



Characterization of semiglobal stability properties for discrete-time models of non-uniformly sampled nonlinear systems

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ABSTRACT

Discrete-time models of non-uniformly sampled nonlinear systems under zero-order hold relate the next state sample to the current state sample, (constant) input value, and sampling interval. The exact discrete-time model, that is, the discrete-time model whose state matches that of the continuous-time nonlinear system at the sampling instants may be difficult or even impossible to obtain. In this context, one approach to the analysis of stability is based on the use of an approximate discrete-time model and a bound on the mismatch between the exact and approximate models. This approach requires three conceptually different tasks: (i) ensure the stability of the (approximate) discrete-time model, (ii) ensure that the stability of the approximate model carries over to the exact model, (iii) if necessary, bound intersample behaviour. Existing conditions for ensuring the stability of a discrete-time model as per task (i) have some or all of the following drawbacks: are only sufficient but not necessary; do not allow for varying sampling rate; cannot be applied in the presence of state-measurement or actuation errors. In this paper, we overcome these drawbacks by providing characterizations of, i.e. necessary and sufficient conditions for, two stability properties: semiglobal asymptotic stability, robustly with respect to bounded disturbances, and semiglobal input-to-state stability, where the (disturbance) input may successfully represent state-measurement or actuation errors. Our results can be applied when sampling is not necessarily uniform.

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1. Introduction

Most of the existing stability results for non-uniformly sampled systems deal with linear systems [1–3]. Some results applicable to different classes of nonlinear systems were given in [4–7]. In [4], stabilization of homogeneous nonlinear systems with sampled-data inputs is analysed in by means of an emulation approach. In [5,6], multi-rate sampled-data stabilization via immersion and invariance for nonlinear systems in feedback form was developed. [7] presents sufficient conditions for uniform input-to-output and input-to-state stability for closed-loop systems with zero-order hold.

Strategies where sampling is inherently non-uniform and which have application to nonlinear systems are those of event- and self-triggered control. In an event-triggered control strategy, the control action is computed based on the continuous-time system model (with the aid of a Lyapunov function, e.g.) and current state or output measurements, applied to the plant, and held constant until a condition that triggers the control action update becomes true [8,9]. The triggering condition requires continuous

monitoring of some system variables, and thus this type of event-triggered control does not exactly constitute a sampled-data strategy. Other event-triggered strategies that verify the condition only periodically have been developed for linear systems [10,11]. Self-triggered control [9,12,13], in addition to computing the current control action based on the continuous-time model, also computes the time instant at which the next control update will occur requiring only sampled measurements.

Some approaches to stability analysis and control design for nonlinear systems under sampling and hold are based on the use of a discrete-time model for the sampled system. These approaches are referred to as Discrete-Time Design (DTD), or Sampled-Data Design (SDD) if, in addition, inter-sample behaviour is taken into account [14,15]. As opposed to the linear-system case, the differential equations that describe a continuous-time nonlinear system's dynamics may be difficult or impossible to solve in closed form, and hence a discrete-time model exactly matching the state of the continuous-time system at the sampling instants is usually unavailable. If the continuous-time system is input-affine then the exact discrete-time model can be approximated to desired accuracy via the procedure in [16,17]. Thus, DTD or SDD for nonlinear systems are usually based on an *approximate* discrete-time model.

Interesting work on DTD for nonlinear systems under uniform sampling appears in [14,18]. The results of [14,18] are of the following conceptual form: given a specific bound on the mismatch

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between the exact and approximate discrete-time models (which can be known without having to compute the exact model) then some stability property on the approximate closed-loop model will carry over (in a practical sense) to the exact model for all sufficiently small sampling periods. These results have been extended to provide input-to-state and integral-input-to-state stability results [19,20], to observer design [21], and to networked control systems [22]. All of these results are specifically suited to the case when sampling is uniform during operation or, in the case of [22], when a nominal sampling period can be defined. Some extensions to the non-uniform sampling case were given in [23–25] that are also based on an approximate discrete-time model. Specifically, [23] gives preliminary results to ensure the practical asymptotic stability of the exact discrete-time model under non-uniform sampling, [25] gives a sufficient condition for the semiglobal practical input-to-state stability of the exact discrete-time model with respect to state-measurement errors, and [24] shows that a global stability property under uniform sampling, namely (β, \mathbb{R}^n) -stability, is equivalent to the analogous property under non-uniform sampling.

The aforementioned DTD approach requires two conceptually different tasks: (i) ensure the stability of the (approximate) discrete-time model, and (ii) ensure that the stability of the approximate model carries over to the exact model. For the SDD approach, the following task should be added: (iii) bound intersample behaviour. The existing conditions for ensuring the stability of a discrete-time model as per task (i) have some or all of the following drawbacks: are only sufficient but not necessary; do not allow for varying sampling rate; cannot be applied in the presence of state-measurement or actuation errors.

This paper addresses stability analysis for discrete-time models of sampled-data nonlinear systems under the aforementioned DTD approach. We characterize, i.e. give necessary and sufficient conditions for, two stability properties: semiglobal asymptotic stability, robustly with respect to bounded disturbances, and semiglobal input-to-state stability, where the (disturbance) input may successfully represent state-measurement or actuation errors, both specifically suited to non-uniform sampling. In this context, the contribution of the current paper is to overcome all of the drawbacks relating to task (i) and mentioned in the previous paragraph. Our results thus apply to a discrete-time model of a sampled nonlinear system, irrespective of how accurate this model may be. If the discrete-time model is only approximate, then our results can be used in conjunction with the results in [25] in order to conclude about the (practical) stability of the (unknown) exact model, as per task (ii).

The organization of this paper is as follows. This section ends with a brief summary of the notation employed. In Section 2 we state the problem and the required definitions and properties. Our main results are given in Section 3. An illustrative example is provided in Section 4 and concluding remarks are presented in Section 5. The Appendix contains the proofs of some of the presented results.

Notation. $\mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{N}$ and \mathbb{N}_0 denote the sets of real, nonnegative real, natural and nonnegative integer numbers, respectively. We write $\alpha \in \mathcal{K}$ if $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is strictly increasing, continuous and $\alpha(0) = 0$. We write $\alpha \in \mathcal{K}_{\infty}$ if $\alpha \in \mathcal{K}$ and α is unbounded. We write $\beta \in \mathcal{KL}$ if $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}$ for all $t \geq 0$, and $\beta(s, \cdot)$ is strictly decreasing asymptotically to 0 for every s . We denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ by $|x|$. We denote an infinite sequence as $\{T_i\} := \{T_i\}_{i=0}^{\infty}$. For any sequences $\{T_i\} \subset \mathbb{R}_{\geq 0}$ and $\{e_i\} \subset \mathbb{R}^m$, and any $\gamma \in \mathcal{K}$, we take the following conventions: $\sum_{i=0}^{-1} T_i = 0$ and $\gamma(\sup_{0 \leq i \leq -1} |e_i|) = 0$. Given a real number $T > 0$ we denote by $\Phi(T) := \{\{T_i\} : \{T_i\} \text{ is such that } T_i \in (0, T) \text{ for all } i \in \mathbb{N}_0\}$ the set of all sequences of real numbers in the open interval $(0, T)$. For a given sequence we denote the norm $\|\{x_i\}\| := \sup_{i \geq 0} |x_i|$.

