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Symmetric implication zroupoids and identities of Bol-Moufang type

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Abstract:	An algebra $\mathbf A = \langle A, \to, 0 \rangle$, where $\to$ is binary and 0 is a constant, is called an {\it implication zroupoid} ($\mathcal I$-zroupoid, for short) if $\mathbf A$ satisfies the identities: (I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, and (I$_{0}$): $ 0'' \approx 0$, where $x' : = x \to 0$. An implication zroupoid is {\it symmetric} if it satisfies the identities: $x'' \approx x$ and $(x \to y')' \approx (y \to x')'$. An identity is of Bol-Moufang type if it contains only one binary operation symbol, one of its three variables occurs twice on each side, each of the other two variables occurs once on each side, and the variables occur in the same (alphabetical) order on both sides of the identity. In this paper we make a systematic analysis of all 60 identities of Bol-Moufang type in the variety $\mathcal S$ of symmetric $\mathcal I$-zroupoids. We show that 47 of the subvarities of $\mathcal S$, defined by the identities of Bol-Moufang type are equal to the variety $\mathcal{SL}$ of $\lor$-semilattices with the least element 0 and, of the remaining, there are only 3 distinct ones. We also give an explicit description of the poset of the (distinct) subvarieties of $\mathcal S$ of Bol-Moufang type.
Section/Category:	Foundations

Symmetric implication zroupoids and identities of Bol-Moufang type

Juan M. Cornejo and Hanamantagouda P. Sankappanavar

Abstract

An algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is binary and 0 is a constant, is called an *implication zroupoid* ($\mathcal{I}$-zroupoid, for short) if $\mathbf{A}$ satisfies the identities: (I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, and ($\mathrm{I}_0$): $0'' \approx 0$, where $x' := x \to 0$. An implication zroupoid is *symmetric* if it satisfies the identities: $x'' \approx x$ and $(x \to y')' \approx (y \to x')'$. An identity is of Bol-Moufang type if it contains only one binary operation symbol, one of its three variables occurs twice on each side, each of the other two variables occurs once on each side, and the variables occur in the same (alphabetical) order on both sides of the identity.

In this paper we make a systematic analysis of all 60 identities of Bol-Moufang type in the variety $\mathcal{S}$ of symmetric $\mathcal{I}$-zroupoids. We show that 47 of the subvarities of $\mathcal{S}$, defined by the identities of Bol-Moufang type are equal to the variety $\mathcal{SL}$ of $\vee$-semilattices with the least element 0 and, of the remaining, there are only 3 distinct ones. We also give an explicit description of the poset of the (distinct) subvarieties of $\mathcal{S}$ of Bol-Moufang type.

1 Introduction

In 1934, Bernstein gave a system of axioms for Boolean algebras in [Be34] using implication alone. Even though his system was not equational, it is not hard to see that one could easily convert it into an equational one by using an additional constant. In 2012, the second author extended this "modified Bernstein's theorem" to De Morgan algebras in [San12]. Indeed, he shows in [San12] that the varieties of De Morgan algebras, Kleene algebras, and Boolean algebras are term-equivalent, respectively, to the varieties, $\mathcal{DM}$,

$\mathcal{KL}$, and $\mathcal{BA}$ (defined below) whose defining axioms use only an implication $\to$ and a constant 0.

The primary role played by the identity (I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, where $x' := x \to 0$, in the axiomatization of each of those new varieties motivated the second author to introduce a new equational class of algebras called "Implication zroupoids" in [San12]. It also turns out that this new variety contains the variety $\mathcal{SL}$ (defined below) which is shown (see [CS16]) to be term-equivalent to the variety of $\vee$-semilattices with the least element 0.

DEFINITION 1.1. *An algebra* $\mathbf{A} = \langle A, \to, 0\rangle$*, where* $\to$ *is binary and* 0 *is a constant, is called a* zroupoid*. A zroupoid* $\mathbf{A} = \langle A, \to, 0\rangle$ *is an* Implication zroupoid ($\mathcal{I}$-zroupoid, for short) if $\mathbf{A}$ satisfies:

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, where $x' := x \to 0$,

(I_0) $0'' \approx 0$.

$\mathcal{I}$ denotes the variety of implication zroupoids. The varieties $\mathcal{DM}$, $\mathcal{KL}$, $\mathcal{BA}$ and $\mathcal{SL}$ are defined relative to $\mathcal{I}$, respectively, by the following identities:

(DM) $(x \to y) \to x \approx x$ (De Morgan Algebras);
(KL) $(x \to x) \to (y \to y) \approx y \to y$ (Kleene algebras);
(BA) $x \to x \approx 0'$ (Boolean algebras);
(SL) $x' \approx x$ and $x \to y \approx y \to x$.

As proved in [San12], the variety $\mathcal{I}$ generalizes the variety of De Morgan algebras and exhibits several interesting properties; for example, the identity $x''' \to y \approx x' \to y$ holds in $\mathcal{I}$. Several new subvarieties of $\mathcal{I}$ are also introduced and investigated in [San12]. The (still largely unexplored) lattice of subvarieties of $\mathcal{I}$ seems to be fairly complex. In fact, Problem 6 of [San12] calls for an investigation of the structure of the lattice of subvarieties of $\mathcal{I}$.

The papers [CS16], [CS16a] and [CS16b] have addressed further, but still partially, the above-mentioned problem by introducing new subvarieties of $\mathcal{I}$ and investigating relationships among them. The (currently known) size of the poset of subvarieties of $\mathcal{I}$ is at least 24; but it is still unknown whether the lattice of subvarieties is finite or infinite. Two of the subvarieties of $\mathcal{I}$ are: $\mathcal{I}_{2,0}$ and $\mathcal{MC}$ which are defined relative to $\mathcal{I}$, respectively, by the following identities, where $x \wedge y := (x \to y')'$:
($\mathrm{I}_{2,0}$) $x'' \approx x$;

(MC) $x \wedge y \approx y \wedge x$.

For a somewhat more detailed summary of the results contained in the above-mentioned papers, we refer the reader to the Intoduction of [CS16c].

DEFINITION 1.2. *Let* $\mathbf{A} \in \mathcal{I}$. $\mathbf{A}$ *is* involutive *if* $A \in \mathcal{I}_{2,0}$. $\mathbf{A}$ *is* meet-commutative *if* $A \in \mathcal{MC}$. $\mathbf{A}$ *is* symmetric *if* $\mathbf{A}$ *is both involutive and meet-commutative. Let* $\mathcal{S}$ *denote the variety of symmetric* $\mathcal{I}$*-zroupoids. In other words,* $\mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{MC}$.

In the present paper we are interested in the subvarieties of $\mathcal{S}$ defined by certain *weak associative laws*, called "Bol-Moufang" laws. A precise definition of a *weak associative law* appears in [Ku96], which is essentially restated below.

DEFINITION 1.3. *Let* $n \in \mathbb{N}$ *and let* $\mathcal{L} := \langle \times \rangle$, *where* $\times$ *is a binary operation symbol. A (groupoid) term in* $\mathcal{L}$ *is of length* n *if the number of occurrences of variables (not necessarily distinct) is* n. *A weak associative law of length* n *in* $\mathcal{L}$ *is an identity of the form* $p \approx q$, *where* p *and* q *are of length* n *and contain the the same variables that occur in the same order in both* p *and* q *(only the bracketing is possibly different).*

We note that the identities of *Bol-Moufang type* investigated in [Fe69] are weak associative laws of length 4 in three distinct variables, with one of them repeated (see below for a more precise definition).

The following (general) problem presents itself naturally.

PROBLEM: Let $\mathcal{V}$ be a given variety of algebras (whose language includes a binary operation symbol, say, $\times$). Investigate the subvarieties of $\mathcal{V}$ defined by weak associative laws (with respect to $\times$) and their mutual relationships.

Special cases of the above problem have already been considered in the literature, wherein the weak associative laws are the identities of Bol-Moufang type, and the variety $\mathcal{V}$ is the variety of quasigroups or the variety of loops (see [Fe69], [Ku96], [PV05a], [PV05b]).

[Fe69] has noted that there are 60 weak associative laws of length 4 in three distinct variables (with one variable repeated). Since the Moufang laws and the Bol identities were among those 60 laws, the author of that paper called such identities as those of Bol-Moufang type. For more information about these identities in the context of quasigroups and loops, see [Ku96, PV05a, PV05b].

In this paper we initiate the systematic study of the identities of Bol-Moufang type in the context of symmetric implication zroupoids.

Without loss of generality, we will assume that the variables in p and q occur alphabetically. So, we can say (more explicitly) that an identity $p \approx q$, in the language $\langle \to \rangle$, is of *Bol-Moufang type* if:

(i) p and q contain the same three variables (chosen alphabetically),

(ii) The length of $p = 4 =$ the length of q,

(iii) The order in which the variables appear in p is exactly the same as the order in which they appear in q.

Let B denote the set of identities of Bol-Moufang type.

The systematic notation for the identities in B, presented below, was developed in [PV05a].

Let x, y, z be all the variables appearing in the identities of B. Without loss of generality, we can assume that they appear in the terms in the alphabetical order. Then, the 6 ways in which the 3 variables can form a word of length 4 are shown below:

A :	xxyz;	B :	xyxz;
C :	xyyz;	D :	xyzx;
E :	xyzy;	F :	xyzz.

