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Conformal Killing–Yano 2-forms

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Riemannian manifolds carrying 2-forms satisfying the Killing–Yano equation and the conformal Killing–Yano equation are natural generalizations of nearly Kähler and Sasakian manifolds. In this article we exhibit new solutions of these equations. We also provide obstructions for their existence on Lie groups, and reduce the study of conformal Killing–Yano 2-forms to a particular class of non degenerate Killing–Yano 2-forms.

1. Introduction

A p-form ω on a Riemannian manifold (M, g) is called *Killing–Yano* if it satisfies the equation

$$(\nabla_X \omega)(Y, X_1, \dots, X_{p-1}) = -(\nabla_Y \omega)(X, X_1, \dots, X_{p-1}) \tag{1}$$

for any vector fields $X, Y, X_1, \dots, X_{p-1}$ on M. In the case of $p = 1$, a Killing–Yano 1-form is dual to a Killing vector field. In this sense, Killing–Yano forms are natural generalizations of Killing vector fields. They were first introduced by K. Yano ([15]), who showed that Killing–Yano forms give rise to quadratic first integrals of the geodesic equation. This was first used by R. Penrose and M. Walker ([13]) to integrate the equation of motion.

In [12] the Killing–Yano equation is studied for the fundamental forms defining a G-structure, for $G = SO(n)$, $SU(n)$, $U(n)$, $Sp(n) \times Sp(1), Sp(n), G_2$ or $Spin(7)$. He proves that if the fundamental form satisfies (1) then, in most cases, it is parallel with respect to the Levi-Civita connection. The case of a compact simply connected symmetric space M has been considered in [5] where it is shown that M carries a non-parallel Killing–Yano p-form, $p \geq 2$, if and only if it is isometric to a Riemannian product $S^k \times N$, where S^k is a round sphere and $k > p$.

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As a further generalization, a p-form ω on an n-dimensional Riemannian manifold (M, g) is called *conformal Killing–Yano* (CKY for short) if it satisfies the following equation:

$$\nabla_X \omega = \frac{1}{p+1}\, \iota_X d\omega - \frac{1}{n-p+1}\, X^* \wedge d^*\omega, \tag{2}$$

for any vector field X on M, where ∇ is the Levi-Civita connection, X^* is the 1-form dual to X and $d^* = (-1)^{n(p+1)+1} * d *$ is the co-differential. Note that a CKY form ω is co-closed, that is $d^*\omega = 0$, if and only if it is Killing–Yano.

We remark that the space of CKY forms is conformally invariant. Indeed, if ω is a CKY p-form on (M, g) and $\tilde{g} := e^{2f} g$ is a conformally equivalent metric, then the form $\tilde{\omega} := e^{(p+1)f}\omega$ is a CKY p-form on $(M, \tilde{g})$ (see [6]). Moreover, this space of CKY forms is invariant by the Hodge-star operator (see [14]).

It was proved in [14] that on a compact 7-manifold with holonomy G_2 any CKY p-form with $p \neq 3, 4$ is parallel. The description of conformal Killing–Yano p-forms on a compact Riemannian product was obtained in [10], proving that such a form is a sum of forms of the following types: parallel forms, pull-back of Killing–Yano forms on the factors, and their Hodge duals.

In this article we will deal with 2-forms which are (conformal) Killing–Yano. We observe that the 2-form ω associated to a nearly Kähler manifold satisfies the Killing–Yano equation (1), while the canonical 2-form of a Sasakian manifold satisfies (2). Our goal is to construct examples of Riemannian manifolds carrying these distinguished 2-forms.

This article is organized as follows. In Section 2 we recall some basic results, and in Section 3 we study Killing–Yano 2-forms. We prove that on the total space of certain Riemannian submersions with totally geodesic fibers, for any $t > 0$ there exists a Riemannian metric g_t admitting Killing–Yano 2-forms, extending results by Nagy on nearly Kähler structures in this class of manifolds ([11]). We also show a method to build Killing–Yano 2-forms on Lie groups with left invariant metrics, starting with a Lie group equipped with such a tensor and a suitable representation of its Lie algebra. In Section 4 we consider invariant conformal Killing–Yano 2-forms on Lie groups with left invariant metrics. One first obstruction obtained is that those forms occur in odd dimensions, provided that they are not Killing–Yano (see Theorem 4.3). Furthermore, imposing certain restrictions on the codifferential of such a form, then the center of the group is 1-dimensional and the quotient of the group by its center inherits a non-degenerate Killing–Yano 2-form (see Theorem 4.6). Using this, we show that skew-symmetric non-degenerate parallel 2-forms give rise to CKY 2-forms on a higher dimensional Lie group by considering central extensions. This leads to the study of such forms in Section 5, where we give some obstructions for their existence (see Theorem 5.1). Finally, in Section 6 we focus on the existence of (conformal) Killing–Yano 2-forms in two special classes of Lie groups with left invariant metrics: (i) Lie groups with flat left invariant metric, and (ii) almost abelian Lie groups, that is, Lie groups such that the corresponding Lie algebra has a codimension 1 abelian ideal.

2. Preliminaries

Let (M, g) be a Riemannian manifold, and $T : TM \to TM$ a skew-symmetric endomorphism of the tangent bundle TM of M with its associated 2-form ω given by $\omega(X, Y) = g(TX, Y)$ for all X, Y vector fields on M.

We denote by N_T the Nijenhuis tensor of T, defined by

$$N_T(X, Y) := [TX, TY] - T([X, TY] + [TX, Y]) + T^2[X, Y] \tag{3}$$

and by ∇ the Levi-Civita connection associated to (M, g). Since

$$(\nabla_X T)\, Y = \nabla_X (TY) - T(\nabla_X Y), \tag{4}$$

one has the following identity:

$$N_T(X,Y) = (\nabla_{TX}T)\,Y - (\nabla_{TY}T)\,X - T\,(\nabla_X T)\,Y + T\,(\nabla_Y T)\,X, \tag{5}$$

for all X, Y vector fields on M.

The endomorphism T is called integrable when $N_T \equiv 0$, and the tensor field T is called parallel with respect to ∇ when $\nabla T = 0$, that is, $(\nabla_X T)\,Y = 0$ for all X, Y vector fields on M (see (4)).

It follows from (5) that if T is parallel then T is integrable.

Recall that given a 2-form ω on M, the exterior derivative $d\omega$ of ω can be computed in terms of $\nabla\omega$ as follows:

$$d\omega(X,Y,Z) = (\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(Z,X) + (\nabla_Z\omega)(X,Y), \tag{6}$$

for all vector fields X, Y, Z on M.

If $T : TM \to TM$ is a skew-symmetric invertible endomorphism of the tangent bundle TM of (M, g) with associated 2-forms ω and μ given by

$$\omega(X,Y) = g(TX,Y), \qquad \mu(X,Y) = g(T^{-1}X,Y)$$

for all X, Y vector fields on M, then after standard computations we obtain the following result:

Lemma 2.1. *For any X, Y, Z vector fields on M, the following identities hold:*

(i) $\nabla_X T^{-1} = -T^{-1}(\nabla_X T)T^{-1}$,
(ii) $T^{-1}N_T(T^{-1}X, T^{-1}Y) = TN_{T^{-1}}(X,Y)$,
(iii) $2g\left((\nabla_X T)\,Y, Z\right) = d\omega(X,Y,Z) + d\mu(X,TY,TZ) - g\left(N_T(Y,Z), T^{-1}X\right)$,
(iv) $d\mu(TX,TY,TZ) = (\nabla_{TX}\omega)(Y,Z) + (\nabla_{TY}\omega)(Z,X) + (\nabla_{TZ}\omega)(X,Y)$.