2. Preliminaries

2.1. Problem statement

We consider discrete-time systems that arise when modelling non-uniformly sampled continuous-time nonlinear systems of the form

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad (1)$$

under zero-order hold, where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and control vectors respectively. We consider that the sampling instants $t_k, k \in \mathbb{N}_0$, satisfy $t_0 = 0$ and $t_{k+1} = t_k + T_k$, where $\{T_k\}_{k=0}^{\infty}$ is the sequence of corresponding sampling periods. As opposed to the uniform sampling case where $T_k = T$ for all $k \in \mathbb{N}_0$, we consider that the sampling periods may vary; we refer to this situation as Varying Sampling Rate (VSR). In addition, we assume that the current sampling period T_k is known or determined at the current sampling instant t_k . This situation arises when the controller determines the next sampling instant according to a certain control strategy, such as in self-triggered control; we refer to this scheme as controller-driven sampling. Due to zero-order hold, the continuous-time control signal u is piecewise constant such that $u(t) = u(t_k) =: u_k$ for all $t \in [t_k, t_{k+1})$. The class of discrete-time systems that arise when modelling (1) in this scheme is thus of the form

$$x_{k+1} = F(x_k, u_k, T_k). \quad (2)$$

Our results apply to this class of systems irrespective of whether the system model accurately describes the behaviour of some continuous-time system at the sampling instants or not. If the discrete-time model employed were the exact discrete-time model for some continuous-time system, then stability of the model could give some indication on the stability of the continuous-time system. Since regrettably the exact discrete-time model is in general impossible to obtain, then approximate models should be used.

Given that the current sampling period T_k is known or determined at the current sampling instant t_k , then the current control action u_k may depend not only on the current state sample x_k but also on T_k . If, in addition, state-measurement or actuation errors exist $e_k \in \mathbb{R}^q$, where the dimension q depends on the type of error (i.e., $q = n$ for state-measurement additive error or $q = m$ for actuation additive error), then the true control action applied will also be affected by such errors

$$u_k = U(x_k, e_k, T_k). \quad (3)$$

Under (3), the closed-loop model becomes

$$x_{k+1} = F(x_k, U(x_k, e_k, T_k), T_k) =: \bar{F}(x_k, e_k, T_k) \quad (4)$$

which is once again on the form (2). We stress that a control action $u_k = \bar{U}(x_k, e_k)$ is also of the form (3) and hence also covered by our results. Sufficient conditions for some stability properties to carry over from an approximate discrete-time model to the exact model were given in [18] under uniform sampling and in [25] for the controller-driven sampling case here considered. These conditions are based on bounds on the mismatch between the exact and approximate models and can be computed without having to compute the exact model.

We will characterize two stability properties for discrete-time models of the form (4): robust semiglobal stability and semiglobal input-to-state stability. For the sake of clarity of the proofs we will use $d_k \in \mathbb{R}^p$ instead of e_k to represent bounded disturbances that do not destroy asymptotic stability. Given $D > 0$, we define $\mathcal{D} := \{\{d_i\} \subset \mathbb{R}^p : |d_i| \leq D, \forall i \in \mathbb{N}_0\}$, the set of all disturbance sequences whose norm is not greater than D . Thus, for our robust stability results, we will consider a discrete-time model of the form

$$x_{k+1} = \bar{F}(x_k, d_k, T_k), \quad \{d_i\} \in \mathcal{D}. \quad (5)$$

2.2. Stability properties for varying sampling rate

The next definitions are extensions of stability properties in [14,19,26–28]. The first one can be seen as a robust and semiglobal (with respect to initial states) version of (β, \mathbb{R}^n) -stability of [14], suitable for the non-uniform sampling case. The second definition presents the discrete-time global, semiglobal and semiglobal practical versions of the input-to-state stability (ISS) for non-uniform sampling.

Definition 2.1. The system (5) is said to be Robustly Semiglobally (Asymptotically) Stable under Varying Sampling Rate (RSS-VSR) if there exists a function $\beta \in \mathcal{KL}$ such that for every $M \geq 0$ there exists $T^\star = T^\star(M) > 0$ such that the solutions of (5) satisfy

$$|x_k| \leq \beta \left(|x_0|, \sum_{i=0}^{k-1} T_i \right) \quad (6)$$

for all $k \in \mathbb{N}_0$, $\{T_i\} \in \Phi(T^\star)$, $|x_0| \leq M$ and $\{d_i\} \in \mathcal{D}$.

Remark 2.2. Without loss of generality, the function $T^\star(\cdot)$ in Definition 2.1 can be taken nonincreasing.

The RSS-VSR property is semiglobal because the bound T^\star on the sampling periods may depend on how far from the origin the initial conditions may be (as quantified by M). If there exists $\beta \in \mathcal{KL}$ and $T^\star > 0$ such that (6) holds for all $k \in \mathbb{N}_0$, $\{T_i\} \in \Phi(T^\star)$ and $x_0 \in \mathbb{R}^n$, then the system is said to be globally Robustly Stable under VSR (RS-VSR). When disturbances are not present ($\mathcal{D} = \{0\}$, i.e. $D = 0$), the RS-VSR property becomes (β, \mathbb{R}^n) -stability under VSR [24]. If in addition to lack of disturbances, uniform sampling is imposed ($T_k = T$ for all $k \in \mathbb{N}_0$), then RS-VSR becomes (β, \mathbb{R}^n) -stability [14]. In [24], it was shown that existence of $\beta \in \mathcal{KL}$ such that a system is (β, \mathbb{R}^n) -stable is equivalent to existence of $\tilde{\beta} \in \mathcal{KL}$ such that it is $(\tilde{\beta}, \mathbb{R}^n)$ -stable under VSR.

Definition 2.3. The system (4) is said to be

1. Semiglobally ISS-VSR (S-ISS-VSR) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $M \geq 0$ and $E \geq 0$ there exists $T^\star = T^\star(M, E) > 0$ such that the solutions of (4) satisfy

$$|x_k| \leq \beta \left(|x_0|, \sum_{i=0}^{k-1} T_i \right) + \gamma \left(\sup_{0 \leq i \leq k-1} |e_i| \right), \quad (7)$$

for all $k \in \mathbb{N}_0$, $\{T_i\} \in \Phi(T^\star)$, $|x_0| \leq M$ and $\|e_i\| \leq E$.

2. Semiglobally Practically ISS-VSR (SP-ISS-VSR) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for every $M \geq 0$, $E \geq 0$ and $R > 0$ there exists $T^\star = T^\star(M, E, R) > 0$ such that the solutions of (4) satisfy

$$|x_k| \leq \beta \left(|x_0|, \sum_{i=0}^{k-1} T_i \right) + \gamma \left(\sup_{0 \leq i \leq k-1} |e_i| \right) + R, \quad (8)$$

for all $k \in \mathbb{N}_0$, $\{T_i\} \in \Phi(T^\star)$, $|x_0| \leq M$ and $\|e_i\| \leq E$.

Note that S-ISS-VSR \Rightarrow SP-ISS-VSR.

3. Main results

In this section, we present characterizations of the RSS-VSR and S-ISS-VSR properties defined in Section 2.2. In Lemma 3.1, ϵ - δ and Lyapunov-type characterizations are given for the RSS-VSR property. A partial proof of this lemma is given in Appendix. For the complete proof see [29].

