CONSTITUTIVE MODELLING AND DISCONTINUOUS BIFURCATION ASSESSMENT IN UNSATURATED SOILS

by

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Abstract

In this work an elastoplastic constitutive theory for unsaturated soils is presented. The proposed material model is formulated in the general framework of the theory of porous media and of the flow theory of plasticity. The model is based on an extension of the well-known MRS Lade model whereby the suction and the effective stress tensor are introduced as additional independent and dependent stress components, respectively. Consequently the cap and cone yield conditions of the MRS Lade model both in hardening and softening as well as the internal evolution laws in these regimes are redefined to include the dependency on the suction. The paper illustrates the predictive capability of the Extended MRS Lade Model for partially saturated soils. Finally, the condition for discontinuous bifurcation in elastoplastic partially saturated porous media as well as the localized failure predictions of the proposed material formulation for different suctions and stress states are also analyzed and discussed.

Partially saturated soils, elastoplasticity, failure behavior, localized failure.

1 Introduction

In the last years significant attention was directed toward the development of constitutive theories for partially saturated soils. These materials are characterized by particular and complex response behaviors which strongly differ from those corresponding to both dry and saturated soils. Actually, the physics and engineering principles involved in dry soils are essentially the same to those involved in saturated soils. The main difference between a completely dry and a completely saturated soil is related to the compressibility of the pore fluid. The water in a saturated soil is basically incompressible. The water becomes compressible as air bubbles appear in the water.

From phenomenological observation we know that below the water table, the pore-water pressures is positive and the soils are, in general, saturated. However, above the water table, the pore-water pressures is, in general, negative. The negative pore-water pressures above the water table are mostly referenced to the pore-air pressure. The difference between the pore-air pressure and pore-water pressure is called the matric suction. Suction in an unsaturated soil is made up of two components, namely , matric suction and osmotic suction. The sum of the two components is called total suction. The osmotic suction is a function of the amount of dissolved salts acting in the pore fluid, and written in terms of a pressure. The matric suction is of primary interest because it is the stress variable which is strongly influenced by environmental changes.

Among the different experimental observation-based elastoplastic models for partially saturated soils in the literature, the model by Alonso et al. [1] is one of the most representative one. They adopted two independent stress variables, i.e. the total stress in excess of pore air pressure and the suction. Similar to this model are the constitutive formulations proposed by Schrefler and Zhan [13], Cui et al. [3], Bolzon et al.[2], Wheeler and Sivakumar [18] and Kohgo et al. [7]. A comprehensive review of the different proposals is given by Gens [4]. Recently, Khalili [6] proposed a constitutive model based on the effective stress concept. However, in their detailed formulation of the model Loret and Khalili [9] included the suction as an independent variable in the yield function and plastic potential, in addition to the effective stress and the suction dependent hardening parameter.

Contrarily to the formulation of constitutive equations for partially saturated soils, the analysis of the conditions for discontinuous bifurcation in the form of localized failure has not received considerable attention so far. Actually, the intrinsic hydro-mechanical coupling of partially saturated porous media and the presence of the suction in the constitutive equations strongly affects the localized failure indicators. As a consequence, both the solutions for discontinuous bifurcation as well as the critical directions for localization depend not only on the mechanical non-linear properties of the material formulation, ie. yield condition, non-associativity, hardening/softening evolution law, etc., but also on the hydraulic features of the deformation history.

In this work an elastoplastic constitutive model for partially saturated soils is proposed. The model is an extension of the MRS-Lade model by Sture et al. [15], and is a further development of Lade's three-invariant model for cohesionless soils. The proposed elastoplatic material model, the Extended MRS-Lade model, is described in the space of the three effective stress invariants and of the suction, which is introduced as a new independent variable.

The constitutive equations of the proposed model are also analyzed with regards to the solutions of discontinuous bifurcation. In this sense, the localized failure predictions and the critical directions for localization of the Extended MRS Lade model are analyzed for different confinement pressures and suctions.