Corollary 2.2. *With notation as above, the skew-symmetric invertible tensor T is parallel if and only if $N_T = 0$ and $d\omega = d\mu = 0$.*

3. Killing–Yano 2-forms

A 2-form ω on a Riemannian manifold (M, g) is called *Killing–Yano* if it satisfies the *Killing–Yano equation*,

$$(\nabla_X\omega)(Y,Z) = -(\nabla_Y\omega)(X,Z), \tag{7}$$

where ∇ is the Levi-Civita connection and X, Y, Z are arbitrary vector fields on M (see [15]).

Using identity (6) the equivalence below follows.

Lemma 3.1. *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection and ω a 2-form on M. The following conditions are equivalent:*

(i) $(\nabla_X\omega)(Y,Z) + (\nabla_Y\omega)(X,Z) = 0$;
(ii) $d\omega(X,Y,Z) = 3(\nabla_X\omega)(Y,Z)$.

Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection and ω a 2-form on M satisfying any of the conditions of Lemma 3.1. Then the skew-symmetric endomorphism T of TM defined by ω and g, that is $\omega(X,Y) = g(TX,Y)$ satisfies

$$(\nabla_X T)Y = -(\nabla_Y T)X \tag{8}$$

for all X, Y vector fields on M. Conversely, if T is a skew-symmetric endomorphism of TM satisfying $(\nabla_X T)X = 0$ then the 2-form $\omega(X,Y) = g(TX,Y)$ satisfies any of the conditions of Lemma 3.1. After this observation we will refer indistinctly to a 2-form ω or a skew symmetric $(1,1)$-tensor T satisfying (7) or (8) as Killing–Yano (KY). Note that if (J, g) is an almost Hermitian structure, then the fundamental 2 form ω given by $\omega(X,Y) = (JX,Y)$ is Killing–Yano if and only if (J, g) is nearly Kähler.

Proposition 3.2. *If T is a Killing–Yano tensor on M, then*

$$\langle N_T(X,Y), Z\rangle = \frac{1}{3}\left(d\omega(TX,Y,Z) + d\omega(X,TY,Z) + 2d\omega(X,Y,TZ)\right).$$

Moreover, if T is invertible, then

$$d\mu(TX,TY,TZ) = \frac{1}{3}\left(d\omega(TX,Y,Z) + d\omega(X,TY,Z) + d\omega(X,Y,TZ)\right).$$

Proof. From Lemma 3.1 (ii), we obtain the following relations:

$$\begin{aligned}
d\omega(X,Y,TZ) &= 3(\nabla_X\omega)(Y,TZ) = 3g((\nabla_X T)Y,TZ),\\
d\omega(Y,X,TZ) &= 3g((\nabla_Y T)X,TZ),\\
d\omega(TX,Y,Z) &= 3g((\nabla_{TX} T)Y,Z),\\
d\omega(TY,X,Z) &= 3g((\nabla_{TY} T)X,Z).
\end{aligned}$$

Now, we apply identity (5) to obtain

$$\begin{aligned}
3g(N_T(X,Y),Z) &= d\omega(TX,Y,Z) - d\omega(TY,X,Z) + d\omega(X,Y,TZ) - d\omega(Y,X,TZ)\\
&= d\omega(TX,Y,Z) + d\omega(X,TY,Z) + 2d\omega(X,Y,TZ),
\end{aligned}$$

and this proves the first identity in the statement.

The second identity follows from Lemma 2.1 (iii) and Lemma 3.1 (ii). □

Corollary 3.3. *If T is an invertible integrable Killing–Yano tensor on M, then T is parallel.*

Proof. Since $N_T \equiv 0$, it follows from Lemma 3.2 that

$$d\omega(TX,Y,Z) + d\omega(X,TY,Z) + 2d\omega(X,Y,TZ) = 0,$$

for all X, Y, Z vector fields on M. Thus

$$\begin{aligned}
d\omega(X,Y,TZ) &= -\frac{1}{2}(d\omega(TX,Y,Z) + d\omega(X,TY,Z))\\
&= -\frac{1}{2}(d\omega(Y,Z,TX) + d\omega(Z,X,TY))\\
&= \frac{1}{4}(d\omega(TY,Z,X) + d\omega(Y,TZ,X) + d\omega(TZ,X,Y) + d\omega(Z,TX,Y))\\
&= \frac{1}{4}(2d\omega(X,Y,TZ) + d\omega(TX,Y,Z) + d\omega(X,TY,Z))\\
&= 0.
\end{aligned}$$

As T is invertible, it follows that ω is closed and therefore parallel (Lemma 3.1). □

Remark 1. Corollary 3.3 is a generalization of the fact that an integrable nearly Kähler structure is actually Kähler.

In the following examples we exhibit new solutions to the Killing–Yano equation (7).

Example 1. Let us consider a Riemannian submersion

$$F \hookrightarrow (M, g) \to N$$

with totally geodesic fibers, and let $TM = \mathcal{V} \oplus \mathcal{H}$ the corresponding decomposition of TM into vertical and horizontal components. Let us also assume that M carries a complex structure J such that (M, g, J) is Kähler and $J(\mathcal{V}) = \mathcal{V}$, $J(\mathcal{H}) = \mathcal{H}$.

For $t > 0$, let g_t be the Riemannian metric on M defined by

$$\begin{aligned} g_t(v, w) &= tg(v, w), \\ g_t(x, y) &= g(x, y), \\ g_t(v, x) &= 0, \end{aligned}$$

for any $v, w \in \mathcal{V}$, $x, y \in \mathcal{H}$. According to [7], $F \hookrightarrow (M, g_t) \to N$ is a Riemannian submersion with totally geodesic fibers.

We consider next the endomorphism $\hat{J}$ of TM defined by

$$\hat{J}|_{\mathcal{V}} = aJ, \qquad \hat{J}|_{\mathcal{H}} = bJ, \tag{9}$$

for some $a, b \in \mathbb{R}$ such that $a^2 + b^2 \neq 0$. Note that $\hat{J}$ is skew-symmetric with respect to g_t for any $t > 0$. For instance, if $a = -1$, $b = 1$, then $(M, g_t, \hat{J})$ is almost Hermitian, and it was proved in [11] that for $t = \frac{1}{2}$ this almost Hermitian manifold is actually nearly Kähler.

Let us denote by ∇^t the Levi-Civita connection associated to g_t, and let A be the O'Neill tensor of the Riemannian submersion $(M, g) \to N$. After standard computations, we obtain

$$(\nabla^t_V \hat{J})W = 0, \quad (\nabla^t_X \hat{J})Y = (b - a)JA_X Y,$$
$$(\nabla^t_X \hat{J})V = (a - b)tJA_X V, \quad (\nabla^t_V \hat{J})X = 2b(1 - t)JA_X V,$$

for V, W vertical vector fields and X, Y horizontal vector fields. Recalling that $A_X Y = -A_Y X$ whenever X and Y are horizontal, it follows that $\hat{J}$ is Killing–Yano if and only if

$$(a - b)t = 2b(t - 1). \tag{10}$$

Therefore, for a fixed $t > 0$, we obtain infinitely many values of a, b satisfying (10), and consequently infinitely many Killing–Yano tensors $\hat{J}$ as in (9).