Lemma 3.1. The following statements are equivalent:

1. The system (5) is RSS-VSR.
2. For every $M \geq 0$ there exists $T^\star = T^\star(M) > 0$ so that
 - (i) for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ (δ is independent of M) such that the solutions of (5) with $|x_0| \leq \min\{\delta, M\}$, $\{T_i\} \in \Phi(T^\star)$ and $\{d_i\} \in \mathcal{D}$ satisfy $|x_k| \leq \epsilon$ for all $k \in \mathbb{N}_0$,
 - (ii) for all $L \geq 0$, there exists $C = C(M, L) \geq 0$ such that the solutions of (5) with $|x_0| \leq M$, $\{T_i\} \in \Phi(T^\star)$ and $\{d_i\} \in \mathcal{D}$ satisfy $|x_k| \leq C$, for all $k \in \mathbb{N}_0$ for which $\sum_{i=0}^{k-1} T_i \leq L$, and
 - (iii) for all $\epsilon > 0$, there exists $\mathcal{T} = \mathcal{T}(M, \epsilon) \geq 0$ such that the solutions of (5) with $|x_0| \leq M$, $\{T_i\} \in \Phi(T^\star)$ and $\{d_i\} \in \mathcal{D}$ satisfy $|x_k| \leq \epsilon$, for all $k \in \mathbb{N}_0$ for which $\sum_{i=0}^{k-1} T_i \geq \mathcal{T}$.
3. There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for every $M \geq 0$ there exist $T^\star = T^\star(M) > 0$ and $V_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

$$\alpha_1(|x|) \leq V_M(x), \quad \forall x \in \mathbb{R}^n, \quad (9a)$$

$$V_M(x) \leq \alpha_2(|x|), \quad \forall |x| \leq M, \quad (9b)$$
 and

$$V_M(\bar{F}(x, d, T)) - V_M(x) \leq -T\alpha_3(|x|) \quad (10)$$
 for all $|x| \leq M$, $|d| \leq D$ and $T \in (0, T^\star)$.

The ϵ - δ characterization in item 2 of Lemma 3.1 contains all the ingredients of an ϵ - δ characterization of uniform global asymptotic stability for a continuous-time system [30] but in semiglobal form and for a discrete-time model. These ingredients are: semiglobal uniform stability in 2(i), semiglobal uniform boundedness in 2(ii), and semiglobal uniform attractivity in 2(iii). The Lyapunov conditions in item 3. have several differences with respect to the Lyapunov-type conditions ensuring (β, \mathbb{R}^n) -stability [14] or (β, \mathbb{R}^n) -stability under VSR [24]. First, note that the Lyapunov-type function V_M may be not the same for each upper bound M on the norm of the state. Second, the functions V_M may take infinite values and it is not required that they satisfy any Lipschitz-type condition. Third, the upper bound given by $\alpha_2 \in \mathcal{K}_\infty$ should only hold for states whose norm is upper bounded by M .

Theorem 3.2 gives necessary and sufficient conditions for a discrete-time model of the form (4) to be S-ISS-VSR. These conditions consist of specific boundedness and continuity requirements and a Lyapunov-type condition. The characterizations given in Lemma 3.1 are used in the proof of Theorem 3.2.

Theorem 3.2. The following statements are equivalent:

1. The system (4) is S-ISS-VSR.
2.
 - (i) There exists $\hat{T} > 0$ so that $\bar{F}(0, 0, T) = 0$ for all $T \in (0, \hat{T})$.
 - (ii) There exists $\hat{T} > 0$ such that for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|\bar{F}(x, e, T)| < \epsilon$ whenever $|x| \leq \delta$, $|e| \leq \delta$ and $T \in (0, \hat{T})$.
 - (iii) For every $M \geq 0$ and $E \geq 0$, there exist $C = C(M, E) > 0$ and $\hat{T} = \hat{T}(M, E) > 0$, with $C(\cdot, \cdot)$ nondecreasing in each variable and $\hat{T}(\cdot, \cdot)$ nonincreasing in each variable, such that $|\bar{F}(x, e, T)| \leq C$ for all $|x| \leq M$, $|e| \leq E$ and $T \in (0, \hat{T})$.
 - (iv) There exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\rho \in \mathcal{K}$ such that for every $M \geq 0$ and $E \geq 0$ there exist $\hat{T} = \hat{T}(M, E) > 0$ and $V = V_{M,E} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

$$\alpha_1(|x|) \leq V(x), \quad \forall x \in \mathbb{R}^n, \quad (11a)$$

$$V(x) \leq \alpha_2(|x|), \quad \forall |x| \leq M, \quad (11b)$$

¹ As explained under "Notation" in Section 1, for $k = 0$ we interpret $\sum_{i=0}^{-1} T_i = 0$ and $\gamma(\sup_{0 \leq i \leq -1} |e_i|) = 0$.

and

$$V(\bar{F}(x, e, T)) - V(x) \leq -T\alpha_3(|x|) \quad (12)$$

for all $\rho(|e|) \leq |x| \leq M$, $|e| \leq E$ and $T \in (0, \tilde{T})$.

Proof of Theorem 3.2. (1 \Rightarrow 2) Let $\beta_0 \in \mathcal{KL}$, $\gamma_0 \in \mathcal{K}_\infty$ and $T^*(\cdot, \cdot)$ characterize the S-ISS-VSR property.

(1) \Rightarrow 2(i) Define $\hat{T} := T^*(0, 0)$. From (4) and (7), we have

$$|\bar{F}(0, 0, T)| \leq \beta_0(0, T) + \gamma_0(0) = 0$$

for all $T \in (0, \hat{T})$.

(1) \Rightarrow 2(ii) Define $\hat{\beta}, \eta \in \mathcal{K}_\infty$ via $\hat{\beta}(s) := \beta_0(s, 0) + s$ and $\eta(s) := \min\{\hat{\beta}^{-1}(s/2), \gamma_0^{-1}(s/2)\}$. Define $\hat{T} := T^*(\eta(1), \eta(1))$. Let $\epsilon > 0$. Choose $\delta = \eta(\min\{\epsilon, 1\}) > 0$. Note that $T^*(\delta, \delta) \geq \hat{T}$ because $\delta \leq \eta(1)$ and T^* is nonincreasing in each variable. Then, using (4) and (7), it follows that for all $|x| \leq \delta$, $|e| \leq \delta$ and $T \in (0, \hat{T})$ we have

$$|\bar{F}(x, e, T)| \leq \beta_0(\delta, T) + \gamma_0(\delta) < \hat{\beta}(\delta) + \gamma_0(\delta) \leq \epsilon. \quad (13)$$

(1) \Rightarrow 2(iii) Let $\tilde{T} = T^*$ and $C(M, E) = \beta_0(M, 0) + \gamma_0(E)$. Then, \tilde{T} is nonincreasing in each variable and C is increasing (and hence nondecreasing) in each variable. Let $M, E \geq 0$. Then, from (4) and (7), for all $|x| \leq M$, $|e| \leq E$ and $T \in (0, \tilde{T}(M, E))$ we have that $|\bar{F}(x, e, T)| \leq \beta_0(M, 0) + \gamma_0(E) = C(M, E)$.

