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Non-Riemmanian geometry, force-free magnetospheres and the generalized Grad-Shafranov equation

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The magnetosphere structure of a magnetar is considered in the context of a theory of gravity with dynamical torsion field beyond the standard General Relativity (GR). To this end, the axially symmetric version of the Grad-Shafranov equation (GSE) is obtained in this theoretical framework. The resulting GSE solution in the case of the magnetosphere corresponds to a stream function containing also a pseudoscalar part. This function solution under axisymmetry presents a complex character that (as in the quantum field theoretical case) could be associated with an axidilaton field. Magnetar–pulsar mechanism is suggested and the conjecture about the origin of the excess energy due the GSE describing the magnetosphere dynamics is claimed. We also show that two main parameters of the electrodynamic processes (as described in GR framework by Goldreich and Julian (GJ) [*Astrophys. J.* **157** (1969) 869]) are modified but the electron-positron pair rate $\dot{N}$ remains invariant. The possible application of our generalized equation (defined in a non-Riemannian geometry) to astrophysical scenarios involving emission of energy by gravitational waves, as described in the context of GR in [S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano and D. Vernieri, *Astrophys. Space Sci.* **333** (2011) 29–35], is briefly discussed.

Keywords: Non-Riemannian geometry; Grad-Shafranov equation; Magnetosphere dynamics; Magnetar model.

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1. Introduction to the Problem

For a long time, attempts have been made to give concrete answers to various astrophysical and cosmological mechanisms, in particular the origin, both of primordial fields of different types, as well as of the stellar and cosmological dynamics. Given that both general relativity (GR) and the standard model (SM) of elementary particles do not finish giving full explanations to these questions, the idea of a reformulation of a unified theory beyond RG and MS seems very attractive. In previous references, the authors introduced a unified model based on a non-Riemannian geometry containing a dynamic antisymmetric torsion that admits not only the same results of GR and SM already proven, but also satisfactorily solves problems that GR and SM present difficulties or inconsistencies. Some of those problems that were satisfactorily treated in the context of this new formulation were the determination of the mass of the axion [12], violation of CP of the neutrino [14], primordial magnetogenesis [10], etc. In this work we calculate the equations controlling the stellar magnetospheres of compact objects in particular, in this new context. To this end, force-free conditions are adopted by deriving the equilibrium conditions depending on a flow function with a pseudoscalar part coming from the torsion.

Actually, a typical example is the axisymmetric force-free magnetosphere in the exterior of a neutron star. Two possibilities are proposed for the energy storage prior to magnetar outbursts to explain the relevant phenomena: storage in the magnetar crust or in the magnetosphere. The latter model is discussed in terms of similarity with solar flares [15, 16]. In the solar flare model e.g. [17], the energy is quasi-statically stored by thermal motion at the surface, and is suddenly released as large-scale eruptive coronal mass ejections. The energy is dissipated via a magnetic reconnection associated with the field reconfiguration. Analogous energy buildup and release processes may be relevant to the magnetar giant flares, although the energy scale differs by many orders. In sum, one must entirely rethink the physics of neutrino cooling, photon emission, and particle emission from a neutron star, when its magnetic field (instead of its rotation) is the main source of free energy. This possibility is completely feasible in the context of the model previously presented in [11] that is based on a geometric (Lagrangian) action that can be considered the non-Riemannian generalization of the Born-Infeld model (see details in [9–11])

$$\mathcal{L}_{\rm gs} = \sqrt{\det\left[\lambda g_{\alpha\beta}\left(1+\frac{R_s}{4\lambda}\right)+\lambda F_{\alpha\beta}\left(1+\frac{R_A}{\lambda}\right)\right]}, \tag{1}$$

$$R_s \equiv g^{\alpha\beta}R_{(\alpha\beta)}; \quad R_A \equiv f^{\alpha\beta}R_{[\alpha\beta]}; \tag{2}$$

(with $f^{\alpha\beta} \equiv \frac{\partial \ln(\det F_{\mu\nu})}{\partial F_{\alpha\beta}}$, $\det F_{\mu\nu} = 2F_{\mu\nu}\widetilde{F}^{\mu\nu}$).

In this model, the torsion $T^{\alpha}_{\beta\gamma}$ has a *dynamic character* (contrary to other models in the literature) and is *totally antisymmetric*, which allows it to be related to its dual vector h_μ. The other important feature that the energy-momentum tensor and fundamental constants (really functions of the spacetime) are geometrically-induced and not imposed "by hand".