We have proved

Theorem 3.4. *Let $F \hookrightarrow (M, g) \to N$ be a Riemannian submersion with totally geodesic fibers. Assume that M admits a complex structure J such that (J, g) is a Kähler structure on M and J preserves the corresponding vertical and horizontal subbundles of TM. Then for any $t > 0$ the Riemannian metric g_t defined above admits Killing–Yano tensors.*

Corollary 3.5. *On the twistor space of any quaternionic-Kähler manifold with positive scalar curvature and for any $t > 0$, there exists a Riemannian metric g_t admitting Killing–Yano tensors.*

A particular case of twistor spaces as in the previous corollary are $\mathbb{C}P^3$ (the twistor space of S^4) and the flag manifold F_3 (the twistor space of $\mathbb{C}P^2$); see for instance [8].

Example 2. On a Lie group G equipped with a left invariant metric, the search for left invariant 2-forms satisfying the Killing–Yano equation (7) reduces to finding skew-symmetric endomorphisms $T : \mathfrak{g} \to \mathfrak{g}$ of its Lie algebra satisfying

$$(\nabla_x T)y = -(\nabla_y T)x, \qquad x, y \in \mathfrak{g}.$$

This endomorphism is called a Killing–Yano tensor on the Lie algebra $\mathfrak{g}$. Some examples of Lie groups with left invariant metric admitting left invariant KY tensors are given in [1,4].

In this example we exhibit a method to obtain new examples of KY tensors on Lie algebras, beginning with a Lie algebra equipped with such a tensor and a suitable representation.

Let $(\mathfrak{h}, \langle\ ,\ \rangle)$ be a Lie algebra with an inner product and let $(V, \langle\ ,\ \rangle)$ be a finite dimensional vector space with an inner product. Given a representation $\pi : \mathfrak{h} \to \mathrm{End}(V)$, set $\mathfrak{g} := \mathfrak{h} \ltimes_\pi V$ equipped with the orthogonal direct sum of the inner products on $\mathfrak{h}$ and V.

If T and E are skew-symmetric operators on $\mathfrak{h}$ and V, respectively, let us define a skew-symmetric operator T_1 on $\mathfrak{g}$ by

$$T_1|_{\mathfrak{h}} = T, \quad T_1|_V = E.$$

We want to determine when T_1 is a KY tensor on $\mathfrak{g}$. In order to do so, we compute first the Levi-Civita connection $\overline{\nabla}$ on $\mathfrak{g}$ in terms of the Levi-Civita connection ∇ on $\mathfrak{h}$ and the representation π. It is readily verified that, whenever $x, y \in \mathfrak{h}$ and $u, v \in V$,

$$\overline{\nabla}_x y = \nabla_x y, \quad \overline{\nabla}_x u = \pi(x)^a u, \quad \overline{\nabla}_u x = -\pi(x)^s u,$$
$$\overline{\nabla}_u v \in \mathfrak{h} \text{ and } \langle \overline{\nabla}_u v, x\rangle = \langle \pi(x)^s u, v\rangle,$$

where $\pi(x)^a$ and $\pi(x)^s$ denote the skew-symmetric and symmetric components of $\pi(x)$, respectively. Using these expressions, we obtain the following relations concerning $\overline{\nabla} T_1$, for $x \in \mathfrak{h}$ and $u \in V$:

- $(\overline{\nabla}_x T_1)x = (\nabla_x T)x$,
- $(\overline{\nabla}_u T_1)u \in \mathfrak{h}$ and $\langle (\overline{\nabla}_u T_1)u, x\rangle = \frac{1}{2}\langle [\pi(x), E]u, u\rangle + \langle \pi(Tx)u, u\rangle$,
- $(\overline{\nabla}_u T_1)x + (\overline{\nabla}_x T_1)u = -\pi(Tx)^s u + E\pi(x)^s u + [\pi(x)^a, E]u$.

As a consequence, we obtain the following result.

Proposition 3.6. *With notation as above, if*

(i) *T is a Killing–Yano tensor on $\mathfrak{h}$,*
(ii) *$\pi(x)$ is skew-symmetric for all $x \in \mathfrak{h}$,*
(iii) *$[\pi(x), E] = 0$ for all $x \in \mathfrak{h}$,*

then T_1 is a Killing–Yano tensor on $\mathfrak{g}$. Furthermore, T_1 is parallel if and only if T is parallel.

For instance, we may take $V = \mathbb{R}^{2n}$, $E = J$ the canonical complex structure on $\mathbb{R}^{2n}$, and π a representation such that $\pi(x)$ is skew symmetric and $[\pi(x), J] = 0$ for all $x \in \mathfrak{h}$, that is, $\pi(x) \in \mathfrak{u}(n)$ for all $x \in \mathfrak{h}$.

Remark 2. If $\mathfrak{h}$ is a solvable Lie algebra, then $\pi(\mathfrak{h})$ is a solvable Lie subalgebra of $\mathfrak{so}(V)$, which is compact. Thus, $\pi(\mathfrak{h})$ is abelian and therefore $\pi(x) = 0$ for $x \in \mathfrak{h}'$. If $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{so}(V)$ such that $\pi(\mathfrak{h}) \subseteq \mathfrak{a}$, then $E \in \mathfrak{a}$.

4. Conformal Killing–Yano 2-forms

A 2-form ω on an n-dimensional Riemannian manifold (M, g) is called *conformal Killing–Yano* (CKY for short) if it satisfies the following equation:

$$(\nabla_X \omega)(Y, Z) = \frac{1}{3} d\omega(X, Y, Z) - \frac{1}{n-1}(X^* \wedge d^*\omega)(Y, Z), \tag{11}$$

for any vector fields X, Y, Z on M where ∇ is the Levi-Civita connection, X^* is the 1-form dual to X and $d^* = (-1)^{n+1} * d *$, is the co-differential. Note that such a form ω is KY if and only if $d^*\omega = 0$.

According to [14, Proposition 2.7], a 2-form ω is a conformal Killing–Yano tensor if and only if there exists a 1-form θ on M such that

$$(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(X, Z) = 2g(X, Y)\theta(Z) - g(X, Z)\theta(Y) - g(Y, Z)\theta(X), \tag{12}$$

for any vector fields X, Y, Z on M. Furthermore, the 1-form θ is given by

$$\theta = -\frac{1}{n-1} d^*\omega. \tag{13}$$

An interesting class of manifolds admitting CKY 2-forms is given by the Sasakian manifolds. We recall that a Riemannian manifold (M, g) is called Sasakian if there exists a unit length Killing vector field ψ such that for any vector field X on M,

$$\nabla_X d\psi^* = -2X^* \wedge \psi^* \tag{14}$$

where $\psi^*(Y) = g(\psi, Y)$. It was proved in [14] that $d\psi^*$ is a CKY 2-form. In [3], many examples of left invariant Sasakian structures on Lie groups were provided using a central extension of a Kähler Lie group. In a similar fashion, we will prove that starting with a Lie group with a special left invariant KY 2-form, a central extension of this group carries a left invariant solution to the CKY equation (11).