(1) \Rightarrow 2(iv) Define $\beta(s, t) := 2\beta_0(s, t)$ and $\gamma(s) := 2\gamma_0(s)$. Define $\alpha \in \mathcal{K}_\infty$ via $\alpha(s) := \beta(s, 0)$ and $\sigma \in \mathcal{K}_\infty$ via $\sigma(s) := \gamma^{-1}(\frac{1}{2}\alpha^{-1}(s))$. Consider the following system:

$$x_{k+1} = \bar{F}(x_k, \sigma(|x_k|)d_k, T_k), \quad \|\{d_i\}\| \leq 1, \quad (14)$$

with $d_k \in \mathbb{R}^q$ for all $k \in \mathbb{N}_0$.

Claim 1. For every $M \geq 0$ there exists $\bar{T} = \bar{T}(M) > 0$, with $\bar{T}(\cdot)$ nonincreasing, such that the solutions of (14) satisfy

$$|x_k| \leq \max \left\{ \beta \left(|x_0|, \sum_{i=0}^k T_i \right), \frac{1}{2} |x_0| \right\} \leq \alpha(|x_0|) \quad (15)$$

for all $k \in \mathbb{N}_0$, whenever $|x_0| \leq M$ and $\{T_i\} \in \Phi(\bar{T})$.

Proof of Claim 1. Given $M \geq 0$, take $\bar{T}(M) = T^*(M, \gamma^{-1}(M/2)) > 0$. Note that \bar{T} is nonincreasing because $\gamma^{-1} \in \mathcal{K}_\infty$ and T^* is nonincreasing in each variable. We establish the result by induction. For $k = 0$, we have $|x_0| \leq \beta_0(|x_0|, 0) = \alpha(|x_0|)$. Suppose that $|x_i| \leq \alpha(|x_0|)$ for all $0 \leq i \leq k$. Then, $|\sigma(|x_i|)d_i| \leq \sigma(|x_i|) \leq \sigma(\alpha(|x_0|)) = \gamma^{-1}(|x_0|/2) \leq \gamma^{-1}(M/2)$ for all $0 \leq i \leq k$. Then, for all $\{T_i\} \in \Phi(\bar{T}(M))$, we have

$$|x_{k+1}| \leq \beta_0 \left(|x_0|, \sum_{i=0}^k T_i \right) + \gamma_0 \left(\sup_{0 \leq i \leq k} |\sigma(|x_i|)d_i| \right) \quad (16)$$

$$\leq \max \left\{ \beta \left(|x_0|, \sum_{i=0}^k T_i \right), \gamma \left(\sup_{0 \leq i \leq k} |\sigma(|x_i|)d_i| \right) \right\}$$

$$\leq \max \left\{ \beta \left(|x_0|, \sum_{i=0}^k T_i \right), \frac{1}{2} |x_0| \right\} \leq \alpha(|x_0|), \quad (17)$$

where in the last inequality we have used the fact that $2s \leq 2\beta_0(s, 0) = \alpha(s)$. By induction, then $|x_k| \leq \alpha(|x_0|)$ for all $k \in \mathbb{N}_0$. \square

We next show that (14) is RSS-VSR by means of Lemma 3.1. Given $M \geq 0$, take $T^*(M) = \bar{T}(M)$.

Condition 2(i) of Lemma 3.1: Let $\epsilon > 0$ and take $\delta = \alpha^{-1}(\epsilon)$. Then, if $|x_0| \leq \min\{\delta, M\}$ and $\{T_i\} \in \Phi(T^*(M))$, by Claim 1 it follows that $|x_k| \leq \alpha(|x_0|) \leq \epsilon$ for all $k \in \mathbb{N}_0$.

Condition 2(ii) of Lemma 3.1: Define $C(M, L) = \alpha(M)$. Then, if $|x_0| \leq M$ and $\{T_i\} \in \Phi(T^*(M))$, by Claim 1 it follows that $|x_k| \leq \alpha(|x_0|) \leq C$ for all $k \in \mathbb{N}_0$.

Condition 2(iii) of Lemma 3.1: For every $j \in \mathbb{N}_0$, define $M_j := \frac{M}{2^j}$, $t_j = t_j(M) > 0$ via $\beta(M_j, t_j) = \frac{1}{2}M_j$ and $\tau_j = \tau_j(M)$ via $\tau_j(M) := jT^*(M) + \sum_{i=0}^j t_i(M)$.

Claim 2. Consider $|x_0| \leq M$ and $\{T_i\} \in \Phi(T^*(M))$. For all $j, k \in \mathbb{N}_0$ for which $\sum_{i=0}^{k-1} T_i \geq \tau_j$, it happens that

$$|x_k| \leq M_{j+1}.$$

Proof of Claim 2. By induction on j . If $\sum_{i=0}^{k-1} T_i \geq \tau_0 = t_0$, then from (15) in Claim 1 and since $M_0 = M$, we have that

$$|x_k| \leq \max\{\beta(M_0, t_0), M_0/2\} = M_1.$$

Hence, our induction hypothesis holds for $j = 0$. Next, suppose that for some $j \in \mathbb{N}_0$ and for all $k \in \mathbb{N}_0$ for which $\sum_{i=0}^{k-1} T_i \geq \tau_j$ it happens that $|x_k| \leq M_{j+1}$. Let $k^* = \min\{k \in \mathbb{N}_0 : \sum_{i=0}^{k-1} T_i \geq \tau_j\}$. Then, $\sum_{i=0}^{k^*-1} T_i < \tau_j + T^*$ and $|x_{k^*}| \leq M_{j+1} \leq M_0 = M$. If $\sum_{i=0}^{k-1} T_i \geq \tau_{j+1} = \tau_j + T^* + t_{j+1}$, then necessarily $\sum_{i=k^*}^{k-1} T_i \geq t_{j+1}$. Note that $\Phi(T^*(M_0)) \subset \Phi(T^*(M_j))$ for all $j \in \mathbb{N}_0$. By Claim 1 and time invariance, it follows that, for all $\{T_i\} \in \Phi(T^*(M_0))$ and all k for which $\sum_{i=0}^{k-1} T_i \geq \tau_{j+1}$, then

$$\begin{aligned} |x_k| &\leq \max \left\{ \beta \left(|x_{k^*}|, \sum_{i=k^*}^{k-1} T_i \right), \frac{1}{2} |x_{k^*}| \right\} \\ &\leq \max \left\{ \beta(M_{j+1}, t_{j+1}), \frac{1}{2} M_{j+1} \right\} = M_{j+2}. \end{aligned}$$

Therefore, our induction hypothesis holds for $j + 1$. \square

Given $\epsilon > 0$ define $p = p(M, \epsilon) \in \mathbb{N}$ and $\mathcal{T}(M, \epsilon)$ as $p(M, \epsilon) := \min\{j \in \mathbb{N}_0 : M_j < 2\epsilon\}$ and $\tau(M, \epsilon) := \tau_p(M)$. By Claim 2, it follows that for all $|x_0| \leq M$, all $\{T_i\} \in \Phi(T^*(M))$ and all $k \in \mathbb{N}_0$ for which $\sum_{i=0}^{k-1} T_i \geq \mathcal{T}(M, \epsilon)$, then $|x_k| \leq M_{p+1} < \epsilon$. Therefore, by Lemma 3.1, the system (14) is RSS-VSR and there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that for every $M \geq 0$ there exist $T^* = T^*(M) > 0$ and $V_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that (9) holds. Consider $E \geq 0$ given and define $\bar{T}(M, E) := T^*(M)$, then (11) holds. Also

$$V_M(\bar{F}(x, \sigma(|x|)d, T)) - V_M(x) \leq -T\alpha_3(|x|), \quad (18)$$

holds for all $|x| \leq M$, all $|d| \leq 1$ and all $T \in (0, \bar{T})$. Select $\rho(s) := \sigma^{-1}(s)$. Then, for all $|e| \leq E$ such that $\rho(|e|) \leq |x|$ we have $|e| \leq \sigma(|x|)$. Therefore all e such that $\rho(|e|) \leq |x|$ can always be written as $e = \sigma(|x|)d$ for some $d \in \mathbb{R}^q$ with $|d| \leq 1$. Then, from (18), we have that (12) holds.