The results in this work demonstrate the proposed model's predictive capability of the response behavior of partially saturated soils. Also the strong influence of the suction on the failure mode and on the critical direction for localized failure is demonstrated.

2 Constitutive stress

Partially saturated soils are generally described in terms of the constitutive or effective stress tensor $\boldsymbol{\sigma'}$ and the suction s as a dependent and an independent stress variable, respectively, where

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{I} p_w = \boldsymbol{\sigma}_n + \boldsymbol{I}\, s \qquad (1)$$

$$s \quad = \quad (p_a - p_w) \tag{2}$$

thereby is $\boldsymbol{\sigma}$ the total stress tensor, $\boldsymbol{\sigma}_n$ the net stress tensor, p_a, p_w the pore air and pore water pressure, respectively, and $\boldsymbol{I}$ the second order identity tensor. The term effective stress is due to Loret and Khalili [9], but as pointed out by Sheng et al. [14] who proposed the name *constitutive stress* instead, it is not an effective stress in Terzaghi's sense.
In many geotechnical applications the air pressure remains constant and, as a consequence, pore water pressure instead of the suction can be treated as a variable in the model formulation. Nevertheless, the constitutive formulation in this work is based on the suction allowing for the most general applications of the model.

3 Flow rule-based elastoplastic equations for partially saturated soils

3.1 The general formulation

Many plasticity models are characterized by yield surfaces, such as Tresca's, Mohr-Coulomb's and a variety of cone-cap criteria. Each convex function $F_i(\boldsymbol{\sigma'}, s, \boldsymbol{\kappa})$, that in case of partially saturated soils are defined in the space of the effective stress tensor and the suction, can be treated as an independent yield function, that depends on the set of hardening/softening variables represented by the array $\boldsymbol{\kappa}$. They are subsequently chosen as scalars κ_i that represent the plastic work or effective plastic strain measures in conjunction with the plasticity models presented in the following sections.
The intersection of all the sets of stresses defined by $F_i \leq 0$ defines the convex set $B\{\boldsymbol{\kappa}\}$ of plastically admissible constitutive stresses $\boldsymbol{\sigma'}$ and suction $\mathbf{s}$

$$B\{\boldsymbol{\kappa}\} = \{\boldsymbol{\sigma'}, s | F_i(\boldsymbol{\sigma'}, s, \boldsymbol{\kappa}) \leq 0,\ i = 1, 2, ..., U\} \tag{3}$$

Moreover, the space $B_{\lambda_i}\{\boldsymbol{\kappa}, \dot{\boldsymbol{\epsilon}}, \dot{s}\}$ can be introduced in the form

$$B_{\lambda_i}\{\boldsymbol{\kappa}, \dot{\boldsymbol{\epsilon}}, \dot{s}\} = \{\boldsymbol{\sigma'}, s | F_i(\boldsymbol{\sigma'}, \mathrm{s}, \boldsymbol{\kappa}) \leq 0,\ \text{ if } \dot{\lambda}_i(\dot{\boldsymbol{\epsilon}}, \dot{s}) > 0,\ i = 1, 2, ..., U\} \tag{4}$$

to account for plastic loading and elastic unloading, where the parameters $\dot{\lambda}_i,\ i = 1, 2, ..., U$, define the surfaces that are active. Plastic loading occurs, when at least some $\dot{\lambda}_i > 0$.
Following Weihe [17], the flow rule can be formulated in terms of the space of sub-differentials ∂F_λ, representing a fan of admissible normals at each corner of the composite failure surface

$$\partial F_{\lambda_i}\{\boldsymbol{\sigma'}, \dot{s}, \boldsymbol{\kappa}, \dot{\boldsymbol{\epsilon}}\} = \{\boldsymbol{a} | (\boldsymbol{\sigma'} - \boldsymbol{\sigma'}_o) : \boldsymbol{a} \geq 0, \forall \boldsymbol{\sigma'}_o, \in B_{\lambda_i}\{\boldsymbol{\kappa}, \dot{\boldsymbol{\epsilon}}, \dot{s}\}\} \tag{5}$$