It is clear that there are exactly 5 ways in which a word of size 4 can be bracketed, as given below:

1: $o \to (o \to (o \to o))$;
2: $o \to ((o \to o) \to o)$;
3: $(o \to o) \to (o \to o)$;
4: $(o \to (o \to o)) \to o$;
5: $((o \to o) \to o) \to o$.

Let (X_{ij}) with $X \in \{\mathrm{A}, \ldots, \mathrm{F}\}$, $1 \leq i < j \leq 5$, denote the identity from B whose left-hand side is bracketed according to i, and whose right-hand side is bracketed according to j. For instance, (A_{12}) is the identity

$x \to (x \to (y \to z)) \approx x \to ((x \to y) \to z)$. We note that B_{15} is a Moufang identity, while E_{25} is a Bol identity.

It is noted in [Fe69] that there are $(6 \times (4+3+2+1) =)$ 60 nontrivial identities in B.

We will denote by $\mathcal{X}_{ij}$ the subvariety of $\mathcal{S}$ defined by the identity (X_{ij}).

We will show in Section 3 that 47 of these 60 varieties coincide with $\mathcal{SL}$ (and hence with each other). In Section 4, we prove our main result that there are 4 nontrivial varieties of Bol-Moufang type that are distinct from each other. Furthermore, we describe explicitly the poset formed by them, together with the variety $\mathcal{BA}$ (which is contained in some of them).

We would like to acknowledge that the software "Prover 9/Mace 4" developed by McCune [Mc] has been useful to us in some of our findings presented in this paper. We have used it to find examples and to check some conjectures.

2 Preliminaries

We refer the reader to the standard references [BD74], [BS81] and [R74] for concepts and results used, but not explained, in this paper.

Recall from [San12] that $\mathcal{SL}$ is the variety of semilattices with a least element 0. It was shown in [CS16] that $\mathcal{SL} = \mathcal{C} \cap \mathcal{I}_{1,0}$.

The two-element algebras $\mathbf{2_s}$, $\mathbf{2_b}$ were introduced in [San12]. Their operations $\to$ are respectively as follows:

$\to$:	0	1
0	0	1
1	1	1

$\to$:	0	1
0	1	1
1	0	1

Notice that $\mathcal{V}(\mathbf{2_b}) = \mathcal{BA}$. Recall also from [CS16, Corollary 11.4] that $\mathcal{V}(\mathbf{2_s}) = \mathcal{SL}$. The following lemma easily follows from the definition of $\wedge$ given earlier in the Introduction.

LEMMA 2.1. $\mathcal{MC} \cap \mathcal{I}_{1,0} \subseteq \mathcal{C} \cap \mathcal{I}_{1,0} = \mathcal{SL}$.

LEMMA 2.2. [San12, Theorem 8.15] *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-zroupoid. Then the following are equivalent:*

(a) $0' \to x \approx x$,

(b) $x'' \approx x$,

(c) $(x \to x')' \approx x$,

(d) $x' \to x \approx x$.

Recall that $\mathcal{I}_{2,0}$ and $\mathcal{MC}$ are the subvarieties defined, respectivaly, relative to $\mathcal{I}$ by the equations

$$x'' \approx x. \tag{$I_{2,0}$}$$

$$x \wedge y \approx y \wedge x. \tag{MC}$$

LEMMA 2.3. **[San12]** *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then*

(a) $x' \to 0' \approx 0 \to x$,

(b) $0 \to x' \approx x \to 0'$.

LEMMA 2.4. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(a) $(x \to 0') \to y \approx (x \to y') \to y$,

(b) $x \to (0 \to x)' \approx x'$,

(c) $(y \to x) \to y \approx (0 \to x) \to y$,

(d) $(0 \to x) \to (0 \to y) \approx x \to (0 \to y)$,

(e) $x \to y \approx x \to (x \to y)$,

(f) $0 \to (0 \to x)' \approx 0 \to x'$,

(g) $0 \to (x' \to y)' \approx x \to (0 \to y')$,

(h) $0 \to (x \to y) \approx x \to (0 \to y)$,

(i) $0 \to (x \to y')' \approx 0 \to (x' \to y)$,

(j) $x \to (y \to x') \approx y \to x'$,

(k) $(x \to y')' \to z \approx x \to (y \to z)$.

Proof. For the proofs of items (a), (b), (c), (f), (g), (h), (i) and (j) we refer the reader to [CS16]. The proofs of items (d) and (e) are given in [CS16a], while the proof of the item (k) can be found in [CS16c]. □

LEMMA 2.5. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$ *such that* $\mathbf{A} \models 0 \to x \approx x$, *then* $\mathbf{A} \models (x \to y)' \approx x' \to y'$.

Proof. Let $a, b \in A$. Hence, we have that

$$\begin{array}{rcll} a' \to b' & = & 0 \to (a' \to b') & \text{by hyphotesis} \\ & = & a \to (0 \to b) & \text{by Lemma 2.4 (h)} \\ & = & 0 \to (a \to b)' & \text{by Lemma 2.4 (g)} \\ & = & (a \to b)' & \text{by hyphotesis.} \end{array}$$

This completes the proof. □

3 Symmetric $\mathcal{I}$-zroupoids of Bol-Moufang type

Recall that the variety $\mathcal{S} = \mathcal{I}_{2,0} \cap \mathcal{MC}$, which was investigated in [CS16]. Throughout this section, $\mathbf{A} \in \mathcal{S}$.

In this section our goal is to prove that 47 of the subvarieties of $\mathcal{S}$ of Bol-Moufang type are equal to $\mathcal{SL}$ and hence are equal to each other. First, we present some new properties of $\mathcal{S}$ which will be useful later in this paper.

LEMMA 3.1. $\mathbf{A}$ *satisfies:*

(a) $x \to (y \to z) \approx y \to (x \to z)$,

(b) $x' \to y \approx y' \to x$.

Proof. (a) Let $a, b, c \in A$. Then

$$\begin{array}{rcll} a \to (b \to c) & = & (a \to b')' \to c & \text{by Lemma 2.4 (k)} \\ & = & (b \to a')' \to c & \text{by (MC)} \\ & = & b \to (a \to c) & \text{by Lemma 2.4 (k).} \end{array}$$

(b) Let $a, b \in A$. Then,

$$\begin{array}{rcll} a' \to b & = & (a' \to b'')'' & \text{by } (I_{2,0}) \\ & = & (b' \to a'')'' & \text{by (MC)} \\ & = & b' \to a & \text{by } (I_{2,0}). \end{array}$$

□

LEMMA 3.2. *Let* $\mathbf{A} \models x \to x \approx x$*. Then* $\mathbf{A} \models x' \approx x$*.*

Proof. Let $a \in A$. Then

$$\begin{array}{rcll} a' & = & (a \to a)' & \text{by hyphotesis} \\ & = & (a \to a'')' & \text{by } (I_{2,0}) \\ & = & (a' \to a')' & \text{by (MC)} \\ & = & a'' & \text{by hyphotesis} \\ & = & a & \text{by } (I_{2,0}). \end{array}$$

□

LEMMA 3.3. *Let* $\mathbf{A} \models x' \approx x \to x$ *and* $\mathbf{A} \models 0' \approx 0$*. Then* $\mathbf{A} \models x' \approx x$*.*

Proof. Let $a \in A$. Then

$$\begin{array}{rcll} a & = & a'' & \\ & = & (a \to a)' & \text{by hypothesis} \\ & = & (a \to a) \to 0 & \\ & = & (a \to a) \to 0' & \text{by hypothesis} \\ & = & \{(0'' \to a) \to (a \to 0')'\}' & \text{by (I)} \\ & = & \{(0 \to a) \to (a \to 0')'\}' & \text{by } (I_{2,0}) \\ & = & \{(0 \to a) \to (a \to 0)'\}' & \text{by hypothesis} \\ & = & \{(0 \to a) \to a''\}' & \\ & = & \{(0 \to a) \to a\}' & \text{by } (I_{2,0}) \\ & = & \{(a \to a) \to a\}' & \text{by Lemma 2.4 (c)} \\ & = & (a' \to a)' & \text{by hypothesis} \\ & = & a' & \text{by Lemma 2.2 (d),} \end{array}$$

completing the proof. □

LEMMA 3.4. *Let* $\mathbf{A} \models 0 \to (x \to x) \approx x \to x$*. Then*
$\mathbf{A} \models (x \to x) \to (y \to z) \approx ((x \to x) \to y) \to z$*.*

Proof. Let $a, b, c \in A$. Hence

$$\begin{aligned}
(a \to a) \to b' &= [0 \to (a \to a)] \to b' && \text{by hyphotesis} \\
&= [b' \to (a \to a)] \to b' && \text{by Lemma 2.4 (c)} \\
&= \{(b'' \to b') \to [(a \to a) \to b']'\}' && \text{by (I)} \\
&= \{b' \to [(a \to a) \to b']'\}' && \text{by Lemma 2.2 (d)} \\
&= \{[(a \to a) \to b'] \to b''\}' && \text{by Lemma 3.1 (b)} \\
&= \{[(a \to a) \to b'] \to b\}' && \text{by } (I_{2,0}) \\
&= \{[(a \to a) \to 0'] \to b\}' && \text{by Lemma 2.4 (a)} \\
&= \{[0 \to (a \to a)'] \to b\}' && \text{by Lemma 2.3 (b)} \\
&= \{[0 \to (a' \to a')] \to b\}' && \text{by Lemma 2.4 (i)} \\
&= \{(a' \to a') \to b\}' && \text{by hyphotesis} \\
&= \{(a \to a) \to b\}' && \text{by Lemma 3.1 (b).}
\end{aligned}$$

Therefore,

$$\mathbf{A} \models (x \to x) \to y' \approx [(x \to x) \to y]'. \tag{1}$$

Hence,

$$\begin{aligned}
((a \to a) \to b) \to c &= c' \to [(a \to a) \to b]' && \text{by Lemma 3.1 (b)} \\
&= c' \to [(a \to a) \to b'] && \text{by (1)} \\
&= (a \to a) \to (c' \to b') && \text{by Lemma 3.1 (a)} \\
&= (a \to a) \to (b \to c) && \text{by Lemma 3.1 (b).}
\end{aligned}$$

This completes the proof. □

LEMMA 3.5. $\mathcal{BA} \subseteq \mathcal{A}_{12}$.