(2 \Rightarrow 1) We aim to prove that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $M_0 \geq 0, E_0 \geq 0$ there exists $T^*(M_0, E_0) > 0$ such that the solutions of (4) satisfy

$$|x_k| \leq \beta \left(|x_0|, \sum_{i=0}^{k-1} T_i \right) + \gamma \left(\sup_{0 \leq i \leq k-1} |e_i| \right) \quad (19)$$

for all $k \in \mathbb{N}_0$, all $\{T_i\} \in \Phi(T^*)$, all $|x_0| \leq M_0$, and all $\| \{e_i\} \| \leq E_0$. Consider $\rho \in \mathcal{K}$ from 2(iv). Define, $\forall s \geq 0$,

$$\mathcal{X}_1(s) := \{x \in \mathbb{R}^n : |x| \leq \rho(s)\} \quad (20)$$

$$\mathcal{E}(s) := \{e \in \mathbb{R}^q : |e| \leq s\} \quad (21)$$

$$\bar{T}(s) := \min\{\hat{T}, \hat{T}, \check{T}(\rho(s), s)\} \quad (22)$$

$$S(s) := \mathcal{X}_1(s) \times \mathcal{E}(s) \times (0, \bar{T}(s)) \quad (23)$$

$$\sigma(s) := \sup_{(x,e,T) \in S(s)} |\bar{F}(x, e, T)|. \quad (24)$$

Claim 3. *There exists $\zeta \in \mathcal{K}_\infty$ such that $\zeta \geq \sigma$.*

Proof of Claim 3. From (20)–(21), we have $\mathcal{X}_1(0) = \{0\}$ and $\mathcal{E}(0) = \{0\}$. From 2(i)–2(iii), then $\tilde{T}(0) > 0$ and $\sigma(0) = 0$. We next prove that σ is right-continuous at zero. Let $\epsilon > 0$ and take $\delta = \delta(\epsilon)$ according to 2(ii). Define $\hat{\delta} := \min\{\delta, \rho^{-1}(\delta)\}$ (if $\delta \notin \text{dom } \rho^{-1}$, just take $\hat{\delta} = \delta$). Then for all $x \in \mathcal{X}_1(\hat{\delta})$ and $e \in \mathcal{E}(\hat{\delta})$ the inequalities $|x| \leq \delta$ and $|e| \leq \delta$ hold. Consequently, by 2(ii), we have $\sigma(s) \leq \epsilon$ for all $0 \leq s \leq \hat{\delta}$. This shows that $\lim_{s \rightarrow 0^+} \sigma(s) = \sigma(0) = 0$.

From 2(iii), it follows that $|\tilde{F}(x, e, T)| \leq C(M, E)$ for all $|x| \leq M$, $|e| \leq E$ and $T \in (0, \tilde{T}(M, E))$. From (20)–(24) and the fact that $C(\cdot, \cdot)$ is nondecreasing in each variable, it follows that $\sigma(s) \leq C(\rho(s), s)$ for all $s \geq 0$. Then, we have $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $\sigma(0) = 0$, σ is right-continuous at zero and bounded by a nondecreasing function. By [31, Lemma 2.5], there exists a function $\zeta \in \mathcal{K}_\infty$ such that $\zeta \geq \sigma$. \square

Define $\eta \in \mathcal{K}_\infty$ via

$$\eta(s) := \max\{\zeta(s), \rho(s)\} \quad \forall s \geq 0. \quad (25)$$

Consider $M_0 \geq 0$ and $E_0 \geq 0$ given and $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ from 2(iv). Select $E := E_0$ and

$$M := \alpha_1^{-1} \circ \alpha_2(\max\{M_0, \eta(E_0)\}). \quad (26)$$

Let 2(iv) generate $\tilde{T} = \tilde{T}(M, E) > 0$ and $V_{M,E} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that (11) and (12) hold. Note that $M \geq \max\{M_0, \eta(E_0)\}$ and that $\tilde{T}(M, E) \leq \tilde{T}(M_0, E_0)$ because \tilde{T} is nonincreasing in each variable. Define $T^* = \min\{\tilde{T}, \tilde{T}(E)\}$ and

$$\mathcal{X}_2(s) := \{x : V_{M,E}(x) \leq \alpha_2(\eta(s))\}. \quad (27)$$

Consider that $0 \leq s \leq \eta^{-1}(M)$. Let $x \in \mathcal{X}_1(s)$, by (20) and (25), we have $|x| \leq \rho(s) \leq \eta(s) \leq M$ for all $0 \leq s \leq \eta^{-1}(M)$. Then, by (11b), $V_{M,E}(x) \leq \alpha_2(|x|) \leq \alpha_2(\eta(s))$ for all $0 \leq s \leq \eta^{-1}(M)$. Therefore, $\mathcal{X}_1(s) \subseteq \mathcal{X}_2(s)$ for all $0 \leq s \leq \eta^{-1}(M)$. Let $x_k := x(k, x_0, \{e_i\}, \{T_i\})$ denote the solution to (4) corresponding to $|x_0| \leq M_0$, $\|\{e_i\}\| \leq E_0$ and $\{T_i\} \in \Phi(T^*)$. From (11b) if x_k satisfies $|x_k| \leq M$ we have $\alpha_2^{-1}(V_{M,E}(x_k)) \leq |x_k|$; using this in (12) then

$$\begin{aligned} V_{M,E}(x_{k+1}) - V_{M,E}(x_k) &\leq -T_k \alpha_3(|x_k|) \leq -T_k \alpha(V_{M,E}(x_k)) \\ \text{if } \rho(|e_k|) &\leq |x_k| \leq M \end{aligned} \quad (28)$$

where $\alpha := \alpha_3 \circ \alpha_2^{-1}$.