The non-associated flow rule-based general constitutive equations for partially saturated soils can, therefore, be expressed as

$$\dot{\boldsymbol{\sigma}}' = \mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p) \tag{6}$$

$$(\boldsymbol{\sigma'} - \boldsymbol{\sigma'}_o) : \mathbf{A} : \dot{\boldsymbol{\epsilon}}_p \geq 0 \ \ \forall \boldsymbol{\sigma'}_o \in B_{\lambda_i}\{\boldsymbol{\kappa}, \dot{\boldsymbol{\epsilon}}, \dot{s}\} \tag{7}$$

$$\dot{\boldsymbol{\kappa}} = h\{\dot{\boldsymbol{\epsilon}}_p\} \tag{8}$$

where $\dot{\boldsymbol{\epsilon}}_p$ is the plastic portion of total strain rate tensor $\dot{\boldsymbol{\epsilon}}$, the function h is a first degree homogeneous vector function and eq.(7) represents an associated flow rule for the transformed plastic strain rate $\mathbf{A} : \dot{\boldsymbol{\epsilon}}_p$ provided the fourth order transformation operator $\mathbf{A}$ exists. An associated flow rule for the plastic strain rate $\dot{\boldsymbol{\epsilon}}_p$ is thus defined by $\mathbf{A} = \mathsf{I}$, being I the forth order identity tensor.

The *variational* form of the non-associated flow rule in eq.(7) can be reformulated using its *rate* form and the Kuhn-Tucker conditions

$$\dot{\boldsymbol{\epsilon}}_p = \sum_{i=1}^{U} \dot{\lambda}_i \boldsymbol{m}_i^\sigma, \;\; \dot{\lambda}_i \geq 0, \;\; F_i \dot{\lambda}_i = 0 \tag{9}$$

where

$$\boldsymbol{m}_i^\sigma = \mathbf{A}^{-1} : \boldsymbol{n}_i^\sigma, \qquad \boldsymbol{n}_i^\sigma = \frac{\partial F_i}{\partial \boldsymbol{\sigma'}} \tag{10}$$

is the direction of the plastic flow associated with the yield function F_i. Thereby, and as indicated in eq.(10), represent $\boldsymbol{m}_i^\sigma$ and $\boldsymbol{n}_i^\sigma$ the gradients to the plastic potential G_i and to the yield surface F_i, respectively, with respect to the constitutive stresses.

3.2 The consistency condition

In elastoplastic constitutive formulations, the consistency condition during plastic loading leads to the explicit form of the continuum material operator. In case of partially saturated soils the consistent condition takes the form

$$\dot{F}_i = \boldsymbol{n}_i^\sigma : \dot{\boldsymbol{\sigma}}' + n_i^s \dot{s} + r_i \dot{\kappa}_i = 0, \qquad i = 1, 2, ..., U \tag{11}$$

with

$$n_i^s = \frac{\partial F_i}{\partial s} \tag{12}$$

$$r_i = \frac{\partial F_i}{\partial \kappa_i} \tag{13}$$

$$\dot{\kappa}_i = \dot{\lambda}_i \, h_i(\boldsymbol{m}_i^\sigma) \tag{14}$$

As compared to the consistency condition of classical or conventional elastoplastic models, eq.(11) has an additional term $n_i^s \dot{s}$ related with the evolution of the suction and the gradient of the yield surface with respect to the suction. As pointed out by Sheng et al. [14], many authors have simply neglected this additional term in their formulations of constitutive equations for partially saturated soils and, as a consequence, the resulting consistency conditions are mathematically not rigorous.