Proof. It is routine to check that $\mathbf{2_b} \in \mathcal{A}_{12}$. The proof is complete since we know $\mathcal{V}(\mathbf{2_b}) = \mathcal{BA}$. □

LEMMA 3.6. *$\mathcal{SL} \subseteq \mathcal{X}_{ij}$, for $\mathcal{X} \in \{A, B, C, D, E, F\}$ and for all $1 \leq i < j \leq 5$.*

Proof. By a routine computation, it is easy to check that $\mathbf{2_s} \in \mathcal{X}_{ij}$ for all $1 \leq i < j \leq 5$. Then the proof is complete, in view of $\mathcal{V}(\mathbf{2_s}) = \mathcal{SL}$. □

Let $\mathbf{M}$ denote the set that contains the subvarieties $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$, $\mathcal{A}_{24}$, $\mathcal{A}_{34}$, $\mathcal{A}_{45}$, $\mathcal{B}_{12}$, $\mathcal{B}_{14}$, $\mathcal{B}_{15}$, $\mathcal{B}_{23}$, $\mathcal{B}_{24}$, $\mathcal{B}_{34}$, $\mathcal{B}_{35}$, $\mathcal{B}_{45}$, $\mathcal{C}_{12}$, $\mathcal{C}_{13}$, $\mathcal{C}_{14}$, $\mathcal{C}_{15}$, $\mathcal{C}_{23}$, $\mathcal{C}_{24}$, $\mathcal{C}_{34}$, $\mathcal{C}_{35}$, $\mathcal{C}_{45}$, $\mathcal{D}_{13}$, $\mathcal{D}_{14}$, $\mathcal{D}_{15}$, $\mathcal{D}_{23}$, $\mathcal{D}_{24}$, $\mathcal{D}_{34}$, $\mathcal{D}_{45}$, $\mathcal{E}_{12}$, $\mathcal{E}_{13}$, $\mathcal{E}_{14}$, $\mathcal{E}_{15}$, $\mathcal{E}_{23}$, $\mathcal{E}_{24}$, $\mathcal{E}_{34}$, $\mathcal{E}_{35}$, $\mathcal{E}_{45}$, $\mathcal{F}_{12}$, $\mathcal{F}_{14}$, $\mathcal{F}_{15}$, $\mathcal{F}_{23}$, $\mathcal{F}_{24}$, $\mathcal{F}_{34}$, $\mathcal{F}_{35}$, $\mathcal{F}_{45}$. Observe that $\mathbf{M}$ has 47 elements.

THEOREM 3.7. *If $\mathcal{X} \in \mathbf{M}$ then $\mathcal{X} = \mathcal{SL}$.*

Proof. In the proof below the following list of statements will be useful.

(∗) The identity $(I_{2,0})$, Lemma 2.2 (a), Lemma 2.2 (d), Lemma 2.4 (a), Lemma 2.4 (b), Lemma 2.4 (e) and Lemma 3.1 (a).

Let $\mathcal{X} \in M$. In view of Lemma 3.6, it suffices to prove that $\mathcal{X} \subseteq \mathcal{SL}$. In fact, by Lemma 2.1, it suffices to prove that $\mathcal{X} \models x' \approx x$. Let $\mathbf{A} \in \mathcal{X}$ and let $a \in \mathbf{A}$.

To facilitate a uniform presentation (and to make the proof shorter), we introduce the following notation, where $x_0, y_0, z_0 \in \mathbf{A}$:

The notation

$$\mathcal{X}/x_0, y_0, z_0$$

denotes the following statement:

"In the identity (X) that defines the variety $\mathcal{X}$, relative to $\mathcal{S}$, if we assign $x := x_0, y := y_0, z := z_0$ (and simplify it using the list (∗)), then $\mathbf{A} \models x \to x \approx x$".

We divide the 47 varieties under consideration into several groups (again, with a view to making this proof shorter) as follows:

Firstly, we consider the varieties associated with the following statements:

1. $\mathcal{A}_{24}/a, a', a$,
2. $\mathcal{A}_{45}/a, 0, 0$,
3. $\mathcal{B}_{12}/a', a, 0$,
4. $\mathcal{B}_{14}/a, a, 0$,
5. $\mathcal{B}_{23}/a', 0, 0$,
6. $\mathcal{B}_{24}/a, 0', 0$,
7. $\mathcal{B}_{34}/0', a, a$,
8. $\mathcal{B}_{35}/0', a, a$,
9. $\mathcal{B}_{45}/a, 0', 0$,
10. $\mathcal{C}_{15}/0', a, 0$,
11. $\mathcal{C}_{23}/0', a, 0$,
12. $\mathcal{C}_{24}/a, 0, a$,
13. $\mathcal{C}_{35}/0', a, 0$,
14. $\mathcal{C}_{45}/a, 0, a$,
15. $\mathcal{D}_{13}/a, a', a'$,
16. $\mathcal{D}_{14}/a, a, a'$,
17. $\mathcal{D}_{23}/a, 0', 0'$,
18. $\mathcal{D}_{24}/a, 0', 0'$,
19. $\mathcal{D}_{45}/a, 0, 0$,
20. $\mathcal{E}_{15}/0', a, 0'$,
21. $\mathcal{E}_{23}/0', a, 0'$,
22. $\mathcal{E}_{34}/0', a, 0'$,
23. $\mathcal{E}_{35}/0', a, 0'$,
24. $\mathcal{F}_{24}/a, a, 0$.

It is routine to verify that each of the above statements is true, from which it follows that, in each case, $\mathcal{X} \models x \to x \approx x$. Then, applying Lemma

3.2, we get that $\mathcal{X} \models x' \approx x$.

The notation, where $x_0, y_0, z_0, x_1, y_1, z_1 \in \mathbf{A}$,

$$\mathcal{X}/x_0, y_0, z_0/x_1, y_1, z_1\ /p \approx q$$

is an abbreviation for the following statement:

"In the identity (X) that defines the variety $\mathcal{X}$, relative to $\mathcal{S}$, if we assign $x := x_0, y := y_0, z := z_0$, (and simplify (X) using the appropriate lemmas from the list $(*)$), we obtain that $\mathbf{A} \models 0' \approx 0$; and then we assign $x := x_1, y := y_1, z := z_1$ in the identity (X) (and simplify it using $0' = 0$ and the list $(*)$), then $\mathbf{A} \models p \approx q$."

Secondly, consider the varieties associated with the following statements:

1. $\mathcal{D}_{15}/0, 0, 0'/0, a, a\ /x \to x \approx x$,
2. $\mathcal{D}_{34}/0, 0, 0/0', a, a\ /x \to x \approx x$,
3. $\mathcal{E}_{13}/0, 0, 0'/a', 0, a'/x \to x \approx x$,
4. $\mathcal{E}_{14}/0, 0, 0/a, a', 0\ /x \to x \approx x$,
5. $\mathcal{E}_{24}/0, 0, 0/a, 0, a\ /x \to x \approx x$,
6. $\mathcal{E}_{45}/0, 0, 0/a, 0, a\ /x \to x \approx x$.

It is straightforward to verify that each of the above statements is true. Hence, it follows that in each case $\mathcal{X} \models x \to x \approx x$. Then, applying Lemma 3.2, we get that $\mathcal{X} \models x' \approx x$.

Thirdly, consider the varieties associated with the following statements:

1. $\mathcal{A}_{14}/0, 0, 0/a, 0', 0\ /x' \approx x$,
2. $\mathcal{F}_{14}/0, 0, 0/a, 0, 0\ /x' \approx x$,
3. $\mathcal{F}_{23}/0', 0, 0/a, 0, 0\ /x' \approx x$,
4. $\mathcal{F}_{34}/0', 0, 0/a, a, 0\ /x' \approx x$,
5. $\mathcal{F}_{35}/0', 0, 0/0', a, 0\ /x' \approx x$,
6. $\mathcal{F}_{45}/0, 0, 0/a, 0, 0\ /x' \approx x$.

It is easy to verify that the above statements are true. Hence, it follows in each of the above cases that $\mathcal{X} \models x' \approx x$.

Lastly, consider the varieties associated with the following statements:

1. $\mathcal{A}_{15}/0, 0', 0/a, 0, 0\ /x' \approx x \to x$,
2. $\mathcal{A}_{34}/0, 0, 0/a, 0, 0\ /x' \approx x \to x$,
3. $\mathcal{F}_{15}/0', 0, 0/a, a', 0\ /x' \approx x \to x$.