Field equations link the dual vector with the electromagnetic field via the following expression:

$$\nabla_\alpha T^{\alpha\beta\gamma} = -\lambda F^{\beta\gamma} \to \nabla_{[\beta} h_{\gamma]} = -\lambda^* F_{\beta\gamma}, \tag{3}$$

which indicates that the magnetic field (in the case of interest here) is related in this theoretical context to the dynamics of the torsion vector h_μ. At the same time, we demonstrate, generalizing the Helmholtz theorem in four dimensions, that the torsion vector admits an unique geometric decomposition of the form

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \gamma_1 \overbrace{\varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta}}^{\text{axial vector}} + \gamma_2 \overbrace{P_\alpha}^{\text{polar vector}}, \tag{4}$$

where Ω, $A_{\gamma\delta}, M_{\beta\gamma\delta}, P_\alpha$ fields can be associated to particles (matter) and physical observables (e.g. vorticity, helicity, etc.). In that same reference, we find via Killing-Yano symmetries, fields and possible physical observables associated to $A_{\gamma\delta}$ and Ω, in Eq. (4). In the $3+1$ decomposition of the spacetime, expression (4) with geometrically admissible fields (Killing-Yano symmetries) takes the form

$$h_0 = \nabla_0 \Omega + \frac{4\pi}{3}[h_M + q_s n_s \overline{u}_s \cdot \overline{B}] + \gamma_1 h_V + \gamma_2 P_0, \tag{5}$$

$$\begin{aligned} h_i = \nabla_i \Omega &+ \frac{4\pi}{3}[-((\overline{A} + q_s n_s \overline{u}_s) \times \overline{E})_i + (\Phi + q_s n_s u_{0s})\overline{B}_i] \\ &+ \gamma_1 [u_0(\overline{\nabla} \times \overline{u}) + (\overline{u} \times \overline{\nabla} u_0) + (\overline{u} \times \dot{\overline{u}})]_i + \gamma_2 P_i. \end{aligned} \tag{6}$$

Note that in h_0 we can recognize the magnetic and vortical helicities where A_μ is the vector potential and q_s is the particle charge, n_s is the number density (in the rest frame) and the four-velocity of species s is u_s^γ. Consequently the simplest mechanism to generate the necessary amount of energy of magnetospheres (even without star rotation) can be described as follows:

(1) The axion and other pseudoscalars and pseudovector particles (contained in h_α) plus all helicities increase the original magnetic field B e.g.: due to the induction (dynamo) linearized expression from Sec. 2, as

$$\nabla \times (\alpha B) = h \times E - h_0 B - (E \cdot \nabla\omega)\overline{m}. \tag{7}$$

(2) B increases, and increases the magnetic helicity H_M defined as (g_3 determinant of the absolute space, see Sec. 4)

$$H_M = \int A \cdot B \sqrt{g_3} d^3x.$$

(3) H_M in turn increases B even more via expressions (3) and (7) through the torsion vector h_α.

(4) Consequently, the total energy in the magnetosphere will be increased to a certain limit (see Sec. 4)

$$E_M = \int \alpha B^2 \sqrt{g_3} d^3x.$$

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(5) After some limit is determined, the excess energy in the magnetosphere is ejected and the process is repeated.

With the above motivation, we will work out the problem of the force-free magnetosphere computing explicitly the Grad-Shafranov equation in the case of a axisymmetric configuration (without rotation, in principle) considering the dual of the torsion tensor h_μ from the gravitational theory based in affine geometry given in [11]. To this end, the force of Lorentz in the context of the unified model will be calculated, the $3+1$ formalism introduced and the geometrically-induced alpha term (with introduction of the physical currents, which intervenes in the equation of the induction producing the dynamo effect) determined. Finally, we will present a concise discussion on the problem of the physics of magnetospheres based on the expressions obtained and the current knowledge regarding the intervention of high energy processes in these scenarios.