After replacing the stress-strain relation

$$\dot{\boldsymbol{\sigma}}' = \mathbf{E} : (\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_p) \tag{15}$$

in eq.(11), the explicit expression of the plastic multiplier $\dot{\lambda}$ can be obtained as

$$\dot{\lambda}_i = \frac{\boldsymbol{n}_i^\sigma : \mathbf{E} : \dot{\boldsymbol{\epsilon}} + \boldsymbol{n}_i^\sigma : \boldsymbol{I} \, \dot{s}}{\boldsymbol{n}_i^\sigma : \mathbf{E} : \boldsymbol{m}_i^\sigma - r_i h_i} \tag{16}$$

Substituting eq.(16) into the *rate* form of the flow rule and then into eq.(15) leads to the compact form of the constitutive equations

$$\dot{\boldsymbol{\sigma}}' = \mathbf{D}_{ep} : \dot{\boldsymbol{\epsilon}}' \tag{17}$$

whereby, and according to Sheng et at. [14], the extended strain rate field $\dot{\boldsymbol{\epsilon}}'$ was introduced. This extended field is composed by the classical strain rate tensor $\dot{\boldsymbol{\epsilon}}$ and the suction rate $\dot{s}$ which is treated as an extra strain rate field

$$\dot{\boldsymbol{\epsilon}}' = \begin{pmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{s}\,\boldsymbol{I} \end{pmatrix} \tag{18}$$

The material operator $\mathbf{D}_{ep}$ in eq.(17) is defined as

$$\mathbf{D}_{ep} = \begin{pmatrix} \mathbf{E}_{ep} & \mathbf{E}_{s} \end{pmatrix} \tag{19}$$

with

$$\mathbf{E}_{ep} = \mathbf{E} - \sum_{i=1}^{U} \left[\frac{\mathbf{E} : \boldsymbol{m}^{\sigma} \otimes \boldsymbol{n}^{\sigma} : \mathbf{E}}{\boldsymbol{n}^{\sigma} : \mathbf{E} : \boldsymbol{m}^{\sigma} + H} \right]_i \tag{20}$$

$$\mathbf{E}_{s} = - \sum_{i=1}^{U} \left[\frac{\mathbf{E} : \boldsymbol{m}^{\sigma} \otimes n^{s}\,\boldsymbol{I}}{\boldsymbol{n}^{\sigma} : \mathbf{E} : \boldsymbol{m}^{\sigma} + H} \right]_i \tag{21}$$

being $i = 1, 2, ..., U$. The hardening/softening modulus H_i is defined as

$$H_i = -r_i\, h_i(\boldsymbol{m}_i^{\sigma}) \tag{22}$$

Replacing eq.(17) in eq.(1) we obtain the evolution of the total stress tensor

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}' + \dot{p}_w \boldsymbol{I} = \begin{cases} \text{General case} & : \; \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} + [-\mathbf{I} + \mathbf{E}_s] : \boldsymbol{I}\dot{s} \\ \dot{p}_a = 0 \text{ case:} & : \; \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} - [-\mathbf{I} + \mathbf{E}_s] : \boldsymbol{I}\dot{p}_w \\ \dot{p}_a = \dot{p}_w = 0 \text{ case:} & : \; \dot{\boldsymbol{\sigma}}' = \mathbf{E}_{ep} : \dot{\boldsymbol{\epsilon}} \end{cases} \tag{23}$$

4 Constitutive model for partially saturated soils

The elastoplastic model proposed in this work for partially saturated soils is an extension of the cap-cone MRS-Lade model [15] for cohesive-frictional drive soils. The main features of the MRS-Lade model for sands are:

- The yield condition is defined by means of two surfaces: a cone and a cap.
- The hardening and softening rules both in the cone and cap regimes are defined in terms of the plastic work rate.
- It includes a non associated flow rule which only affects the volumetric component of the plastic strain in the cone regime.

These are also the features of the Extended MRS-Lade model for partially saturated soils in this work. In the following the governing equations of the proposed model for unsaturated soils-like porous media are presented.