4.1. Left invariant CKY 2-forms on Lie groups

Let g be a left invariant Riemannian metric on the Lie group G, and let $\mathfrak{g}$ be its Lie algebra. Many geometric invariants can be computed at the Lie algebra level. In particular, the Levi-Civita connection ∇ associated to g, when applied to left invariant vector fields, is given by:

$$2\langle \nabla_x y, z\rangle = \langle [x, y], z\rangle - \langle [y, z], x\rangle + \langle [z, x], y\rangle, \qquad x, y, z \in \mathfrak{g}, \tag{15}$$

where $\langle \ , \ \rangle$ is the inner product induced by g on $\mathfrak{g}$. Note that $\nabla g = 0$ implies that ∇_x is a skew-symmetric endomorphism of $\mathfrak{g}$ for any $x \in \mathfrak{g}$.

A left invariant 2-form ω on G is a 2-form such that $L_a^* \omega = \omega$ for all $a \in G$, where L_a is left translation by $a \in G$. We will consider left invariant 2-forms ω on (G, g) satisfying (11). Since $\nabla\omega$, $d\omega$ and $d^*\omega$ are left invariant as well, we will study $\omega \in \bigwedge^2 \mathfrak{g}^*$ satisfying (11) for $x \in \mathfrak{g}$, or equivalently, ω satisfying (12) for $x, y, z \in \mathfrak{g}$ and $\theta \in \mathfrak{g}^*$. We will call such a form a conformal Killing–Yano 2-form on $(\mathfrak{g}, \langle \ , \ \rangle)$.

From now on, we will consider left invariant CKY 2-forms on (G, g), where g is a left invariant metric. With respect to g, any left invariant 2-form ω on G gives rise to a skew-symmetric endomorphism T of the Lie algebra $\mathfrak{g}$ defined by $\omega(x, y) = \langle Tx, y\rangle$, for any $x, y \in \mathfrak{g}$. If ω is a left invariant CKY 2-form on (G, g), the associated endomorphism T of $\mathfrak{g}$ will be called a conformal Killing–Yano tensor on $\mathfrak{g}$.

All known examples of CKY tensors (not KY) on Lie algebras occur only in odd dimensions. We will prove next that this is always the case.

Proposition 4.1. *If T is a conformal Killing–Yano tensor on the Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$, with associated 1-form θ defined by (12), then $\theta \circ T = 0$.*

Proof. From (12), taking $x = y$, we obtain

$$\langle(\nabla_x T)x, z\rangle = \|x\|^2\theta(z) - \langle x, z\rangle\theta(x), \tag{16}$$

for any $x, z \in \mathfrak{g}$. If we consider $z = Tx$, we obtain, since T is skew-symmetric,

$$\theta(Tx) = \frac{\langle(\nabla_x T)x, Tx\rangle}{\|x\|^2} \tag{17}$$

for any $0 \neq x \in \mathfrak{g}$. Now, using (15), we get

$$\langle(\nabla_x T)x, Tx\rangle = \langle[T^2x, x], x\rangle. \tag{18}$$

There is an orthonormal basis $\{e_i, f_i, u_j\}$ of $\mathfrak{g}$ such that $Te_i = a_i f_i$, $Tf_i = -a_i e_i$ and $Tu_j = 0$, for some $a_i \in \mathbb{R}^\times$ and for $1 \le i \le r$, $1 \le j \le s$. Therefore $T^2 e_i = -a_i^2 e_i$, $T^2 f_i = -a_i^2 f_i$. Using this in (17) and (18), we obtain $\theta(Te_i) = \theta(Tf_i) = \theta(Tu_j) = 0$ for all i and j, hence $\theta \circ T = 0$. $\square$

Corollary 4.2. *With the same hypothesis as in the previous result, if $\xi \in \mathfrak{g}$ is the unique element of $\mathfrak{g}$ that satisfies $\theta(x) = \langle\xi, x\rangle$ for all $x \in \mathfrak{g}$, then $T\xi = 0$.*

Remark 3. It follows from [1, Lemma 2.3] that when $\mathfrak{g}$ is an n-dimensional unimodular Lie algebra, the vector ξ from Corollary 4.2 is given by $\xi = -\frac{1}{2(n-1)}\sum_{i=1}^n [Te_i, e_i]$, where $\{e_1, \dots, e_n\}$ is any orthonormal basis of $\mathfrak{g}$. In particular, $\xi \in \mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$.

In the following result we provide a strong restriction to the existence of a CKY tensor which is not KY, namely, the dimension of the Lie algebra has to be odd and the associated 2-form has maximal rank.

Theorem 4.3. *Let T be a conformal Killing–Yano tensor on the Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$, with associated 1-form θ defined by (12). If $\theta \neq 0$, then $\dim\mathfrak{g}$ is odd and $T|_{\xi^\perp} : \xi^\perp \to \xi^\perp$ is a linear isomorphism. Moreover, $(\nabla_\xi T)\xi = \nabla_\xi \xi = 0$, and $\xi^\perp$ is stable by the operator ad_ξ.*

Proof. Let $x \in \ker T \cap \xi^\perp$, then from (16) with $z = \xi$, we obtain

$$\langle(\nabla_x T)x, \xi\rangle = \|x\|^2\|\xi\|^2.$$

Using that $Tx = 0$ and $T\xi = 0$, we see that the left-hand side of the previous equation is 0, hence $x = 0$. This means that $\ker T$ is generated by ξ, and the restriction $T|_{\xi^\perp} : \xi^\perp \to \xi^\perp$ is a skew-symmetric isomorphism, therefore $\dim \xi^\perp = 2m$ and $\dim \mathfrak{g} = 2m + 1$.

Since $\nabla_\xi T$ is skew-symmetric, we have $\langle(\nabla_\xi T)\xi, \xi\rangle = 0$, and for $x \in \xi^\perp$, from (16) we have

$$\langle(\nabla_\xi T)\xi, x\rangle = \|\xi\|^2\langle\xi, x\rangle - \langle\xi, x\rangle\|\xi\|^2 = 0,$$

so that $(\nabla_\xi T)\xi = 0$. Moreover, $0 = \langle(\nabla_\xi T)\xi, x\rangle = -\langle T\nabla_\xi \xi, x\rangle = \langle \nabla_\xi \xi, Tx\rangle$. Since T is an isomorphism on $\xi^\perp$ and $\langle \nabla_\xi \xi, \xi\rangle = 0$, we obtain $\nabla_\xi \xi = 0$.

To prove the last assertion, we have for any $x \in \mathfrak{g}$,

$$0 = \langle \nabla_\xi \xi, x\rangle = -\langle [\xi, x], \xi\rangle,$$

therefore $\mathrm{ad}_\xi(x) \in \xi^\perp$ for any $x \in \mathfrak{g}$, in particular, $\xi^\perp$ is ad_ξ-stable. □

Corollary 4.4. *Let T be a conformal Killing–Yano tensor on the n-dimensional Lie algebra $(\mathfrak{g}, \langle\ ,\ \rangle)$, with associated 1-form θ defined by (12). If n is even, then $\theta = 0$, that is, T is a KY tensor on $\mathfrak{g}$.*

Corollary 4.5. *If $\dim \mathfrak{g} = 4$, then any conformal Killing–Yano tensor is parallel.*

Proof. Let T be a conformal Killing–Yano tensor on $\mathfrak{g}$, and let ω denote the corresponding CKY 2-form. According to Corollary 4.4, both T and ω are Killing–Yano, so that $d^*\omega = 0$. It follows from [14] that $*\omega$ is a CKY 2-form on $\mathfrak{g}$. Again, due to Corollary 4.4, $*\omega$ is Killing–Yano, so that $d^*(*\omega) = 0$. But this implies $*(d\omega) = 0$, thus $d\omega = 0$. Since ω is a closed KY 2-form, it is parallel. □

Remark 4. More generally, it was proved in [2] that on a 4-dimensional Riemannian manifold, any conformal Killing–Yano 2-form of constant length is parallel.