It is clear that each of the above statements is true. Hence in each case $\mathcal{X} \models x' \approx x \to x$. Then, applying Lemma 3.3, we get that $\mathcal{X} \models x' \approx x$.

Thus, the varieties still left to consider are $\mathcal{A}_{13}$, $\mathcal{B}_{15}$, $\mathcal{C}_{12}$, $\mathcal{C}_{13}$, $\mathcal{C}_{14}$, $\mathcal{C}_{34}$, $\mathcal{E}_{12}$ and $\mathcal{F}_{12}$.

From Lemma 3.1 (a) we can easily verify that $\mathcal{C}_{13} = \mathcal{B}_{12}$, $\mathcal{C}_{14} = \mathcal{B}_{14}$, $\mathcal{C}_{34} = \mathcal{B}_{24}$, $\mathcal{E}_{12} = \mathcal{D}_{13}$, and $\mathcal{F}_{12} = \mathcal{E}_{13}$, from which we have, in view of earlier conclusions that $\mathcal{C}_{13} = \mathcal{C}_{14} = \mathcal{C}_{34} = \mathcal{E}_{12} = \mathcal{E}_{14} = \mathcal{F}_{12} = \mathcal{SL}$.

Now, we are left with $\mathcal{A}_{13}$, $\mathcal{B}_{15}$, and $\mathcal{C}_{12}$ to consider. But, it can be easily seen, using Lemma 3.1, that $\mathcal{C}_{12} = \mathcal{A}_{13}$. Thus, it only remains to verify that $\mathcal{A}_{13} = \mathcal{SL}$ and $\mathcal{B}_{15} = \mathcal{SL}$.

First, we show that $\mathcal{A}_{13} = \mathcal{SL}$. Let $\mathbf{A} \in \mathcal{A}_{13}$ and let $a \in A$. Then

$$\begin{array}{rcll} a & = & 0' \to a & \text{by Lemma 2.2 (a)} \\ & = & (0 \to 0) \to a & \\ & = & (0 \to 0) \to (a' \to a) & \text{by Lemma 2.2 (d)} \\ & = & 0 \to (0 \to (a' \to a)) & \text{by } (A_{13}) \\ & = & 0 \to (0 \to a) & \text{by Lemma 2.2 (d)} \\ & = & 0 \to a & \text{by Lemma 2.4 (e).} \end{array}$$

Hence
$$\mathbf{A} \models x \approx 0 \to x. \tag{2}$$
Therefore

$$\begin{array}{rcll} a & = & a' \to a & \text{by Lemma 2.2 (d)} \\ & = & a' \to (a' \to a) & \text{by Lemma 2.4 (e)} \\ & = & a' \to (a' \to a'') & \text{by } (I_{2,0}) \\ & = & a' \to (a' \to (a' \to 0)) & \\ & = & (a' \to a') \to (a' \to 0) & \text{by } (A_{13}) \\ & = & (a' \to a') \to a'' & \\ & = & (a' \to 0') \to a'' & \text{by Lemma 2.4 (a)} \\ & = & (0 \to a'') \to a'' & \text{by Lemma 2.3 (b)} \\ & = & (0 \to a) \to a & \text{by } (I_{2,0}) \\ & = & a \to a & \text{by (2).} \end{array}$$

Using Lemma 3.2 we have that $\mathbf{A} \in \mathcal{SL}$, hence $\mathcal{A}_{13} \in \mathcal{SL}$, implying $\mathcal{A}_{13} = \mathcal{SL}$.

Next, we prove that $\mathcal{B}_{15} = \mathcal{SL}$. Let $\mathbf{A} \in \mathcal{B}_{15}$ and let $a \in A$. Then

$$
\begin{aligned}
a &= a'' && \text{by } (I_{2,0})\\
&= (a'' \to a')' && \text{by Lemma 2.2 (d)}\\
&= ((a' \to 0) \to a')' &&\\
&= ((a' \to 0) \to a') \to 0 &&\\
&= a' \to (0 \to (a' \to 0)) && \text{by } (B_{15})\\
&= a' \to (0 \to a'') &&\\
&= a' \to (0 \to a) && \text{by } (I_{2,0})\\
&= 0 \to (a' \to a) && \text{by Lemma 3.1 (a)}\\
&= 0 \to a && \text{by Lemma 2.2 (d).}
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= a'' && \text{by } (I_{2,0})\\
&= (a \to a')' && \text{by Lemma 2.2 (d)}\\
&= (a \to (a \to 0))' &&\\
&= (a \to (a \to (a \to 0)))' && \text{by Lemma 2.4 (e)}\\
&= ((a \to a) \to a)'' && \text{by } (B_{15})\\
&= (a \to a) \to a && \text{by } (I_{2,0})\\
&= (0 \to a) \to a && \text{by Lemma 2.4 (c)}\\
&= a \to a && \text{by previous calculus.}
\end{aligned}
$$

Thus,

$$\mathbf{A} \models x \approx x \to x.$$

By Lemma 3.2 we have that $\mathbf{A} \models x' \approx x$, implying that $\mathcal{B}_{15} \in \mathcal{SL}$. So, $\mathcal{B}_{15} = \mathcal{SL}$, completing the proof. □

The following corollary is immediate from the above theorem.

COROLLARY 3.8. $\mathbf{M} = \{\mathcal{SL}\}$.

Thus, there are 47 Bol-Moufang subvarieties each of which is equal to $\mathcal{SL}$; and hence they are equal to each other.

In order to be able to compare the remaining (possibly distinct) subvarieties with each other, the following lemmas will be useful.

LEMMA 3.9. *Let* $\mathbf{A} \in \mathcal{A}_{23}$. *Then* $\mathbf{A}$ *satisfies the identities:*

(a) $0 \to x \approx x$,

(b) $(x \to x) \to y \approx x \to ((x \to 0') \to y)$,

(c) $(x \to x) \to y \approx x \to (x \to y')'$.

Proof. (a) Let $a \in A$. Then

$$\begin{aligned} a &= 0' \to a && \text{by Lemma 2.2 (a)} \\ &= (0 \to 0) \to a \\ &= (0 \to 0) \to a'' && \text{by } (I_{2,0}) \\ &= (0 \to 0) \to (a' \to 0) \\ &= 0 \to (0 \to a')' && \text{by } (A_{23}) \\ &= 0 \to a'' && \text{by Lemma 2.4 (f)} \\ &= 0 \to a && \text{by } (I_{2,0}). \end{aligned}$$

(b) Let $a, b \in A$. Then, using (A_{23}) and Lemma 2.2 (a), we have that $a \to ((a \to 0') \to b) = (a \to a) \to (0' \to b) = (a \to a) \to b$.

(c) Let $a, b \in A$. By (A_{23}) we have that $(a \to a) \to b = (a \to a) \to b'' = (a \to a) \to (b' \to 0) = a \to (a \to b')'$.

□

LEMMA 3.10.

(1) *Let* $\mathbf{A} \in \mathcal{A}_{25} \cup \mathcal{C}_{25} \cup \mathcal{D}_{25} \cup \mathcal{E}_{25}$. *Then* $\mathbf{A}$ *satisfies the identities*

(a) $0 \to x \approx x$,

(b) $(x \to y)' \approx x' \to y'$.

(2) *Let* $\mathbf{A} \in \mathcal{A}_{25}$. *Then* $\mathbf{A} \models x \to (y \to (y' \to z)) \approx y \to ((y \to x) \to z)$.

Proof. (1) (a) Let $\mathbf{A} \in \mathcal{A}_{25}$ and $a \in A$. Then

$$\begin{aligned} a &= a'' && \text{by } (I_{2,0}) \\ &= a' \to 0 \\ &= (0' \to a') \to 0 && \text{by Lemma 2.2 (a)} \\ &= ((0 \to 0) \to a') \to 0 \\ &= 0 \to ((0 \to a') \to 0) && \text{by } (A_{25}) \\ &= 0 \to a'' && \text{by Lemma 2.4 (f)} \\ &= 0 \to a && \text{by } (I_{2,0}). \end{aligned}$$

Let $\mathbf{A} \in \mathcal{C}_{25}$ and $a \in A$. Then

$$\begin{aligned} 0 \to a &= 0 \to (0' \to a) && \text{by Lemma 2.2 (a)} \\ &= 0 \to ((0' \to 0') \to a) && \text{by Lemma 2.2 (a)} \\ &= ((0 \to 0') \to 0') \to a && \text{by } (C_{25}) \\ &= (0' \to 0') \to a && \text{by Lemma 2.2 (d)} \\ &= 0' \to a && \text{by Lemma 2.2 (a)} \\ &= a && \text{by Lemma 2.2 (a).} \end{aligned}$$

Let $\mathbf{A} \in \mathcal{D}_{25}$ and $a \in A$. Then

$$\begin{array}{rcll} 0 \to a & = & 0 \to a'' & \text{by } (I_{2,0}) \\ & = & 0 \to (0 \to a')' & \text{by Lemma 2.4 (b)} \\ & = & 0 \to ((0 \to a') \to 0) & \\ & = & ((0 \to 0) \to a') \to 0 & \text{by } (D_{25}) \\ & = & (0' \to a') \to 0 & \\ & = & a' \to 0 & \text{by Lemma 2.2 (a)} \\ & = & a & \text{by } (I_{2,0}). \end{array}$$