4.1 Yield condition

The yield condition, see [11], is defined in terms of the first invariant of the effective stress tensor p', of the second and third invariant of the deviatoric stress tensor q and θ, respectively, and of the hardening/softening variables in the cone region κ_{cone}. Defining the effective pressure in terms of the net mean stress p_n and the suction s the generic shape of the cone takes the form

$$F_{cone} = F(p_n, q, \theta, s, \kappa_{cone}) = f(q, \theta) - \eta_{cone}(\kappa_{cone})[p_n + s - p_c] = 0 \quad (24)$$

with

$$f(q, \theta) = q[1 + \frac{q}{q_a}]^m g(\theta) \quad (25)$$

being m a material constant controlling the curvature of the cone in the meridian (p_n, q) planes, with $0 \leq m \leq 1$, q_a a positive reference deviator stress, η_{cone} the angle of internal friction, and

$$p_n = \frac{I_{n\,1}}{3} \quad (26)$$

$$q = \sqrt{3J_2} \quad (27)$$

$$\cos\theta = \frac{3\sqrt{3}}{2}\frac{J_3}{\sqrt{J_2}} \quad (28)$$

Thereby is $I_{n\,1}$ the first invariant of the net stress tensor and J_2 and J_3 the second and third invariants of the deviatoric stress tensor, respectively. Finally, $g(\theta)$ is the Willam & Warnke [19] factor which assures a continuous and smooth variation of the shear strength in the deviatoric plane as long as the so-called eccentricity parameter e fulfills the condition $1/2 \leq e \leq 1$. Figure 1 illustrates the projection of the Extended MRS Lade model's yield surface in the meridian plane $\pi/3$ and its variation with the suction. The intersection of the yield surface with the $p_n - s$ plane defines a Loading Collapse (LC) yield curve which accounts for the increase of the elastic regime with the increment of s while reducing this regime to its minimum when $s = 0$ (saturated soil).

The formulation of the Extended MRS-Lade model as the original one, implies that a plastic flow is associated for the capped yield surface. The cap surface is defined by

$$F_{cap}(p_n, q, \theta, s, \kappa_{cap}) = \left(\frac{p_n - p_m}{p_r}^2\right) + \left(\frac{f}{f_r}\right)^2 - 1 \quad (29)$$

with

$$p_r = \frac{(1-\alpha)[\psi(1-\alpha)+\alpha]}{2\psi(1-\alpha)+\alpha} p_{cap}(\kappa_{cap}) \quad (30)$$

$$p_m = \frac{\alpha^2 + \psi(1-\alpha^2)}{2\psi(1-\alpha)+\alpha} p_{cap}(\kappa_{cap}) \quad (31)$$

$$f_r = \eta_{cone}[\psi(1-\alpha)+\alpha]\left[\frac{\alpha}{2\psi(1-\alpha)+\alpha}\right]^{\frac{1}{2}} \quad (32)$$

Figure 1. Extended MRS-Lade model's failure envelope in compressive meridian.

$$\psi = \frac{\eta_{cap}}{\eta_{cone}} \quad \text{and} \quad -\frac{\alpha}{2(1-\alpha)} < \psi \tag{33}$$

and

$$p_{cap}(\kappa_{cap}) = p_{cap,0}(1 + (\kappa_{cap})^{1/r}) \tag{34}$$
$$p_{cap,0} = p_{0^*} + i \cdot s \tag{35}$$

whereby the dependency of the pressure $p_{cap,0}$ on the suction in the last equation is due to Schrefler & Bolzon [12]. This function does fully define the dependency of the cap yield surface on the suction. The parameter p_{0^*} in eq.(35) represents the pre-consolidation pressure for saturated condition while αp_{cap} the pressure corresponding to the intersection between the cap and the cone yield surfaces, and i is a model parameter that takes into account the yield surface growth as increasing suction values.