In the rest of this section we will study the case when the $(2n+1)$-dimensional Lie algebra $\mathfrak{g}$ admits a CKY 2-form ω such that the associated 1-form θ is dual to a non zero central element. In this case, θ turns out to be a contact form, that is, $\theta \wedge (d\theta)^n \neq 0$. Compare with the Sasakian case, where if the center is different from zero, then the center is generated by the Reeb vector (see [3]).

Theorem 4.6. *Let T be a CKY tensor on $(\mathfrak{g}, \langle\ ,\ \rangle)$, with ω the associated 2-form and $\xi \in \mathfrak{g}$ as in Corollary 4.2. If $\xi \in \mathfrak{z}$, then $\mathfrak{z} = \mathbb{R}\xi$ and $\mathfrak{h} := \xi^\perp$ admits a Lie bracket $[\,,]'$ such that $S := T|_{\mathfrak{h}}$ is an invertible KY tensor on $\mathfrak{h}$, and the 2-form μ on $(\mathfrak{h}, [\,,]')$ defined by $\mu(x, y) = \langle S^{-1}x, y\rangle$ is closed and co-closed, hence harmonic.*

Moreover, the 1-form θ is a contact form on $\mathfrak{g}$.

Proof. Let us prove first that $\mathfrak{z} = \mathbb{R}\xi$. Let $x \in \mathfrak{z}$, $\langle x, \xi\rangle = 0$, then from (16) with $z = \xi$, we obtain

$$\langle(\nabla_x T)x, \xi\rangle = \|x\|^2\|\xi\|^2.$$

Using that $x \in \mathfrak{z}$ and $T\xi = 0$, we see that the left-hand side of the previous equation is 0, hence $x = 0$, so that $\mathfrak{z}$ is generated by ξ.

As a consequence, we can decompose $\mathfrak{g}$ as the orthogonal sum $\mathfrak{g} = \mathbb{R}\xi \oplus \xi^\perp$. Let us denote $\mathfrak{h} := \xi^\perp$. For any $x, y \in \mathfrak{h}$, we then have a decomposition

$$[x, y] = [x, y]' - 2\mu(x, y)\xi, \qquad [x, y]' \in \mathfrak{h},\ \mu \in {\textstyle\bigwedge^2} \mathfrak{h}^*. \tag{19}$$

Computing $\langle [x, y], \xi\rangle = \theta([x, y]) = -d\theta(x, y)$ and using (19), we obtain the following expression for μ:

$$\mu(x, y) = \frac{d\theta(x, y)}{2\|\xi\|^2}. \tag{20}$$

We will prove next that $[\,,]'$ defines a Lie bracket on $\mathfrak{h}$ and that μ is a closed 2-form on $\mathfrak{h}$ with this Lie algebra structure. Indeed, for $x, y, z \in \mathfrak{h}$, we consider the Jacobi identity for $[\,,]$: $[[x, y], z] + [[y, z], x] +$

$[[z, x], y] = 0$. Expanding this expression according to the decomposition (19), and using that $\xi \in \mathfrak{z}$, we obtain

$$[[x, y]', z]' + [[y, z]', x]' + [[z, x]', y]' = 0, \text{ and } \mu([x, y]', z) + \mu([y, z]', x) + \mu([z, x]', y) = 0.$$

This means that $[\,,]'$ is a Lie bracket on $\mathfrak{h}$ and $d'\mu = 0$, where d' is the differential associated to this Lie bracket on $\mathfrak{h}$.

Next, we will show that $S := T|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{h}$ is a Killing–Yano tensor on $\mathfrak{h}$. Note that according to Theorem 4.6 this endomorphism is invertible. The Levi-Civita connection ∇' associated to $\langle\,,\rangle|_{\mathfrak{h}\times\mathfrak{h}}$ on $(\mathfrak{h}, [\,,]')$ is given by the $\mathfrak{h}$-component of ∇. Using this, and the fact that $\ker\theta = \mathfrak{h}$, it follows that

$$\langle(\nabla'_x S)y, z\rangle = \frac{1}{3}d\omega(x, y, z) = \frac{1}{3}d'\omega(x, y, z), \qquad x, y, z \in \mathfrak{h}$$

hence S is a Killing–Yano tensor on $\mathfrak{h}$.

Finally, we will find another expression for the 2-form μ, one that does not involve the 1-form θ. For $x, y \in \mathfrak{h}$, it is easy to see that

$$-d\theta(x, Ty) = \theta([x, Ty]) = 2\langle(\nabla_x T)y, \xi\rangle = 2(\nabla_x\omega)(y, \xi).$$

Now, using that ω is a conformal Killing–Yano 2-form and $d\omega(x, y, \xi) = 0$ (since $\xi \in \mathfrak{z}$ and $T\xi = 0$), we compute

$$\begin{aligned}(\nabla_x\omega)(y, \xi) &= \frac{1}{3}d\omega(x, y, \xi) + x^* \wedge \theta(y, \xi) \quad \text{(from (11) and (13))}\\ &= x^* \wedge \theta(y, \xi)\\ &= \langle x, y\rangle\|\xi\|^2 - \langle\xi, \xi\rangle\theta(y)\\ &= \langle x, y\rangle\|\xi\|^2.\end{aligned}$$

Therefore, since $S = T|_{\mathfrak{h}}$ is invertible, we have $d\theta(x, y) = 2\langle S^{-1}x, y\rangle\|\xi\|^2$, thus, replacing in (20), we obtain

$$\mu(x, y) = \langle S^{-1}x, y\rangle, \quad x, y \in \mathfrak{h}.$$

Next, we note that

$$d\theta(x, y) = -2\omega(T^{-1}x, T^{-1}y)\|\xi\|^2, \quad x, y \in \mathfrak{h}.$$

Therefore, since ω is non degenerate on $\mathfrak{h}$, we have that θ is a contact form on $\mathfrak{g}$.

It remains to show that the 2-form μ is co-closed. This will be a consequence of the following lemma, and the proof is complete. □

Lemma 4.7. *If S is an invertible Killing–Yano tensor on $\mathfrak{g}$ and μ is the 2-form associated to S^{-1} then $d^*\mu = 0$.*

Proof. Recall that $\mu(x, y) = \langle S^{-1}x, y\rangle$, for $x, y \in \mathfrak{g}$. We compute

$$d^*\mu(x) = -\sum_{i=1}^{n}(\nabla_{e_i}\mu)(e_i, x) = -\sum_{i=1}^{n}(\nabla_{e_i}\mu)(SS^{-1}e_i, SS^{-1}x) \tag{21}$$

Using Lemma 2.1 and the fact that S is a Killing–Yano tensor it then follows

$$d^*\mu(x) = -\sum_{i=1}^{n}(\nabla_{S^{-1}x}\omega)(e_i, S^{-1}e_i) = \sum_{i=1}^{n} g(S^{-1}(\nabla_{S^{-1}x}S)e_i, e_i)$$
$$= \operatorname{tr}(S^{-1}(\nabla_{S^{-1}x}S))$$
$$= 0,$$

as claimed. □

Remark 5. It can be seen that in the family of examples of KY tensors given in Theorem 3.4, the 2-form μ associated to the tensor $\hat{J}^{-1}$ is also co-closed.