Let $\mathbf{A} \in \mathcal{E}_{25}$ and $a \in A$. Then

$$\begin{array}{rcll} 0 \to a & = & 0 \to a'' & \text{by } (I_{2,0}) \\ & = & 0 \to (0 \to a')' & \text{by Lemma 2.4 (b)} \\ & = & 0 \to ((0 \to a') \to 0) & \\ & = & ((0 \to 0) \to a') \to 0 & \text{by } (E_{25}) \\ & = & (0' \to a') \to 0 & \\ & = & a' \to 0 & \text{by Lemma 2.2 (a)} \\ & = & a & \text{by } (I_{2,0}). \end{array}$$

(b) By (1a), $\mathbf{A} \models x \approx 0 \to x$. Then by Lemma 2.5, we have

$$\mathbf{A} \models (x \to y)' \approx x' \to y'.$$

(2) Let $a, b, c \in A$. Then

$$\begin{array}{rcll} a \to (b \to (b' \to c)) & = & a \to ((b \to b)' \to c) & \text{by } (A_{25}) \\ & = & a \to (c' \to (b \to b)) & \text{by Lemma 3.1 (b)} \\ & = & a \to (b \to (c' \to b)) & \text{by Lemma 3.1 (a)} \\ & = & b \to (a \to (c' \to b)) & \text{by Lemma 3.1 (a)} \\ & = & b \to (c' \to (a \to b)) & \text{by Lemma 3.1 (a)} \\ & = & b \to ((a \to b)' \to c) & \text{by Lemma 3.1 (b)} \\ & = & b \to ((a' \to b') \to c) & \text{by (1b)} \\ & = & b \to ((b \to a) \to c) & \text{by Lemma 3.1 (b).} \end{array}$$

This completes the proof. □

LEMMA 3.11. *Let* $\mathbf{A} \in \mathcal{A}_{12} \cup \mathcal{D}_{12} \cup \mathcal{D}_{35}$ *then* $\mathbf{A}$ *satisfies the identities*

(a) $0 \to x' \approx x \to x \approx 0 \to x$,

(b) $0 \to (x \to y) \approx 0 \to (y \to x)$,

(c) $0 \to (x \to (y \to z)) \approx 0 \to ((x \to y) \to z)$.

Proof. Let $a, b, c \in A$.

(a) Suppose $\mathbf{A} \in \mathcal{A}_{12}$. Then

$$\begin{array}{rcll} 0 \to a' & = & a \to 0' & \text{by Lemma 2.3 (b)} \\ & = & a \to (a \to 0') & \text{by Lemma 2.4 (e)} \\ & = & a \to (a \to (0 \to 0)) & \\ & = & a \to ((a \to 0) \to 0) & \text{by } (A_{12}) \\ & = & a \to a'' & \\ & = & a \to a & \text{by } (I_{2,0}). \end{array}$$

Thus, we have

$$\mathbf{A} \models 0 \to x' \approx x \to x. \tag{3}$$

Hence,

$$\begin{array}{rcll} 0 \to a & = & 0 \to a'' & \text{by } (I_{2,0}) \\ & = & a' \to a' & \text{by (3)} \\ & = & a \to a & \text{by Lemma 3.1 (b).} \end{array}$$

Consequently,

$$\mathbf{A} \models 0 \to x \approx x \to x.$$

Next, assume that $\mathbf{A} \in \mathcal{D}_{12}$. Then

$$\begin{array}{rcll} a \to a & = & a \to (0' \to a) & \text{by Lemma 2.2 (a)} \\ & = & a \to ((0 \to 0) \to a) & \\ & = & a \to (0 \to (0 \to a)) & \text{by } (D_{12}) \\ & = & 0 \to (0 \to (a \to a)) & \text{by Lemma 3.1 (a) twice} \\ & = & 0 \to (a \to a) & \text{by Lemma 2.4 (e)} \\ & = & 0 \to (a'' \to a'') & \text{by } (I_{2,0}) \\ & = & 0 \to (a' \to a')' & \text{by Lemma 2.4 (i)} \\ & = & 0 \to ((a' \to a') \to 0) & \\ & = & 0 \to (a' \to (a' \to 0)) & \text{by } (D_{12}) \\ & = & 0 \to (a' \to a) & \text{by } (I_{2,0}) \\ & = & 0 \to a & \text{by Lemma 2.2 (d).} \end{array}$$

Hence

$$\mathbf{A} \models 0 \to x \approx x \to x. \tag{4}$$

Now,

$$\begin{aligned} 0 \to a' &= a' \to a' && \text{by (4)} \\ &= a \to a && \text{by Lemma 3.1 (b).} \end{aligned}$$

Consequently,

$$\mathbf{A} \models 0 \to x' \approx x \to x.$$

Finally, assume that $\mathbf{A} \in \mathcal{D}_{35}$. Then

$$\begin{aligned} 0 \to a &= a' \to (0 \to a) && \text{by Lemma 2.4 (j)} \\ &= (a \to 0) \to (0 \to a) && \\ &= ((a \to 0) \to 0) \to a && \text{by } (D_{35}) \\ &= a'' \to a && \\ &= a \to a && \text{by } (I_{2,0}). \end{aligned}$$

Hence

$$\mathbf{A} \models 0 \to x \approx x \to x. \tag{5}$$

$$\begin{aligned} 0 \to a' &= a' \to a' && \text{by (5)} \\ &= a \to a && \text{by Lemma 3.1 (b).} \end{aligned}$$

Consequently.

$$\mathbf{A} \models 0 \to x' \approx x \to x.$$

(b) Observe that

$$\begin{aligned} 0 \to (a \to b) &= 0 \to (b' \to a') && \text{by Lemma 3.1 (b)} \\ &= (0 \to b') \to (0 \to a') && \text{by Lemma 2.4 (h) and (d)} \\ &= (0 \to b) \to (0 \to a) && \text{by (a)} \\ &= 0 \to (b \to a) && \text{by Lemma 2.4 (h) and (d).} \end{aligned}$$

(c)

$$\begin{aligned} 0 \to (b \to (c \to a)) &= (0 \to b) \to (0 \to (c \to a)) \\ & \quad \text{by Lemma 2.4 (h) and (d)} \\ &= (0 \to b) \to (0 \to (a \to c)) && \text{by (b)} \\ &= 0 \to (b \to (a \to c)) \\ & \quad \text{by Lemma 2.4 (h) and (d)} \\ &= 0 \to (a \to (b \to c)) && \text{by Lemma 3.1 (a)} \\ &= 0 \to ((b \to c) \to a) && \text{by (b).} \end{aligned}$$

This completes the proof. $\square$

4 Distinct Varieties of Symmetric $\mathcal{I}$-zroupoids of Bol-Moufang type

Recall that 47 of the 60 subvarieties of $\mathcal{S}$ were shown to be equal to the variety $\mathcal{SL}$. In this section we will investigate the relationships among the remaining 13 varieties: $\mathcal{A}_{12}, \mathcal{A}_{23}, \mathcal{A}_{25}, \mathcal{A}_{35}, \mathcal{B}_{13}, \mathcal{B}_{25}, \mathcal{C}_{25}, \mathcal{D}_{12}$, $\mathcal{D}_{25}, \mathcal{D}_{35}, \mathcal{E}_{25}, \mathcal{F}_{13}$ and $\mathcal{F}_{25}$. We still need to determine which of these are distinct from each other. The following theorem throws more light on this issue.

THEOREM 4.1. *We have*

(a) $\mathcal{A}_{23} = \mathcal{A}_{25} = \mathcal{C}_{25} = \mathcal{D}_{25} = \mathcal{E}_{25}$,

(b) $\mathcal{A}_{12} = \mathcal{B}_{13} = \mathcal{D}_{12} = \mathcal{D}_{35} = \mathcal{F}_{13}$,

(c) $\mathcal{A}_{35} = \mathcal{F}_{25}$.