Claim 4. *If $|x_0| \leq M_0$ then $|x_k| \leq M$ for all $k \in \mathbb{N}_0$.*

Proof of Claim 4. See [29]. \square

Claim 5. *Consider $\|\{e_i\}\| \leq E_0$. If $x_\ell \in \mathcal{X}_2(\|\{e_i\}\|)$ for some $\ell \in \mathbb{N}_0$ then x_k remains in $\mathcal{X}_2(\|\{e_i\}\|)$ for all $k \geq \ell$.*

Proof of Claim 5. See [29]. \square

Let $|x_0| \leq M_0$ and $t_k = \sum_{i=0}^{k-1} T_i$ for every $k \in \mathbb{N}_0$. Consider the function

$$y(t) := V_{M,E}(x_k) + \frac{t - t_k}{T_k} [V_{M,E}(x_{k+1}) - V_{M,E}(x_k)] \quad \text{if } t \in [t_k, t_{k+1}), \quad (29)$$

which depends on the initial condition x_0 , on the sampling period sequence $\{T_i\}$, on the disturbance sequence $\{e_i\}$ and on the given constants M, E (through the fact that V depends on the latter constants) and satisfies $y(0) = V_{M,E}(x_0) \geq 0$. From (29) we have that

$$\dot{y}(t) = \frac{V(x_{k+1}) - V(x_k)}{T_k} \quad \forall t \in (t_k, t_{k+1}), \quad \forall k \in \mathbb{N}_0 \quad (30)$$

and

$$y(t) \leq V_{M,E}(x_k), \quad \forall t \in [t_k, t_{k+1}). \quad (31)$$

By Claim 4 and (27), we have that (28) holds for all $x_k \notin \mathcal{X}_2(\|\{e_i\}\|)$ for all $k \in \mathbb{N}_0$. Using (28) and (30), for all $x_k \notin \mathcal{X}_2(\|\{e_i\}\|)$, we have

$$\dot{y}(t) \leq -\alpha(V_{M,E}(x_k)) \leq -\alpha(y(t)). \quad (32)$$

Hence (32) holds for almost all $t \in [0, t_{k^*})$ with $t_{k^*} = \inf\{t_k : x_k \in \mathcal{X}_2(\|\{e_i\}\|)\}$. Note that the function $\alpha = \alpha_3 \circ \alpha_2^{-1}$ does not depend on any of the following quantities: $x_0, \{T_i\}, \{e_i\}, M_0$ or E_0 . Using Lemma 4.4 of [32], there exists $\beta_1 \in \mathcal{K}\mathcal{L}$ such that, for all $t \in [0, t_{k^*})$ we have

$$y(t) \leq \beta_1(y(0), t). \quad (33)$$

From (29), $y(t_k) = V_{M,E}(x_k)$ for all $k \in \mathbb{N}_0$. Evaluating (33) at $t = t_k$, then

$$V_{M,E}(x_k) \leq \beta_1 \left(V_{M,E}(x_0), \sum_{i=0}^{k-1} T_i \right), \quad k = 0, 1, \dots, k^* - 1. \quad (34)$$

From Claim 5 and (27) if $x_k \in \mathcal{X}_2(\|\{e_i\}\|)$ then $V_{M,E}(x_k) \leq \alpha_2 \circ \eta(\|\{e_i\}\|)$. Combining the latter with (34), then

$$V_{M,E}(x_k) \leq \beta_1 \left(V_{M,E}(x_0), \sum_{i=0}^{k-1} T_i \right) + \alpha_2(\eta(\|\{e_i\}\|)), \quad \forall k \in \mathbb{N}_0.$$

Define $\beta \in \mathcal{K}\mathcal{L}$ via $\beta(s, \tau) := \alpha_1^{-1}(2\beta_1(\alpha_2(s), \tau))$, and $\gamma \in \mathcal{K}_\infty$ via $\gamma(s) := \alpha_1^{-1}(2\alpha_2(\eta(s)))$. Using the fact that $\chi(a+b) \leq \chi(2a) + \chi(2b)$ for every $\chi \in \mathcal{K}$ and (11) it follows that

$$|x_k| \leq \beta \left(|x_0|, \sum_{i=0}^{k-1} T_i \right) + \gamma(\|\{e_i\}\|) \quad (35)$$

for all $|x_0| \leq M_0$, all $\|\{e_i\}\| \leq E_0$ and $\{T_i\} \in \Phi(T^*)$. We have thus established that (4) is S-ISS-VSR. \blacksquare

Theorem 3.2 shows that there is no loss of generality in the search of a Lyapunov function for a S-ISS-VSR discrete-time model since its existence is a necessary condition. The fact that S-ISS-VSR implies SP-ISS-VSR then shows that Theorem 3.2 also provides sufficient, although not necessary, conditions for SP-ISS-VSR. In [25, Theorem 3.2] we provided checkable sufficient conditions for a discrete-time model of the form (2) and (4) to be SP-ISS-VSR. The conditions in items (i), (ii) and (iii) for [25, Theorem 3.2] and Theorem 3.2 are identical. The main difference between these theorems reside in the Lyapunov-type condition: the quantity $R > 0$ that defines the practical nature of the SP-ISS-VSR property does not exist here and the Lyapunov function of the current theorem is only upper bounded in a compact set defined by $M \geq 0$. The existence of necessary and sufficient conditions of the kind of Theorem 3.2 for the SP-ISS-VSR property remains an open problem.

4. Example

Consider the Euler (approximate) discrete-time model of the Example A of [25]:

$$x_{k+1} = x_k + T_k(x_k^3 + u_k) =: F(x_k, u_k, T_k). \quad (36)$$

This open-loop Euler model was fed back with the control law $u_k = U(\hat{x}_k, T_k) = -\hat{x}_k - 3\hat{x}_k^3$ and additive state-measurement errors e_k were considered, so that $\hat{x}_k = x_k + e_k$. The resulting approximate closed-loop model $\tilde{F}(x, e, T) = F(x, U(x + e, T), T)$ is

$$\tilde{F}(x, e, T) = x - T[2x^3 + 9ex^2 + (9e^2 + 1)x + 3e^3 + e]. \quad (37)$$

In Example A of [25] we established that (37) is SP-ISS-VSR with respect to input e . We will prove that (37) is not only SP-ISS-VSR but also S-ISS-VSR. We make use of Theorem 3.2. The continuity and boundedness assumptions 2(i), 2(ii) and 2(iii) of Theorem 3.2 are easy to verify for (37). To prove assumption 2(iv)

define $\alpha_1, \alpha_2, \alpha_3, \rho \in \mathcal{K}_\infty$ via $\alpha_1(s) = \alpha_2(s) = s^2$, $\alpha_3(s) = 3s^4 + s^2$ and $\rho(s) = s/K$ with $K > 0$ to be selected. Let $M \geq 0$ and $E \geq 0$ be given and define $V(x) = x^2$. Then (11) is satisfied. We have

$$V(\bar{F}(x, e, T)) - V(x) = [h(x, e) + g(x, e)T]T, \quad (38)$$

$$h(x, e) = -2x[2x^3 + 9ex^2 + (9e^2 + 1)x + (3e^3 + e)], \quad (39)$$

$$g(x, e) = [2x^3 + 9ex^2 + (9e^2 + 1)x + (3e^3 + e)]^2. \quad (40)$$

Expanding $g(x, e)$, taking absolute values on sign indefinite terms and noting that whenever $\rho(|e|) \leq |x|$ we have $|e| \leq K|x|$ we can bound $g(x, e)$ as

$$g(x, e) \leq a(K)x^6 + b(K)x^4 + c(K)x^2, \quad \text{if } |e| \leq K|x|,$$

where $a(K) = 9K^6 + 135K^4 + 174K^3 + 117K^2 + 90K + 4$, $b(K) = 6K^4 + 24K^3 + 36K^2 + 22K + 4$ and $c(K) = K^2 + 2K + 1$. Expanding $h(x, e)$, taking absolute values on sign indefinite terms and bounding $|e| \leq K|x|$, it follows that