We will show next that the converse of Theorem 4.6 holds.

Theorem 4.8. *Let S be an invertible Killing–Yano tensor on a Lie algebra $(\mathfrak{h}, [\ ,\ ]', \langle\ ,\ \rangle)$ such that the 2-form defined by $\mu(x, y) = \langle S^{-1}x, y\rangle$ is closed. Set $\mathfrak{g} := \mathfrak{h} \oplus \mathbb{R}\xi$ with Lie bracket $[\ ,\ ]$ given by*

$$[\mathfrak{h}, \xi] = 0, \quad [x, y] = [x, y]' - 2\mu(x, y)\xi, \quad x, y \in \mathfrak{h}$$

and inner product obtained by extending the one on $\mathfrak{h}$ such that $\langle\mathfrak{h}, \xi\rangle = 0$, $\|\xi\| > 0$ arbitrary. Then the endomorphism T of $\mathfrak{g}$ given by $T|_{\mathfrak{h}} = S$, $T\xi = 0$, is a conformal Killing–Yano tensor on $\mathfrak{g}$.

Proof. It is readily verified that $(\mathfrak{g}, [\ ,\ ])$ is indeed a Lie algebra.

The Levi-Civita connection ∇ on $\mathfrak{g}$ is related to the Levi-Civita connection ∇' on $\mathfrak{h}$ in the following way:

$$\nabla_x y = \nabla'_x y - \mu(x, y)\xi, \qquad \nabla_\xi x = \nabla_x \xi = \|\xi\|^2 S^{-1}x, \qquad \nabla_\xi \xi = 0, \tag{22}$$

for any $x, y \in \mathfrak{h}$.

Using (22) and the fact that S is a Killing–Yano tensor on $\mathfrak{h}$, it is straightforward to verify that T is a conformal Killing–Yano tensor on $\mathfrak{g}$. □

Remark 6. The Lie algebra $\mathfrak{g}$ constructed in Theorem 4.8 is known as the 1-dimensional central extension of $\mathfrak{h}$ by the cocycle (-2μ).

Corollary 4.9. *For any invertible parallel tensor S on a metric Lie algebra $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$, the 1-dimensional central extension of $\mathfrak{h}$ by the 2-cocycle (-2μ), with $\mu(x, y) = \langle S^{-1}x, y\rangle$, carries a conformal Killing–Yano tensor.*

Remark 7. The only known examples to us of invertible Killing–Yano tensors S on Lie algebras such that 2-form μ associated to S^{-1} is closed are the parallel ones.

5. Skew-symmetric invertible parallel tensors on Lie groups

According to Corollary 4.9, skew-symmetric invertible parallel tensors on a Lie algebra may be used to build CKY tensors on a higher dimensional Lie algebra. For that reason in this section we analyze such tensors and discuss their existence and properties.

The main result of this section is the following theorem, which provides an obstruction for the existence of skew-symmetric parallel tensors.

Theorem 5.1. *If $\mathfrak{g}$ is a Lie algebra with an inner product $\langle\, ,\rangle$ such that $\mathfrak{g}' \cap \mathfrak{z} \neq \{0\}$, then there is no skew-symmetric invertible parallel tensor on $\mathfrak{g}$. In particular, this holds for any inner product on a non-abelian nilpotent Lie algebra.*

This theorem will follow easily from the next lemma.

Lemma 5.2. *Let $\mathfrak{g}$ be a Lie algebra such that its center $\mathfrak{z}$ is non-zero, and let $\langle\, ,\rangle$ denote an inner product on $\mathfrak{g}$. If T is a skew-symmetric invertible endomorphism of $\mathfrak{g}$ such that $\nabla T = 0$, where ∇ denotes the Levi-Civita connection associated to $\langle\, ,\rangle$, then:*

(i) $\mathfrak{z} + T\mathfrak{z} \subseteq (\mathfrak{g}')^{\perp}$,
(ii) $\mathfrak{z} + T\mathfrak{z}$ *is an abelian Lie subalgebra of* $\mathfrak{g}$,
(iii) ad_{Tz} *is skew-symmetric for any* $z \in \mathfrak{z}$,
(iv) $[T, \mathrm{ad}_{Tz}] = 0$ *for any* $z \in \mathfrak{z}$.

Proof. Let $0 \neq z \in \mathfrak{z}$. We will prove this theorem computing $\rho(Tz)$, the Ricci curvature in the direction of Tz. Recall from [9] that if $u \in (\mathfrak{g}')^{\perp}$, then $\rho(u) \leq 0$, and moreover, $\rho(u) = 0$ if and only if ad_u is skew-symmetric.

Since T is parallel, the 2-form ω defined by $\omega(x, y) = \langle Tx, y\rangle$, $x, y \in \mathfrak{g}$, is closed, and then we compute

$$0 = d\omega(z, x, y) = -\omega([x, y], z) = -\langle T[x, y], z\rangle = \langle [x, y], Tz\rangle,$$

showing that $Tz \in (\mathfrak{g}')^{\perp}$. Therefore, it follows that $\rho(Tz) \leq 0$ and $\rho(Tz) = 0$ if and only if ad_{Tz} is skew-symmetric.

Let us now assume that the following claim holds (a proof will be given below):

Claim. *For any $0 \neq z \in \mathfrak{z}$, the Ricci curvature $\rho(Tz)$ in the direction of Tz satisfies $\rho(Tz) \geq 0$. Moreover, $\rho(Tz) = 0$ if and only if $z \in (\mathfrak{g}')^{\perp}$.*

Therefore, we have that $\rho(Tz) = 0$, thus $z \in (\mathfrak{g}')^{\perp}$, $Tz \in (\mathfrak{g}')^{\perp}$ and ad_{Tz} is skew-symmetric, proving (i) and (iii).

In order to prove (ii), we recall that if T is parallel, then T is integrable, so that $N_T(x, y) = 0$ for all $x, y \in \mathfrak{g}$. We only have to check that $[Tz_1, Tz_2] = 0$ for any $z_1, z_2 \in \mathfrak{z}$. Indeed,

$$[Tz_1, Tz_2] = T([Tz_1, z_2] + [z_1, Tz_2]) - T^2[z_1, z_2] = 0,$$

and (ii) is proved.

In order to prove (iv), it is sufficient to check that if $x \in \mathfrak{g}'$ and ad_x is skew-symmetric then $\nabla_x T = [\mathrm{ad}_x, T]$, and this is straightforward.

Proof of Claim. Let $\{e_1, \dots, e_{2n}\}$ an orthonormal basis of $\mathfrak{g}$ such that

$$Te_{2i-1} = a_i e_{2i}, \quad Te_{2i} = -a_i e_{2i-1}, \tag{23}$$

for some $a_i \in \mathbb{R}^{\times}$, $i = 1, \dots, n$. Using the expression $R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]}$ for the curvature tensor, we compute the Ricci curvature in the direction of Tz:

$$\rho(Tz) = \sum \langle R(Tz, e_i)e_i, Tz\rangle = \sum \langle R(Te_i, z)z, Te_i\rangle,$$

where in the second equality we have used that T is skew-symmetric and parallel, and the symmetries of the curvature tensor R. Using (23), we obtain

$$\rho(Tz) = \sum_{i=1}^{n} a_i^2 \left(\langle R(e_{2i-1}, z)z, e_{2i-1}\rangle + \langle R(e_{2i}, z)z, e_{2i}\rangle\right).$$

Since $z \in \mathfrak{z}$, it is straightforward that $\langle R(x,z)z, x\rangle = \|\nabla_z x\|^2$ for any $x \in \mathfrak{g}$, and using this in the expression above, we get

$$\rho(Tz) = \sum_{i=1}^{n} a_i^2 (\|\nabla_z e_{2i-1}\|^2 + \|\nabla_z e_{2i}\|^2), \tag{24}$$

and therefore, $\rho(Tz) \geq 0$.