Proof. (a) Let $\mathbf{A} \in \mathcal{A}_{23}$ and $a, b, c \in A$. We have that

$$\begin{aligned}
c' \to (a \to (b \to a)) &= a \to (c' \to (b \to a)) && \text{by Lemma 3.1 (a)} \\
&= a \to (b \to (c' \to a)) && \text{by Lemma 3.1 (a)} \\
&= a \to (b \to (a' \to c)) && \text{by Lemma 3.1 (b)} \\
&= a \to (a' \to (b \to c)). && \text{by Lemma 3.1 (a).}
\end{aligned}$$

Hence $\mathbf{A}$ satisfies the identity

$$z' \to (x \to (y \to x)) \approx x \to (x' \to (y \to z)). \tag{6}$$

Observe

$$\begin{array}{rcll}
[a \to (a \to b')']' & = & [(a \to b') \to a']' & \text{by Lemma 3.1 (b)} \\
& = & [(a'' \to a) \to (b' \to a')']'' & \text{by (I)} \\
& = & (a \to a) \to (b' \to a')' & \text{by } (I_{2,0}) \\
& = & (a \to a) \to ((b \to 0) \to a')' & \\
& = & (a \to a) \to [(a'' \to b) \to (0 \to a')']'' & \text{by (I)} \\
& = & (a \to a) \to [(a \to b) \to (0 \to a')'] & \text{by } (I_{2,0}) \\
& = & (a \to a) \to [(a \to b) \to a''] & \text{by Lemma 3.9 (a)} \\
& = & (a \to a) \to [(a \to b) \to a] & \\
& = & (a \to a) \to [(0 \to b) \to a] & \text{by Lemma 2.4 (c)} \\
& = & (a \to a) \to (b \to a) & \text{by Lemma 3.9 (a)} \\
& = & a \to ((a \to b) \to a) & \text{by } (A_{23}) \\
& = & a \to ((0 \to b) \to a) & \text{by Lemma 2.4 (c)} \\
& = & a \to (b \to a) & \text{by Lemma 3.9 (a).}
\end{array}$$

Hence, using (6), we have that **A** satisfies

$$z' \to [x \to (x \to y')']' \approx x \to (x' \to (y \to z)).$$

Therefore, by Lemma 3.9 (a),

$$\mathbf{A} \models z' \to [x \to (x \to y')']' \approx x \to ((0 \to x') \to (y \to z)). \tag{7}$$

From

$$\begin{array}{rcll}
a \to ((0 \to a') \to (b \to c)) & = & a \to ((a \to 0') \to (b \to c)) & \text{by Lemma 2.3 (b)} \\
& = & (a \to a) \to (b \to c) & \text{by Lemma 3.9 (b)} \\
& = & a \to ((a \to b) \to c) & \text{by } (A_{23}),
\end{array}$$

and the equation (7), we obtain

$$\mathbf{A} \models z' \to [x \to (x \to y')']' \approx x \to ((x \to y) \to z). \tag{8}$$

Finally,

$$\begin{array}{rcll}
a \to ((a \to b) \to c) & = & c' \to [a \to (a \to b')']' & \text{by (8)} \\
& = & c' \to [(a \to a) \to b]' & \text{by Lemma 3.9 (c)} \\
& = & [(a \to a) \to b] \to c & \text{by Lemma 3.1 (b),}
\end{array}$$

proving that the identity (A_{25}) holds in **A**. Therefore, we have that

$$\mathcal{A}_{23} \subseteq \mathcal{A}_{25}. \tag{9}$$

Let $\mathbf{A} \in \mathcal{A}_{25}$ and $a, b, c \in A$. Then

$$\begin{aligned}
((a \to b) \to b) \to c &= [((a \to b) \to b) \to c]'' \quad \text{by } (I_{2,0}) \\
&= [c' \to ((a \to b) \to b)']'' \quad \text{by Lemma 3.1 (b)} \\
&= [c' \to (b' \to (a \to b)')']'' \quad \text{by Lemma 3.1 (b)} \\
&= [c' \to (b \to (a \to b))]'' \quad \text{by Lemma 3.10 (1b)} \\
&= [c'' \to (b \to (a \to b))']' \quad \text{by Lemma 3.10 (1b)} \\
&= [c \to (b \to (a \to b))']' \quad \text{by } (I_{2,0}) \\
&= [c \to (b \to (b' \to a'))']' \quad \text{by Lemma 3.1 (b)} \\
&= [c \to (b' \to (b' \to a')')]' \quad \text{by Lemma 3.10 (1b)} \\
&= [c \to (b' \to (b \to a))]' \quad \text{by Lemma 3.10 (1b)} \\
&= [c \to (b \to (b' \to a))]' \quad \text{by Lemma 3.1 (a)} \\
&= [b \to ((b \to c) \to a)]' \quad \text{by Lemma 3.10 (2)} \\
&= [((b \to b) \to c) \to a]' \quad \text{by } (A_{25}) \\
&= [a' \to ((b \to b) \to c)']' \quad \text{by Lemma 3.1 (b)} \\
&= a \to ((b \to b) \to c) \quad \text{by Lemma 3.10 (1b).}
\end{aligned}$$

Therefore we have that

$$\mathcal{A}_{25} \subseteq \mathcal{C}_{25}. \tag{10}$$

Let $\mathbf{A} \in \mathcal{C}_{25}$ and $a, b, c \in A$. We have that

$$\begin{aligned}
((a \to b) \to c) \to a &= ((a \to b) \to c)'' \to a && \text{by } (I_{2,0}) \\
&= ((a \to b)' \to c')' \to a && \text{by Lemma 3.10 (1b)} \\
&= ((a' \to b') \to c')' \to a && \text{by Lemma 3.10 (1b)} \\
&= a' \to ((a' \to b') \to c') && \text{by Lemma 3.1 (b)} \\
&= a' \to (c \to (a' \to b')') && \text{by Lemma 3.1 (b)} \\
&= c \to (a' \to (a' \to b')') && \text{by Lemma 3.1 (a)} \\
&= c \to (a' \to (a \to b)) && \text{by Lemma 3.10 (1b).}
\end{aligned}$$

Hence,

$$\begin{aligned}
((a \to b) \to c) \to a &= c \to (a' \to (b' \to a')) && \text{by Lemma 3.1 (b)}\\
&= c \to (b' \to (a' \to a')) && \text{by Lemma 3.1 (a)}\\
&= c \to ((a' \to a')' \to b) && \text{by Lemma 3.1 (b)}\\
&= c \to ((a \to a) \to b) && \text{by Lemma 3.10 (1b)}\\
&= ((c \to a) \to a) \to b && \text{by } (C_{25})\\
&= b' \to ((c \to a) \to a)' && \text{by Lemma 3.1 (b)}\\
&= b' \to ((c \to a)' \to a') && \text{by Lemma 3.10 (1b)}\\
&= b' \to (a \to (c \to a)) && \text{by Lemma 3.1 (b)}\\
&= b' \to (c \to (a \to a)) && \text{by Lemma 3.1 (a)}\\
&= b' \to ((a \to a)' \to c') && \text{by Lemma 3.1 (b)}\\
&= (a \to a)' \to (b' \to c') && \text{by Lemma 3.1 (a)}\\
&= (b' \to c')' \to (a \to a) && \text{by Lemma 3.1 (b)}\\
&= (b \to c) \to (a \to a) && \text{by Lemma 3.10 (1b)}\\
&= a \to ((b \to c) \to a) && \text{by Lemma 3.1 (a).}
\end{aligned}$$

Thus,

$$\mathcal{C}_{25} \subseteq \mathcal{D}_{25}. \tag{11}$$

Let $\mathbf{A} \in \mathcal{D}_{25}$ and $a, b, c \in A$. Then

$$\begin{aligned}
((a \to b) \to c) \to b &= b' \to ((a \to b) \to c)' && \text{by Lemma 3.1 (b)}\\
&= b' \to (c' \to (a \to b)')' && \text{by Lemma 3.1 (b)}\\
&= b' \to (c' \to (a' \to b'))' && \text{by Lemma 3.10 (1b)}\\
&= b' \to (c \to (a' \to b')') && \text{by Lemma 3.10 (1b)}\\
&= b' \to (c \to (a \to b)) && \text{by Lemma 3.10 (1b)}
\end{aligned}$$

$$\begin{aligned}
((a \to b) \to c) \to b &= b' \to (c \to (b' \to a')) && \text{by Lemma 3.1 (b)}\\
&= b' \to (b' \to (c \to a')) && \text{by Lemma 3.1 (a)}\\
&= b' \to (c \to a') && \text{by Lemma 2.4 (e)}\\
&= b' \to (a \to c') && \text{by Lemma 3.1 (b)}\\
&= a \to (b' \to c') && \text{by Lemma 3.1 (a)}\\
&= a \to (b \to c)' && \text{by Lemma 3.10 (1b)}\\
&= (b \to c) \to a' && \text{by Lemma 3.1 (b)}\\
&= (b \to (b \to c)) \to a' && \text{by Lemma 2.4 (e)}\\
&= ((b \to c)' \to b') \to a' && \text{by Lemma 3.1 (b)}\\
&= ((b \to c) \to b)' \to a' && \text{by Lemma 3.10 (1b)}\\
&= a \to ((b \to c) \to b) && \text{by Lemma 3.1 (b).}
\end{aligned}$$

Consequently,

$$\mathcal{D}_{25} \subseteq \mathcal{E}_{25}. \tag{12}$$

Let $\mathbf{A} \in \mathcal{E}_{25}$ and $a, b, c \in A$. Then

$$\begin{array}{rcll}
(a \to a) \to (b \to c) & = & b \to ((a \to a) \to c) & \text{by Lemma 3.1 (a)} \\
& = & b \to (c' \to (a \to a)') & \text{by Lemma 3.1 (b)} \\
& = & b \to (c' \to (a' \to a')) & \text{by Lemma 3.10 (1b)} \\
& = & b \to (a' \to (c' \to a')) & \text{by Lemma 3.1 (a).} \\
& = & b \to ((c' \to a')' \to a) & \text{by Lemma 3.1 (b)} \\
& = & b \to ((c \to a) \to a) & \text{by Lemma 3.10 (1b)} \\
& = & ((b \to c) \to a) \to a & \text{by } (E_{25}) \\
& = & a' \to ((b \to c) \to a)' & \text{by Lemma 3.1 (b)} \\
& = & a' \to (a' \to (b \to c)')' & \text{by Lemma 3.1 (b)} \\
& = & a' \to (a' \to (b' \to c'))' & \text{by Lemma 3.10 (1b)} \\
& = & a' \to (a' \to (c \to b))' & \text{by Lemma 3.1 (b)} \\
& = & a' \to (a \to (c \to b)') & \text{by Lemma 3.10 (1b)} \\
& = & a \to (a' \to (c \to b)') & \text{by Lemma 3.1 (a)} \\
& = & a \to (a \to (c \to b))' & \text{by Lemma 3.10 (1b)} \\
& = & a \to (c \to (a \to b))' & \text{by Lemma 3.1 (a)} \\
& = & a \to (c \to (a' \to b')')' & \text{by Lemma 3.10 (1b)} \\
& = & a \to ((a' \to b') \to c')' & \text{by Lemma 3.1 (b)} \\
& = & a \to ((a \to b)' \to c')' & \text{by Lemma 3.10 (1b)} \\
& = & ((a \to b)' \to c') \to a' & \text{by Lemma 3.1 (b)} \\
& = & ((a \to b) \to c)' \to a' & \text{by Lemma 3.10 (1b)} \\
& = & a \to ((a \to b) \to c) & \text{by Lemma 3.1 (b).}
\end{array}$$