$$\begin{aligned} h(x, e) &\leq -4x^4 - 2x^2 + 18Kx^4 + (6K^3|x|^3 + 2K|x|)|x| \\ &\leq -4x^4 - 2x^2 + d(K)x^4 + 2Kx^2 \end{aligned}$$

where $d(K) = (6K^2 + 18)K$. Select $K = 0.025$ and $\tilde{T} = \min\left\{\frac{1}{2b(K)}, \frac{1}{2(a(K)M^4 + c(K))}\right\}$. Then, for all $\frac{|e|}{K} \leq |x| \leq M$, we can bound (38) as

$$\begin{aligned} h(x, e) + g(x, e)T &\leq -4x^4 - 2x^2 + d(K)x^4 + 2Kx^2 \\ &\quad + (a(K)x^6 + b(K)x^4 + c(K)x^2)T \\ &= -3x^4 - x^2 \\ &\quad + x^2((b(K)T + d(K) - 1)x^2 + a(K)x^4T + c(K)T + 2K - 1) \\ &\leq -\alpha_3(|x|) \\ &\quad + x^2((b(K)\tilde{T} + d(K) - 1)x^2 + (a(K)M^4 + c(K))\tilde{T} + 2K - 1) \\ &\leq -\alpha_3(|x|). \end{aligned}$$

The last inequality holds because, for the chosen values of K and \tilde{T} , the expression between parentheses is less than zero. Thus, assumption 2(iv) of Theorem 3.2 is satisfied and the system (37) is S-ISS-VSR.

5. Conclusions

We have given necessary and sufficient conditions for two stability properties especially suited to discrete-time models of nonlinear systems under non-uniform sampling. We have given ϵ - δ and Lyapunov-based characterizations of robust semiglobal stability (RSS-VSR) and a Lyapunov-type characterization of semiglobal input-to-state stability (S-ISS-VSR), both under non-uniform sampling. We have illustrated the application of the results on a numerical example for an approximate closed-loop discrete-time model with additive state measurement disturbances. The provided results can be combined with previous results to ensure stability properties for closed-loop systems whose control law has been designed based on an approximate model.

Appendix. Sketch of the proof of Lemma 3.1

Due to space limitations, we prove $2. \Rightarrow 1. \Rightarrow 3.$ See [29] for the full proof.

($2. \Rightarrow 1.$) Let $M \geq 0$ and $T^*(M) > 0$ be such that 2(i)–2(iii) hold. Let $\bar{\delta}(\epsilon) := \sup\{\delta : \delta \text{ corresponds to } \epsilon \text{ as in 2(i)}\}$, the supremum of all applicable δ . Then $\bar{\delta}(\epsilon) \leq \epsilon$ for all $\epsilon > 0$, and $\bar{\delta} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is positive and non-decreasing. Let $0 < c < 1$. Then, there exists $\alpha \in \mathcal{K}$ such that $\alpha(s) \leq c\bar{\delta}(s)$. Define $c_1 = \lim_{s \rightarrow \infty} \alpha(s)$, then $\alpha^{-1} : [0, c_1] \rightarrow \mathbb{R}_{\geq 0}$. From 2(i), we know that $|x_0| \leq \alpha(\epsilon) \leq \min\{c\bar{\delta}(\epsilon), M\} < \bar{\delta}(\epsilon) \Rightarrow |x_k| \leq \epsilon$, for all $k \in \mathbb{N}_0$, all

$\{T_i\} \in \Phi(T^*(M))$ and all $\{d_i\} \in \mathcal{D}$. Choosing $\epsilon = \alpha^{-1}(|x_0|)$ when $|x_0| < c_1$, it follows that whenever $|x_0| < c_1$ and $|x_0| \leq M$, $k \in \mathbb{N}_0$, $\{T_i\} \in \Phi(T^*(M))$ and $\{d_i\} \in \mathcal{D}$, then

$$|x_k| \leq \alpha^{-1}(|x_0|). \quad (A.1)$$

Next, define

$$\underline{C}(M, L) := \inf\{C : C \text{ corresponds to } M, L \text{ as in 2(ii)}\},$$

$$\underline{T}(M, \epsilon) := \inf\{T : T \text{ corresponds to } M, \epsilon \text{ as in 2(iii)}\},$$

the infima over all applicable C and T from conditions 2(ii) and 2(iii), respectively. Then, \underline{C} is nonnegative and nondecreasing in each variable, and $\underline{T}(M, \epsilon)$ is nonnegative, nondecreasing in M for every fixed $\epsilon > 0$, and nonincreasing in ϵ for every fixed $M > 0$. Given $s > 0$, consider $\underline{T}(s, 1/s)$ and $\underline{C}(s, \underline{T}(s, 1/s))$. If x_k is a solution to (5) corresponding to an initial condition satisfying $|x_0| \leq s$, $\{T_i\} \in \Phi(T^*(s))$ and $\{d_i\} \in \mathcal{D}$, then

$$|x_k| \leq \begin{cases} \underline{C}(s, \underline{T}(s, 1/s)), & \text{whenever } \sum_{i=0}^{k-1} T_i < \underline{T}(s, 1/s), \\ \frac{1}{s}, & \text{otherwise.} \end{cases} \quad (A.2)$$

Define $\hat{C} : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ via $\hat{C}(s) := \max\{\underline{C}(s, \underline{T}(s, 1/s)), 1/s\}$. By the monotonicity properties of \underline{C} , \underline{T} and $1/s$, there exists $p > 0$ such that \hat{C} decreases over $(0, p)$ and is nondecreasing over (p, ∞) . Therefore there exists $\bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\bar{\alpha}(s) \geq \begin{cases} \alpha^{-1}(s), & \text{if } 0 \leq s < \frac{c_1}{2}, \\ \hat{C}(s), & \text{if } \frac{c_1}{2} \leq s. \end{cases} \quad (A.3)$$

Then, if $\{T_i\} \in \Phi(T^*(|x_0|))$ and $\{d_i\} \in \mathcal{D}$, by (A.1)–(A.3), we have $|x_k| \leq \bar{\alpha}(|x_0|)$ for all $k \in \mathbb{N}_0$. Consequently, if $\{T_i\} \in \Phi(T^*(M))$ and $\{d_i\} \in \mathcal{D}$

$$\bar{\alpha}(M) \leq \epsilon \text{ and } |x_0| \leq M \Rightarrow |x_k| \leq \epsilon, \quad \forall k \in \mathbb{N}_0, \quad (A.4)$$

and, by 2(iii),

$$\sum_{i=0}^{k-1} T_i > \underline{T}(M, \epsilon) \text{ and } |x_0| \leq M \Rightarrow |x_k| \leq \epsilon. \quad (A.5)$$

Define $\tilde{\beta} : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ via

$$\tilde{\beta}(r, t) := \inf\{\epsilon : \underline{T}(r, \epsilon) \leq t\}. \quad (A.6)$$

Claim 6. There exists $\beta \in \mathcal{KL}$ such that $\beta \geq \tilde{\beta}$.

Proof of Claim 6. See [29]. \square

From (A.5) and (A.6) then if $|x_0| \leq M$ we have, for all $k \in \mathbb{N}_0$ such that $\sum_{i=0}^{k-1} T_i \geq t$ with $\{T_i\} \in \Phi(T^*(M))$ and $\{d_i\} \in \mathcal{D}$, that $|x_k| \leq \tilde{\beta}(M, t) \leq \beta(M, t)$. Consequently, $|x_k| \leq \beta(|x_0|, \sum_{i=0}^{k-1} T_i)$, for all $k \in \mathbb{N}_0$, all $\{T_i\} \in \Phi(T^*(M))$, all $|x_0| \leq M$ and all $\{d_i\} \in \mathcal{D}$, which establishes that the system (5) is RSS-VSR.