If $\rho(Tz) = 0$, then it follows from (24) that $\nabla_z \equiv 0$, and replacing this in (15) we obtain that $z \in (\mathfrak{g}')^\perp$, and the claim is proved. □

Remark 8. Theorem 5.1 is a generalization of the fact that there are no left-invariant Kähler structures on nilpotent Lie groups, unless the group is abelian.

Remark 9. The examples of non degenerate KY tensors on 2-step nilpotent Lie algebras given in [4] cannot be parallel, according to Theorem 5.1. Choosing appropriate representations of these nilpotent Lie algebras and applying Proposition 3.6, we may produce many non parallel KY tensors on Lie algebras, not necessarily nilpotent.

6. Examples and non-examples

In this section we will focus on the existence of left invariant KY and CKY tensors on special classes of Lie groups equipped with left invariant metrics, namely: (i) Lie groups such that the left invariant metric is flat, and (ii) almost abelian Lie groups. We will show that in most cases the KY and CKY tensors are in fact parallel.

6.1. Flat Lie groups

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We recall from [9] that an inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ induces a flat left invariant metric on G if and only if the following conditions are satisfied:

(i) there exists an abelian ideal $\mathfrak{u}$ of $\mathfrak{g}$ such that its orthogonal complement $\mathfrak{a} = \mathfrak{u}^\perp$ is an abelian subalgebra,
(ii) ad_x is skew-symmetric for any $x \in \mathfrak{a}$.

Moreover, $\nabla_u = 0$ for all $u \in \mathfrak{u}$, and $\dim \mathfrak{g}' \geq 2$ if $\mathfrak{g}$ is not abelian.

Lemma 6.1. *Let G be a non abelian Lie group with a flat left invariant metric. If T is a left invariant CKY tensor, then T is parallel.*

Proof. We prove first that T is KY. Let ξ be as in Corollary 4.2, which, according to Remark 3, belongs to $\mathfrak{g}' \subset \mathfrak{u}$, since $\mathfrak{g}$ is clearly unimodular.

If $\xi \neq 0$, take $v \in \mathfrak{u}$ with $\langle v, \xi\rangle = 0$. Then, it follows from (16) with $z = \xi$ that

$$\langle (\nabla_v T)v, \xi\rangle = \|v\|^2\|\xi\|^2.$$

Since $\nabla_v = 0$, we have that the left-hand side of the equation above is 0, and therefore $v = 0$. This implies that $\mathfrak{u} = \mathbb{R}\xi$, which is not possible since $\dim \mathfrak{g}' \geq 2$. As a consequence, $\xi = 0$ and T is a KY tensor.

We will prove next that T is parallel. Since $\nabla_u = 0$ for any $u \in \mathfrak{u}$, we only have to prove that $(\nabla_a T)u = 0$ and $(\nabla_a T)b = 0$ for all $u \in \mathfrak{u}$ and $a, b \in \mathfrak{a}$. For the former, $(\nabla_a T)u = -(\nabla_u T)a = 0$ since T is KY. For the latter, we compute for $c \in \mathfrak{a}$,

$$\langle(\nabla_a T)b, c\rangle = \langle\nabla_a Tb, c\rangle + \langle\nabla_a b, Tc\rangle = 0,$$

using (15), since $\mathfrak{a}$ is abelian and orthogonal to $\mathfrak{g}'$. Next we compute, for $u \in \mathfrak{u}$,

$$\langle(\nabla_a T)b, u\rangle = -\langle b, (\nabla_a T)u\rangle = 0.$$

It follows that T is parallel. $\square$

6.2. Almost abelian Lie groups

Let us recall that a Lie group is called almost abelian if its Lie algebra has a codimension one abelian ideal. We will prove below that any left invariant KY tensor on such a Lie group (equipped with a left invariant metric) has to be parallel, while if there exists a left invariant CKY tensor which is not KY, then the dimension of the Lie group has to be 3.

In order to do so, we will set first some notation. Let G be an almost abelian Lie group with Lie algebra $\mathfrak{g}$, and let us denote by $\mathfrak{u}$ the codimension one abelian ideal in $\mathfrak{g}$. Choosing a unit vector b orthogonal to $\mathfrak{u}$ set $L = \operatorname{ad}_b$, L^* its adjoint transformation, $S = \frac{1}{2}(L + L^*)$ and $A = \frac{1}{2}(L - L^*)$ the self adjoint and skew adjoint components of L. Note that $Ab = Sb = 0$. It follows from [9] that

$$\nabla_b b = 0, \quad \nabla_b u = Au, \quad \nabla_u b = -Su, \quad \nabla_u v = \langle Su, v\rangle b, \tag{25}$$

where $u, v \in \mathfrak{u}$.

Theorem 6.2. *Let $\mathfrak{g}$ be an almost abelian Lie algebra (that is, $\mathfrak{g}$ has a codimension one abelian ideal) equipped with an inner product $\langle\ ,\ \rangle$ admitting a conformal Killing–Yano tensor T, and let θ be the associated 1-form given by (12).*

(i) *If $\theta \neq 0$, then $\dim\mathfrak{g} = 3$ and $\mathfrak{g}$ is isomorphic either to the Heisenberg Lie algebra $\mathfrak{h}_3$ or to $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$, where $\mathfrak{aff}(\mathbb{R})$ denotes the only non abelian 2-dimensional Lie algebra.*
(ii) *If $\theta = 0$, then T is parallel.*

Proof. With notation as above, let us suppose that $\dim\mathfrak{u} = n$. We prove first that $\theta(b) = 0$. Let ω denote the 2-form associated with T, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $\mathfrak{u}$. Then we compute

$$\begin{aligned}
d^*\omega(b) &= -(\nabla_b\omega)(b, b) - \sum_{i=1}^n (\nabla_{e_i}\omega)(e_i, b) \\
&= \sum_{i=1}^n \{\omega(\nabla_{e_i}e_i, b) + \omega(e_i, \nabla_{e_i}b)\} \\
&= \sum_{i=1}^n \{\omega(\langle Se_i, e_i\rangle b, b) - \omega(e_i, Se_i)\} \quad \text{using (25)} \\
&= -\sum_{i=1}^n \langle Te_i, Se_i\rangle \\
&= \operatorname{tr}(TS) = -\operatorname{tr}(ST) \\
&= 0.
\end{aligned}$$

Therefore, using (13), we obtain $\theta(b) = 0$. Let us now compute θ on $\mathfrak{u}$. Setting $x = b$ in (16), we have

$$\begin{aligned}\theta(z) &= \langle (\nabla_b T)b, z\rangle \\ &= \langle \nabla_b Tb, z\rangle + \langle \nabla_b b, Tz\rangle \\ &= \langle ATb, z\rangle \quad \text{using (25)}.\end{aligned}$$

From this and $\theta(b) = 0$, it follows that $\xi = ATb$, where ξ is given as in Corollary 4.2.