Therefore,

$$\mathcal{E}_{25} \subseteq \mathcal{A}_{23}. \tag{13}$$

From (9), (10), (11), (12) and (13), we conclude that

$$\mathcal{A}_{23} = \mathcal{A}_{25} = \mathcal{C}_{25} = \mathcal{D}_{25} = \mathcal{E}_{25}.$$

(b) Using Lemma 3.1 (a) we can easily verify that $\mathcal{A}_{12} = \mathcal{B}_{13}$ and $\mathcal{F}_{13} = \mathcal{D}_{12}$.

Let $\mathbf{A} \in \mathcal{A}_{12}$ and $a, b, c \in A$. Then

$$\begin{array}{rcll}
a \to (b \to (c \to a)) & = & b \to (a \to (c \to a)) & \text{by Lemma 3.1 (a)} \\
& = & b \to (c \to (a \to a)) & \text{by Lemma 3.1 (a)} \\
& = & b \to (c \to (0 \to a)) & \text{by Lemma 3.11 (a)} \\
& = & b \to (0 \to (c \to a)) & \text{by Lemma 3.1 (a)} \\
& = & 0 \to (b \to (c \to a)) & \text{by Lemma 3.1 (a)} \\
& = & 0 \to ((b \to c) \to a) & \text{by Lemma 3.11 (c)} \\
& = & (b \to c) \to (0 \to a) & \text{by Lemma 3.1 (a)} \\
& = & (b \to c) \to (a \to a) & \text{by Lemma 3.11 (a)} \\
& = & a \to ((b \to c) \to a) & \text{by Lemma 3.1 (a).}
\end{array}$$

Therefore,

$$\mathcal{A}_{12} \subseteq \mathcal{D}_{12}. \tag{14}$$

Let $\mathbf{A} \in \mathcal{D}_{12}$ and $a, b, c \in A$. Then

$$\begin{array}{cll}
& ((a \to b) \to c) \to a & \\
= & [(a' \to (a \to b)) \to (c \to a)']' & \text{by (I)} \\
= & [(a \to (a' \to b)) \to (c \to a)']' & \text{by Lemma 3.1 (a)} \\
= & [(a \to (b' \to a)) \to (c \to a)']' & \text{by Lemma 3.1 (b)} \\
= & [(b' \to (a \to a)) \to (c \to a)']' & \text{by Lemma 3.1 (a)} \\
= & [(b' \to (0 \to a)) \to (c \to a)']' & \text{by Lemma 3.11 (a)} \\
= & [(0 \to (b' \to a)) \to (c \to a)']' & \text{by Lemma 3.1 (a)} \\
= & [(0 \to (a' \to b)) \to (c \to a)']' & \text{by Lemma 3.1 (b)} \\
= & [(0 \to (0 \to (a' \to b))) \to (c \to a)']' & \text{by Lemma 2.4 (e)} \\
= & [((c \to a)' \to (0 \to (a' \to b))) \to (c \to a)']' & \text{by Lemma 2.4 (c)} \\
= & [(0 \to ((c \to a)' \to (a' \to b))) \to (c \to a)']' & \text{by Lemma 3.1 (a)}
\end{array}$$

and

$$
\begin{aligned}
& [(0 \to ((c \to a)' \to (a' \to b))) \to (c \to a)']' \\
= \; & [[(0 \to (c \to a)') \to (0 \to (a' \to b))] \to (c \to a)']' \\
& \text{by Lemma 2.4 (h) and (d)} \\
= \; & [[(0 \to (c \to a)) \to (0 \to (a' \to b))] \to (c \to a)']' && \text{by Lemma 3.11 (a)} \\
= \; & [[0 \to ((c \to a) \to (a' \to b))] \to (c \to a)']' \\
& \text{by Lemma 2.4 (h) and (d)} \\
= \; & [[0 \to (((c \to a) \to a') \to b)] \to (c \to a)']' && \text{by Lemma 3.11 (c)} \\
= \; & [[(0 \to ((c \to a) \to a')) \to (0 \to b)] \to (c \to a)']' \\
& \text{by Lemma 2.4 (h) and (d)} \\
= \; & [[(0 \to (a' \to (c \to a))) \to (0 \to b)] \to (c \to a)']' && \text{by Lemma 3.11 (b)} \\
= \; & [[(0 \to (c \to a)) \to (0 \to b)] \to (c \to a)']' && \text{by Lemma 2.4 (j)} \\
= \; & [[(0 \to (c \to a)') \to (0 \to b)] \to (c \to a)']' && \text{by Lemma 3.11 (a)} \\
= \; & [[0 \to ((c \to a)' \to b)] \to (c \to a)']' \\
& \text{by Lemma 2.4 (h) and (d)} \\
= \; & [[(c \to a)' \to (0 \to b)] \to (c \to a)']' && \text{by Lemma 3.1 (a)} \\
= \; & [[0 \to (0 \to b)] \to (c \to a)']' && \text{by Lemma 2.4 (c)} \\
= \; & [[0 \to (0 \to b)'] \to (c \to a)']' && \text{by Lemma 3.11 (a)} \\
= \; & [[(0 \to b) \to 0'] \to (c \to a)']' && \text{by Lemma 2.3 (b)} \\
= \; & [[(0 \to b) \to (c \to a)''] \to (c \to a)']' && \text{by Lemma 2.4 (a)} \\
= \; & [[(0 \to b) \to (c \to a)] \to (c \to a)']' && \text{by } (I_{2,0}) \\
= \; & [[(c \to a)' \to (0 \to b)'] \to (c \to a)']' && \text{by Lemma 3.1 (b)} \\
= \; & [[((c \to a) \to 0) \to (0 \to b)'] \to (c \to a)']' \\
= \; & [[((c \to a) \to 0) \to (0 \to b)'] \to (0' \to (c \to a))']' && \text{by Lemma 2.2 (a)} \\
= \; & [(0 \to b)' \to 0'] \to (c \to a) && \text{using (I)} \\
= \; & [0 \to (0 \to b)] \to (c \to a) && \text{by Lemma 2.3 (b)} \\
= \; & (0 \to b) \to (c \to a) && \text{by Lemma 2.4 (e)} \\
= \; & c \to ((0 \to b) \to a) && \text{by Lemma 3.1 (a)} \\
= \; & c \to ((a \to b) \to a) && \text{by Lemma 2.4 (c)} \\
= \; & (a \to b) \to (c \to a) && \text{by Lemma 3.1 (a).}
\end{aligned}
$$

Hence

$$\mathcal{D}_{12} \subseteq \mathcal{D}_{35}. \tag{15}$$

Next, let $\mathbf{A} \in \mathcal{D}_{35}$ and $a, b, c \in A$. Then

$$
\begin{aligned}
&a \to ((a \to b) \to c) \\
&\quad = (a \to b) \to (a \to c) && \text{by Lemma 3.1 (a)} \\
&\quad = [[(a \to c)' \to a] \to [b \to (a \to c)]']' && \text{by (I)} \\
&\quad = [[((a \to c) \to 0) \to a] \to [b \to (a \to c)]']' \\
&\quad = [[(a \to c) \to (0 \to a)] \to [b \to (a \to c)]']' && \text{by } (D_{35}) \\
&\quad = [[0 \to ((a \to c) \to a)] \to [b \to (a \to c)]']' && \text{by Lemma 3.1 (a)} \\
&\quad = [[0 \to ((0 \to c) \to a)] \to [b \to (a \to c)]']' && \text{by Lemma 2.4 (c)} \\
&\quad = [[(0 \to c) \to (0 \to a)] \to [b \to (a \to c)]']' && \text{by Lemma 3.1 (a)} \\
&\quad = [[0 \to (c \to a)] \to [b \to (a \to c)]']' \\
&\quad \text{by Lemma 2.4 (h) and (d)} \\
&\quad = [[0 \to (a \to c)] \to [b \to (a \to c)]']' && \text{by Lemma 3.11 (b)} \\
&\quad = [[(a \to c)' \to 0'] \to [b \to (a \to c)]']' && \text{by Lemma 2.3 (b)} \\
&\quad = (0' \to b) \to (a \to c) && \text{using (I)} \\
&\quad = b \to (a \to c) && \text{by Lemma 2.2 (a)} \\
&\quad = a \to (b \to c) && \text{by Lemma 3.1 (a)} \\
&\quad = a \to (a \to (b \to c)) && \text{by Lemma 2.4 (e).}
\end{aligned}
$$