4.2 Hardening/Softening relations

The hardening and softening parameters κ_{cone} and κ_{cap} are defined in terms of the accumulated plastic work w^p that is dissipated during loading along the actual stress path

$$\dot{w}^p = \int \boldsymbol{\sigma}_n : \dot{\boldsymbol{\epsilon}}_n^p dt \tag{36}$$

and are given in terms of the rate laws

$$\dot{\kappa}_{cone} = \frac{1}{c_{cone} p_a} \left(\frac{p + s - p_c}{p_a} \right)^{-l} \dot{w}^p \tag{37}$$
$$\dot{\kappa}_{cap} = \frac{1}{c_{cap} p_a \left(\frac{p_{cap,0}}{p_a}^{r} \right)} \dot{w}^p \tag{38}$$

where $c_{cone}, c_{cap}, p_a, l$ and r are material constants. The hardening variables directly influence the yield surface, as described in [14]. Schematically, the relations are defined as $\eta_{cone} = \eta_{cone}(\kappa_{cone})$ and $p_{cap} = p_{cap}(\kappa_{cap})$, according to eq.(34), such that the surface exhibits a smooth transition from the elastic to the plastic regime, that eventually leads to softening behavior.

4.3 Flow rule

The flow rules devised for the Extended MRS-Lade model assume, similarly to the original formulation by Sture et al. [15], non-associated flow for the cone which only affects the volumetric flow. However, in the present formulation the level of volumetric non-associativity in the cone regime is defined in terms of the suction in order to reproduce the tendency to associated flow of porous media, i.e. volumetric dilatancy reduction, with decreasing suction. In the present formulation the plastic potential function in the cone regime is defined in the form

$$G_{cone}(p,q,\theta,s) = f(q,\theta) - \left[n + (1-n)\left(\frac{s_{max}-s}{s_{max}}\right)^{t} \right] \eta_{cone}(\kappa_{cone})(p+s-p_c) \tag{39}$$

where $f(q,\theta)$ is defined in eq.(25), n is a scalar parameter such that $0 \leq n \leq 1$, s_{max} is the maximum suction (water pressure) of the soil, and t is a parameter controlling the rate that the plastic flow approaches the normality condition when $s \rightarrow 0$, with $t \geq 1$. From the last equation follows that the associated flow $G_{cone} = F_{cone}$ is obtained when $s = 0$ and that the maximum level of volumetric non-associativity is reached when $s = s_{max}$, i.e. for dry soils.

5 Discontinuous bifurcation condition

In this section the condition for localized failure modes in the form of discontinuous bifurcation is defined for unsaturated media. Discontinuous bifurcation or localized failure mode is detected by the formation of spatial discontinuities or jumps in the kinematic field across singularity surfaces that emerge in a stressed body. The analysis of localization leads to the same format and consequent relations as the propagation condition for plane acoustic waves in solid, see e.g. Thomas [16] and Hill [5]. The formation of a *weak discontinuity* assumes that a second order singularity appears in the strain rate field, while the displacement rates are still continuous

$$[[\dot{\boldsymbol{u}}]] = \dot{\boldsymbol{u}}^{+} - \dot{\boldsymbol{u}}^{-} = 0 \tag{40}$$

$$[[\boldsymbol{\nabla}_x \dot{\boldsymbol{u}}]] = \boldsymbol{\nabla}_x \dot{\boldsymbol{u}}^{+} - \boldsymbol{\nabla}_x \dot{\boldsymbol{u}}^{-} \neq 0 \tag{41}$$

here the double brackets indicate the jump. Applying Maxwell's theorem [8], the jump condition of the velocity gradient must be a rank-one tensor

$$[[\boldsymbol{\nabla}_x \dot{\boldsymbol{u}}]] = \dot{\gamma} \boldsymbol{M} \otimes \boldsymbol{N} \tag{42}$$

where $\boldsymbol{N}$ is the normal to the discontinuity surface, $\boldsymbol{M}$ defines the jump direction, and $\dot{\gamma}$ the jump magnitude. Using the strain definition of classical continua, the strain rate jump takes the form

$$[[\dot{\boldsymbol{\epsilon}}]] = \frac{1}{2}\dot{\gamma}\left(\boldsymbol{N} \otimes \boldsymbol{M} + \boldsymbol{M} \otimes \boldsymbol{N}\right) \tag{43}$$

Under the assumption of a continuous water pressure field, i.e. $[[\dot{s}]] = 0$ and assuming that at the onset of localization both sides of the singularity surface are in plastic loading state, the jump of the total stress state follows from the elastoplastic constitutive law (39) and the strain rate jump (43) as

$$[[\dot{\boldsymbol{\sigma}}]] = \left[\left[\dot{\boldsymbol{\sigma}}'\right]\right] = \dot{\gamma} \mathbf{E}_{ep} : (\boldsymbol{N} \otimes \boldsymbol{M}) \tag{44}$$

Figure 2. Model predictions of Plain Strain Passive (PSP) Tests at different Suctions.