(i) Assume $\theta \neq 0$ so that $ATb \neq 0$. Let us take now $x \in \mathfrak{u} \cap \xi^{\perp}$, it follows from (16) with $z = \xi$ that

$$\langle (\nabla_x T)x, \xi\rangle = \|x\|^2\|\xi\|^2.$$

Using that $T\xi = 0$, $\mathfrak{u}$ is abelian and (15) in the left-hand side of the previous equation, we obtain

$$\langle x, Tb\rangle\langle Sx, \xi\rangle = \|x\|^2\|\xi\|^2. \tag{26}$$

As a consequence we have that if $\langle x, Tb\rangle = 0$ then $x = 0$. This shows that $\mathfrak{g}$ is a 3-dimensional Lie algebra spanned by $\{b, Tb, \xi = ATb\}$.

We determine next the Lie brackets. If we take $x = Tb$ in (26), we get $\langle STb, \xi\rangle = \|\xi\|^2$. With this, we prove that ξ is in the center $\mathfrak{z}$ of $\mathfrak{g}$. Indeed, $\langle [\xi, b], \xi\rangle = 0$ since $\operatorname{ad}_\xi : \xi^{\perp} \to \xi^{\perp}$ (Theorem 4.3), and $\langle [\xi, b], b\rangle = 0$ since $\mathfrak{u}$ is an ideal. Finally,

$$\langle [\xi, b], Tb\rangle = -\langle A\xi + S\xi, Tb\rangle = \langle \xi, ATb\rangle - \langle \xi, STb\rangle = \|\xi\|^2 - \|\xi\|^2 = 0.$$

Setting $f_1 := b$, $f_2 := \frac{Tb}{\|Tb\|}$, $f_3 := \frac{\xi}{\|\xi\|}$, it follows that $\{f_1, f_2, f_3\}$ is an orthonormal basis of $\mathfrak{g}$ and the Lie brackets in this basis are given by

$$[f_1, f_2] = \alpha f_2 + 2\|\xi\|^2 f_3, \quad [f_1, f_3] = 0, \quad [f_2, f_3] = 0,$$

for some $\alpha \in \mathbb{R}$. Note that $\alpha = \operatorname{tr}\operatorname{ad}_b$, and therefore, if $\alpha = 0$ then $\mathfrak{g}$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_3$, while if $\alpha \neq 0$ then it is easily seen that $\mathfrak{g}$ is isomorphic to $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$.

(ii) Assume now $\theta = 0$, so that T is a Killing–Yano tensor on $\mathfrak{g}$, i.e., $(\nabla_x T)x = 0$ for all $x \in \mathfrak{g}$. Using (25), we have that this is equivalent to the following conditions:

(C1) $ATb = 0$,
(C2) $\langle Tu, b\rangle Su + \langle Su, u\rangle Tb = 0$ for all $u \in \mathfrak{u}$, and
(C3) $\langle ATu, v\rangle - \langle TAu, v\rangle + \langle TSu, v\rangle = 0$ for all $u, v \in \mathfrak{u}$.

We will consider two cases: (a) $Tb \neq 0$, (b) $Tb = 0$.

(a) $Tb \neq 0$: In (C2) set $u = Tb$. It follows that $\|Tb\|^2 STb = \langle STb, Tb\rangle Tb$ hence Tb is an eigenvector of S. From (C1), $ATb = 0$. Thus, $\mathfrak{v}$, the orthogonal complement of the span of $\{b, Tb\}$ is preserved by S and A. Let $\{u_i\}$ an orthonormal basis of eigenvectors of S on $\mathfrak{v}$. If in (C2) we take $u = u_i$ one obtains $Su_i = 0$, therefore $S|_{\mathfrak{v}} = 0$.

Note that if Tb is an eigenvector of S with eigenvalue 0 then ad_b is skew-symmetric and the metric is flat; therefore T is parallel according to Lemma 6.1. Suppose next that $STb = \delta Tb$ with $\delta \neq 0$, we will prove that T preserves $\mathfrak{v}$ and hence also $\{b, Tb\}$.

- If in (C3) we set $u \in \mathfrak{v}$ and $v = Tb$, then using $S|_{\mathfrak{v}} = 0$ and (C1) we obtain $\langle T^2 b, Au\rangle = 0$.
- If in (C3) we set $u = Tb$ and $v \in \mathfrak{v}$, then using (C1) and the item above, we get $\langle T^2 b, v\rangle = 0$.

Thus, T preserves $\mathfrak{v}$ as asserted. Using this fact together with $S|_{\mathfrak{v}} = 0$ in (C3), one obtains that $[A, T] = 0$.

The matrices of A, S and T with respect to the basis $\{b, Tb/\|Tb\|, u_i\}$ are

$$T = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & T|_{\mathfrak{v}} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A|_{\mathfrak{v}} \end{pmatrix},$$

and

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for some $a, \delta \in \mathbb{R}$.

We show next that T is parallel. It is easy to see that $\nabla_b T = [A, T] = 0$. Since T is Killing–Yano, we only have to check that $(\nabla_{Tb}T)u = 0$ and $(\nabla_u T)v = 0$ for any $u, v \in \mathfrak{v}$. We compute

$$(\nabla_{Tb}T)u = \nabla_{Tb}Tu - T\nabla_{Tb}u = \langle STb, Tu\rangle b - \langle STb, u\rangle Tb = \delta(\langle Tb, Tu\rangle b - \langle Tb, u\rangle Tb) = 0,$$

and

$$(\nabla_u T)v = \langle Su, Tv\rangle - \langle Su, v\rangle Tb = 0.$$

Thus, T is parallel.

(b) $Tb = 0$: Conditions (C1) and (C2) are trivially satisfied. Since $Tb = 0$, we have that $T : \mathfrak{u} \to \mathfrak{u}$, and it follows from (C3) that $[A, T] = -TS$. Hence, TS is skew-symmetric and as a consequence, $TS = ST$. It follows that:

- T preserves the eigenspaces $\mathfrak{g}_\lambda$ of the symmetric operator S;
- $[A, T^n] = -nT^nS$ for all $n \in \mathbb{N}$.

On each eigenspace $\mathfrak{g}_\lambda$ of S with $\lambda \neq 0$, the equation above becomes $[A, T^n] = -\lambda nT^n$. If $T^n \neq 0$ for all $n \in \mathbb{N}$, the operator $\mathrm{ad}_A : \mathrm{End}(\mathfrak{g}_\lambda) \to \mathrm{End}(\mathfrak{g}_\lambda)$ would have infinitely many eigenvalues, which is impossible. Therefore, $T^k = 0$ for some $k \in \mathbb{N}$, so that T is nilpotent on $\mathfrak{g}|_\lambda$. Since T is also skew-symmetric, we have $T|_{\mathfrak{g}_\lambda} = 0$.

On the eigenspace $\mathfrak{g}_0$, we have trivially that $TS = 0$. Thus, we obtain that $TS = 0$ on $\mathfrak{g}$, therefore $[A, T] = 0$ on $\mathfrak{g}$.

Since $(\nabla_b T)u = [A, T]u$ and $(\nabla_u T)v = \langle u, STv\rangle$ (since $Tb = 0$) for any $u, v \in \mathfrak{u}$, we have that T is parallel. □

Remark 10. The existence of CKY 2-forms on 3-dimensional Lie groups was considered in [1], where a complete classification was given.

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