As a consequence, we have that

$$\mathcal{D}_{35} \subseteq \mathcal{A}_{12}. \tag{16}$$

Therefore, using (14), (15) and (16) we get that

$$\mathcal{A}_{12} = \mathcal{D}_{12} = \mathcal{D}_{35}.$$

(c) Let $\mathbf{A} \in \mathcal{A}_{35}$ and $a, b, c \in A$. Then

$$
\begin{aligned}
((a \to b) \to c) \to c &= [(c' \to (a \to b)) \to (c \to c)']' && \text{by (I)} \\
&= [(c \to c) \to (c' \to (a \to b))']' && \text{by Lemma 3.1 (b)} \\
&= [(c \to c) \to (c' \to (a \to b))]'' && \text{by } (A_{35}) \\
&= (c \to c) \to (c' \to (a \to b)) && \text{by } (I_{2,0}) \\
&= (c \to c) \to (a \to (c' \to b)) && \text{by Lemma 3.1 (a)} \\
&= a \to ((c \to c) \to (c' \to b)) && \text{by Lemma 3.1 (a)} \\
&= a \to ((c \to c) \to (c' \to b)'') && \text{by } (I_{2,0}) \\
&= a \to ((c \to c) \to (c' \to b)')' && \text{by } (A_{35}) \\
&= a \to ((c' \to b) \to (c \to c)')' && \text{by Lemma 3.1 (b)} \\
&= a \to ((b \to c) \to c) && \text{by (I).}
\end{aligned}
$$

Then

$$\mathcal{A}_{35} \subseteq \mathcal{F}_{25}.$$

For the converse, suppose $\mathbf{A} \in \mathcal{F}_{25}$ and $a, b, c \in A$. Then we have that

$$\begin{aligned}
0 \to (a \to a) &= 0 \to ((0' \to a) \to a) && \text{by Lemma 2.2 (a)} \\
&= ((0 \to 0') \to a) \to a && \text{by } (F_{25}) \\
&= ((0'' \to 0') \to a) \to a && \\
&= a \to a && \text{by Lemma 2.2 (d).}
\end{aligned}$$

Hence,

$$\mathbf{A} \models 0 \to (x \to x) \approx x \to x.$$

Using Lemma 3.4 we have that $\mathbf{A} \in \mathcal{A}_{35}$.

Thus,

$$\mathcal{F}_{25} \subseteq \mathcal{A}_{35}.$$

This completes the proof. □

Since $\mathcal{SL}$ coincides with some of Bol-Moufang varieties, we regard it as a Bol-Moufang type variety.

We are now ready to present our main result of this paper.

THEOREM 4.2. *There are 4 nontrivial varieties of Bol-Moufang type that are distinct from each other: $\mathcal{SL}$, $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and $\mathcal{F}_{25}$; and they satisfy the following inclusions:*

(a) $\mathcal{SL} \subset \mathcal{A}_{23} \subset \mathcal{F}_{25}$,

(b) $\mathcal{SL} \subset \mathcal{A}_{12}$,

(c) $\mathcal{BA} \subset \mathcal{A}_{12} \subset \mathcal{F}_{25}$,

(d) $\mathcal{F}_{25} \subset \mathcal{I}_{2,0} \cap \mathcal{MC}$,

(e) $\mathcal{SL} = \mathcal{A}_{23} \cap \mathcal{A}_{12}$.

Proof. (a) By Theorem 3.6 we know that $\mathcal{SL} \subseteq \mathcal{A}_{23}$.

The following example shows that the inclusion is proper:

$\to$:	0	1	2	3
0	0	1	2	3
1	2	3	2	3
2	1	1	3	3
3	3	3	3	3

Next, we wish to show that $\mathcal{A}_{23} \subseteq \mathcal{F}_{25}$. So, let $\mathbf{A} \in \mathcal{A}_{23}$ and $a \in A$. Then

$$\begin{aligned}
0 \to (a \to a) &= (0 \to a) \to (0 \to a) && \text{by Lemma 2.4 (h) and (d)} \\
&= 0 \to ((0 \to a) \to a) && \text{by Lemma 3.1 (a)} \\
&= (0 \to 0) \to (a \to a) && \text{by } (A_{23}) \\
&= a \to a && \text{by Lemma 2.2 (a).}
\end{aligned}$$

By Lemma 3.4 and Theorem 4.1 (c), we have $\mathcal{A}_{23} \subseteq \mathcal{F}_{25}$.

The following example shows that the inclusion is proper:

$\to$:	0	1
0	1	1
1	0	1

(b) By Theorem 3.6 we know that $\mathcal{SL} \subseteq \mathcal{A}_{12}$.
The following example shows that the inclusion is proper:

$\to$:	0	1
0	1	1
1	0	1

(c) By Theorem 3.5 we know that $\mathcal{BA} \subseteq \mathcal{A}_{12}$.
The following example shows that the inclusion is proper:

$\to$:	0	1
0	0	1
1	1	1

To prove $\mathcal{A}_{12} \subseteq \mathcal{F}_{25}$, we let $\mathbf{A} \in \mathcal{A}_{12}$ and $a \in A$. Then

$$\begin{array}{rcll} 0 \to (a \to a) & = & (0 \to a) \to (0 \to a) & \text{by Lemma 2.4 (h) and (d)} \\ & = & 0 \to (0 \to a) & \text{by Lemma 3.11 (a)} \\ & = & 0 \to a & \text{by Lemma 2.4 (e)} \\ & = & a \to a & \text{by Lemma 3.11 (a).} \end{array}$$

By Lemma 3.4 and Theorem 4.1 (c), $\mathcal{A}_{12} \subseteq \mathcal{F}_{25}$.

The following example shows that the inclusion is proper:

$\to$:	0	1	2	3
0	0	1	2	3
1	2	3	2	3
2	1	1	3	3
3	3	3	3	3

(d) The following example shows that the inclusion is proper:

$\to$:	0	1	2
0	2	2	2
1	1	1	2
2	0	1	2

(e) Let $\mathbf{A} \in \mathcal{A}_{23} \cap \mathcal{A}_{12}$ and $a \in A$. Then

$$\begin{array}{rcll} a & = & a'' & \\ & = & a' \to 0 & \\ & = & 0' \to (a' \to 0) & \text{by Lemma 2.2 (a)} \\ & = & (0 \to 0) \to (a' \to 0) & \\ & = & 0 \to ((0 \to a') \to 0) & \text{by } (A_{23}) \\ & = & 0 \to (0 \to (a' \to 0)) & \text{by } (A_{12}) \\ & = & 0 \to (a' \to 0) & \text{by Lemma 2.4 (e)} \\ & = & 0 \to a'' & \\ & = & 0 \to a. & \end{array}$$

Hence

$$\mathbf{A} \models x \approx 0 \to x. \tag{17}$$

Consequently,

$$\mathbf{A} \models 0 \approx 0'. \tag{18}$$

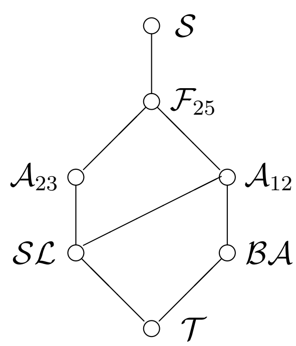
Therefore,

$$\begin{array}{rcll} a \to a & = & (a \to a)'' & \\ & = & [(a \to a) \to 0]' & \\ & = & [(a \to a) \to 0']' & \text{by (18)} \\ & = & [(a \to a) \to (0 \to 0)]' & \\ & = & [a \to ((a \to 0) \to 0)]' & \text{by } (A_{23}) \\ & = & [a \to (a \to (0 \to 0))]' & \text{by } (A_{12}) \\ & = & [a \to (0 \to 0)]' & \text{by Lemma 2.4 (e)} \\ & = & [a \to 0']' & \\ & = & [0 \to a']' & \text{by Lemma 2.3 (b)} \\ & = & [a']' & \text{by (17)} \\ & = & a. & \end{array}$$

Thus, $\mathbf{A} \models x \to x \approx x$. Hence, from Lemma 3.2 and Lemma 2.1, $\mathbf{A} \in \mathcal{SL}$. In view of Lemma 3.6 the proof is complete.

□

From the preceding theorem, it is easy to see that the poset (in fact, $\wedge$-semilattice) of varieties of Bol-Moufang type, together with the variety $\mathcal{BA}$ of Boolean algebras, under inclusion is as shown below:



5 Concluding Remarks

It is interesting to point out that our investigations into Bol-Moufang identities relative to $\mathcal{S}$, have revealed three new varieties, namely $\mathcal{A}_{12}$, $\mathcal{A}_{23}$ and

$\mathcal{F}_{25}$, hitherto unknown. We **conjecture** that the variety $\mathcal{A}_{23}$ covers $\mathcal{SL}$ in the lattice of subvarieties of Bol-Moufang type of $\mathcal{S}$.

In [CS16d], we investigate all the remaining weak associative laws of size ≤ 4 which will complement the results of this paper.

We conclude with the remark that it would be of interest to investigate the identities of Bol-Moufang type relative to other (important) subvarieties of $\mathcal{I}$ (see the Problem mentioned in the Introduction).

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