Figure 3. Model predictions of Plain Strain Active (PSA) Tests at different Suctions.

According to Cauchy's lemma, the traction rate vector $\dot{\boldsymbol{t}}$ has to remain continuous across the singularity surface in the interior of a solid. Therefore, the localization condition takes the form

$$\mathbf{0} = \left[\left[\dot{\boldsymbol{t}}\right]\right] = \boldsymbol{N} \cdot [[\dot{\boldsymbol{\sigma}}]] = (\boldsymbol{N} \cdot \mathsf{E}_{ep} \cdot \boldsymbol{N}) \cdot (\dot{\gamma}\boldsymbol{M}) = \boldsymbol{Q}_{ep} \cdot (\dot{\gamma}\boldsymbol{M}) \tag{45}$$

whereby

$$\boldsymbol{Q}_{ep} = \boldsymbol{N} \cdot \mathsf{E}_{ep} \cdot \boldsymbol{N} \tag{46}$$

is the localization tensor and $\boldsymbol{M}$ is the eigenvector that defines the direction of the strain rate jump. Thus, the discontinuity bifurcation begins when the localization tensor turns singular, i.e. when

$$\det\left(\boldsymbol{Q}_{ep}\right) = 0 \tag{47}$$

The last equality represents the localization condition in the present theory of partially saturated soils and, for the particular case of a continuous water pressure field considered here, coincides with that of classical continua. In the next section the localization condition will be analyzed during deformation histories of soils with different degrees of saturation.

6 Model predictions

The model predictions of the failure response behavior of partially saturated soils are analyzed for different stress paths and deformation histories. Particularly, the influence of the suction in the response behavior is evaluated and discussed.

The numerical analyses in this section illustrate the predictive capability of the proposed model for the plane strain passive, plane strain active and uniaxial compression tests.

Figure 2 shows the model predictions of the plane strain passive tests (PSP) for different suction levels in terms of the constitutive vertical stress vs. the vertical and lateral strain component plots. This numerical analysis were performed in plain strain condition and under mixed control, i.e. the vertical

Figure 4. Model predictions of Uniaxial Compression Tests. Axial symmetric stress state (ASS).

Figure 5. Localization analysis at 90% of PSP test's peak stresses.

strain increments, the constant lateral confinement stress and the (null) out-of-plane strain are known while the vertical stress, the lateral strain and the out-of-plain stress are unknown. The results in figure 2 clearly illustrate the strong influence of the suction in the response behavior of partial saturated soils in terms of the limit stress, the ductility and the lateral strain. Particularly, we observe that with increasing suction a reduction of the ductility takes place together with an increment of the peak stress and of the lateral strain. This agrees very well with the features of the partially saturated soils response behavior.

Figure 3 illustrates the prediction of the model for the plane strain active tests (PSA) with different level of the suction in terms of q and the second invariant of the deviatoric strain tensor e. These tests were also performed under mixed control. However, and contrary to the PSP case, the applied vertical strain in the PSA tests is in the tensile direction. The results in figure 3 as the previous ones in figure 3 demonstrate the significant influence of the suction in the response behavior of partially saturated soils under plane strain condition.