2. The Model, Generalized Lorentz Force and α-Term

As we see before in [9–11], the geometrically-induced Lorentz force that we have been obtained from the model in the linear limit was

$$(\overline{h} \cdot B + \rho_e)E + J \times B = (E \cdot B)\overline{h}, \tag{8}$$

consequently, in the case of force-free condition with non-vanishing torsion field implies: $(E \cdot B) = 0$. General assumptions for $3+1$ splitting in axisymmetrical spacetimes can be introduced in a standard form (e.g.: j, E and B can be treated as 3-vectors in spacelike hypersurfaces). In terms of these 3-vectors, the nonlinear equations of the original model can be linearized and consequently written in a Maxwellian fashion as

$$\nabla \cdot \mathbb{E} = -h \cdot \mathbb{B} + 4\pi\rho_e, \tag{9}$$

$$\nabla \cdot B = 0, \tag{10}$$

$$\nabla \times (\alpha E) = (B \cdot \nabla\omega)\overline{m}, \tag{11}$$

$$\nabla \times (\alpha\mathbb{B}) = h \times \mathbb{E} - h_0\mathbb{B} - (\mathbb{E} \cdot \nabla\omega)\overline{m}. \tag{12}$$

The derivatives in these equations are covariant derivatives with respect to the metric of the absolute space γ_{ij} being α, β: lapse and shift functions respectively and $\mathbb{E} = \frac{\partial \mathcal{L}_{\rm gs}}{\partial E}$ and $\mathbb{B} = \frac{\partial \mathcal{L}_{\rm gs}}{\partial B}$. Because it is unified model, we need to replace $\overline{h} \times \overline{E}$ in order to introduce the physical currents as follows. From the above equations in the exact form, the geometrical current induced by the non-Riemannian framework is

$$J \equiv +h \times \mathbb{E} - h_0 \mathbb{B}, \tag{13}$$

consequently,

$$J \times \mathbb{B} \Rightarrow (h \cdot \mathbb{B})\mathbb{E} = J \times \mathbb{B} + (\mathbb{B} \cdot \mathbb{E})h, \tag{14}$$

then $h \times \mathbb{E}$

$$h \times \mathbb{E} = h \times \left[\frac{(\mathbb{B} \cdot \mathbb{E})h + J \times \mathbb{B}}{(h \cdot \mathbb{B})}\right] = J - \frac{(h \cdot J)\mathbb{B}}{(h \cdot \mathbb{B})}, \tag{15}$$

consequently, the relation with the physical scenario can be implemented as follows:

$$J - \frac{(h \cdot J)\mathbb{B}}{(h \cdot \mathbb{B})} \to \alpha_g \left(j_{\mathrm{ph}} - \frac{(h \cdot j_{\mathrm{ph}})\mathbb{B}}{(h \cdot \mathbb{B})}\right), \tag{16}$$

transforming the set (9)–(12) at the linear level, namely $\mathbb{E} \to E$ and $\mathbb{B} \to B$, to

$$\nabla \cdot E = -h \cdot B + 4\pi\rho_e, \tag{17}$$

$$\nabla \cdot B = 0, \tag{18}$$

$$\nabla \times (\alpha E) = (B \cdot \nabla\omega)\overline{m}, \tag{19}$$

$$\nabla \times (\alpha B) = \alpha_g \left(j_{\mathrm{ph}} - \frac{(h \cdot j_{\mathrm{ph}})B}{(h \cdot B)}\right) - h_0 B - (E \cdot \nabla\omega)\overline{m}, \tag{20}$$

$$= \alpha_g j_{\mathrm{ph}} - \left[h_0 + \frac{(h \cdot j_{\mathrm{ph}})}{(h \cdot B)}\right] B - (E \cdot \nabla\omega)\varpi^2. \tag{21}$$

From the begining of radio pulsar studies, three main parameters determining the key electrodynamic processes were defined: from the calculations above, we will demonstrate in a simple way that the quantities defined from the density are all altered. The said alteration comes from the dynamics of h, being able to accentuate or even annul the effect that the rotation has on that density. The first was the electric charge density that is needed to screen the longitudinal electric field near the neutron star surface, namely $\rho_{\mathrm{GJ}} = -\frac{\Omega \cdot B}{2\pi c}$, This quantity, introduced by Goldreich and Julian (GJ) in 1969 [5] was used to determine the characteristic particle number density $n_{\mathrm{GJ}} = \frac{|\rho_{\mathrm{GJ}}|}{|e|}$ (of the order of $10^{-12}\,\mathrm{cm}^3$ near the neutron star surface). Here, as h must be considered from Eq. (17) (we concentrate on the linearized version to simplify the analysis), the corresponding charge density to that of GJ is

$$\rho_{\mathrm{UFT}} = -\frac{(\Omega + h) \cdot B}{2\pi c} \equiv \rho_{\mathrm{GJ}} + \rho_h$$

(subindices GJ indicate here the corresponding GJ quantity) consequently, the characteristic charge density can only be determined through the knowledge of h, and the corresponding characteristic number density that will be

$$n_{\mathrm{UFT}} = \left| -\frac{(\Omega + h) \cdot B}{2\pi c} \right| \Big/ |e|.$$

Also, the characteristic current density is modified as

$$j_{\mathrm{UFT}} = c\rho_{\mathrm{UFT}},$$

which is much more important as indicated in [6] because in such approaches it is the longitudinal electric current circulating in the magnetosphere that will play the key role.

