mathcal{N}'\Psi_\beta \otimes 1,\ \beta > 0,$
- (ii) $\Phi_0 = 1,\ \beta = 0,$
- (iii) $\Phi_\beta = \mathcal{N}'\check{\Psi}_\beta \otimes 1,\ \beta < 0;$

(2) *for* $\lambda \neq 0$ *and* $\alpha \in \mathbb{R}$,

$$\Phi_{\lambda,\alpha} = e^{i\lambda + n/2} \Gamma(1-\mu)\Gamma(\mu)\, {}_1F_1(\mu; 1, -i\lambda s) + 2e^{i\lambda t} \mathfrak{Re}\big(e^{i\lambda s/2} \Gamma(\mu) G(\mu; 1, -i\lambda s)\big)$$

where $\mu = 1/2 - i\alpha$, $s = xy$ *and* ${}_1F_1, G$ *correspond to the classical independent solutions of the confluent hypergeometric equation.*

Acknowledgements. Most of this work was done along a series of reciprocal visits between Prof. Fulvio Ricci of SNS Pisa and the authors. We are indebted to him for the generous contribution of ideas. We would also like to express our thanks to the referee whose observations helped improve our work significantly.

This research was partly supported by CONICET and SecytUNC.

References

[vD] G. van Dijk, *Group representations on spaces of distributions*, Russian J. Math. Phys. 2 (1994), 57–68.

[F] J. Faraut, *Distributions sphériques sur les spaces hyperboliques*, J. Math. Pures Appl. 58 (1979), 369–444.
[K] T. Kobayashi, *Multiplicity free representations and visible actions on complex manifolds*, Publ. RIMS Kyoto Univ. 41 (2005), 497–549.
[L] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publ., 1972.
[LS1] F. Levstein and L. Saal, *Generalized Gelfand pairs associated to Heisenberg type groups*, J. Lie Theory 18 (2008), 503–515.
[LS2] F. Levstein and L. Saal, *Spherical distributons of some generalized Gelfand pairs attached to the Heisenberg group*, in: Contemp. Math. 537, Amer. Math. Soc., 2011, 241–253.
[Ma] G. Mackey, *Unitary Group Representation in Physics, Probability, and Number Theory*, Benjamin/Cummings, 1978.
[MT] K. Mokni et E. G. F. Thomas, *Paires de Gelfand généralisées associées au groupe d'Heisenberg*, J. Lie Theory 8 (1998), 325–334.
[Mo] V. F. Molchanov, *Spherical functions on hyperboloids*, Mat. Sb. 99 (1976), 139–161 (in Russian); English transl.: Math. USSR-Sb. 28 (1976), 119–139.
[Ta] M. Taylor, *Noncommutative Harmonic Analysis*, Math. Surveys Monogr. 22, Amer. Math. Soc., 1986.
[Te] A. Tengstrand, *Distributions invariant under an orthogonal group of arbitrary signature*, Math. Scand. 8 (1960), 201–218.
[V] V. S. Varadarajan, *Lie Groups, Lie Algebras and Their Representations*, Grad. Texts in Math. 102, Springer, 1984.

Fernando Levstein, Linda Saal
Facultad de Matemática Astronomía y Física
Universidad Nacional de Córdoba
Córdoba, Argentina
E-mail: levstein@gmail.com