Finally, figure 4, depicts the proposed model predictions of the uniaxial compression tests in axial symmetric stress condition for different suction levels. Although a very ductile behavior is observed in the analysis with the lowest suction, $s = 10kPa$, the response behavior exhibit a peak load and a softening regime. This is a remarkable difference with the plane strain compression analysis in figure 2 whereby a continuous hardening regime without axial stress degradation can be observed when $s = 10kPa$. Actually, the more ductile behaviors in the PSP tests as compared to those of the uniaxial compression tests in axial symmetric stress state is mainly due to the presence of the confinement stress (σ_1) in the first ones. The comparative analysis between the results in figures 2 and 4 leads to the conclusion that the influence of the suction in the overall response behavior reduces with increasing confinement pressure, i.e. see the stronger dependency of the peak load on the suction in the tests without confinement pressure in figure 4 when compared to the results in figure 2.

Figure 6. Localization analysis at peak stress of PSP tests.

Figure 7. Localization analysis at residual stress of PSP tests.

7 Localized failure predictions

Under the assumption of continuous water pressure, $[[\dot{p}_w]] = 0$, the localized failure condition was analyzed at different stress states along the PSP, the PSA and the uniaxial compression tests of figures 2, 3 and 4, respectively. Figure 5 illustrates the localized failure analysis at 90% of the peak load performed in the PSP tests. At this level of the axial stress the conditions for localized failure are fulfilled for the first time in the PSP test with the largest suction $s = 400kPa$ as can be observed in figure 6. The other two tests with $s = 100kPa$ and $s = 10kPa$ indicates diffuse failure as the localization tensor remains non-singular. The results in figure 5 also indicate that the critical direction for localization varies with the suction. In other words, both the failure mode (diffuse or localized) as well as the orientation of the potential or critical shear band depend on the saturation degree of the soil. The localization analysis in the PSP tests was also performed at peak and at the residual stress and the corresponding results are depicted in figures 6 and 7. We observe in these figures that the localization tensor remains positive defined up to the final stage in the test with the minimum suction $s = 10kPa$, while in the other two tests with larger suction the localization condition was fulfilled. Therefore, we conclude that the reduction of the suction suppresses localization or discontinuous bifurcation and lead to diffused or continuous failure modes. Moreover, the fact that already for $s = 10kPa$ no localized failure is obtained in the PSP test also demonstrates that the stabilizing effect of the reducing suction takes place before full saturation of the soils.
The results of the localization analysis at residual stress of the PSA tests are shown in figure 8. As before, we observe that both the failure mode and the critical localization direction depends on the saturation degree of the soil. However, in the PSA tests the stabilizing effect of the reducing suction with regards to discontinuous bifurcation takes place for considerable lower values of the suction, corresponding almost to fully saturated soils. With other words, for the same level of confining pressure the dependency of the failure mode in the suction is more relevant in the PSP tests than in the PSA ones.
Finally, figure 9 describes the performance of the localized failure indicator at peak load of the uniaxial compression tests under axial symmetric stress state with $s = 10kPa$, $s = 100kPa$ and $s = 400kPa$. In this figure also the performance of the localization indicator corresponding to the classical

Figure 8. Localization analysis at final stage of PSA tests.

Figure 9. Localization analysis at peak. Uniaxial compression test in axial symmetric state.

MRS-Lade model is shown. These results agree with the observations by Peric [10] in the sense that the classical MRS-Lade model does not lead to localized failure in axial symmetric stress state. However, they show that even in this stress state the destabilizing effect associated with the increasing suction is important and lead in the extreme case, i.e. when $s = 400kPa$, to localized failure mode.

8 Conclusions

An elastoplastic constitutive theory for partially saturated soils is proposed on the basis of the well-known MRS Lade model. The proposed model is defined in the space of the effective stresses and of the suction which strongly influence the shape of the maximum strength envelope and of the yield conditions. The flow rule of the so-called Extended MRS Lade model is based on a restricted non-associated whereby only the volumetric plastic flow in the cone regime is non-associate.

The predictive analysis of the proposed model demonstrates its capability to reproduce the most relevant features of partially saturated soil response behaviors. On the other hand, the localization analysis performed with the model demonstrates that the increment of the suction is related with a destabilizing effect as discontinuous bifurcations in the form of localized failure take place instead of diffuse or continuous failure modes. The results also illustrate the relevant influence of the suction in the critical directions for localization.

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