The second parameter is the particle multiplication defined currently as $\lambda_{\mathrm{GJ}} = n_e/n_{\mathrm{GJ}}$, which shows how much the secondary particle number density exceeds the critical number density n_{GJ}. Also, this parameter is affected according to our work, as

$$\lambda_{\mathrm{UFJ}} = n_e/n_{\mathrm{UFT}},$$

that is evidently greater than the same GJ quantity. In the above expression, we have n_h, in which the secondary particle number density must be greater than in the GJ case to exceed the new critical number density n_{UFT}. Finally, the third relevant quantity is the hydrodynamic particle flow that now is $\dot{N}_{\mathrm{UFT}} m_e c^2 \Gamma$ (Γ here and below denotes the hydrodynamic Lorentz factor of the outflowing plasma) with the electron–positron pair injection rate

$$\dot{N}_{\mathrm{UFT}} = c\pi\lambda_{\mathrm{UFT}} R_0^2 n_{\mathrm{UFT}} = \dot{N},$$

that, with these definitions, it is not modified.

3. Force-Free Magnetospheres: Generalized Grad-Shafranov Equation

The consistent theoretical description of gravitational magnetohydrodynamics (MHD) equilibria is of fundamental importance for understanding the phenomenology of accretion disks (AD) around compact objects (black holes, neutron stars, etc.). The very existence of these equilibria is actually suggested by observations, which not only show the evidence of quiescent and essentially non-relativistic, AD plasmas close to compact stars, but also the dynamical interplay with high energy processes involving the magnetospheres of compact objects, in particular pulsars, quasars and magnetars. The electromagnetic (EM) fields involved, in particular the electric field, may locally be extremely intense [5], so several standard processes such as electron positron pair creation occur, but several exotic interactions involving neutrinos with axions and other dark matter candidates must also be taken into account. This suggests therefore that such equilibria (if it certainly exists) should be described in the framework of unified field theory beyond general relativity (GR) [9–12] and beyond the standard model (SM). Extending previous approaches, holding for compact objects/black hole axisymmetric geometries having into account effect of space-time curvature, the purpose of this work is the formulation of a generalized Grad-Shafranov (GGS) [2, 1, 3, 4] equation based in a non-Riemannian geometry with dynamical torsion field suitable for the investigation of accretion, jets and winds and other astrophysical effects when high energy effects (exotic or not) are present. Now we will calculate the GSE with axisymmetry. Arguments and procedures for calculating GSE in this model are similar in form to works well known in the context of GR [6–8] (we use through the work [8] notation) to have a reasonable comparison parameter with those results.

4. Magnetic Fields and Currents

From the electrodynamic equations in 3+1 formulation of curved spacetimes, under axisymmetry, the magnetic field is split as $\overline{B} = B_p + B_t$, where

$$\begin{aligned} B_p &= \frac{1}{2\pi\varpi^2}(\nabla\psi + \overline{m} \times \nabla h_0) \times \overline{m} \\ &= \frac{1}{2\pi\varpi}(\nabla\psi \times e_{\widehat{\phi}} + \varpi\nabla h_0) = \frac{\nabla\psi \times \overline{m}}{2\pi\varpi^2} + \nabla\chi, \\ B_t &= -\frac{2I(\psi,\chi)}{\alpha\varpi c} e_{\widehat{\phi}}, \end{aligned} \tag{22}$$

with $e_{\widehat{\phi}}$ unitary toroidal vector, h_0 pseudoscalar field that we redefine as χ(zero component of the dual of the antisymmetric torsion field) and $\overline{m} \cdot \overline{m} = g_{\phi\phi} = \varpi^2$. The expression for the toroidal magnetic field coming from the Ampere law and the currents enclosed by the surface A, namely I (depending on ψ and χ), are obtained similarly to the magnetic flux assuming the form: $\nabla I = \nabla I(\psi) + \overline{m} \times \nabla I(\chi)$. Consequently, due that $(E \cdot B) = 0$ the Eq. (22) brings the force-free condition as