($1. \Rightarrow 3.$) Let $\beta \in \mathcal{KL}$ and $T^*(\cdot)$ be given by the RSS-VSR property. Without loss of generality, suppose that T^* is nonincreasing (recall Remark 2.2). It follows from [33, Lemma 7] that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\beta(s, t)) \leq \alpha_2(s)e^{-3t}, \quad \forall s \geq 0, t \geq 0. \quad (A.7)$$

Define $\alpha_3 := \alpha_1$, let $\bar{T} > 0$ be such that

$$T \leq 1 - e^{-2T}, \quad \forall T \in (0, \bar{T}), \quad (A.8)$$

and define $T^* : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ via $T^*(s) := \min\{T^*(\beta(s, 0)), \bar{T}\}$. Since $s \leq \beta(s, 0)$ for all $s \geq 0$ and T^* is nonincreasing, it follows that $T^*(s) \leq T^*(\beta(s, 0))$, and hence $\Phi(T^*(s)) \subset \Phi(T^*(\beta(s, 0)))$, for all $s > 0$. Let $x(k, \xi, \{d_i\}, \{T_i\})$ denote the solution of (5) at instant $k \in \mathbb{N}_0$ that corresponds to a sampling period sequence $\{T_i\}$, the disturbance sequence $\{d_i\}$ and satisfies $x(0, \xi, \{d_i\}, \{T_i\}) = \xi$. Note that

$x(k, \xi, \{d_i\}, \{T_i\})$ may be not defined for arbitrary $(k, \xi, \{d_i\}, \{T_i\})$. For $M \geq 0$ and $\xi \in \mathbb{R}^n$, define

$$S(M, \xi) := \{(k, \{d_i\}, \{T_i\}) \in \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M)) : \\ x(k, \xi, \{d_i\}, \{T_i\}), \text{ is defined}\}, \quad (\text{A.9})$$

and the function $V_M : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ via

$$V_M(\xi) := \sup_{(k, \{d_i\}, \{T_i\}) \in S(M, \xi)} \alpha_1(|x(k, \xi, \{d_i\}, \{T_i\})|) e^{2 \sum_{i=0}^{k-1} T_i}. \quad (\text{A.10})$$

Note that if $\{d_i\} \in \mathcal{D}$ and $\{T_i\} \in \Phi(T^*(M))$, then $(0, \{d_i\}, \{T_i\}) \in S(M, \xi)$ for all $\xi \in \mathbb{R}^n$ because $x(0, \xi, \{d_i\}, \{T_i\}) = \xi$ is defined. Hence, $S(M, \xi) \neq \emptyset$ for all $M \geq 0$ and $\xi \in \mathbb{R}^n$. We next show that item (3) is satisfied with $\alpha_1, \alpha_2, \alpha_3, T^*$ and V_M as defined.

From (A.10), it follows that

$$V_M(\xi) \geq \alpha_1(|x(0, \xi, \{d_i\}, \{T_i\})|) = \alpha_1(|\xi|), \quad (\text{A.11})$$

holds for all $\xi \in \mathbb{R}^n$. Then (9a) follows. Consider $M \geq 0$. For $|\xi| \leq M$ and $(k, \{d_i\}, \{T_i\}) \in S(M, \xi)$, it follows that $\{T_i\} \in \Phi(T^*(M)) \subset \Phi(T^*(M))$. Therefore, $|\xi| \leq M$ and $(k, \{d_i\}, \{T_i\}) \in S(M, \xi)$ imply that (6) is satisfied. Using (6) and (A.7) in (A.10) then, for all $|\xi| \leq M$ we have

$$V_M(\xi) \leq \sup_{(k, \{d_i\}, \{T_i\}) \in S(M, \xi)} \alpha_1 \left[\beta \left(|\xi|, \sum_{i=0}^{k-1} T_i \right) \right] e^{2 \sum_{i=0}^{k-1} T_i} \\ \leq \sup_{(k, \{d_i\}, \{T_i\}) \in S(M, \xi)} \alpha_2(|\xi|) e^{-\sum_{i=0}^{k-1} T_i} = \alpha_2(|\xi|), \quad (\text{A.12})$$

whence (9b) is established. Let $M \geq 0$ and define $\tilde{M} := \beta(M, 0)$. By the definition of RSS-VSR and the fact that $\Phi(T^*(M)) \subset \Phi(T^*(M)) \subset \Phi(T^*(M))$, it follows that $|x(j, \xi, \{d_i\}, \{T_i\})| \leq M$ for all $j \in \mathbb{N}_0$ whenever $|\xi| \leq M$, $\{d_i\} \in \mathcal{D}$ and $\{T_i\} \in \Phi(T^*(M))$. Therefore, $S(M, x(j, \xi, \{d_i\}, \{T_i\})) = \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M))$ for all $j \in \mathbb{N}_0$ whenever $|\xi| \leq M$, $\{d_i\} \in \mathcal{D}$ and $\{T_i\} \in \Phi(T^*(M))$. Thus, for all $|\xi| \leq M$, all $|d| \leq D$ and all $T \in (0, T^*(M))$, we have $S(M, \bar{F}(\xi, d, T)) = \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M))$, and

$$V_M(\bar{F}(\xi, d, T)) = \\ = \sup_{(k, \{d_i\}, \{T_i\}) \in \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M))} \alpha_1 \left(|x(k, \bar{F}(\xi, d, T), \{d_i\}, \{T_i\})| \right) e^{2 \sum_{i=0}^{k-1} T_i} \\ = \sup_{(\ell, \{d_i\}, \{T_i\}) \in \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M))} \alpha_1 \left(|x(\ell, \xi, \{d_i\}, \{T_i\})| \right) e^{2 \sum_{i=0}^{\ell-1} T_i} \\ \leq e^{-2T} \sup_{(\ell, \{d_i\}, \{T_i\}) \in \mathbb{N}_0 \times \mathcal{D} \times \Phi(T^*(M))} \alpha_1(|x(\ell, \xi, \{d_i\}, \{T_i\})|) e^{2 \sum_{i=0}^{\ell-1} T_i} \\ \leq V_M(\xi) e^{-2T} \quad (\text{A.13})$$

where we have used the facts that $e^{2 \sum_{i=0}^{\ell-1} T_i} = e^{2(T + \sum_{i=0}^{\ell-1} T_i)} e^{-2T}$, $\{d_i\} \in \mathcal{D}$ and $\{T_i\} \in \Phi(T^*(M))$. From (A.8), (A.13), and the fact that $T \in (0, T^*(M)) \subset (0, \bar{T})$, then for $|\xi| \leq M$,

$$V_M(\bar{F}(\xi, d, T)) \leq V_M(\xi) - V_M(\xi)(1 - e^{-2T}), \quad \text{hence}$$

$$V_M(\bar{F}(\xi, d, T)) - V_M(\xi) \leq -TV_M(\xi) \leq -T\alpha_3(|\xi|),$$

where we have used (A.11). Then (10) follows and (1. \Rightarrow 3.) is established. ■

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