$$\begin{aligned} j_p &= \frac{1}{2\pi\alpha\varpi}(e_{\widehat{\phi}} \times \nabla I) = -\frac{\nabla I(\psi) \times \overline{m}}{2\pi\alpha\varpi^2} + \frac{\nabla I(\chi)}{2\pi}, \\ &= -\frac{1}{\alpha}\frac{dI}{d\zeta}B_p, \end{aligned} \tag{23}$$

where we have defined the multi-vector $\zeta \equiv \psi + \overline{m}\chi$ ($\overline{m} \equiv \varpi e_{\widehat{\phi}}$) (and consequently under action of exterior derivative: $d\zeta = d\psi + \overline{m} \times d\chi$ and $\nabla\zeta = \nabla\psi + \overline{m} \times \nabla\chi$). Thus,

$$B_p \cdot dA = d\zeta, \tag{24}$$

$$\oint \alpha j_p \cdot dA = -\oint \frac{dI}{d\zeta} B_p \cdot dA = -\int_0^\zeta \frac{dI}{d\zeta} B_p \cdot dA, \tag{25}$$

$$I(0) = I(\zeta), \tag{26}$$

$$j_t = -\frac{1}{8\pi}\left[\frac{\varpi c}{\alpha}\nabla \cdot \left(\frac{\alpha}{\varpi^2}\underbrace{(\nabla\psi + \overline{m} \times \nabla\chi)}_{\equiv\nabla\zeta}\right) + \frac{\varpi(\Omega_F - \omega)}{\alpha^2 c}(\nabla\psi + \overline{m} \times \nabla\chi) \cdot \nabla\omega\right], \tag{27}$$

$$v_F = \frac{1}{\alpha}(\Omega_F - \omega)\varpi, \tag{28}$$

$$E_p = -\frac{v_F}{c}(e_{\widehat{\phi}} \times B_p) = -\frac{1}{2\pi c\alpha}(\Omega_F - \omega)\nabla\zeta(\text{force-free}), \tag{29}$$

$$\rho_e = \frac{1}{4\pi}(\nabla \cdot E_p + \overline{h} \cdot B) = \frac{1}{4\pi}\nabla \cdot E_p. \tag{30}$$

then

$$\rho_e = -\frac{1}{8\pi^2}\frac{\varpi^2(\Omega_F - \omega)}{\alpha^2 c}\left\{\nabla \cdot \left(\frac{\alpha}{\varpi^2}\nabla\zeta\right) + \frac{\alpha}{\varpi^2}\nabla\zeta \cdot \nabla \ln\left[\frac{\varpi^2(\Omega_F - \omega)}{\alpha^2 c}\right]\right\}, \tag{31}$$

as usual we can eliminate the first factor: $\nabla \cdot (\frac{\alpha}{\varpi^2}(\nabla\Psi + \overline{m} \times \nabla\chi))$ between expressions, (27) and (31), consequently

$$\rho_e - \frac{\varpi(\Omega_F - \omega)}{\alpha c}\frac{j_t}{c} = -\frac{1}{8\alpha\pi^2}\left\{\left(\frac{\varpi(\Omega_F - \omega)}{\alpha c}\right)^2 \nabla\zeta \cdot \nabla\omega - \frac{c\alpha^2}{\varpi^2}\nabla\zeta \cdot \nabla\left(\frac{\varpi^2(\Omega_F - \omega)}{\alpha^2 c}\right)\right\}. \tag{32}$$

In this case with dynamical torsion field, the transfield component of the momentum equation for the force-free case, namely

$$(\overline{h} \cdot B + \rho_e)E + J \times B = 0, \tag{33}$$

becomes

$$\frac{j_t}{c} - \frac{\varpi(\Omega_F - \omega)}{\alpha c}\rho_e = \frac{1}{2\alpha^2\varpi c^2}\frac{IdI}{d\zeta}. \tag{34}$$

Solving for ρ_e and j_t

$$8\pi^2\rho_e = \frac{\frac{\varpi(\Omega_F - \omega)}{\alpha c}}{1 - (\frac{\varpi(\Omega_F - \omega)}{\alpha c})^2}\left[\frac{8\pi^2}{2\alpha^2\varpi c}\frac{IdI}{d\zeta} + \frac{\varpi(\Omega_F - \omega)}{\alpha^2 c}\nabla\zeta \cdot \nabla\omega - \frac{c}{\varpi}\nabla\zeta \cdot \nabla \ln\left(\frac{\varpi^2(\Omega_F - \omega)}{\alpha^2 c}\right)\right], \tag{35}$$

$$8\pi^2 j_t = \frac{1}{1 - (\frac{\varpi(\Omega_F - \omega)}{\alpha c})^2}\left[\frac{8\pi^2}{2\alpha^2\varpi c}\frac{IdI}{d\zeta} + \frac{\varpi(\Omega_F - \omega)}{\alpha^2 c}\nabla\zeta \cdot \nabla\omega - \frac{c}{\varpi}\nabla\zeta \cdot \nabla \ln\left(\frac{\varpi^2(\Omega_F - \omega)}{\alpha^2 c}\right)\right]$$

(it is due to Ampere equation (fourth Maxwell equation above) projected toroidally). (36)

Consequently, from (35) and (36), we obtain

$$\nabla \cdot \left(\frac{\alpha}{\varpi^2}\nabla\zeta\right) = \nabla \cdot \left(\frac{(\Omega_F - \omega)^2}{\alpha c^2}\nabla\zeta\right) - \frac{(\Omega_F - \omega)}{\alpha c^2}\frac{d\Omega_F}{d\zeta}|\nabla\zeta|^2 - \frac{8\pi^2}{2\alpha(\varpi c)^2}\frac{IdI}{d\zeta}. \tag{37}$$

Equations (35), (36) and (37) are the Grad Shafranov ones, note that this can be seen as a two-dimensional Poisson type equation (axisymmetry) of a complex variable $\zeta \equiv \psi + \overline{m}\chi \to \psi + i\chi$ with a source term $\propto \frac{IdI}{d\zeta}$.

5. Discussion

In order to help the reader and on the possible physical scenario in which we could study our model with respect to the energy limits and the possible emission mechanisms, a good analysis using the Post-Newtonian method in the context of GR standard was carried out in [18].

In [18], authors have calculated the electromagnetic corrections to the gravitational waves emitted by a coalescing binary system as a contribution to the total energy–momentum tensor (EMT) of a dipolar electromagnetic field. Consequently, the goal in that case was the determination of the correction to the emission of standard gravitational energy by a gravitomagnetic term that becomes null when the magnetic field becomes zero.

In our case (as it is easy to see in [10, 11]) the source of the gravitational field and the electromagnetic field is given, in a unified way, by the geometrically-induced EMT it will be very interesting in a future work to develop in the context of our model, the same procedure as in [18] to study the same physical scenario.

6. Concluding Remarks and Outlook

As we saw in previous works, the physical currents are linked to the torsion vector h by means of its only decomposition in fields of matter (particles) and observables. Being a pseudoscalar field playing the role of axion and magnetic, vortex and mixed helices respectively. Also, P_0 is an arbitrary polar vector with γ_2 pseudoscalar quantity that we will put equal to zero, in principle. Note that geometrically the vector torsion field can be uniquely decomposed as

$$h_0 = \nabla_0 a + \varepsilon_{\alpha}^{\gamma\delta\rho} \frac{4\pi}{3} [h_M + q_s n_s \overline{u}_s \cdot \overline{B}] + \gamma_1 h_V + \gamma_2 P_0, \tag{38}$$

being a pseudoscalar field playing the role of axion and h_M magnetic, h_V vortex (γ_1 scalar) and $\overline{u}_s \cdot \overline{B}$ mixed helicities, respectively. Also, P_0 is an arbitrary polar vector with γ_2 pseudoscalar quantity that we will put equal to zero, in principle. Consequently, the complex flow function contains the dynamics of the axion (candidate of dark matter) and the helicities correspond to the term alpha in the equation of induction that generates the astrophysical dynamo effect e.g.:

$$\nabla\zeta = \nabla\psi + \overline{m} \times \nabla \left(\nabla_0 a + \varepsilon_{\alpha}^{\gamma\delta\rho} \frac{4\pi}{3} [h_M + q_s n_s \overline{u}_s \cdot \overline{B}] + \gamma_1 h_V + \gamma_2 P_0 \right). \tag{39}$$

It is interesting to note that, in contrast to this work that is of first principles, the helicities in the alpha term, that causes the anomalous current proportional to the magnetic field B, were suggested in recent works of astrophysics [13] (pulsars, magnetars, gravastars) and placed "by hand".

Points to be considered in future work will be the different types of accretion with dark matrix and effects in jets and mechanisms of accretion in pulsars and effects of emission of gravitational waves in compact objects and black holes. The exotic interactions and charge separation due to the pseudovectorial character of the torsion field h_μ will also be considered.

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